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Local existence and uniqueness for the n-dimensional Helfrich flow as a projected gradient flow

Takeyuki Nagasawa and Taekyung Yi

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Abstract. The gradient flow associated to the Helfrich variational problem, called the Helfrich flow is considered. Here the n-dimensional Helfrich flow is investigated for any n, as a projected gradient flow. A result of local existence is proved. The uniqueness is shown for the cases (i) for the initial hypersurface with non-zero Gramian when $n \geq 2$, (ii) for every initial curve when $n = 1$.

Key words: Helfrich variational problem, gradient flow, constraints.

1. Introduction

Let Σ be a compact closed immersed orientable hypersurface in $\mathbb{R}^{n+1}$. The vectors $\boldsymbol{f}$ and $\boldsymbol{\nu}$ are the position vector of a point on Σ and the unit normal vector there respectively. We denote the mean curvature H, and dS stands for surface element. Functionals $\mathcal{W}, \mathcal{A}$ and $\mathcal{V}$ are defined by

$$\mathcal{W}(\Sigma) = \frac{n}{2}\int_\Sigma (H - c_0)^2 dS, \quad \mathcal{A}(\Sigma) = \int_\Sigma dS, \quad \mathcal{V}(\Sigma) = -\frac{1}{n+1}\int_\Sigma \boldsymbol{f}\cdot\boldsymbol{\nu} dS.$$

Here, c_0 is a given constant. $\mathcal{A}(\Sigma)$ is the area of Σ. $\mathcal{V}(\Sigma)$ is the enclosed volume, when Σ is an embedded hypersurface and $\boldsymbol{\nu}$ is the inner normal.

For given constants $\mathcal{A}_0$ and $\mathcal{V}_0$, consider critical points of $\mathcal{W}(\cdot)$ under the constrains $\mathcal{A}(\Sigma) = \mathcal{A}_0$, $\mathcal{V}(\Sigma) = \mathcal{V}_0$. This problem is called the **Helfrich variational problem**. This problem was firstly proposed by Helfrich [5] as a model of shape transformation theory of human red blood cells. For this case $n = 2$, and c_0 is the spontaneous curvature which is determined by the molecular structure of cell membrane. The surface Σ stands for the cell membrane.

For $n = 1$, the functional $\mathcal{W}$ is

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$$\frac{1}{2}\int_{\Sigma} H^2 dS - c_0 \int_{\Sigma} H ds + \frac{1}{2} c_0^2 \mathcal{A}(\Sigma).$$

If we consider the variational problem under the constrain of length $\mathcal{A}$ among curves with fixed rotation number, then we can replace the functional with the first integral $\frac{1}{2}\int_{\Sigma} H^2 dS$. Because the second and third integrals are respectively constant multiples of rotation number and the length, which are invariant for our problem. According to [2], a shape transformation of a closed loop of plastic tape between two parallel flat plates is governed by the one-dimensional Helfrich variational problem. This problem is also related with the spectral optimization problem for plain domains. Let Ω be a bounded plane domain, and Σ be its boundary. The function $G(x,y,t)$ is the Green function for the heat equation in $\Omega \times (0,T)$ under the Dirichlet condition. The asymptotic expansion

$$\int_{\Omega} G(x,x,t)dx = \frac{1}{4\pi t}\left(a_0 + a_1 t^{1/2} + a_2 t + a_3 t^{3/2} + \cdots\right) \quad \text{as} \quad t \to +0$$

is well-known as the trace formula. Here

$$a_0 = \mathcal{V}(\Sigma), \quad a_1 = -\frac{\sqrt{\pi}}{2}\mathcal{A}(\Sigma), \quad a_2 = \frac{1}{3}\int_{\Sigma} H dS \quad a_3 = \frac{\sqrt{\pi}}{64}\int_{\Sigma} H^2 dS.$$

a_2 is determined by the topology of Ω. Hence the one-dimensional Helfrich problem is equivalent to the following problem: For given a_0, a_1 and a_2 find the domain Ω which minimize a_3. This problem was proposed and investigated by Watanabe [11], [12].

In this paper, we consider the associated gradient flow. Let $\{\Sigma(t)\}_{t\geq 0}$ be one-parameter family of hypersurfaces, and let V be the normal velocity of deformation. The equation of flow is

$$V(t) = -\delta\mathcal{W}(\Sigma(t)) - \lambda_1(\Sigma(t))\delta\mathcal{A}(\Sigma(t)) - \lambda_2(\Sigma(t))\delta\mathcal{V}(\Sigma(t)). \tag{1.1}$$

A solution is called the **Helfrich flow**. Here δ means the first variation, and λ_j's are Lagrange multipliers. The multipliers are unknown functions determined from the solution itself. It is natural that they are determined so that $\mathcal{A}(\Sigma(t)) \equiv \mathcal{A}_0$, $\mathcal{V}(\Sigma(t)) \equiv \mathcal{V}_0$. Let $\langle \cdot, \cdot \rangle$ denote the $L^2(\Sigma)$-inner product. Since

$$\frac{d}{dt}\mathcal{A}(\Sigma(t)) = \langle \delta\mathcal{A}(\Sigma(t)), V(t)\rangle, \quad \frac{d}{dt}\mathcal{V}(\Sigma(t)) = \langle \delta\mathcal{V}(\Sigma(t)), V(t)\rangle,$$

we obtain

$$\begin{pmatrix} \langle \delta\mathcal{A}(\Sigma(t)), \delta\mathcal{A}(\Sigma(t))\rangle & \langle \delta\mathcal{V}(\Sigma(t)), \delta\mathcal{A}(\Sigma(t))\rangle \\ \langle \delta\mathcal{A}(\Sigma(t)), \delta\mathcal{V}(\Sigma(t))\rangle & \langle \delta\mathcal{V}(\Sigma(t)), \delta\mathcal{V}(\Sigma(t))\rangle \end{pmatrix} \begin{pmatrix} \lambda_1(\Sigma(t)) \\ \lambda_2(\Sigma(t)) \end{pmatrix}$$
$$= - \begin{pmatrix} \langle \delta\mathcal{A}(\Sigma(t)), \delta\mathcal{W}(\Sigma(t))\rangle \\ \langle \delta\mathcal{V}(\Sigma(t)), \delta\mathcal{W}(\Sigma(t))\rangle \end{pmatrix} \tag{1.2}$$

by calculating the product of (1.1) with $\delta\mathcal{A}(\Sigma(t))$ and $\delta\mathcal{V}(\Sigma(t))$. Put

$$G(\Sigma(t)) = \det \begin{pmatrix} \langle \delta\mathcal{A}(\Sigma(t)), \delta\mathcal{A}(\Sigma(t))\rangle & \langle \delta\mathcal{V}(\Sigma(t)), \delta\mathcal{A}(\Sigma(t))\rangle \\ \langle \delta\mathcal{A}(\Sigma(t)), \delta\mathcal{V}(\Sigma(t))\rangle & \langle \delta\mathcal{V}(\Sigma(t)), \delta\mathcal{V}(\Sigma(t))\rangle \end{pmatrix}.$$

This is a Gramian of $\delta\mathcal{A}(\Sigma(t))$ and $\delta\mathcal{V}(\Sigma(t))$. When $G(\Sigma(t)) \neq 0$, the multipliers $\lambda_j(\Sigma)$ are uniquely determined from $\Sigma(t)$, and the equation is settled. When $G(\Sigma(t)) = 0$, they are not uniquely determined, but we can show that the linear combination $\lambda_1(\Sigma(t))\delta\mathcal{A}(\Sigma(t)) + \lambda_2(\Sigma(t))\delta\mathcal{V}(\Sigma(t))$ is uniquely determined. As a result, we have the following.

Theorem 1.1 *Let $P(\Sigma(t))$ be the orthogonal projection from $L^2(\Sigma(t))$ to $(\mathrm{span}_{L^2(\Sigma(t))}\{\delta\mathcal{A}(\Sigma(t)), \delta\mathcal{V}(\Sigma(t))\})^{\perp}$. Then the equation of Helfrich flow can be written as*

$$V(t) = -P(\Sigma(t))\delta\mathcal{W}(\Sigma(t)) \quad \textit{for} \quad t > 0. \tag{1.3}$$

Solutions of the equation satisfy

$$\frac{d}{dt}\mathcal{W}(\Sigma(t)) \equiv -\|V(t)\|^2_{L^2(\Sigma(t))}, \quad \frac{d}{dt}\mathcal{A}(\Sigma(t)) \equiv 0, \quad \frac{d}{dt}\mathcal{V}(\Sigma(t)) \equiv 0. \tag{1.4}$$

In Section 3, we shall give its proof.

We get a result on the existence and uniqueness of the initial value problem for the equation in Theorem 1.1. Let h^{α} be the little Hölder space.

Theorem 1.2

(i) *Assume that Σ_0 is in the class $h^{3+\alpha} (0 < \alpha < 1)$, and that $G(\Sigma_0) \neq 0$. Then there exists $T > 0$ such that there uniquely exists the solution*

$\{\Sigma(t)\}_{0\le t<T}$ of (1.3) satisfying $\Sigma(0)=\Sigma_0$.

(ii) *Assume that $G(\Sigma_0)=0$. Let H_0 and R_0 be the mean curvature and the scalar curvature of Σ_0 respectively. Put*

$$\overline{H_0}=\frac{1}{A_0}\int_{\Sigma_0}H_0dS,\quad \tilde{R_0}=R_0-\frac{1}{A_0}\int_{\Sigma_0}R_0dS.$$

If $(\overline{H_0}-c_0)\tilde{R_0}\equiv 0$, then there exists a global solution $\{\Sigma(t)\}_{t\ge 0}$ of (1.3) satisfying $\Sigma(0)=\Sigma_0$.

Remark 1.1 For (ii), we do not know uniqueness of solutions for $n\ge 2$. When $n=1$, however, the uniqueness holds. See Theorem 4.1.

The low-dimensional Helfrich flow has been considered in [6] (for $n=2$) and in [7] (for $n=1$).

In [6], λ_1,λ_2 are not determined as above, but given as known constants. That is, for given $\{\lambda_1,\lambda_2,\Sigma_0\}$ as the data, solutions of (1.1) with $\Sigma(0)=\Sigma_0$ were constructed. Of course, solutions do not satisfy $\frac{d}{dt}\mathcal{A}(\Sigma(t))\equiv 0$, $\frac{d}{dt}\mathcal{V}(\Sigma(t))\equiv 0$, and we cannot expect the global existence. Indeed, there exist solutions blowing up in finite/infinite time. The problem is shifted to find triples $\{\lambda_1,\lambda_2,\Sigma_0\}$ so that the solution can extend globally in time. In [6], the existence of such triples was shown near spheres. Furthermore, such triples form a finite dimensional center manifold. The class of initial surfaces is $h^{2+\alpha}$ for some $\alpha\in(0,1)$, which is wider than ours. In our formulation $\nabla_g H$ appears in the explicit expression of λ_1, λ_2 and therefore we need extra regularity of Σ_0 than [6].

In [7], we did not treat (1.1) (or (1.2)) directly. The gradient flow $\{\Sigma(\varepsilon,t)\}$ associated with the functional

$$\mathcal{W}(\Sigma)+\frac{1}{2\varepsilon}(\mathcal{A}(\Sigma)-\mathcal{A}_0)^2+\frac{1}{2\varepsilon}(\mathcal{V}(\Sigma)-\mathcal{V}_0)^2\quad(\varepsilon>0)$$

was constructed. The solution of (1.1) was obtained as the limit of $\{\Sigma(\varepsilon,t)\}$ as $\varepsilon\to+0$. This is a global solution, and satisfies (1.3). The class of initial curve is C^∞, but the uniqueness was uncertain.

This paper consists four sections. Following Introduction we calculate the first variation of the functional and we express (1.1) with geometrical quantity of $\Sigma(t)$ in Section 2. In Section 3, we show Theorem 1.1. In Section

4, following the method of [6], we regard $\Sigma(t)$ as the perturbation of Σ_0 in normal direction with $\rho(t)$, and using $\rho(t)$, we write down (1.1). Using theory of quasi-linear parabolic equations [1], we shall show Theorem 1.2.

2. The derivation of equation

In this section, we write down (1.1) explicitly in terms of geometrical quantities of $\Sigma(t)$. To do this, we need the first variation formulas of $\mathcal{W}$, $\mathcal{A}$ and $\mathcal{V}$. Those of $\mathcal{A}$ and $\mathcal{V}$ are well-known. That of $\mathcal{W}$ is essentially found in [3], however, we give it here again. Let

$$\Sigma = \left\{ \boldsymbol{f} = \boldsymbol{f}(s^1, \dots, s^n) \in \mathbb{R}^{n+1} \mid (s^1, \dots, s^n) \text{ is a local coordinate system} \right\}$$

be a hypersurface. Let $\boldsymbol{\nu}$ denote the unit normal vector field on Σ.

The vector $\boldsymbol{\nu}$ is given by

$$\boldsymbol{\nu} = \frac{\boldsymbol{f}_1 \wedge \boldsymbol{f}_2 \wedge \cdots \wedge \boldsymbol{f}_n}{\|\boldsymbol{f}_1 \wedge \boldsymbol{f}_2 \wedge \cdots \wedge \boldsymbol{f}_n\|}, \quad \boldsymbol{f}_i = \frac{\partial \boldsymbol{f}}{\partial s^i}. \tag{2.1}$$

Put

$$g_{ij} = \boldsymbol{f}_i \cdot \boldsymbol{f}_j, \quad g = \det(g_{ij}), \quad \boldsymbol{\nu}_i = \frac{\partial \boldsymbol{\nu}}{\partial s^i}.$$

It is easy to see

$$\boldsymbol{f}_i \cdot \boldsymbol{\nu} = \boldsymbol{f}_j \cdot \boldsymbol{\nu} = \boldsymbol{\nu} \cdot \boldsymbol{\nu}_i = \boldsymbol{f} \cdot \boldsymbol{\nu}_j = 0, \quad \|\boldsymbol{f}_1 \wedge \boldsymbol{f}_2 \wedge \cdots \wedge \boldsymbol{f}_n\| = \sqrt{g}. \tag{2.2}$$

The first fundamental form is given by

$$\mathrm{I} = d\boldsymbol{f} \cdot d\boldsymbol{f} = g_{ij} ds^i ds^j. \tag{2.3}$$

Put

$$\mathrm{II} = -d\boldsymbol{\nu} \cdot d\boldsymbol{f} = \boldsymbol{\nu} \cdot d^2\boldsymbol{f} = h_{ij} ds^i ds^j, \quad h_{ij} = -\boldsymbol{\nu}_i \cdot \boldsymbol{f}_j = -\boldsymbol{\nu}_j \cdot \boldsymbol{f}_i, \tag{2.4}$$

which is the second fundamental form. Let (g^{ij}) denote the inverse matrix of (g_{ij}). The mean curvature and the surface element are given by

$$H = \frac{1}{n} g^{ij} h_{ij}, \tag{2.5}$$

$$dS = \sqrt{g}\, ds^1 \cdots ds^n. \tag{2.6}$$

By (2.2)–(2.4), we have $\boldsymbol{f}_{ij} \cdot \boldsymbol{\nu} = -\boldsymbol{f}_i \cdot \boldsymbol{\nu}_j = h_{ij}$, and

$$\boldsymbol{f}_{ij} = \frac{\partial^2 \boldsymbol{f}}{\partial s^i \partial s^j} = \Gamma^k_{ij} \boldsymbol{f}_k + h_{ij} \boldsymbol{\nu}, \tag{2.7}$$

where

$$\Gamma^k_{i\ell} = \frac{g^{kj}}{2} \left(\frac{\partial g_{ij}}{\partial s^\ell} + \frac{\partial g_{j\ell}}{\partial s^i} - \frac{\partial g_{i\ell}}{\partial s^j} \right)$$

is called the Christoffel symbol. By the Weingarten equation

$$\boldsymbol{\nu}_i = -h^j_i \boldsymbol{f}_j, \quad h^j_i = g^{jk} h_{ki}, \tag{2.8}$$

we obtain

$$\boldsymbol{\nu}_i \cdot \boldsymbol{\nu}_j = h^k_i h^l_j \boldsymbol{f}_k \cdot \boldsymbol{f}_l = h^k_i h^l_j g_{kl} = h^k_i h_{jk}.$$

For a smooth function φ on Σ, consider the normal variation

$$\Sigma_t = \{ \boldsymbol{f}(t) = \boldsymbol{f} + t\varphi\boldsymbol{\nu} \in \mathbb{R}^{n+1} \}.$$

If $|t|$ is sufficiently small, Σ_t becomes a hypersurface. The first variation $\delta\mathcal{F}$ of functional $\mathcal{F}$ to the direction φ is given by

$$\langle \delta\mathcal{F}(\Sigma), \varphi \rangle = \frac{d}{dt} \mathcal{F}(\Sigma_t) \bigg|_{t=0}.$$

If $\langle \delta\mathcal{F}(\Sigma), \varphi \rangle = 0$ for arbitrary φ, we write $\delta\mathcal{F}(\Sigma) = 0$ and Σ is called critical. We calculate the first variation concretely here. We use the notation δ not only for functionals but also for geometrical quantities to mean $\frac{d}{dt}|_{t=0}$. Then we obtain

$$\delta \boldsymbol{f} = \varphi \boldsymbol{\nu}, \quad \delta \boldsymbol{f}_i = \varphi_i \boldsymbol{\nu} + \varphi \boldsymbol{\nu}_i \tag{2.9}$$

$$\delta g_{ij} = -2\varphi h_{ij}, \quad \delta g^{ij} = 2\varphi g^{ik} h_k^j, \tag{2.10}$$

$$\delta\sqrt{g} = -n\varphi H\sqrt{g}. \tag{2.11}$$

By (2.7) and (2.8), we get

$$\begin{aligned}
\delta \boldsymbol{f}_{ij} &= \varphi_{ij}\boldsymbol{\nu} + \varphi_i\boldsymbol{\nu}_j + \varphi_j\boldsymbol{\nu}_i + \varphi\boldsymbol{\nu}_{ij} \\
&= \varphi_{ij}\boldsymbol{\nu} + \varphi_i\boldsymbol{\nu}_j + \varphi_j\boldsymbol{\nu}_i - \varphi\big\{\big(h_i^k\big)_j \boldsymbol{f}_k + h_i^k \boldsymbol{f}_{kj}\big\} \\
&= \varphi_{ij}\boldsymbol{\nu} + \varphi_i\boldsymbol{\nu}_j + \varphi_j\boldsymbol{\nu}_i - \varphi\big\{\big(h_i^k\big)_j \boldsymbol{f}_k + h_i^k \big(\Gamma_{kj}^{\ell}\boldsymbol{f}_\ell + h_{kj}\boldsymbol{\nu}\big)\big\}.
\end{aligned} \tag{2.12}$$

Using (2.2), we obtain

$$\boldsymbol{\nu}\cdot\delta\boldsymbol{f}_{ij} = \varphi_{ij} - \varphi h_i^k h_{kj}. \tag{2.13}$$

Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n, \boldsymbol{u}_{n+1}$ be vectors in $\mathbb{R}^{n+1}$. The scalar product of the vector $\boldsymbol{u}_{n+1}$ and the vector $\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_n$ are given by

$$\boldsymbol{u}_{n+1}\cdot\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_n = \det(\boldsymbol{u}_1, \dots, \boldsymbol{u}_n, \boldsymbol{u}_{n+1}). \tag{2.14}$$

It follows from (2.1), (2.5), (2.7), (2.8), (2.9), (2.11), and (2.14) that

$$\boldsymbol{f}_{ij}\cdot\delta\boldsymbol{\nu} = -\varphi_k\Gamma_{ij}^k. \tag{2.15}$$

Therefore, by (2.13) and (2.15), we obtain

$$\delta h_{ij} = \varphi_{ij} - \varphi_k\Gamma_{ij}^k - \varphi h_i^k h_{kj} = \nabla_i\varphi_j - \varphi h_i^k h_{kj}. \tag{2.16}$$

Here, $\nabla_i\varphi_j = \varphi_{ij} - \varphi_k\Gamma_{ij}^k$ is the convariant derivative of φ_j. By direct computation together with (2.13) and (2.16), we obtain

$$n(\delta H) = \Delta_g\varphi + \varphi h_j^i h_i^j. \tag{2.17}$$

Here, Δ_g is the Laplacian-Beltrami operator defined by

$$\begin{aligned}
\Delta_g\varphi &= g^{i\ell}\nabla_i\varphi_\ell = g^{i\ell}\varphi_{i\ell} - g^{i\ell}\Gamma_{i\ell}^k\varphi_k \\
&= g^{ij}\varphi_{ij} + \frac{1}{\sqrt{g}}\big(\sqrt{g}g^{kj}\big)_j\varphi_k = \frac{1}{\sqrt{g}}\big(\sqrt{g}g^{ij}\varphi_i\big)_j.
\end{aligned}$$

The scalar curvature R is given by

$$R = n^2 H^2 - h^i_j h^j_i. \tag{2.18}$$

Combining (2.17) and (2.18), we obtain

$$n(\delta H) = \Delta_g \varphi + (n^2 H^2 - R)\varphi. \tag{2.19}$$

Put $\mathcal{W}_p(\Sigma) = \int_\Sigma H^p dS$. Thus by using (2.6), (2.11) and (2.19), we can prove that

$$\delta \mathcal{W}_p(\Sigma)[\varphi] = \int_\Sigma \left[\frac{p}{n} H^{p-1} \Delta_g \varphi + \left\{ n(p-1) H^{p+1} - \frac{p}{n} H^{p-1} R \right\} \varphi \right] dS. \tag{2.20}$$

When Σ is closed, using integration by parts, we obtain from (2.20)

$$\delta \mathcal{W}_p(\Sigma)[\varphi] = \int_\Sigma \left\{ \frac{p}{n} \Delta_g H^{p-1} + n(p-1) H^{p+1} - \frac{p}{n} H^{p-1} R \right\} \varphi dS. \tag{2.21}$$

Since

$$\mathcal{W}(\Sigma) = \frac{n}{2} \int_\Sigma \big(H^2 - 2c_0 H + {c_0}^2\big) dS = \frac{n}{2} \big(\mathcal{W}_2(\Sigma) - 2c_0 \mathcal{W}_1(\Sigma) + {c_0}^2 \mathcal{W}_0(\Sigma)\big),$$

we obtain

$$\delta \mathcal{W}(\Sigma)[\varphi] = \int_\Sigma \left(\Delta_g H + \frac{n^2}{2} H^3 - HR + c_0 R - \frac{n^2}{2} {c_0}^2 H \right) \varphi dS.$$

As well known, we have

$$\delta \mathcal{A}(\Sigma)[\varphi] = -\int_\Sigma nH\varphi dS, \quad \delta \mathcal{V}(\Sigma)[\varphi] = -\int_\Sigma \varphi dS.$$

As a result the equation (1.1) of Helfrich flow becomes

$$\begin{aligned} V(t) = {} & -\Delta_{g(t)} H(t) - \frac{n^2}{2} H^3(t) + H(t)R(t) - c_0 R(t) + \frac{n^2}{2} {c_0}^2 H(t) \\ & + \lambda_1(\Sigma(t)) nH(t) + \lambda_2(\Sigma(t)). \end{aligned} \tag{2.22}$$

3. The Helfrich flow as a projected gradient flow

In this section, we show the following.

Theorem 3.1 *If $\lambda_1(\Sigma(t))$ and $\lambda_2(\Sigma(t))$ are determined so that $\frac{d}{dt}\mathcal{A}(\Sigma(t)) \equiv 0$, $\frac{d}{dt}\mathcal{V}(\Sigma(t)) \equiv 0$ in the equation* (1.1) *of Helfrich flow, then it can be written as*

$$V(t) = -P(\Sigma(t))\delta\mathcal{W}(\Sigma(t)) \quad (t > 0). \tag{3.1}$$

Here $P(\Sigma(t))$ is the orthogonal projection from $L^2(\Sigma(t))$ to the subspace $(\mathrm{span}_{L^2(\Sigma(t))}\{\delta\mathcal{A}(\Sigma(t)), \delta\mathcal{V}(\Sigma(t))\})^{\perp}$.

Conversely solutions to (3.1)*, if exist, satisfy*

$$\frac{d}{dt}\mathcal{W}(\Sigma(t)) \equiv -\|V(t)\|^2_{L^2(\Sigma(t))}, \quad \frac{d}{dt}\mathcal{A}(\Sigma(t)) \equiv 0, \quad \frac{d}{dt}\mathcal{V}(\Sigma(t)) \equiv 0. \tag{3.2}$$

Proof. The following is a special case of theory of projected gradient flows [9]. We denote $V(t), \Sigma(t)$ simply by V, Σ respectively. $\|\cdot\|$ stands for the $L^2(\Sigma)$-norm. Put

$$\tilde{H} = H - \frac{1}{A}\int_\Sigma H dS, \quad H_* = \begin{cases} \frac{\tilde{H}}{\|\tilde{H}\|} & (\tilde{H} \not\equiv 0) \\ 0 & (\tilde{H} \equiv 0) \end{cases}, \quad 1_* = \frac{1}{\|1\|}.$$

Note that $\langle H_*, 1_*\rangle = 0$. Since $\delta\mathcal{A}(\Sigma) = -nH$, $\delta\mathcal{V}(\Sigma) = -1$, we have

$$\mathrm{span}_{L^2(\Sigma)}\{\delta\mathcal{A}(\Sigma), \delta\mathcal{V}(\Sigma)\} = \mathrm{span}_{L^2(\Sigma)}\{H, 1\} = \mathrm{span}_{L^2(\Sigma)}\{H_*, 1_*\}.$$

Hence the equation (1.1) becomes

$$V = -\delta\mathcal{W}(\Sigma) - \lambda_1\delta\mathcal{A}(\Sigma) - \lambda_2\delta\mathcal{V}(\Sigma) = -\delta\mathcal{W}(\Sigma) - \mu_1 1_* - \mu_2 H_* \tag{3.3}$$

for some μ_j. It follows from $\frac{d\mathcal{A}(\Sigma)}{dt} = \frac{d\mathcal{V}(\Sigma)}{dt} = 0$ that $\langle H, V\rangle = \langle 1, V\rangle = 0$. This implies

$$\langle 1_*, V\rangle = \langle H_*, V\rangle = 0.$$

Taking the $L^2(\Sigma)$-inner product (3.3) and 1_*, H_*, we get

$$0=\langle 1_*,V\rangle=-\langle 1_*,\delta\mathcal{W}(\Sigma)\rangle-\mu_1,\quad 0=\langle H_*,V\rangle=-\langle H_*,\delta\mathcal{W}(\Sigma)\rangle-\mu_2\|H_*\|^2.$$

In spite of $H_*=0$ or not, it holds that

$$-\mu_1 1_*-\mu_2 H_*=\langle 1_*,\delta\mathcal{W}(\Sigma)\rangle 1_*+\langle H_*,\delta\mathcal{W}(\Sigma)\rangle H_*.$$

Hence (3.3) is

$$V=-\delta\mathcal{W}(\Sigma)+\langle 1_*,\delta\mathcal{W}(\Sigma)\rangle 1_*+\langle H_*,\delta\mathcal{W}(\Sigma)\rangle H_*=-P(\Sigma)\delta\mathcal{W}(\Sigma).$$

Consequently we obtain (3.1).

Conversely it holds for solution to (3.1) that

$$\begin{aligned}\frac{d}{dt}\mathcal{W}(\Sigma)&=\langle\delta\mathcal{W}(\Sigma),V\rangle=\langle\delta\mathcal{W}(\Sigma),-P(\Sigma)\delta\mathcal{W}(\Sigma)\rangle\\&=-\|P(\Sigma)\delta\mathcal{W}(\Sigma)\|^2=-\|V\|^2.\end{aligned}$$

Since $V\in(\mathrm{span}_{L^2(\Sigma)}\{\delta\mathcal{A}(\Sigma),\delta\mathcal{V}(\Sigma)\})^\perp$, we have

$$\frac{d}{dt}\mathcal{A}(\Sigma)=\langle\delta\mathcal{A}(\Sigma),V\rangle=0,\quad\frac{d}{dt}\mathcal{V}(\Sigma)=\langle\delta\mathcal{V}(\Sigma),V\rangle=0.\qquad\square$$

4. The existence

In this section we prove Theorem 1.2. Firstly we consider the case $G(\Sigma_0)\neq 0$. If the Herfrich flow with $\Sigma(0)=\Sigma_0$ exists, it holds that $G(\Sigma(t))\neq 0$ for sufficiently small $t>0$. We denote $\Sigma(t)$ simply by Σ. It follows from (1.2) that

$$\begin{aligned}\begin{pmatrix}\lambda_1(\Sigma)\\\lambda_2(\Sigma)\end{pmatrix}&=-\begin{pmatrix}\langle\delta\mathcal{A}(\Sigma),\delta\mathcal{A}(\Sigma)\rangle&\langle\delta\mathcal{V}(\Sigma),\delta\mathcal{A}(\Sigma)\rangle\\\langle\delta\mathcal{A}(\Sigma),\delta\mathcal{V}(\Sigma)\rangle&\langle\delta\mathcal{V}(\Sigma),\delta\mathcal{V}(\Sigma)\rangle\end{pmatrix}^{-1}\begin{pmatrix}\langle\delta\mathcal{A}(\Sigma),\delta\mathcal{W}(\Sigma)\rangle\\\langle\delta\mathcal{V}(\Sigma),\delta\mathcal{W}(\Sigma)\rangle\end{pmatrix}\\&=-\frac{1}{G(\Sigma)}\begin{pmatrix}\langle\delta\mathcal{V}(\Sigma),\delta\mathcal{V}(\Sigma)\rangle&-\langle\delta\mathcal{V}(\Sigma),\delta\mathcal{A}(\Sigma)\rangle\\-\langle\delta\mathcal{A}(\Sigma),\delta\mathcal{V}(\Sigma)\rangle&\langle\delta\mathcal{A}(\Sigma),\delta\mathcal{A}(\Sigma)\rangle\end{pmatrix}\\&\quad\times\begin{pmatrix}\langle\delta\mathcal{A}(\Sigma),\delta\mathcal{W}(\Sigma)\rangle\\\langle\delta\mathcal{V}(\Sigma),\delta\mathcal{W}(\Sigma)\rangle\end{pmatrix}.\end{aligned}\tag{4.1}$$

By results of Section 2, we have

$$
\begin{aligned}
\langle \delta\mathcal{A}(\Sigma), \delta\mathcal{A}(\Sigma)\rangle &= \int_\Sigma n^2 H^2 dS,\\
\langle \delta\mathcal{A}(\Sigma), \delta\mathcal{V}(\Sigma)\rangle &= \int_\Sigma nH dS,\\
\langle \delta\mathcal{V}(\Sigma), \delta\mathcal{V}(\Sigma)\rangle &= \int_\Sigma dS,\\
\langle \delta\mathcal{A}(\Sigma), \delta\mathcal{W}(\Sigma)\rangle &= -\int_\Sigma nH\Big(\Delta_g H + \frac{n^2}{2}H^3 - HR + c_0R - \frac{n^2}{2}{c_0}^2 H\Big)dS\\
&= \int_\Sigma \Big(n|\nabla_g H|^2 - \frac{n^3}{2}H^4 + nH^2R - nc_0HR + \frac{n^3}{2}{c_0}^2H^2\Big)dS,\\
\langle \delta\mathcal{V}(\Sigma), \delta\mathcal{W}(\Sigma)\rangle &= -\int_\Sigma \Big(\Delta_g H + \frac{n^2}{2}H^3 - HR + c_0R - \frac{n^2}{2}{c_0}^2 H\Big)dS\\
&= \int_\Sigma \Big(-\frac{n^2}{2}H^3 + HR - c_0R + \frac{n^2}{2}{c_0}^2 H\Big)dS,\\
G(\Sigma) &= \int_\Sigma n^2H^2dS\int_\Sigma dS - \Big(\int_\Sigma nHdS\Big)^2 = n^2\mathcal{A}\int_\Sigma \tilde{H}^2 dS.
\end{aligned}
\tag{4.2}
$$

Here

$$
\tilde{H} = H - \bar{H}, \quad \bar{H} = \frac{1}{\mathcal{A}}\int_\Sigma H dS.
$$

Inserting these into (4.1), we have the explicit expression of $\lambda_j(\Sigma)$'s in the case $G(\Sigma) \neq 0$.

Proposition 4.1 *When $G(\Sigma) \neq 0$, $\lambda_j(\Sigma)$'s are given by*

$$
\begin{aligned}
\lambda_1(\Sigma) &= \frac{n\mathcal{A}}{G(\Sigma)}\int_\Sigma \Big\{-|\nabla_g H|^2 + \tilde{H}\Big(\frac{n^2}{2}H^3 - HR + c_0R - \frac{n^2}{2}c_0^2H\Big)\Big\}dS,\\
\lambda_2(\Sigma) &= \frac{n^2}{G(\Sigma)}\int_\Sigma \Big\{\mathcal{A}\bar{H}|\nabla_g H|^2\\
&\quad + \Big(\int_\Sigma \tilde{H}^2dS - \mathcal{A}\bar{H}\tilde{H}\Big)\Big(\frac{n^2}{2}H^3 - HR + c_0R - \frac{n^2}{2}c_0^2H\Big)\Big\}dS.
\end{aligned}
$$

In particular they depend on

$$\int_\Sigma |\nabla_g H|^2 dS, \quad \int_\Sigma H^p dS \ (p = 0, 1, 2, 3, 4), \quad \int_\Sigma H^q R dS \ (q = 0, 1, 2),$$

analytically.

In order to prove Theorem 1.2 (i), we regard $\Sigma(t)$ as the perturbation of Σ_0 in normal direction with signed distance $\rho(t)$. This is in a similar manner to [6]. We can write down the Laplace-Beltrami operater, the mean curvature, the scalar curvature, and the Lagrange multipliers in term of the function ρ and its derivatives, denoted $\Delta_\rho, H(\rho), R(\rho), \lambda_1(\rho)$, and $\lambda_2(\rho)$ respectively. Let $\bigcup_{\ell=1}^m U_\ell$ be the open covering of Σ_0. We denote the inner unit normal vector field of Σ_0 by $\boldsymbol{\nu}_0$. The mapping $X_\ell : U_\ell \times (-a, a) \ni (\boldsymbol{s}, r) \to \boldsymbol{s} + r\boldsymbol{\nu}_0(\boldsymbol{s}) \in \mathbb{R}^{n+1}$ is a C^∞-diffeomorphism from $U_\ell \times (-a, a)$ to $\mathcal{R}_\ell = \mathrm{Im}(X_\ell)$ provided $a > 0$ is sufficiently small. Let us denote the inverse mapping X_ℓ^{-1} by (S_ℓ, Λ_ℓ), where $S_\ell(X_\ell(\boldsymbol{s}, r)) = \boldsymbol{s} \in U_\ell$, and $\Lambda_\ell(X_\ell(\boldsymbol{s}, r)) = r \in (-a, a)$.

When $\Sigma(t)$ is close to Σ_0 for small $t > 0$, we can represent it as a graph of a function on Σ_0 as

$$\Sigma_{\rho(t)} = \Sigma(t) = \bigcup_{\ell=1}^m \mathrm{Im}(X_\ell(\cdot, \rho(\cdot, t)) : U_\ell \to \mathbb{R}^n, \ [\boldsymbol{s} \mapsto X_\ell(\boldsymbol{s}, \rho(\boldsymbol{s}, t))]).$$

Conversely, for a given function $\rho : \Sigma_0 \times [0, T) \to (-a, a)$ we define the mapping $\Phi_{\ell,\rho}$ from $\mathcal{R}_\ell \times [0, T)$ to $\mathbb{R}$ by

$$\Phi_{\ell,\rho}(x, t) = \Lambda_\ell(x) - \rho(S_\ell(x), t). \tag{4.3}$$

Then $\Phi_{\ell,\rho}(\cdot, t)^{-1}(0)$ gives the surface $\Sigma_{\rho(t)}$.

The velocity in the direction of the inner normal vector field of $\Sigma = \{\Sigma_{\rho(t)} : t \in [0, T)\}$ at $(x, t) = (X_\ell(\boldsymbol{s}, \rho(\boldsymbol{s}, t)), t)$ is given by

$$V(\boldsymbol{s}, t) = -\frac{\partial_t \Phi_{\ell,\rho}(x, t)}{\|\nabla_x \Phi_{\ell,\rho}(x, t)\|}\bigg|_{x = X_\ell(\boldsymbol{s}, \rho(\boldsymbol{s}, t))} = \frac{\partial_t \rho(\boldsymbol{s}, t)}{\|\nabla_x \Phi_{\ell,\rho}(x, t)\|}\bigg|_{x = X_\ell(\boldsymbol{s}, \rho(\boldsymbol{s}, t))}.$$

The equation (1.1) is represented as

$$\partial_t \rho = L_\rho \Big(-\Delta_\rho H(\rho) - \frac{n^2}{2} H^3(\rho) + H(\rho)R(\rho) - c_0 R(\rho) + \frac{n^2}{2} {c_0}^2 H(\rho) + \lambda_1(\rho) n H(\rho) + \lambda_2(\rho) \Big)$$

where $L_\rho = \|\nabla_x \Phi_{\ell,\rho}(x,t)\|_{x = X_\ell(\boldsymbol{s}, \rho(\boldsymbol{s},t))}$.

Let K_j be the fundamental function of order j of the principal curvatures $\kappa_1, \kappa_2, \ldots, \kappa_n$, that is,

$$K_1 = \sum_i \kappa_i, \ K_2 = \sum_{i<j} \kappa_i \kappa_j, \ K_3 = \sum_{i<j<k} \kappa_i \kappa_j \kappa_k, \ \ldots, \ K_n = \kappa_1 \kappa_2 \ldots \kappa_n.$$

The mean curvature H, the scalar curvature R, and the Gaussian curvature K are given by

$$H = \frac{K_1}{n}, \quad R = 2K_2, \quad K = K_n.$$

To get expressions of $H(\rho)$ and $R(\rho)$, we need those of K_j in term of derivatives of $\Phi_{\ell,\rho}$. We denote $\Phi_{\ell,\rho}$ simply by Φ.

Lemma 4.1 *Assume that a hypersurface is defined by $\{x \in \mathbb{R}^{n+1} : \Phi(x) = 0\}$ locally, and that $\nabla_x \Phi \neq 0$ everywhere near the hypersurface. Then K_j is given by*

$$K_j = \frac{1}{(n-j)!} \frac{d^{n-j}}{d\epsilon^{n-j}} \mathcal{G}(\nabla_x \Phi, \mathrm{Hess}_x \Phi, \epsilon) \Big|_{\epsilon = 0, \{x : \Phi(x) = 0\}},$$

where

$$\mathcal{G}(\boldsymbol{p}, X, \epsilon) = \det_{n+1}\big(\|\boldsymbol{p}\|^{-1} (I_{n+1} - \bar{\boldsymbol{p}} \otimes \bar{\boldsymbol{p}}) X (I_{n+1} - \bar{\boldsymbol{p}} \otimes \bar{\boldsymbol{p}}) + \bar{\boldsymbol{p}} \otimes \bar{\boldsymbol{p}} + \epsilon E^t E \big),$$

$$E = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) \in M_{n+1}(\mathbb{R}), \quad \bar{\boldsymbol{p}} = \|\boldsymbol{p}\|^{-1} \boldsymbol{p} \ \textit{for} \ \boldsymbol{p} \in \mathbb{R}^{n+1}$$

Proof. We prove the assertion by the adapted argument of [6, Lemma 5.1]. We may assume $x = 0$. Then there exists a neighborhood of U of $0 \in \mathbb{R}^n$ such that $\Phi(x) = 0$ is a graph of a function of $f : U \to \mathbb{R}$. Let $\tilde{x} = (x^1, \ldots, x^n)$ and $x = (\tilde{x}, x^{n+1})$. Since principal curvature at 0 are eigenvalues of $\mathrm{Hess}_{\tilde{x}} f(0)$, it holds that

$$\det_n(\mathrm{Hess}_{\tilde{x}} f(0) + \epsilon I_n) = \sum_{j=0}^{n} K_j \epsilon^{n-j},$$

where $K_0 = 1$. Consequently

$$K_j = \frac{1}{(n-j)!} \frac{d^{n-j}}{d\epsilon^{n-j}} \det_n(\mathrm{Hess}_{\tilde{x}} f(0) + \epsilon I_n)\bigg|_{\epsilon=0}.$$

As shown in [6, Lemma 5.1], putting $\boldsymbol{p} = \nabla_x \Phi(0)$, $X = \mathrm{Hess}_x \Phi(0)$, we have

$$\mathrm{Hess}_{\tilde{x}} \tilde{f}(0) = \|\boldsymbol{p}\|^{-1} {}^t E X E.$$

Hence

$$\begin{aligned}
&\det_n(\mathrm{Hess}_{\tilde{x}} f(0) + \epsilon I_n) \\
&\quad = \det_{n+1} \begin{pmatrix} \|\boldsymbol{p}\|^{-1} {}^t E X E + \epsilon I_n & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad = \det_{n+1} \left(\|\boldsymbol{p}\|^{-1} E^t E X E^t E + \epsilon E^t E + \boldsymbol{e}_{n+1} {}^t \boldsymbol{e}_{n+1} \right) \\
&\quad = \det_{n+1} \left(\|\boldsymbol{p}\|^{-1} (I_{n+1} - \boldsymbol{p} \otimes \boldsymbol{p}) X (I_{n+1} - \boldsymbol{p} \otimes \boldsymbol{p}) + \boldsymbol{p} \otimes \boldsymbol{p} + \epsilon E^t E \right). \quad \square
\end{aligned}$$

It follows from (4.3) that $\nabla_x \Phi$ and $\mathrm{Hess}_x \Phi$ can be written in terms of derivatives of ρ up to the 2nd order, and therefore so do $H(\rho)$ and $R(\rho)$. By Proposition 4.1 we find that $\lambda_j(\Sigma)$'s depend analytically on derivatives of ρ up to the 3rd order near $\rho = 0$. Consequently the equation (2.21) is in the form

$$\rho_t + L_\rho \Delta_\rho H(\rho) + \Phi(\rho, \partial\rho, \partial^2\rho, \partial^3\rho) = 0.$$

Now we study precisely where the third derivarive $\partial^3\rho$ appears. There are no terms including it other than $L_\rho \mathit{\Delta}_\rho H(\rho)$ and $\lambda_j(\rho)$. The analysis of the principal term $L_\rho \Delta_\rho H(\rho)$ is in the same as [4] and [6], and $\partial^3\rho$ appears linearly there. We have found $|\nabla_g H|^2$ $(= |\nabla_\rho H(\rho)|^2)$ in the numerator of the expression of $\lambda_j(\Sigma)$ $(= \lambda_j(\rho))$ in Proposition 4.1. It follows from [6, Lemma 2.1] that $\nabla_\rho H(\rho)$ is linear in $\partial^3\rho$. The denominator $G(\Sigma)$ of $\lambda_1(\Sigma)$ does not depend on $\nabla_g H$. Hence we have a term including $\partial^3\rho$ quadrically from $\lambda_1(\rho)$.

An argument similar to [4, Lemma 2.1] and [10, Lemma 2.1] gives the following. Let $h^\gamma(\Sigma_0)$ be the little Hölder space on Σ_0 of order γ. We fix $0 < \alpha < \beta < 1$. For $\beta_0 \in (\alpha, \beta)$, put

$$\mathcal{U} = \left\{\rho \in h^{3+\beta_0}(\Sigma_0) : \|\rho\|_{C^2(\Sigma_0)} < a\right\}.$$

For two Banach spaces E_0 and E_1 satisfying $E_1 \hookrightarrow E_0$ the set $\mathcal{H}(E_1, E_0)$ is the class of $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as an unbounded operater in E_0, generates a strongly continuous analytic semigroup on E_0.

Proposition 4.2 *There exist* $Q \in C^\infty(\mathcal{U}, \mathcal{H}(h^{4+\alpha}(\Sigma_0), h^\alpha(\Sigma_0)))$, *and* $F \in C^\infty(\mathcal{U}, h^{\beta_0}(\Sigma_0))$ *such that the equation* (2.21) *is in the form*

$$\rho_t + Q(\rho)\rho + F(\rho) = 0.$$

Applying [1, Theorem 12.1] with $X_\beta = \mathcal{U}$, $E_1 = h^{4+\alpha}(\Sigma_0)$, $E_0 = h^\alpha(\Sigma_0)$, and $E_\gamma = h^{\beta_0}(\Sigma_0)$, we get an existence and uniqueness result for the Helfrich flow in case $G(\Sigma_0) \neq 0$.

Remark 4.1 The equation dealt with in [6] is a similar forth-order equation, but linear with respect to the third order derivatives of ρ. The term $Q(\rho)\rho$ includes such parts, and $F(\rho)$ does not include the third order derivatives. Therefore it was solvable for initial data in the class $h^{2+\alpha}$. In our case, the terms with $\partial^3\rho$, which are not linear with respect to it, are excluded from $Q(\rho)\rho$, and they are included into $F(\rho)$. This is why we need extra regularity than the result in [6].

Now consider the assertion (ii) in Theorem 1.2. Before going to prove, we see an example of Σ_0 satisfying $G(\Sigma_0) = 0$ and $(\overline{H}_0 - c_0)\tilde{R}_0 \equiv 0$. A typical example is a sphere. Indeed, spheres have constant mean curvature, and there for $G(\Sigma_0) = 0$ (see (4.2)). Since the scalar curvature is also constant, we have $\tilde{R}_0 = 0$. Furthermore spheres are stationary solution to (3.1).

To show the assertion (ii), it is enough to see that Σ_0 is a stationary solution.

Assume that $G(\Sigma) = 0$. It follows from (4.2) that Σ has a constant mean curvature $H = \overline{H}$. Hence

$$\text{span}_{L^2(\Sigma)}\{\delta\mathcal{A}, \delta\mathcal{V}\} = \text{span}_{L^2(\Sigma)}\{1\},$$

and

$$P(\Sigma)\phi = \phi - \frac{1}{\mathcal{A}(\Sigma)} \int_\Sigma \phi dS$$

for $\phi \in L^2(\Sigma)$. Therefore at the time when $G(\Sigma(t)) = 0$, the equation (3.1) becomes

$$\begin{aligned} V(t) &= -\delta\mathcal{W}(\Sigma(t)) + \frac{1}{\mathcal{A}(\Sigma)} \int_\Sigma \delta\mathcal{W}(\Sigma(t))dS \\ &= -\Delta_g \overline{H} - \frac{1}{2}\overline{H}^3 + \overline{H}R - c_0 R + \frac{1}{2}n^2 c_0^2 \overline{H} \\ &\quad + \frac{1}{\mathcal{A}(\Sigma)} \int_\Sigma \left(\frac{1}{2}\overline{H}^3 - \overline{H}R + c_0 R - \frac{1}{2}n^2 c_0^2 \overline{H} \right) dS \\ &= -(\overline{H} - c_0)\tilde{R}, \end{aligned}$$

where

$$\tilde{R} = R - \frac{1}{\mathcal{A}(\Sigma)} \int_\Sigma R dS.$$

Consequently if the hypersurface Σ_0 satisfies $G(\Sigma_0) = 0$ and $(\overline{H} - c_0)\tilde{R} \equiv 0$, then it is a stationary of solution (3.1).

Thus we complete the proof of Theorem 1.2. □

We do not know the uniqueness in case of Theorem 1.2 (ii), expect for $n = 1$.

Theorem 4.1 *Consider the one-dimensional Helfrich flow. If Σ_0 satisfies $G(\Sigma_0) = 0$, then $\{\Sigma(t) \equiv \Sigma_0\}$ is the unique global solution with $\Sigma(0) = \Sigma_0$.*

Remark 4.2 When $n = 1$, the scalar curvature is zero by its definition, and therefore the condition $(\overline{H} - c_0)\tilde{R} \equiv 0$ is automatically satisfied.

Proof. When $n = 1$, the integral $\int_\Sigma H dS$ is a constant multiple of the rotation number. Therefore it does not depend on t. Consequently we have

$$\frac{d}{dt}G(\Sigma) = \mathcal{A}_0 \frac{d}{dt}\int_\Sigma H^2 dS = 2\mathcal{A}_0 \frac{d}{dt}\mathcal{W} = -2\mathcal{A}_0\|V\|^2 \leq 0.$$

Combining this with $G(\Sigma) \geq 0$ (see (4.2)), it hold that $G(\Sigma) \equiv 0$ provided $G(\Sigma_0) = 0$. Using the above relation again, we have $V \equiv 0$, that is, $\Sigma(t) \equiv \Sigma(0)$. □

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Takeyuki Nagasawa
Department of Mathematics
Graduate School of Science and Engineering
Saitama University
Saitama 338-8570, Japan
E-mail: tnagasaw@rimath.saitama-u.ac.jp

Taekyung Yi
Department of Mathematics
Graduate School of Science and Engineering
Saitama University
Saitama 338-8570, Japan
E-mail: s08dm004@mail.saitama-u.ac.jp