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Extremal odd unimodular lattices in dimensions 44, 46 and 47

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Abstract. In this note, extremal odd unimodular lattices in dimensions 44, 46 and 47 are constructed from self-dual codes over $\mathbb{Z}_4$ and $\mathbb{Z}_6$ by Construction A. The lattices in dimensions 46 and 47 seem to be the first explicit examples for such lattices.

Key words: unimodular lattices, self-dual codes, Construction A.

1. Introduction

A (Euclidean) lattice L is *integral* if $L \subseteq L^*$ where L^* is the dual lattice under the standard inner product $\langle x, y\rangle$. An integral lattice with $L = L^*$ is called *unimodular*. The minimum norm $\min(L)$ of L is the smallest norm among all nonzero vectors of L. Rains and Sloane [8] show that the minimum norm μ of an n-dimensional unimodular lattice is bounded by

$$\mu \leq 2\left[\frac{n}{24}\right] + 2 \tag{1}$$

unless $n = 23$ when $\mu \leq 3$. We say that an n-dimensional (odd) unimodular lattice meeting the bound is called *extremal*. It is a fundamental problem to determine if such a lattice exists for each dimension (cf. [3] and [9]). Conway and Sloane [3] gave the exact bound for the minimum norm of a unimodular lattice of dimensions up to 33. Their work is extended to dimensions 45 except 37, 41, 43 (cf. [9]).

In this note, extremal odd unimodular lattices in dimensions 44, 46 and 47 are constructed by Construction A from self-dual $\mathbb{Z}_6$-codes of length 44, $\mathbb{Z}_4$-codes of lengths 46, 47, respectively. These codes are obtained by considering subtracting from some known extremal self-dual codes of larger lengths. Our lattices in dimensions 46 and 47 seem to be the first explicit examples of extremal ones in these dimensions (cf. [9]).

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2. Definitions and basic facts

Let $\mathbb{Z}_{2k}$ $(= \{0, 1, 2, \ldots, 2k-1\})$ be the ring of integers modulo $2k$. A code C of length n over $\mathbb{Z}_{2k}$ (or a $\mathbb{Z}_{2k}$-code of length n) is a $\mathbb{Z}_{2k}$-submodule of $\mathbb{Z}_{2k}^n$. We define the inner product on $\mathbb{Z}_{2k}^n$ by $x \cdot y = x_1y_1 + \cdots + x_ny_n$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. The dual code $C^\perp$ of C is defined as $C^\perp = \{x \in \mathbb{Z}_{2k}^n \mid x \cdot y = 0 \text{ for all } y \in C\}$. A code C is *self-dual* if $C = C^\perp$. The Euclidean weight of a codeword x is $\sum_{i=1}^n \min\{x_i^2, (2k - x_i)^2\}$. The minimum Euclidean weight d_E of C is the smallest Euclidean weight among all nonzero codewords of C. If C is a self-dual $\mathbb{Z}_{2k}$-code with minimum Euclidean weight d_E then

$$A_{2k}(C) = \frac{1}{\sqrt{2k}}\{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \\ (x_1 \pmod{2k}, \ldots, x_n \pmod{2k}) \in C\},$$

is a unimodular lattice with minimum norm $\min\{d_E/2k, 2k\}$ (cf. [1]). $A_{2k}(C)$ is called the lattice constructed from C by Construction A.

Let L be an odd unimodular lattice and let L_0 denote its subset of even norms vectors. The set L_0 is a sublattice of L of index 2 in L [3]. Let L_2 be that unique nontrivial coset of L_0 into L. Then L_0^* can be written as a union of cosets of L_0: $L_0^* = L_0 \cup L_2 \cup L_1 \cup L_3$. The shadow lattice of L is defined to be $S = L_1 \cup L_3$ (cf. [3]).

3. The theta series of extremal odd unimodular lattices in dimensions 44, 46 and 47

We consider the theta series of extremal odd unimodular lattices in dimensions 44, 46 and 47. The theta series $\theta_L(q)$ of a lattice L is the formal power series $\theta_L(q) = \sum_{x \in L} q^{\langle x,x \rangle}$.

Conway and Sloane [3] show that if the theta series of an odd unimodular lattice L is written as

$$\theta_L(q) = \sum_{j=0}^{\lfloor n/8 \rfloor} a_j \theta_3(q)^{n-8j} \Delta_8(q)^j = \sum_i A_i q^i \quad \text{(say)} \tag{2}$$

then the theta series of the shadow lattice S is written as

$$\theta_S(q) = \sum_{j=0}^{\lfloor n/8 \rfloor} \frac{(-1)^j}{16^j} a_j \theta_2(q)^{n-8j} \theta_4(q^2)^{8j} = \sum_i B_i q^i \quad \text{(say)} \tag{3}$$

where $\Delta_8(q) = q\prod_{m=1}^{\infty}(1-q^{2m-1})^8(1-q^{4m})^8$ and $\theta_2(q)$, $\theta_3(q)$ and $\theta_4(q)$ are the Jacobi theta series [4]. As the additional conditions, we have that there is at most one nonzero B_r for $r < (\mu+2)/2$; $B_r = 0$ for $r < \mu/4$; and $B_r \leq 2$ for $r < \mu/2$ where μ is the minimum norm of L.

In the case $n = 46$, since $\mu = 4$, $a_0, \ldots, a_3$ in (2) and (3) are determined as follows: $a_0 = 1$, $a_1 = -92$, $a_2 = 2116$ and $a_3 = -9200$. By the additional condition, we have that $a_5 = 0$ or -2^{15}. Moreover, since the coefficients in the shadow lattice must be non-negative integers, a_4 is divisible by 4, so we put $a_4 = 4\beta$. Then we have the possible theta series $\theta_{46,i}(L)$ and $\theta_{46,i}(S)$ of an extremal odd unimodular lattice and its shadow lattice:

$$\begin{cases} \theta_{46,1}(L) = 1 + (2300+4\beta)q^4 + (582912-16\beta)q^5 \\ \qquad\qquad\qquad\qquad\qquad\qquad + (31905600-144\beta)q^6 + \cdots \\ \theta_{46,1}(S) = \beta q^{7/2} + (9420800-50\beta)q^{11/2} + \cdots \end{cases}$$

$$\begin{cases} \theta_{46,2}(L) = 1 + (2300+4\beta)q^4 + (550144-16\beta)q^5 \\ \qquad\qquad\qquad\qquad\qquad\qquad + (32823104-144\beta)q^6 + \cdots \\ \theta_{46,2}(S) = 2q^{3/2} + (-148+\beta)q^{7/2} + (9426110-50\beta)q^{11/2} + \cdots . \end{cases}$$

In the case $n = 47$, we have that $a_0 = 1$, $a_1 = -94$, $a_2 = 2256$, $a_3 = -10904$, $a_5 = 0$ or -2^{14} we can put $a_4 = 2\beta$ in (2) and (3). Hence we have the possible theta series $\theta_{47,i}(L)$ and $\theta_{47,i}(S)$ of a 47-dimensional extremal odd unimodular lattice and its shadow lattice:

$$\begin{cases} \theta_{47,1}(L) = 1 + (94+2\beta)q^4 + (484288-4\beta)q^5 \\ \qquad\qquad\qquad\qquad\qquad\qquad + (29111424-88\beta)q^6 + \cdots \\ \theta_{47,1}(S) = \beta q^{15/4} + (22331392-49\beta)q^{23/4} + \cdots \end{cases}$$

$$\begin{cases} \theta_{47,2}(L) = 1 + (94+2\beta)q^4 + (320448-4\beta)q^5 \\ \qquad\qquad\qquad\qquad\qquad\qquad + (33371264-88\beta)q^6 + \cdots \\ \theta_{47,2}(S) = 2q^{7/4} + (-146+\beta)q^{15/4} + (22336554-49\beta)q^{23/4} + \cdots . \end{cases}$$

Similarly to the cases $n = 46, 47$, we have the possible theta series $\theta_{44,i}(L)$ and $\theta_{44,i}(S)$ of a 44-dimensional extremal odd unimodular lattice and its shadow lattice:

$$\begin{cases} \theta_{44,1}(L) = 1 + (6600 + 16\beta)q^4 + (811008 - 128\beta)q^5 \\ \qquad\qquad\qquad\qquad\qquad + (37171200 - 128\beta)q^6 + \cdots \\ \theta_{44,1}(S) = \beta q^3 + (1622016 - 52\beta)q^5 + \cdots \end{cases}$$

$$\begin{cases} \theta_{44,2}(L) = 1 + (6600 + 16\beta)q^4 + (679936 - 128\beta)q^5 \\ \qquad\qquad\qquad\qquad\qquad + (41365504 - 128\beta)q^6 + \cdots \\ \theta_{44,2}(S) = 2q + (-152 + \beta)q^3 + (1627628 - 52\beta)q^5 + \cdots . \end{cases}$$

In this case, we have $a_0 = 1$, $a_1 = -88$, $a_2 = 1848$, $a_3 = -6336$, $a_5 = 0$ or -2^{17} and we can put $a_4 = 16\beta$ in (2) and (3).

Note that the theta series of an extremal odd unimodular lattice in dimension 44, 46 or 47 is uniquely determined by the numbers of vectors of norms 4 and 5.

4. An extremal odd unimodular lattice in dimensions 46 and 47

The cyclic $\mathbb{Z}_4$-code with generator polynomial

$$x^{23} + 2x^{21} + x^{19} + x^{18} + 2x^{16} + x^{14} + 3x^{13} + 3x^{12} + 2x^{11} \\ + 3x^{10} + 3x^9 + x^7 + 3x^6 + 3x^5 + 2x^4 + x^3 + x^2 + 3x + 3$$

is called the quadratic residue $\mathbb{Z}_4$-code of length 47 [2]. The extended quadratic residue $\mathbb{Z}_4$-code QR_{48} of length 48 which is obtained by adding 1's in the last coordinate of the generators is a self-dual code with minimum Euclidean weight 24 having Euclidean weights divisible by eight [6].

The following codes

$$\overline{QR_{48}} = \{(x_2, \ldots, x_{48}) \mid (x_1, x_2, \ldots, x_{48}) \in QR_{48},\ x_1 \in \{0, 2\}\}$$
$$\overline{(\overline{QR_{48}})} = \{(x_3, \ldots, x_{48}) \mid (x_1, x_2, \ldots, x_{48}) \in QR_{48},\ (x_1, x_2) \in C_2\}$$

are self-dual codes of lengths 47 and 46, respectively, where C_2 is the self-dual code of length 2. This construction are called subtracting. It is known that $PSL(2, 47)$ acts on the coordinates of QR_{48} (cf. [2] and [6]). Hence it is sufficient to consider codes subtracting the first coordinate x_1 and the first and second coordinates (x_1, x_2), respectively.

Since QR_{48} has minimum Euclidean weight 24, $\overline{QR_{48}}$ and $\overline{(\overline{QR_{48}})}$ are self-dual codes of lengths 47 and 46 with minimum Euclidean weight ≥ 20 and ≥ 16, respectively.

Hence the unimodular lattices $A_4(\overline{QR_{48}})$ and $A_4(\overline{(\overline{QR_{48}})})$ obtained by Construction A from $\overline{QR_{48}}$ and $\overline{(\overline{QR_{48}})}$ are extremal.

Proposition 1 *For dimensions* 46 *and* 47, *there is an extremal odd unimodular lattice.*

Remark The result stated in the above proposition was already announced in [9] (see [9] for explicit generator matrices of these lattices).

It is known that the minimum Euclidean weight d_E of a self-dual $\mathbb{Z}_4$-code of length n is upper bounded

$$d_E \leq 8[n/24] + 8 \tag{4}$$

except when $n \equiv 23 \pmod{24}$ in which case the bound is $d_E \leq 8[n/24] + 12$ [7, Theorem 35]. Thus $\overline{QR_{48}}$ and $\overline{(\overline{QR_{48}})}$ have minimum Euclidean weights 20 and 16, respectively.

Table 1. A_4 and A_5 in $A_4(\overline{QR_{48}})$ and $A_4(\overline{(\overline{QR_{48}})})$.

Lattices	A_4	A_5
$A_4(\overline{(\overline{QR_{48}})})$	2300	582912
$A_4(\overline{QR_{48}})$	94	484288

The numbers A_4 (resp. A_5) of vectors of norm 4 (resp. 5) in $A_4(\overline{QR_{48}})$ and $A_4(\overline{(\overline{QR_{48}})})$ are listed in Table 1. Since the minimum Euclidean weight of $\overline{QR_{48}}$ is 20, the vectors of norm 4 in $A_4(\overline{QR_{48}})$ are of the form $(\pm 2, 0^{46})$, that is, each vector contains only one ± 2 and 46 0's. Hence the kissing number is 94. The other numbers are calculated by MAGMA. Therefore $A_4(\overline{QR_{48}})$ has the theta series $\theta_{47,1}(L)$ with $\beta = 0$ and $A_4(\overline{(\overline{QR_{48}})})$ has the theta series $\theta_{46,1}(L)$ with $\beta = 0$.

5. Extremal odd unimodular lattices in dimension 44

The first example of extremal odd unimodular lattices in dimension 44 has been discovered in [5] and it is denoted by $L_T(T_{44})$ in [5]. In this section, we construct new lattices.

Let C be the extremal Type II $\mathbb{Z}_6$-code $C_{48,Q}$ of length 48 found in [5].

Note that C has a generator matrix of the form $\begin{pmatrix} I, & M \end{pmatrix}$. Let D be the self-dual code of length 4 with generator matrix $\begin{pmatrix} 1014 \\ 0145 \end{pmatrix}$. Let $1 \leq i_1 < i_2 < i_3 < i_4 \leq 24$. Define

$$C'^{(i_1,i_2,i_3,i_4)} = \{(x_1, x_2, \ldots, x_{48}) \in C \mid (x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) \in D\}.$$

Let $C^{(i_1,i_2,i_3,i_4)}$ be the code of length 44 obtained from $C'^{(i_1,i_2,i_3,i_4)}$ deleting the four coordinates i_1, i_2, i_3, i_4. It is easy to see that $C^{(i_1,i_2,i_3,i_4)}$ is a self-dual code.

Here we investigate self-dual codes $C^{(1,2,3,i)}$ for the case $i = 4, \ldots, 24$.

Table 2. Extremal odd unimodular lattices in dimension 44.

Lattices	A_4	A_5	θ_L
$A_6(C^{(1,2,3,4)})$	8104	798976	$\theta_{44,1}(L)$ $(\beta = 94)$
$A_6(C^{(1,2,3,5)})$	8232	797952	$\theta_{44,1}(L)$ $(\beta = 102)$
$A_6(C^{(1,2,3,6)})$	8232	797952	$\theta_{44,1}(L)$ $(\beta = 102)$
$A_6(C^{(1,2,3,7)})$	8104	798976	$\theta_{44,1}(L)$ $(\beta = 94)$
$A_6(C^{(1,2,3,8)})$	8232	797952	$\theta_{44,1}(L)$ $(\beta = 102)$
$A_6(C^{(1,2,3,9)})$	8072	799232	$\theta_{44,1}(L)$ $(\beta = 92)$
$A_6(C^{(1,2,3,10)})$	8200	798208	$\theta_{44,1}(L)$ $(\beta = 100)$
$A_6(C^{(1,2,3,11)})$	8264	797696	$\theta_{44,1}(L)$ $(\beta = 104)$
$A_6(C^{(1,2,3,12)})$	8232	797952	$\theta_{44,1}(L)$ $(\beta = 102)$
$A_6(C^{(1,2,3,13)})$	8232	797952	$\theta_{44,1}(L)$ $(\beta = 102)$
$A_6(C^{(1,2,3,14)})$	8136	798720	$\theta_{44,1}(L)$ $(\beta = 96)$
$A_6(C^{(1,2,3,15)})$	8168	798464	$\theta_{44,1}(L)$ $(\beta = 98)$
$A_6(C^{(1,2,3,16)})$	8104	798976	$\theta_{44,1}(L)$ $(\beta = 94)$
$A_6(C^{(1,2,3,17)})$	8168	798464	$\theta_{44,1}(L)$ $(\beta = 98)$
$A_6(C^{(1,2,3,18)})$	8136	798720	$\theta_{44,1}(L)$ $(\beta = 96)$
$A_6(C^{(1,2,3,19)})$	8168	798464	$\theta_{44,1}(L)$ $(\beta = 98)$
$A_6(C^{(1,2,3,20)})$	8168	798464	$\theta_{44,1}(L)$ $(\beta = 98)$
$A_6(C^{(1,2,3,21)})$	8232	797952	$\theta_{44,1}(L)$ $(\beta = 102)$
$A_6(C^{(1,2,3,22)})$	8168	798464	$\theta_{44,1}(L)$ $(\beta = 98)$
$A_6(C^{(1,2,3,23)})$	8136	798720	$\theta_{44,1}(L)$ $(\beta = 96)$
$A_6(C^{(1,2,3,24)})$	8136	798720	$\theta_{44,1}(L)$ $(\beta = 96)$
$L_T(T_{44})$ in [5]	8104	798976	$\theta_{44,1}(L)$ $(\beta = 94)$