Hokkaido Mathematical Journal Vol. 41 (2012) p. 157–183

Prediction of fractional processes with long-range dependence

Akihiko Inoue and Vo V. Anh

(Received March 26, 2009; Revised October 31, 2011)

Abstract. We introduce a class of Gaussian processes with stationary increments which exhibit long-range dependence. The class includes fractional Brownian motion with Hurst parameter $H > 1/2$ as a typical example. We establish infinite and finite past prediction formulas for the processes in which the predictor coefficients are given explicitly in terms of the MA(∞) and AR(∞) coefficients.

Key words: Predictor coefficients, prediction, fractional Brownian motion, long-range dependence.

1. Introduction

Let $(X(t) : t \in \mathbf{R})$ be a centered Gaussian process with stationary increments, defined on a probability space $(\Omega, \mathcal{F}, P)$, that admits the *moving-average* representation

$$X(t) = \int_{-\infty}^{\infty} \{g(t-s) - g(-s)\} dW(s), \qquad t \in \mathbf{R}, \tag{1.1}$$

where $(W(t) : t \in \mathbf{R})$ is a Brownian motion, and $g(t)$ is a function of the form

$$g(t) = \int_0^t c(s) ds, \qquad t \in \mathbf{R}, \tag{1.2}$$

$$c(t) := I_{(0,\infty)}(t) \int_0^\infty e^{-ts} \nu(ds), \qquad t \in \mathbf{R}, \tag{1.3}$$

with some Borel measure ν on $(0, \infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s} \nu(ds) < \infty. \tag{1.4}$$

We will also assume some extra conditions such as

2010 Mathematics Subject Classification : Primary 60G25; Secondary 60G15.

$$\lim_{t\to 0+} c(t) = \infty, \tag{1.5}$$

$$g(t) \sim t^{H-(1/2)}\ell(t) \cdot \frac{1}{\Gamma(\frac{1}{2}+H)}, \qquad t \to \infty, \tag{1.6}$$

where $\ell(t)$ is a slowly varying function at infinity and H is a constant such that

$$1/2 < H < 1. \tag{1.7}$$

In (1.6), and throughout the paper, $a(t) \sim b(t)$ as $t \to \infty$ means $\lim_{t\to\infty} a(t)/b(t) = 1$. We call $c(t)$ (as well as $g(t)$) the *MA*(∞) *coefficient* of $(X(t))$. We remark that, in the prediction formulas for $(X(t))$ which we consider in this paper, $c(t)$ becomes more relevant than $g(t)$.

A typical example of ν is

$$\nu(ds) = \frac{\sin\{\pi(H-\frac{1}{2})\}}{\pi} s^{(1/2)-H} ds \quad \text{on } (0,\infty) \tag{1.8}$$

with (1.7). For this ν, $g(t)$ becomes

$$g(t) = I_{(0,\infty)}(t) t^{H-(1/2)} \frac{1}{\Gamma(\frac{1}{2}+H)}, \qquad t \in \mathbf{R}, \tag{1.9}$$

and $(X(t))$ reduces to *fractional Brownian motion* $(B_H(t))$ with *Hurst parameter* H (see Example 2.3 below). Fractional Brownian motion, abbreviated fBm, was introduced by Kolmogorov [K]. For $1/2 < H < 1$, fBm has both *self-similarity* and *long-range dependence* (Samorodnitsky and Taqqu [ST]), and plays an important role in various fields such as network traffic (see, e.g., Mikosch et al. [MRRS]) and finance (see, e.g., Hu et al. [HOS]); see also Taqqu [T] and other papers in the same volume. Because of its importance, stochastic calculus for fBm has been developed by many authors; see, e.g., Decreusefond and Üstünel [DU], and Nualart [N]. Grecksch and Anh [GA] introduced Hilbert space-valued fBm and the corresponding stochastic calculus. Duncan et al. [DMP] and Tindel et al. [TTV] studied stochastic evolution equations with fBm in Hilbert spaces. Other important examples of $(X(t))$ are the processes with long-range dependence which, unlike fBm, have two different indices H_0 and H describing the local properties (path properties) and long-time behavior of $(X(t))$, respectively (see

Example 2.4 below).

Let t_0, t_1 and T be real constants such that

$$-\infty < -t_0 \leq 0 \leq t_1 < T < \infty, \qquad -t_0 < t_1. \tag{1.10}$$

For $I = (-\infty, t_1]$ or $[-t_0, t_1]$, we write $P_I X(T)$ for the predictor of the future value $X(T)$ based on the observable $(X(s) : s \in I)$ (see Section 3 below). One of the fundamental prediction problems for $(X(t))$ is to express $P_I X(T)$ using the segment $(X(s) : s \in I)$ and some deterministic quantities. Another is to express the variance of the prediction error $P_I^{\perp} X(T) := X(T) - P_I X(T)$. Results of this type become important tools in the analysis of non-Markovian processes and systems modulated by them (see, e.g., Norros et al. [NVV], Anh et al. [AIK], Inoue et al. [INA] and Inoue and Nakano [IN]). One of our main purposes here is to derive such results for $(X(t))$.

We establish the following infinite and finite past prediction formulas for $(X(t))$ (see Theorems 3.8 and 4.12 below):

$$P_{(-\infty,t_1]} X(T) = X(t_1) + \int_{-\infty}^{t_1} \left\{ \int_0^{T-t_1} b(t_1 - s, \tau) d\tau \right\} dX(s), \tag{1.11}$$

$$P_{[-t_0,t_1]} X(T) = X(t_1) + \int_{-t_0}^{t_1} \left\{ \int_0^{T-t_1} h(s + t_0, u) du \right\} dX(s). \tag{1.12}$$

The significance of (1.11) and (1.12) is that the predictor coefficients $b(t, s)$ and $h(t, s)$ are given explicitly in terms of the MA(∞) coefficient $c(t)$ and AR(∞) coefficient $a(t)$, to be defined in Section 3.1, of $(X(t))$. The integral of $a(t)$ is in fact the coefficient of an AR(∞)-type equation describing $(X(t))$ (see Section 5). We will find that $a(t)$ has a nice integral representation similar to (1.3) (see (3.3) below). It turns out that the existence of such a nice AR(∞) coefficient, in addition to the nice MA(∞) coefficient, is a key to the solution to the prediction problems above.

For fBm with $1/2 < H < 1$, the predictor coefficients $b(t, s)$ and $h(t, s)$ are given in Gripenberg and Norros [GN]. See [NVV] and [NP] for different proofs. Fractional Brownian motion has a variety of nice properties, and the methods of proof of [GN], [NVV], [NP] naturally rely on such special properties of fBm, hence are not applicable to $(X(t))$. The method of this paper is based on the *alternating projections to the past and future* (see Section 4.1 below). As for fBm with $0 < H < 1/2$, its infinite and finite past prediction formulas also exist, and are due to Yaglom [Y] and Nuzman and Poor [NP], respectively (see also Anh and Inoue [AI1]).

In Inoue and Anh [IA], a class of processes $(\tilde{X}(t))$ of the same form

$$\tilde{X}(t) = \int_{-\infty}^{\infty} \{\tilde{c}(t-s) - \tilde{c}(-s)\} dW(s), \qquad t \in \mathbf{R}, \tag{1.13}$$

as (1.1) are introduced. Unlike $g(t)$ in (1.1), however, the kernel $\tilde{c}(t)$ itself is assumed to be of the form

$$\tilde{c}(t) = I_{(0,\infty)}(t) \int_0^{\infty} e^{-ts} \tilde{\nu}(ds), \qquad t \in \mathbf{R}, \tag{1.14}$$

with a Borel measure $\tilde{\nu}$ on $(0,\infty)$ satisfying some suitable conditions. This class of $(\tilde{X}(t))$ includes fBm with $H \in (0, 1/2)$ as a typical example. Notice that $\tilde{c}(t)$ in (1.14) (resp., $g(t)$ in (1.1)) is decreasing (resp., increasing) on $(0,\infty)$ as $t^{H-(1/2)}$ with $H \in (0, 1/2)$ (resp., $(1/2,1)$) is. In [IA], prediction formulas for $(\tilde{X}(t))$ are proved, extending the results for fBm with $H \in (0, 1/2)$ stated above. These prediction formulas for $(\tilde{X}(t))$, including those for fBm with $H \in (0, 1/2)$, have different forms from (1.11) and (1.12), in that no stochastic integrals appear there.

We provide the basic properties and examples of $(X(t))$ in Section 2. We consider the infinite and finite past prediction problems for $(X(t))$ in Sections 3 and 4, respectively. Finally in Section 5, we remark on the AR(∞)-type equations describing $(X(t))$ and $(\tilde{X}(t))$.

2. Basic properties and examples

In this section, we assume (1.2)–(1.4) and

$$\int_1^{\infty} c(t)^2 dt < \infty. \tag{2.1}$$

Then, as in [IA, Lemma 2.1], we have $\int_{-\infty}^{\infty} |g(t-s) - g(-s)|^2 ds < \infty$ for $t \in \mathbf{R}$. Therefore, for a one-dimensional standard Brownian motion $(W(t) : t \in \mathbf{R})$ with $W(0) = 0$, we may define the centered stationary-increment Gaussian process $(X(t) : t \in \mathbf{R})$ by (1.1).

For $s > 0$ and $t \in \mathbf{R}$, we put $\Delta_s X(t) := X(t+s) - X(t)$. Then, by definition, $(\Delta_s X(t) : t \in \mathbf{R})$ is a stationary process.

Lemma 2.1 *Let $s \in (0,\infty)$. We assume* (1.6) *and* (1.7). *Then*

$$E[\Delta_s X(t)\cdot\Delta_s X(0)] \sim t^{2H-2}\ell(t)^2\cdot\frac{s^2\Gamma(2-2H)\sin\{(H-\frac{1}{2})\pi\}}{\pi}, \qquad t\to\infty.$$

Since $-1 < 2H-2 < 0$ in Lemma 2.1, we see from this lemma that $(\Delta_s X(t))$, whence $(X(t))$, has long-range dependence.

We put $\sigma(t) := E[|X(t+s)-X(s)|^2]^{1/2}$ for $t \geq 0$ and $s \in \mathbf{R}$.

Lemma 2.2 *Let $H_0 \in (1/2,1)$ and $\ell_0(\cdot)$ a slowly varying function at infinity. We assume*

$$g(t) \sim t^{H_0-(1/2)}\ell_0(1/t)\cdot\frac{1}{\Gamma(\frac{1}{2}+H_0)}, \qquad t\to 0+\,. \tag{2.2}$$

Then

$$\sigma(t) \sim t^{H_0}\ell_0(1/t)\sqrt{v(H_0)}, \qquad t\to 0+\,,$$

where $v(H_0) := \Gamma(2-2H_0)\cos(\pi H_0)/\{\pi H_0(1-2H_0)\}$. In particular, we have

$$H_0 = \sup\big\{\beta : \sigma(t) = o(t^\beta),\ t\to 0+\big\} = \inf\big\{\beta : t^\beta = o(\sigma(t)),\ t\to 0+\big\}.$$

From Lemma 2.2, we see that the index H_0 describes the path properties of $(X(t))$ (see Adler [A, Section 8.4]).

By the monotone density theorem (cf. Bingham et al. [BGT, Theorem 1.7.5]), (1.6) with (1.7) implies

$$c(t) \sim t^{H-(3/2)}\ell(t)\cdot\frac{1}{\Gamma(H-\frac{1}{2})}, \qquad t\to\infty. \tag{2.3}$$

Similarly, (2.2) implies

$$c(t) \sim t^{H_0-(3/2)}\ell_0(1/t)\cdot\frac{1}{\Gamma(H_0-\frac{1}{2})}. \qquad t\to 0+\,. \tag{2.4}$$

Lemmas 2.1 and 2.2 follow from (2.3) and (2.4), respectively, by standard arguments. However, since we do not use these results, we omit the details.

Example 2.3 For $H \in (1/2,1)$, let ν be as in (1.8). Then we have (1.9);

and so all the conditions above are satisfied. The resulting process $(X(t))$ is fBm $(B_H(t))$:

$$B_H(t) = \frac{1}{\Gamma(\frac{1}{2}+H)} \int_{-\infty}^{\infty} \{((t-s)_+)^{H-(1/2)} - ((-s)_+)^{H-(1/2)}\} dW(s), \quad (2.5)$$

where $(x)_+ := \max(0,x)$ for $x \in \mathbf{R}$. The representation (2.5) of fBm is due to the pioneering work of Mandelbrot and Van Ness [MV].

Example 2.4 Let $f(\cdot)$ be a nonnegative, locally integrable function on $(0,\infty)$. For $H_0, H \in (1/2, 1)$ and slowly varying functions $\ell_0(\cdot)$ and $\ell(\cdot)$ at infinity, we assume

$$f(s) \sim \frac{\sin\{\pi(H_0 - \frac{1}{2})\}}{\pi} s^{(1/2)-H} \ell(1/s), \qquad s \to 0+,$$

$$f(s) \sim \frac{\sin\{\pi(H_0 - \frac{1}{2})\}}{\pi} s^{(1/2)-H_0} \ell_0(s), \qquad s \to \infty.$$

Let $\nu(ds) = f(s)ds$. Then, by Abelian theorems for Laplace transforms (cf. [BGT, Section 1.7]), we have (2.3), whence (1.6). Similarly, we have (2.4), whence (2.2). Thus all the conditions above are satisfied. As we have seen above, the indices H_0 and H describe the path properties and long-time behavior of $(X(t))$, respectively.

3. Infinite past prediction problems

In this section, we assume (1.1)–(1.5), (2.1) and

$$\lim_{t\to\infty} g(t) = \infty. \quad (3.1)$$

Notice that, for the processes $(X(t))$ in Examples 2.3 and 2.4, all these conditions are satisfied. We also assume (1.10).

We write $M(X)$ for the real Hilbert space spanned by $\{X(t) : t \in \mathbf{R}\}$ in $L^2(\Omega, \mathcal{F}, P)$, and $\|\cdot\|$ for its norm. Let I be a closed interval of $\mathbf{R}$ such as $[-t_0, t_1]$, $(-\infty, t_1]$, and $[-t_0, \infty)$. Let $M_I(X)$ be the closed subspace of $M(X)$ spanned by $\{X(t) : t \in I\}$. We write P_I for the orthogonal projection operator from $M(X)$ to $M_I(X)$, and $P_I^\perp$ for its orthogonal complement: $P_I^\perp Z = Z - P_I Z$ for $Z \in M(X)$. Note that, since $(X(t))$ is a Gaussian process, we have $P_I Z = E[Z|\sigma(X(s) : s \in I)]$.

3.1. MA and AR coefficients

The conditions (1.5) and (3.1) imply $\nu(0,\infty)=\infty$ and $\int_0^\infty s^{-1}\nu(ds)=\infty$, respectively. Therefore, by [IA, Theorem 3.2], there exists a unique Borel measure μ on $(0,\infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s}\mu(ds)<\infty, \quad \mu(0,\infty)=\infty, \quad \int_0^\infty \frac{1}{s}\mu(ds)=\infty$$

and

$$-iz\Big\{\int_0^\infty e^{izt}c(t)dt\Big\}\Big\{\int_0^\infty e^{izt}\alpha(t)dt\Big\}=1, \qquad \Im z>0, \tag{3.2}$$

with

$$\alpha(t):=\int_0^\infty e^{-st}\mu(ds), \qquad t>0.$$

We define

$$a(t):=-\frac{d\alpha}{dt}(t)=\int_0^\infty e^{-st}s\mu(ds), \qquad t>0. \tag{3.3}$$

We call $a(t)$ (as well as $\alpha(t)$) the $AR(\infty)$ *coefficient* of $(X(t))$ (see Section 5 for background). We define the positive kernel $b(t,s)$ by

$$b(t,s):=\int_0^s c(u)a(t+s-u)du, \qquad t,s>0.$$

Then, by [IA, Lemma 3.4], the following equalities hold:

$$\int_0^\infty b(t,s)dt=1, \qquad s>0, \tag{3.4}$$

$$c(t+s)=\int_0^t c(t-u)b(u,s)du, \qquad t,s>0. \tag{3.5}$$

3.2. Stochastic integrals

Let I be a closed interval of $\mathbf{R}$. We define

$$\mathcal{H}_I(X) := \left\{ f : \begin{array}{l} f \text{ is a real-valued measurable function on } I \text{ such} \\ \text{that } \int_{-\infty}^{\infty}\{\int_I |f(u)|c(u-s)du\}^2 ds < \infty. \end{array} \right\}.$$

This is the class of functions f for which we can define the stochastic integral $\int_I f(s)dX(s)$. We notice that, by Lemma 5.2 below, the function $c(t)$, whence $\mathcal{H}_I(X)$, is uniquely determined by $(X(t))$. We define a subclass $\mathcal{H}_I^0$ of $\mathcal{H}_I(X)$ by

$$\mathcal{H}_I^0 := \left\{ \sum_{k=1}^{m} a_k I_{(t_{k-1},t_k]}(s) : \begin{array}{l} m \in \mathbf{N},\ -\infty < t_0 < t_1 < \cdots < t_m < \infty \\ \text{with } (t_0, t_m] \subset I,\ a_k \in \mathbf{R}\ (k = 1, \dots, m) \end{array} \right\}.$$

Each member of $f \in \mathcal{H}_I^0$ is a *simple function* on I.

Definition 3.1 For $f = \sum_{k=1}^m a_k I_{(t_{k-1},t_k]} \in \mathcal{H}_I^0$, we define

$$\int_I f(s)dX(s) := \sum_{k=1}^{m} a_k\{X(t_k) - X(t_{k-1})\}.$$

We see that $\int_I f(s)dX(s) \in M_I(X)$ for $f \in \mathcal{H}_I^0$.

Proposition 3.2 *For $f \in \mathcal{H}_I^0$, we have*

$$\int_I f(s)dX(s) = \int_{-\infty}^{\infty}\left\{\int_I f(u)c(u-s)du\right\}dW(s). \tag{3.6}$$

Proof. For $-\infty < a < b < \infty$ with $(a, b] \subset I$, we have

$$X(b) - X(a) = \int_{-\infty}^{\infty}\left\{\int_I I_{(a,b]}(u)c(u-s)du\right\}dW(s),$$

which implies (3.6) for $f = I_{(a,b]}$. The general case follows easily from this. □

Proposition 3.3 *Let $f \in \mathcal{H}_I(X)$ such that $f \geq 0$, and let f_n $(n = 1, 2, \dots)$ be a sequence of simple functions on I such that $0 \leq f_n \uparrow f$ a.e. Then, in $M(X)$,*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(s)dX(s) = \int_{-\infty}^{\infty}\left\{\int_I f(u)c(u-s)du\right\}dW(s).$$

Proof. By Proposition 3.2 and the monotone convergence theorem, we have

$$\left\| \int_I f_n(s)dX(s) - \int_{-\infty}^{\infty} \left\{ \int_I f(u)c(u-s)du \right\} dW(s) \right\|^2$$
$$\leq \int_{-\infty}^{\infty} \left\{ \int_I (f(u) - f_n(u))c(u-s)du \right\}^2 ds \ \downarrow\ 0, \qquad n \to \infty.$$

Thus the proposition follows. □

For a real-valued function f on I, we write $f(x) = f^+(x) - f^-(x)$, where

$$f^+(x) := \max(f(x), 0), \quad f^-(x) := \max(-f(x), 0), \qquad x \in I.$$

Definition 3.4 For $f \in H_I(X)$, we define

$$\int_I f(s)dX(s) := \lim_{n\to\infty} \int_I f_n^+(s)dX(s) - \lim_{n\to\infty} \int_I f_n^-(s)dX(s) \quad \text{in } M(X),$$

where $\{f_n^+\}$ and $\{f_n^-\}$ are arbitrary sequences of non-negative simple functions on I such that $f_n^+ \uparrow f^+$, $f_n^- \uparrow f^-$, as $n \to \infty$, a.e.

From the definition above, we see that $\int_I f(s)dX(s) \in M_I(X)$ for $f \in \mathcal{H}_I(X)$. The next proposition follows immediately from Proposition 3.3.

Proposition 3.5 *The equality* (3.6) *also holds for* $f \in \mathcal{H}_I(X)$.

3.3. Infinite past prediction formulas

We denote by $\mathcal{D}(\mathbf{R})$ the space of all $\phi \in C^\infty(\mathbf{R})$ with compact support, endowed with the usual topology. For a random distribution Y (cf. [I2, Section 2] and [AIK, Section 2]), we write DY for its derivative. For $t \in \mathbf{R}$, we write $M_{(-\infty,t]}(Y)$ for the closed linear hull of $\{Y(\phi) : \phi \in \mathcal{D}(\mathbf{R}),\ \operatorname{supp}\phi \subset (-\infty, t]\}$ in $L^2(\Omega, \mathcal{F}, P)$. Notice that $M_I(X)$ here coincides with that defined above.

As in [IA, Proposition 2.4], we have the next proposition.

Proposition 3.6 *The derivative DX of $(X(t))$ is a purely nondeterministic stationary random distribution, and $(W(t) : t \in \mathbf{R})$ is a canonical Brownian motion of DX in the sense that $M_{(-\infty,t]}(DX) = M_{(-\infty,t]}(DW)$ for every $t \in \mathbf{R}$.*

See Section 5 for the proof.

Here is the infinite past prediction formula for $\int_t^\infty f(s)dX(s)$.

Theorem 3.7 *For $t \in [0,\infty)$ and $f \in \mathcal{H}_{[t,\infty)}(X)$, the following assertions hold*:

(a) $\int_0^\infty b(t-\cdot,\tau)f(t+\tau)d\tau \in \mathcal{H}_{(-\infty,t]}(X)$.
(b) $P_{(-\infty,t]}\int_t^\infty f(s)dX(s) = \int_{-\infty}^t \{\int_0^\infty b(t-s,\tau)f(t+\tau)d\tau\}dX(s)$.

Proof. Since $f \in \mathcal{H}_{[t,\infty)}(X)$ iff $|f| \in \mathcal{H}_{[t,\infty)}(X)$, we may assume $f \geq 0$. Since

$$c(u) = 0, \qquad t \leq 0, \tag{3.7}$$

it follows from (3.5) and the Fubini–Tonelli theorem that, for $s < t$,

$$\begin{aligned}\int_t^\infty f(u)c(u-s)du &= \int_0^\infty d\tau f(t+\tau)\int_0^{t-s} c(t-s-u)b(u,\tau)du \\ &= \int_{-\infty}^t duc(u-s)\int_0^\infty b(t-u,\tau)f(t+\tau)d\tau. \end{aligned}\tag{3.8}$$

Thus we obtain (a). By Proposition 3.6 and [AIK, Proposition 2.3 (2)], we have

$$M_{(-\infty,t]}(X) = M_{(-\infty,t]}(DW). \tag{3.9}$$

This and Proposition 3.5 yield

$$P_{(-\infty,t]}\int_t^\infty f(s)dX(s) = \int_{-\infty}^t \left\{\int_t^\infty f(u)c(u-s)du\right\}dW(s).$$

By (3.7), (3.8) and Proposition 3.5, the integral on the right-hand side is

$$\begin{aligned}&\int_{-\infty}^t \left\{\int_{-\infty}^t duc(u-s)\int_0^\infty b(t-u,\tau)f(t+\tau)d\tau\right\}dW(s) \\ &= \int_{-\infty}^t \left\{\int_0^\infty b(t-s,\tau)f(t+\tau)d\tau\right\}dX(s).\end{aligned}$$

Thus (b) follows. □

By putting $f(s) = I_{(t_1,T]}(s)$ in Theorem 3.7 (b), we immediately obtain the next infinite past prediction formula for $(X(t))$.

Theorem 3.8 *Let* $0 \le t_1 < T < \infty$. *Then* $\int_0^{T-t_1} b(t_1 - \cdot, \tau)d\tau \in \mathcal{H}_{(-\infty,t_1]}(X)$ *and the infinite past prediction formula* (1.11) *holds.*

Using the Hilbert space isomorphism $\theta : M(X) \to M(X)$ characterized by $\theta(X(t)) = X(-t)$ for $t \in \mathbf{R}$, we obtain the next theorem from Theorem 3.7 (see the proof of [AIK, Theorem 3.6]).

Theorem 3.9 *For* $t \in [0,\infty)$ *and* $f \in \mathcal{H}_{[t,\infty)}(X)$, *the following assertions hold*:

(a) $\int_0^\infty b(t + \cdot, \tau)f(t+\tau)d\tau \in \mathcal{H}_{[-t,\infty)}(X)$.
(b) $P_{[-t,\infty)} \int_{-\infty}^{-t} f(-s)dX(s) = \int_{-t}^{\infty}\{\int_0^\infty b(t+s,\tau)f(t+\tau)d\tau\}dX(s)$.

As in [AIK, Definition 2.2], we define another Brownian motion $(W^*(t) : t \in \mathbf{R})$ by

$$W^*(t) := \theta(W(-t)), \qquad t \in \mathbf{R}. \tag{3.10}$$

Proposition 3.10 *Let* I *be a closed interval of* $\mathbf{R}$ *and let* $f \in \mathcal{H}_I(X)$. *Then*

$$\int_I f(s)dX(s) = \int_{-\infty}^{\infty}\left\{\int_I f(u)c(s-u)du\right\}dW^*(s).$$

The proof of Proposition 3.10 is the same as that of [AIK, Proposition 3.5], whence we omit it. We need Theorem 3.9 and Proposition 3.10 in the next section.

Example 3.11 As in Example 2.3, we consider fBm $(B_H(t))$ with $1/2 < H < 1$. Then the MA(∞) coefficient $c(t)$ is given by

$$c(t) = t^{H-(3/2)}\frac{1}{\Gamma(H-\frac{1}{2})}, \qquad t > 0, \tag{3.11}$$

so that $\int_0^\infty e^{izt}c(t)dt = (-iz)^{(1/2)-H}$ for $\Im z > 0$. From (3.2), we have

$$\int_0^\infty e^{izt}\alpha(t)dt = (-iz)^{H-(3/2)}.$$

Hence, $\alpha(t) = t^{(1/2)-H}/\Gamma(\frac{3}{2}-H)$, so that the AR($\infty$) coefficient $a(t)$ is given by

$$a(t) = t^{-(H+(1/2))}\frac{H-\frac{1}{2}}{\Gamma(\frac{3}{2}-H)}, \qquad t>0. \tag{3.12}$$

By the change of variable $u = sv$, $\int_0^s (s-u)^{H-(3/2)}(t+u)^{-H-(1/2)}du$ becomes

$$\begin{aligned} &s^{H-(1/2)}t^{-H-(1/2)}\int_0^1 (1-v)^{H-(3/2)}\{1+(s/t)v\}^{-H-(1/2)}dv \\ &= \frac{1}{(H-\frac{1}{2})}\left(\frac{s}{t}\right)^{H-(1/2)}\frac{1}{t+s}, \end{aligned}$$

where we have used the equality

$$\int_0^1 (1-v)^{p-1}(1+xv)^{-p-1}dv = \frac{1}{p(x+1)}, \qquad p>0,\ x>-1.$$

Thus

$$b(t,s) = \frac{\sin\{\pi(H-\frac{1}{2})\}}{\pi}\left(\frac{s}{t}\right)^{H-(1/2)}\frac{1}{t+s}, \qquad t>0,\ s>0; \tag{3.13}$$

and so, from Theorem 3.8, we see that, for $0 \le t < T$,

$$\begin{aligned} &E[B_H(T) \mid \sigma(B_H(s) : -\infty < s \le t)] \\ &= B_H(t) + \frac{\sin\{\pi(H-\frac{1}{2})\}}{\pi}\int_{-\infty}^t \left\{\int_0^{T-t}\left(\frac{\tau}{t-s}\right)^{H-(1/2)}\frac{1}{t-s+\tau}d\tau\right\}dB_H(s). \end{aligned}$$

This prediction formula was obtained in [GN, Theorem 3.1] by a different method.

4. Finite past prediction problems

In this section, we assume (1.1)–(1.7) and (1.10). Notice that (1.6) with (1.7) implies (3.1) as well as (2.3), whence (2.1). For t_0, t_1, and T in (1.10), we put

$$t_2 := t_0 + t_1, \quad t_3 := T - t_1.$$

4.1. Alternating projections to the past and future

For $n \in \mathbf{N}$, we define the orthogonal projection operator P_n by

$$P_n := \begin{cases} P_{(-\infty,t_1]}, & n = 1, 3, 5, \ldots, \\ P_{[-t_0,\infty)}, & n = 2, 4, 6, \ldots. \end{cases}$$

It should be noted that $\{P_n\}_{n=1}^{\infty}$ is merely an alternating sequence of projection operators, first to $M_{(-\infty,t_1]}(X)$, then to $M_{[-t_0,\infty)}(X)$, and so on. This sequence plays a key role in the proof of the finite past prediction formula for $(X(t))$.

For $t, s \in (0, \infty)$ and $n \in \mathbf{N}$, we define $b_n(t, s) = b_n(t, s; t_2)$ iteratively by

$$\begin{cases} b_1(t, s) := b(t, s), \\ b_n(t, s) := \int_0^{\infty} b(t, u) b_{n-1}(t_2 + u, s) du, \quad n = 2, 3, \ldots. \end{cases} \tag{4.1}$$

Proposition 4.1 *For $f \in \mathcal{H}_{[t_1,\infty)}(X)$, the following assertions hold:*

(a) $\int_0^{\infty} b_n(t_1 - \cdot, \tau) f(t_1 + \tau) d\tau \in \mathcal{H}_{(-\infty,t_1]}(X)$ *for* $n = 1, 3, 5, \ldots$.
(b) $\int_0^{\infty} b_n(t_0 + \cdot, \tau) f(t_1 + \tau) d\tau \in \mathcal{H}_{[-t_0,\infty)}(X)$ *for* $n = 2, 4, 6, \ldots$.

Proof. We may assume that $f \geq 0$. By Theorem 3.7, (a) holds for $n = 1$. By the Fubini–Tonelli theorem, we have, for $s > -t_0$,

$$\int_0^{\infty} du b(t_0 + s, u) \int_0^{\infty} b_1(t_2 + u, \tau) f(t_1 + \tau) d\tau$$
$$= \int_0^{\infty} b_2(t_0 + s, \tau) f(t_1 + \tau) d\tau.$$

Hence, by Theorem 3.9, we have (b) for $n = 2$. Repeating this procedure, we obtain the proposition. □

Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. By Proposition 4.1, we may define the random variables $G_n(f)$ by

$$G_n(f) := \begin{cases} \int_{-t_0}^{t_1} \left\{ \int_0^\infty b_n(t_1 - s, \tau) f(t_1 + \tau) d\tau \right\} dX(s), & n = 1, 3, \ldots, \\ \int_{-t_0}^{t_1} \left\{ \int_0^\infty b_n(t_0 + s, \tau) f(t_1 + \tau) d\tau \right\} dX(s), & n = 2, 4, \ldots. \end{cases}$$

We may also define the random variables $\epsilon_n(f)$ by $\epsilon_0(f) := \int_{t_1}^\infty f(s)dX(s)$ and

$$\epsilon_n(f) := \begin{cases} \int_{-\infty}^{-t_0} \left\{ \int_0^\infty b_n(t_1 - s, \tau) f(t_1 + \tau) d\tau \right\} dX(s), & n = 1, 3, \ldots, \\ \int_{t_1}^{\infty} \left\{ \int_0^\infty b_n(t_0 + s, \tau) f(t_1 + \tau) d\tau \right\} dX(s), & n = 2, 4, \ldots. \end{cases}$$

Proposition 4.2 *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$ and $n \in \mathbf{N}$. Then*

$$P_n P_{n-1} \cdots P_1 \int_{t_1}^\infty f(s) dX(s) = \epsilon_n(f) + \sum_{k=1}^n G_k(f). \tag{4.2}$$

We can prove (4.2) using Proposition 4.1 and the facts

$$M_{[-t_0,t_1]}(X) \subset M_{(-\infty,t_1]}(X) \cap M_{[-t_0,\infty)}(X), \tag{4.3}$$

$$G_k \in M_{[-t_0,t_1]}(X), \qquad k = 1, 2, \ldots. \tag{4.4}$$

Since the proof is similar to that of [AIK, Proposition 4.4], we omit the details.

We are about to investigate the limit of (4.2) as $n \to \infty$ (see Lemma 4.9 below).

For $f \in \mathcal{H}_{[t_1,\infty)}(X)$ and $s > 0$, we define $D_n(s, f) = D_n(s, f; t_1, t_2)$ by

$$D_n(s, f) := \begin{cases} \int_0^\infty c(u) f(t_1 + s + u) du, & n = 0, \\ \int_0^\infty du\, c(u) \int_0^\infty b_n(t_2 + u + s, \tau) f(t_1 + \tau) d\tau, & n = 1, 2, \ldots. \end{cases}$$

From the proof of the next proposition, we see that these integrals converge absolutely. Recall $(W^*(t))$ from (3.10).

Proposition 4.3 *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. Then*

$$P_{n+1}^{\perp}\epsilon_n(f) = \begin{cases} \int_{t_1}^{\infty} D_n(s-t_1,f)dW(s), & n=0,2,4,\dots, \\ \int_{-\infty}^{-t_0} D_n(-t_0-s,f)dW^*(s), & n=1,3,5,\dots. \end{cases}$$

Proof. By (3.9) and Proposition 3.5,

$$P_1^{\perp}\epsilon_0(f) = \int_{t_1}^{\infty}\left\{\int_s^{\infty} f(u)c(u-s)du\right\}dW(s) = \int_{t_1}^{\infty} D_0(s-t_1,f)dW(s).$$

Thus the assertion holds for $n=0$. Let $n=1,3,\dots$. Then, by Proposition 3.10,

$$\epsilon_n(f) = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{-t_0} duc(s-u)\int_0^{\infty} b_n(t_1-u,\tau)f(t_1+\tau)d\tau\right\}dW^*(s).$$

Hence, using [AIK, Proposition 2.3 (7)] and (3.7),

$$\begin{aligned} &P_{n+1}^{\perp}\epsilon_n(f) \\ &= \int_{-\infty}^{-t_0}\left\{\int_{-\infty}^{s} duc(s-u)\int_0^{\infty} b_n(t_1-u,\tau)f(t_1+\tau)d\tau\right\}dW^*(s) \\ &= \int_{-\infty}^{-t_0}\left\{\int_0^{\infty} duc(u)\int_0^{\infty} b_n(t_2+u-t_0-s,\tau)f(t_1+\tau)d\tau\right\}dW^*(s) \\ &= \int_{-\infty}^{-t_0} D_n(-t_0-s,f)dW^*(s). \end{aligned}$$

Thus we obtain the assertion for $n=1,3,\dots$. The proof for $n=2,4,\dots$ is similar; and so we omit it. □

From Propositions 4.2 and 4.3, we immediately obtain the next proposition (cf. the proof of [AIK, Proposition 4.9]).

Proposition 4.4 *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. Then the following assertions hold:*

(a) $\|P_1^{\perp}\int_{t_1}^{\infty} f(s)dX(s)\|^2 = \int_0^{\infty} D_0(s,f)^2 ds.$

(b) $\|P_{n+1}^{\perp}P_nP_{n-1}\cdots P_1\int_{t_1}^{\infty}f(s)dY(s)\|^2 = \int_0^{\infty}D_n(s,f)^2ds$ *for* $n = 1,2,\ldots$.

We write Q for the orthogonal projection operator from $M(X)$ onto the intersection $M_{(-\infty,t_1]}(X)\cap M_{[-t_0,\infty)}(X)$. Then, by von Neumann's alternating projection theorem (see, e.g., [P, Theorem 9.20]), we have $Q = \text{s-lim}_{n\to\infty}P_nP_{n-1}\cdots P_1$. Using this, (4.3) and Proposition 4.4, we immediately obtain the next proposition (cf. the proof of [AIK, Proposition 4.9 (3)]).

Proposition 4.5 *Let* $f\in\mathcal{H}_{[t_1,\infty)}(X)$. *Then* $\lim_{n\to\infty}\int_0^{\infty}D_n(s,f)^2ds = 0$.

We need the next proposition.

Proposition 4.6 *Let* $f\in\mathcal{H}_{[t_1,\infty)}(X)$. *Then, for* $t>0$ *and* $n=0,1,\ldots,$ *we have*

$$\int_0^{\infty}b_{n+1}(t,\tau)f(t_1+\tau)d\tau = \int_0^{\infty}a(t+u)D_n(u,f)du.$$

Proof. We may assume $f\geq 0$. By the Fubini–Tonelli theorem, we have, for $t>0$,

$$\begin{aligned}\int_0^{\infty}b_1(t,\tau)f(t_1+\tau)d\tau &= \int_0^{\infty}\left\{\int_0^{\tau}c(\tau-u)a(t+u)du\right\}f(t_1+\tau)d\tau\\ &= \int_0^{\infty}a(t+u)\left\{\int_0^{\infty}c(\tau)f(t_1+u+\tau)d\tau\right\}du\\ &= \int_0^{\infty}a(t+u)D_0(u,f)du.\end{aligned}$$

Thus the assertion holds for $n=0$. Now we assume that $n\geq 1$. Since we have

$$b_{n+1}(t,\tau) = \int_0^{\infty}a(t+v)\left\{\int_0^{\infty}c(u)b_n(t_2+u+v,\tau)du\right\}dv,\quad t,\tau>0,$$

we obtain the assertion, again using the Fubini–Tonelli theorem. □

For $t,s>0$, we define $k(t,s)=k(t,s;t_2)$ by

$$k(t,s) := \int_0^\infty c(t+u)a(t_2+u+s)du.$$

Notice that $k(t,s) < \infty$ for $t, s > 0$ since $k(t,s) \le c(t)\int_{t_2+s}^\infty a(u)du$.

Proposition 4.7 *Let $f \in \mathcal{H}_{[t_1,\infty)}(X)$. Then*

$$P_{n+1}\epsilon_n(f) = \begin{cases} \int_{-\infty}^{t_1} \left\{ \int_0^\infty k(t_1-s,u)D_{n-1}(u,f)du \right\} dW(s), & n = 2,4,\ldots, \\ \int_{-t_0}^{\infty} \left\{ \int_0^\infty k(t_0+s,u)D_{n-1}(u,f)du \right\} dW^*(s), & n = 1,3,\ldots. \end{cases}$$

Proof. We assume $n = 2, 4, \ldots$. Then, by Propositions 3.5 and 4.6, we have

$$\begin{aligned} &P_{n+1}\epsilon_n(f) \\ &= \int_{-\infty}^{t_1} \left\{ \int_{t_1}^\infty duc(u-s) \int_0^\infty b_n(t_0+u,\tau)f(t_1+\tau)d\tau \right\} dW(s) \\ &= \int_{-\infty}^{t_1} \left\{ \int_0^\infty dvc(t_1-s+v) \int_0^\infty a(t_2+v+u)D_{n-1}(u,f)duv \right\} dW(s) \\ &= \int_{-\infty}^{t_1} \left\{ \int_0^\infty k(t_1-s,u)D_{n-1}(u,f)du \right\} dW(s). \end{aligned}$$

The proof of the case $n = 1, 3, \ldots$ is similar. □

We need the next L^2-boundedness theorem.

Theorem 4.8 *Let $p \in (0, 1/2)$ and let $\ell(\cdot)$ be a slowly varying function at infinity. Let $C(\cdot)$ and $A(\cdot)$ be nonnegative and decreasing functions on $(0,\infty)$. We assume $C(\cdot) \in L^1_{\mathrm{loc}}[0,\infty)$ and $A(0+) < \infty$. We also assume*

$$A(t) \sim t^{-(1+p)}\ell(t)p, \qquad t \to \infty,$$

$$C(t) \sim \frac{t^{-(1-p)}}{\ell(t)} \cdot \frac{\sin(p\pi)}{\pi}, \qquad t \to \infty,$$

and put $K(x,y) := \int_0^\infty C(x+u)A(u+y)du$ for $x, y > 0$. Then

$$\sup_{x>0}\int_0^\infty K(x,y)(x/y)^{1/2}dy<\infty,\qquad \sup_{y>0}\int_0^\infty K(x,y)(y/x)^{1/2}dx<\infty.$$

In particular, the integral operator K defined by $(Kf)(x) := \int_0^\infty K(x,y)f(y)dy$ for $x>0$ is a bounded operator on $L^2((0,\infty),dy)$.

We omit the proof of Theorem 4.8 which is similar to that of [IA, Theorem 5.1].

By putting $z=iy$ in (3.2), we get

$$y\left\{\int_0^\infty e^{-yt}c(t)dt\right\}\left\{\int_0^\infty e^{-yt}\alpha(t)dt\right\}=1,\qquad y>0.$$

By Karamata's Tauberian theorem (cf. [BGT, Theorem 1.7.6]) applied to this, (2.3) implies $\alpha(t)\sim t^{-(H-\frac{1}{2})}/\{\ell(t)\Gamma((3/2)-H)\}$ as $t\to\infty$. This and the monotone density theorem give

$$a(t)\sim\frac{t^{-(H+1/2)}}{\ell(t)}\cdot\frac{(H-\frac{1}{2})}{\Gamma(\frac{3}{2}-H)},\qquad t\to\infty. \tag{4.5}$$

The next lemma is a key to our arguments.

Lemma 4.9 *Let $f\in\mathcal{H}_{[t_1,\infty)}(X)$. Then $\|\epsilon_n(f)\|\to 0$ as $n\to\infty$.*

Proof. It follows from (2.3), (4.5) and Theorem 4.8 below that the integral operator K defined by $Kf(t) := \int_0^\infty k(t,s)f(s)ds$ is a bounded operator on $L^2((0,\infty),ds)$. Hence, by Propositions 4.3, 4.5 and 4.7, we have

$$\begin{aligned}\|\epsilon_n(f)\|^2 &= \int_0^\infty D_n(s,f)^2ds+\int_0^\infty\left\{\int_0^\infty k(s,u)D_{n-1}(u,f)du\right\}^2ds\\ &\le\int_0^\infty D_n(s,f)^2ds+\|K\|^2\int_0^\infty D_{n-1}(s,f)^2ds\to 0,\qquad n\to\infty.\end{aligned}$$

Thus the lemma follows. □

We can now state the conclusions of the arguments above.

Theorem 4.10 *The following assertions hold*:

(a) $M_{[-t_0,t_1]}(X)=M_{(-\infty,t_1]}(X)\cap M_{[-t_0,\infty)}(X)$.
(b) $P_{[-t_0,t_1]}=\text{s-lim}_{n\to\infty}P_nP_{n-1}\cdots P_1$.

(c) $\|P_{[-t_0,t_1]}^{\perp}Z\|^2 = \|P_1^{\perp}Z\|^2 + \sum_{n=1}^{\infty}\|(P_{n+1})^{\perp}P_n\cdots P_1 Z\|^2$ *for* $Z \in M(X)$.

We can prove Theorem 4.10 using Proposition 4.2 and Lemma 4.9. Since the proof is similar to that of [AIK, Theorem 4.6], we omit the details.

4.2. Finite past prediction formulas

We define $h(s,u) = h(s,u;t_2)$ by

$$h(s,u) := \sum_{k=1}^{\infty}\{b_{2k-1}(t_2-s,u) + b_{2k}(s,u)\}, \qquad 0 < s < t_2,\ u > 0. \quad (4.6)$$

Here is the finite past prediction formula for $\int_{t_1}^{\infty} f(s)dX(s)$.

Theorem 4.11 *Let* $f \in \mathcal{H}_{[t_1,\infty)}(X)$. *Then the following assertions hold*:

(a) $\int_0^{\infty} h(t_0+\cdot,u)f(t_1+u)du \in \mathcal{H}_{[-t_0,t_1]}(X)$.
(b) $P_{[-t_0,t_1]}\int_{t_1}^{\infty} f(s)dX(s) = \int_{-t_0}^{t_1}\{\int_0^{\infty} h(t_0+s,u)f(t_1+u)du\}dX(s)$.
(c) $\|P_{[-t_0,t_1]}^{\perp}\int_{t_1}^{\infty} f(s)dX(s)\|^2 = \sum_{n=0}^{\infty}\int_0^{\infty} D_n(s,f)^2 ds$.

Proof. We may assume that $f \geq 0$. By Theorem 4.10 (b), Proposition 4.2 and Lemma 4.9, we have, in $M(X)$,

$$\begin{aligned} P_{[-t_0,t_1]}\int_{t_1}^{\infty} f(s)dX(s) &= \lim_{n\to\infty} P_n P_{n-1}\cdots P_1 \int_{t_1}^{\infty} f(s)dX(s) \\ &= \lim_{n\to\infty}\int_{-t_0}^{t_1}\left\{\int_0^{\infty} h_n(t_0+u,v)f(t_1+v)dv\right\}dX(s), \end{aligned}$$

where, for $0 < s < t_2$ and $u > 0$, we define $h_n(s,u) = h_n(s,u;t_2)$ by

$$h_n(s,u) = \begin{cases} b_1(t_2-s,u) + b_2(s,u) + \cdots + b_n(t_2-s,u), & n = 1,3,5,\ldots, \\ b_1(t_2-s,u) + b_2(s,u) + \cdots + b_n(s,u), & n = 2,4,6,\ldots. \end{cases}$$

Since $h_n(s,u) \uparrow h(s,u)$, we obtain (a) and (b) using the monotone convergence theorem. Finally, (c) follows from Theorem 4.10 (c) and Proposition 4.4. □

For $s, u > 0$, we define $D_n(s) = D_n(s;t_2,t_3)$ by

$$D_n(s) := \int_0^{\infty} du c(u)\int_0^{t_3} b_n(t_2+u+s,\tau)d\tau, \qquad n = 1,2,\ldots.$$

Here are the solutions to the finite past prediction problems for $(X(t))$.

Theorem 4.12 *The finite past prediction formula* (1.12) *and the following equality for the mean-square prediction error hold*:

$$\left\|P^{\perp}_{[-t_0,t_1]}X(T)\right\|^2 = \int_0^{T-t_1} g(s)^2 ds + \sum_{n=1}^{\infty}\int_0^{\infty} D_n(s)^2 ds.$$

Proof. We put $f(s) = I_{(t_1,T]}(s)$. Then $\int_{t_1}^{\infty} f(s)dX(s) = X(T) - X(t_1)$ and

$$\int_0^{\infty} h(t_0+s,u)f(t_1+u)du = \int_0^{t_3} h(t_0+s,u)du, \qquad -t_0 < s < t_1.$$

We also have $D_n(s,f) = D_n(s)$ for $n = 1,2,\dots$ and $D_0(s,f) = g(t_3 - s)$. Thus the theorem follows from Theorem 4.11. □

5. AR(∞)-type equations

In this section, we consider the AR(∞)-type equations for $(X(t))$ in (1.1) and $(\tilde{X}(t))$ in (1.13). For a Borel measure τ on $(0,\infty)$ satisfying $\int_0^{\infty}(1+s)^{-1}\tau(ds) < \infty$, we write

$$F_{\tau}(z) := \int_0^{\infty} \frac{1}{\lambda - iz}\tau(d\lambda), \qquad \Im z \geq 0.$$

First, we consider the process $X = (X(t))$ in (1.1) with (1.2)–(1.5), (2.1) and (3.1). Let $f_t(s) := g(t-s) - g(-s) = \int_{-s}^{t-s} c(u)du$ for $t, s \in \mathbf{R}$.

Lemma 5.1 *Let $t \in \mathbf{R}$. Then the Fourier transform of $f_t(\cdot)$ in the L^2-sense is equal to $(i\xi)^{-1}(1-e^{-it\xi})F_{\nu}(\xi)$*:

$$\frac{(1-e^{-it\xi})}{i\xi}F_{\nu}(\xi) = \underset{M\to\infty}{\text{l.i.m.}}\int_{-M}^{M} e^{-is\xi}f_t(s)ds. \tag{5.1}$$

Proof. Since $\int_{-\infty}^{\infty}|f_t(s)|^2 ds < \infty$, the limit on the right-hand side of (5.1) exists. Therefore, it is enough to justify the following point-wise convergence:

$$\frac{(1-e^{-it\xi})}{i\xi}F_\nu(\xi)=\lim_{M\to\infty}\int_{-M}^{M}e^{-is\xi}f_t(s)ds,\qquad \xi\neq 0. \tag{5.2}$$

Now, if $-M\le t\le M$, then

$$\begin{aligned}&\int_{-M}^{M}e^{-is\xi}f_t(s)ds\\&=\int_{-M}^{M}dse^{-is\xi}\int_0^t c(u-s)du=\int_0^t du\int_{-M}^{M}e^{-is\xi}c(u-s)ds\\&=\int_0^t due^{-iu\xi}\int_{u-M}^{u+M}e^{iv\xi}c(v)dv=\int_0^t due^{-iu\xi}\int_0^{u+M}e^{iv\xi}c(v)dv\end{aligned}$$

because $u-M\le 0\le u+M$ for u between 0 and t, and $c(s)=0$ for $s\le 0$. However,

$$\begin{aligned}&\int_0^t due^{-iu\xi}\int_0^{u+M}e^{is\xi}c(s)ds\\&\quad=\int_0^t due^{-iu\xi}\int_0^\infty\frac{1-e^{(i\xi-\lambda)(u+M)}}{\lambda-i\xi}\nu(d\lambda)\\&\quad=\frac{(1-e^{-it\xi})}{i\xi}F_\nu(\xi)-e^{i\xi M}\int_0^t du\int_0^\infty\frac{e^{-\lambda(u+M)}}{\lambda-i\xi}\nu(d\lambda),\end{aligned}$$

so that, for $\xi\neq 0$,

$$\left|\frac{(1-e^{-it\xi})}{i\xi}F_\nu(\xi)-\int_{-M}^{M}e^{-is\xi}f_t(s)ds\right|\le t\int_0^\infty\frac{e^{-\lambda M}}{|\lambda-i\xi|}\nu(d\lambda)\downarrow 0,\quad M\to\infty.$$

Thus, (5.2) holds. □

For the Brownian motion $W=(W(t))$ in (1.1), let $DW(\phi)=\int_{-\infty}^{\infty}\hat{\phi}(\xi)Z_{DW}(d\xi)$ with $\phi\in\mathcal{D}(\mathbf{R})$ be the spectral decomposition of DW as a stationary random distribution, where $\hat{\phi}(\xi):=\int_{-\infty}^{\infty}e^{-it\xi}\phi(\xi)d\xi$ and Z_{DW} is the associated complex-valued random measure such that $E[Z_{DW}(A)\overline{Z_{DW}(B)}]=(2\pi)^{-1}\int_{A\cap B}d\xi$ (see Itô [It]). By Lemma 5.1 and the Parseval-type formula for the homogeneous random measure Z_{DW}, we obtain $X(t)=\int_{-\infty}^{\infty}[(1-e^{-it\xi})/(i\xi)]F_\nu(\xi)Z_{DW}(d\xi)$, whence

$$DX(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(\xi) F_\nu(\xi) Z_{DW}(d\xi), \qquad \phi \in \mathcal{D}(\mathbf{R}). \tag{5.3}$$

Let ρ_{DX} be the spectral measure of DX: $E[X(\phi)\overline{X(\psi)}] = \int_{-\infty}^{\infty} \hat{\phi}(\xi)\overline{\hat{\psi}}(\xi)$ $\rho_{DX}(d\xi)$. Then, from (5.3), we see that $\rho_{DX}(d\xi) = (2\pi)^{-1}|F_\nu(\xi)|^2 d\xi$. Thus, DX has the spectral density $\Delta_{DX}(\xi) := (2\pi)^{-1}|F_\nu(\xi)|^2$. Since, for $z = x+iy$ with $y > 0$, we have

$$\Re\{F_\nu(z)\} = \int_0^\infty \frac{s+y}{(s+y)^2 + x^2} \nu(ds) > 0,$$

the function $F_\nu(z)$ is an outer function on the upper half plane $\Im z > 0$:

$$F_\nu(z) = \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+\xi z}{\xi - z} \cdot \frac{\log|F_\nu(\xi)|}{1+\xi^2} d\xi\right\}, \qquad \Im z > 0. \tag{5.4}$$

In particular, Proposition 3.6 follows from this and (5.3).

We also have the next lemma.

Lemma 5.2 *The following equality holds*:

$$\int_0^\infty e^{izt} c(t) dt = \sqrt{2\pi} \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1+\xi z}{\xi - z} \cdot \frac{\log|\Delta_{DX}(\xi)|}{1+\xi^2} d\xi\right\}, \qquad \Im z > 0.$$

Proof. Since $F_\nu(z) = \int_0^\infty e^{izt} c(t) dt$ and $|F_\nu(\xi)| = \{2\pi\Delta_{DX}(\xi)\}^{1/2}$, the lemma follows from (5.4). □

From Lemma 5.2, we see that the kernel $c(\cdot)$ is uniquely determined by DX, whence $(X(t))$, as claimed in Section 3.2.

Let $D^2X := D(DX)$. For the AR(∞) kernel $\alpha(\cdot)$ in Section 3.1, we define the convolution $\alpha * D^2X$, which is also a stationary random distribution, by

$$(\alpha * D^2X)(\phi) := \operatorname*{l.i.m.}_{M\to\infty} \int_0^M \alpha(u) D^2X(\tau_u \phi) du, \qquad \phi \in \mathcal{D}(\mathbf{R}), \tag{5.5}$$

where $\tau_u\phi(t) := \phi(t+u)$ and the integral on the right-hand side is an $M(X)$-valued Bochner integral. Then, by [I2, Proposition 2.3] and (5.3), we have

$$(\alpha * D^2X)(\phi) = -\int_{-\infty}^{\infty} i\xi F_\mu(\xi)F_\nu(\xi)\hat{\phi}(\xi)Z_{DW}(d\xi).$$

However, since (3.2) implies $-i\xi F_\mu(\xi)F_\nu(\xi) = 1$ for $\xi \neq 0$, we see that X satisfies

$$\alpha * D^2X = DW. \tag{5.6}$$

More precisely, we have the next theorem.

Theorem 5.3 *The process $(X(t))$ is the only stationary-increment process with $X(0) = 0$ satisfying the following two conditions*:

(1) *the stationary random distribution DX is purely nondeterministic*;
(2) *$(X(t))$ satisfies* (5.6).

The proof of Theorem 5.3 is similar to that of [AI2, Theorem 2.6], whence we omit it. Notice that (5.6) can be written formally as the following AR(∞)-type equation:

$$\int_{-\infty}^{t} \alpha(t-s)\frac{d^2X}{ds^2}(s)ds = \frac{dW}{dt}(t). \tag{5.7}$$

Example 5.4 Let $(B_H(t))$ be the fBm in (2.5) with $1/2 < H < 1$. Then, by Example 3.11, we have $\alpha(t) = t^{(1/2)-H}/\Gamma(\frac{3}{2} - H)$ for $t > 0$, whence (5.7) becomes

$$\frac{1}{\Gamma\left(\frac{3}{2} - H\right)} \int_{-\infty}^{t} \frac{1}{(t-s)^{H-(1/2)}} \cdot \frac{d^2B_H}{ds^2}(s)ds = \frac{dW}{dt}(t).$$

Next, we turn to $\tilde{X} = (\tilde{X}(t))$ in (1.13) with (1.14). We assume that $\tilde{\nu}$ is a Borel measure on $(0, \infty)$ satisfying the following conditions:

$$\int_0^\infty \frac{1}{1+s}\tilde{\nu}(ds) < \infty, \quad \tilde{\nu}((0,\infty)) = \int_0^\infty \frac{1}{s}\tilde{\nu}(ds) = \infty, \quad \int_0^1 \tilde{c}(t)^2dt < \infty.$$

By [IA, Theorem 3.2], there exists a unique Borel measure $\tilde{\mu}$ on $(0, \infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s}\tilde{\mu}(ds) < \infty, \quad \tilde{\mu}((0,\infty)) = \int_0^\infty \frac{1}{s}\tilde{\mu}(ds) = \infty,$$

and $-izF_{\tilde{\nu}}(z)F_{\tilde{\mu}}(z) = 1$ for $\Im z > 0$. If we define

$$\tilde{\alpha}(t) := \int_0^\infty e^{-st}\tilde{\mu}(ds), \qquad t > 0,$$

then the last equality becomes

$$-iz\left\{\int_0^\infty e^{izt}\tilde{c}(t)dt\right\}\left\{\int_0^\infty e^{izt}\tilde{\alpha}(t)dt\right\} = 1, \qquad \Im z > 0. \tag{5.8}$$

By [IA, (2.3)], we have

$$D\tilde{X}(\phi) = \int_{-\infty}^\infty \hat{\phi}(\xi)(-i\xi)F_{\tilde{\nu}}(\xi)Z_{DW}(d\xi), \qquad \phi \in \mathcal{D}(\mathbf{R}), \tag{5.9}$$

whence, in the same way as the proof of [I1, Proposition 5.1], we get

$$(\tilde{\alpha} * D\tilde{X})(\phi) = -\int_{-\infty}^\infty i\xi F_{\tilde{\mu}}(\xi)F_{\tilde{\nu}}(\xi)\hat{\phi}(\xi)Z_{DW}(d\xi), \qquad \phi \in \mathcal{D}(\mathbf{R}),$$

where the convolution $\tilde{\alpha} * D\tilde{X}$ is defined in the same way as (5.5). However, since $-i\xi F_{\tilde{\mu}}(\xi)F_{\tilde{\nu}}(\xi) = 1$ for $\xi \neq 0$, we see that $(\tilde{X}(t))$ satisfies

$$\tilde{\alpha} * D\tilde{X} = DW. \tag{5.10}$$

Notice that the equation (5.10) can be written formally as the following AR(∞)-type equation:

$$\int_{-\infty}^t \tilde{\alpha}(t-s)\frac{d\tilde{X}}{ds}(s)ds = \frac{dW}{dt}(t). \tag{5.11}$$

We can also prove an analogue of Theorem 5.3 for $(\tilde{X}(t))$, which we omit in this paper.

Example 5.5 Let $(B_H(t))$ be the fBm in (2.5) with $0 < H < 1/2$. Then, by [IA, Example 3.9], we have $\tilde{\alpha}(t) = t^{-(1/2)-H}/\Gamma(\frac{1}{2} - H)$ for $t > 0$, whence (5.11) becomes

$$\frac{1}{\Gamma(\frac{1}{2}-H)}\int_{-\infty}^t \frac{1}{(t-s)^{H+(1/2)}} \cdot \frac{dB_H}{ds}(s)ds = \frac{dW}{dt}(t).$$

Acknowledgements We would like to express our gratitude to an anonymous referee for useful suggestions.

References

[A] Adler R. J., *The Geometry of random fields*. John Wiley and Sons, New York, 1981.

[AI1] Anh V. V. and Inoue A., *Prediction of fractional Brownian motion with Hurst index less than* 1/2. Bull. Austral. Math. Soc. **70** (2004), 321–328.

[AI2] Anh V. V. and Inoue A., *Financial markets with memory I: Dynamic models*. Stochastic Anal. Appl. **23** (2005), 275–300.

[AIK] Anh V. V., Inoue A. and Kasahara Y., *Financial markets with memory II: Innovation processes and expected utility maximization*. Stochastic Anal. Appl. **23** (2005), 301–328.

[BGT] Bingham N. H., Goldie C. M. and Teugels J. L., *Regular Variation*. 2nd ed., Cambridge Univ. Press, Cambridge, 1989.

[DU] Decreusefond L. and Üstünel A. S., *Stochastic analysis of the fractional Brownian motion*. Potential Analysis **10** (1999), 177–214.

[DMP] Duncan T. E., Maslowski B. and Pasik-Duncan B., *Fractional Brownian motion and stochastic equations in Hilbert spaces*. Stoch. Dyn. **2** (2002), 225–250.

[GA] Grecksch W. and Anh V. V., *A parabolic stochastic differential equation with fractional Brownian motion input*. Statist. Probab. Letters **41** (1999), 337–345.

[GN] Gripenberg G. and Norros I., *On the prediction of fractional Brownian motion*. J. Appl. Probab. **33** (1996), 400–410.

[HOS] Hu Y., Øksendal B. and Sulem A., *Optimal consumption and portfolio in a Black-Scholes market driven by fractional Brownian motion*. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6** (2003), 519–536.

[I1] Inoue A., *On the equations of stationary processes with divergent diffusion coefficients*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **40** (1993), 307–336.

[I2] Inoue A., *Regularly varying correlation functions and KMO-Langevin equations*. Hokkaido Math. J. **26** (1997), 1–26.

[IA] Inoue A. and Anh V. V., *Prediction of fractional Brownian motion-type processes*. Stoch. Anal. Appl. **25** (2007), 641–666.

[IN] Inoue A. and Nakano Y., *Optimal long-term investment model with memory*. Appl. Math. Optim. **55** (2007), 93–122.

[INA] Inoue A., Nakano Y. and Anh V. V., *Linear filtering of systems with* *memory and application to finance*. J. Appl. Math. Stoch. Anal. 2006, Art. ID 53104, 26 pp.

[It] Itô K., *Stationary random distributions*. Mem. Coll. Sci. Univ. Kyoto **28** (1954), 209–223.

[K] Kolmogorov A., *Wienersche spiralen und einige andere interessante Kurven in Hilbertschen Raum*. C.R. (Dokl.) Acad. Sci. USSR (N.S.) **26** (1940), 422–437.

[MV] Mandelbrot B. and Van Ness J., *Fractional Brownian motions, fractional noises and applications*. SIAM Rev. **10** (1968), 422–437.

[MRRS] Mikosch T., Resnick S., Rootzen H. and Stegeman A., *Is network traffic approximated by stable Levy motion or fractional Brownian motion?* Ann. Appl. Probab. **12** (2002), 23–68.

[NVV] Norros I., Valkeila E. and Virtamo J., *An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions*. Bernoulli **5** (1999), 571–587.

[N] Nualart D., *Stochastic integration with respect to fractional Brownian motion and applications*. Stochastic models, Mexico City 2002, Contemp. Math., vol. 336, Amer. Math. Soc., Providence, RI, 2003, pp. 3–39.

[NP] Nuzman C. J. and Poor H. V., *Linear estimation of self-similar processes via Lamperti's transformation*. J. Appl. Probab. **37** (2000), 429–452.

[P] Pourahmadi M., *Foundations of Time Series Analysis and Prediction Theory*. Wiley-Interscience, New York, 2001.

[ST] Samorodnitsky G. and Taqqu M. S., *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman and Hall, New York, 1994.

[T] Taqqu M., *Fractional Brownian motion and long-range dependence*. Theory and Applications of Long-Range Dependence, P. Doukhan and M. Taqqu (Eds.), Birkhäuser, Boston, 2003, pp. 5–38.

[TTV] Tindel S., Tudor C. A. and Viems F., *Stochastic evolution equations with fractional Brownian motion*. Probab. Theory Relat. Fields **127** (2003), 186–204.

[Y] Yaglom A. M., *Correlation theory of processes with random stationary nth increments*. Mat. Sb. N.S. **37** (1955), 141–196 (in Russian). English translation in Am. Math. Soc. Translations Ser. (2) **8** (1958), 87–141.

Akihiko Inoue
Department of Mathematics
Hiroshima University
Higashi-Hiroshima 739-8526, Japan
E-mail: inoue100@hiroshima-u.ac.jp

Vo V. Anh
School of Mathematical Sciences
Queensland University of Technology
GPO Box 2434, Brisbane, Queensland 4001
Australia
E-mail: v.anh@qut.edu.au