Hokkaido Mathematical Journal Vol. 41 (2012) p. 135–155

Limits of iterations of complex maps and hypergeometric functions

(Dedicated to Professor Keizo Yamaguchi on his sixtieth birthday)

Keiji MATSUMOTO* and Takashi OIKAWA

(Received October 4, 2010)

Abstract. We consider the limit of the iteration of a map $z \mapsto m(z)$ from a complex domain D to D. For two kinds of maps m, we show that each iteration $m^n(z)$ of $m(z)$ converges for any $z \in D$ as $n \to \infty$ and that this limit is expressed by the hypergeometric function. These are analogs of the expression of the arithmetic-geometric mean by the Gauss hypergeometric function.

Key words: limit of iteration, hypergeometric function

1. Introduction

The arithmetic-geometric mean of a and b is defined by the limit of the iteration of a map consisting of the arithmetic and geometric means:

$$(a,b) \mapsto \left(\frac{a+b}{2}, \sqrt{ab}\right).$$

The limit is classically known to be expressed as

$$\frac{a}{F(1/2,1/2,1;1-b^2/a^2)},$$

where $F(\alpha,\beta,\gamma;z)$ is the Gauss hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(\alpha,n)(\beta,n)}{(\gamma,n)(1,n)} z^n, \qquad \begin{array}{l} |z|<1,\ \gamma \neq 0,-1,-2,\dots, \\ (\alpha,n)=\alpha(\alpha+1)\cdots(\alpha+n-1). \end{array}$$

Recently, several analogs of this expression of the arithmetic-geometric mean are obtained from transformation formulas of hypergeometric functions,

---

2000 Mathematics Subject Classification : Primary 30D05; Secondary 33C05, 33C65.

* This author was supported by Max Planck Institute for Mathematics.

refer to [BB1], [C], [HKM], [KM], [KS1], [KS2], and [M].

In this paper, we consider the limit of the iteration of a complex map

$$m : D \ni z \mapsto m(z) \in D.$$

The map m is given as follows:

(1) $D = \left\{ (z_1, z_2) \in (\mathbb{C}^*)^2 \mid |\arg(z_2/z_1)| < \frac{\pi}{3} \right\}$,

$$m(z_1, z_2) = \left(\sqrt[3]{\left(\frac{z_1 - \omega z_2}{1-\omega}\right)^2 \left(\frac{z_2 - \omega z_1}{1-\omega}\right)},\ \sqrt[3]{\left(\frac{z_1 - \omega z_2}{1-\omega}\right) \left(\frac{z_2 - \omega z_1}{1-\omega}\right)^2} \right),$$

(2) $D = \left\{ (z_1, z_2, z_3) \in (\mathbb{C}^*)^3 \mid |\arg(z_2/z_1)|, |\arg(z_3/z_1)| < \frac{\pi}{3} \right\}$,

$$m(z_1, z_2, z_3) = \big(m_1(z),\ \sqrt[3]{m_1(z)^3 - m_\omega(z)^3},\ \sqrt[3]{m_1(z)^3 - m_{\bar\omega}(z)^3}\big),$$

$$m_1(z) = \frac{z_1 + z_2 + z_3}{3}, \quad m_\omega(z) = \frac{z_1 + \omega z_2 + \bar\omega z_3}{3},$$

$$m_{\bar\omega}(z) = \frac{z_1 + \bar\omega z_2 + \omega z_3}{3},$$

where $\mathbb{C}^* = \mathbb{C} - \{0\}$, $\omega = (-1 + \sqrt{-3})/2$ and $\bar\omega$ is its complex conjugate. We assign branches of the cubic roots in m, and show that the iteration $m^n(z) = \overbrace{m \circ \cdots \circ m}^{n}(z)$ of $m(z)$ converges for any $z \in D$ as $n \to \infty$. By using a complex map version of the invariant principle in [BB2] together with a transformation formula for the hypergeometric function in [KS1], [MO] and [V], we express this limit by the hypergeometric function.

In the study of the case (1), we have Theorem 1 which states that the limit is expressed by the Gauss hypergeometric functions with parameters $(\alpha, \beta, \gamma) = (1/3, 2/3, 4/3)$. Its monodromy group is isomorphic to the triangle group $(3, 3, 3)$. In [HKM], several analogs of the arithmetic-geometric mean are studied and the triangle groups (r_1, r_2, r_3) with $1/r_1 + 1/r_2 + 1/r_3 = 1$ except $(3, 3, 3)$ appear. Theorem 1 completes the correspondence between analogs of the arithmetic-geometric mean and the triangle groups acting on Euclidean space $\mathbb{C}$.

The limit for the case (2) is studied in [KS1] when z_1, z_2 and z_3 are positive real numbers. By improving their results on the domain of the variables, we have Theorem 2.

2. The limit of the iteration of a complex map

Let $\mathbb{C}^*$ be the multiplicative group $\mathbb{C} - \{0\}$ and let D be a domain of $(\mathbb{C}^*)^k$ including $(t, \ldots, t)$ for any $t \in \mathbb{C}^*$. We consider a holomorphic map

$$m : D \ni z = (z_1, \ldots, z_k) \mapsto (m_1(z_1, \ldots, z_k), \ldots, m_k(z_1, \ldots, z_k)) \in D$$

satisfying

$$m(t, \ldots, t) = (t, \ldots, t) \quad \text{for any } t \in \mathbb{C}^*. \tag{1}$$

If the domain D and the map m satisfy

(1-a) $t \cdot z = (t \cdot z_1, \ldots, t \cdot z_k) \in D$ for any $t \in \mathbb{C}^*$ and $z = (z_1, \ldots, z_k) \in D$,
(1-b) $m(t \cdot z) = t \cdot m(z)$ for any $t \in \mathbb{C}^*$ and $z \in D$,
(1-c) $m(1, \ldots, 1) = (1, \ldots, 1)$,

then m satisfies (1).

Suppose that for any $z = (z_1, \ldots, z_k) \in D$ there exists $\alpha \in \mathbb{C}^*$ such that

$$\lim_{n \to \infty} m^n(z_1, \ldots, z_k) = \lim_{n \to \infty} \overbrace{m \circ \cdots \circ m}^{n}(z) = (\alpha, \ldots, \alpha).$$

This limit value $\alpha \in \mathbb{C}^*$ is denoted by $m_*^\infty(z) = m_*^\infty(z_1, \ldots, z_k)$. It is characterized by the following proposition, which is a complex map version of the invariant principle in [BB2].

Proposition 1 (Invariant principle) *If a holomorphic function μ on D satisfies*

(i) $\mu(t, \ldots, t) = t$ *for any* $t \in \mathbb{C}^*$,
(ii) $\mu(m(z_1, \ldots, z_k)) = \mu(z_1, \ldots, z_k)$ *for any* $z = (z_1, \ldots, z_k) \in D$,

then $m_*^\infty(z_1, \ldots, z_k) = \mu(z_1, \ldots, z_k)$.

Proof. By using the condition (ii), we have

$$\mu(z) = \mu(m(z)) = \cdots = \mu(m^n(z)).$$

Since μ is continuous and $m^n(z)$ converges to $(m_*^\infty(z), \dots, m_*^\infty(z))$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \mu(m^n(z)) = \mu(m_*^\infty(z), \dots, m_*^\infty(z)).$$

Thus we obtain

$$\mu(z) = \mu(m_*^\infty(z), \dots, m_*^\infty(z)) = m_*^\infty(z)$$

by the condition (i). $\square$

3. Limit of a pair of sequences

In this section, we consider a map

$$m : (z_1, z_2) \mapsto (m_1(z_1, z_2), m_2(z_1, z_2)) = \left(\sqrt[3]{w_1^2 w_2},\ \sqrt[3]{w_1 w_2^2}\right),$$
$$w_1 = \frac{z_1 - \omega z_2}{1-\omega}, \quad w_2 = \frac{z_2 - \omega z_1}{1-\omega}, \qquad \omega = \frac{-1+\sqrt{-3}}{2},$$

on a domain

$$D = \left\{(z_1, z_2) \in (\mathbb{C}^*)^2 \mid |\arg(z_2/z_1)| < \frac{\pi}{3}\right\},$$

which satisfies $(1,1) \in D$ and $t \cdot (z_1, z_2) \in D$ for any $t \in \mathbb{C}^*$ and $(z_1, z_2) \in D$. By assigning branches of the functions m_1 and m_2, we study the iteration of the map m.

3.1. A pair of sequences

We express the functions m_1 and m_2 as

$$m_1(z_1, z_2) = w_1 \sqrt[3]{w_2/w_1}, \quad m_2(z_1, z_2) = w_1(\sqrt[3]{w_2/w_1})^2,$$

where the branches of the cubic roots are determined by the property $m_1(z_1, z_1) = m_2(z_1, z_1) = z_1$ for any $z_1 \in \mathbb{C}^*$. Note that w_1 and w_2 can be expressed as

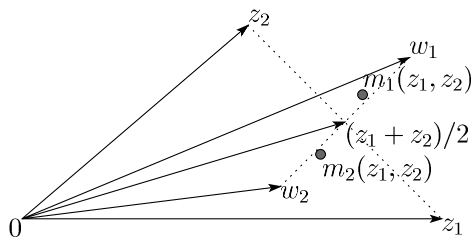

Figure 1. Geometrical interpretation of the map m.

$$\begin{aligned}(w_1,w_2) &= \frac{1}{1-\omega}(z_1,z_2)\begin{pmatrix}1 & -\omega\\ -\omega & 1\end{pmatrix}\\ &= \left(\frac{z_1+z_2}{2}-\frac{1}{\sqrt{-3}}\frac{z_1-z_2}{2},\ \frac{z_1+z_2}{2}+\frac{1}{\sqrt{-3}}\frac{z_1-z_2}{2}\right);\end{aligned}$$

refer to Figure 1 for the relationship between the variables z_1, z_2 and w_1, w_2.

Lemma 1 *We have $m(D) \subset D$.*

Proof. Put $\theta = \arg(z_2/z_1) \in (-\frac{\pi}{3}, \frac{\pi}{3})$ for $(z_1,z_2) \in D$. We have

$$|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos\theta,$$
$$|z_1-z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos\theta,$$

and

$$|z_1+z_2|^2 - |z_1-z_2|^2 = 4|z_1||z_2|\cos\theta > 2|z_1||z_2| > 0.$$

Thus

$$|w_i| \geq \frac{1}{2}|z_1+z_2| - \frac{1}{2\sqrt{3}}|z_1-z_2| > 0$$

for $i = 1, 2$, which means that $w_1, w_2 \in \mathbb{C}^*$. We have a triangle with vertices w_1,w_2 and 0. Note that the mid point of w_1 and w_2 coincides with that of z_1 and z_2, see Figure 1. If z_2 is near to z_1 then the equality

$$\arg(w_2/w_1) = \arg\left(\frac{z_1+z_2}{2}\frac{1}{w_1}\right) + \arg\left(w_2\frac{2}{z_1+z_2}\right)$$

holds with taking values near to 0. When (z_1, z_2) belongs to D, the inequalities

$$\tan\left|\arg\left(\frac{z_1+z_2}{2}\frac{1}{w_1}\right)\right|,\ \tan\left|\arg\left(w_2\frac{2}{z_1+z_2}\right)\right| < \frac{|z_1-z_2|}{\sqrt{3}|z_1+z_2|} < \frac{1}{\sqrt{3}}$$

hold, which imply that

$$\left|\arg\left(\frac{z_1+z_2}{2}\frac{1}{w_1}\right)\right|,\ \left|\arg\left(w_2\frac{2}{z_1+z_2}\right)\right| < \frac{\pi}{6}$$

for any $(z_1, z_2) \in D$. Hence we have

$$|\arg(m_2(z_1,z_2)/m_1(z_1,z_2))| = \frac{1}{3}|\arg(w_2/w_1)| < \frac{\pi}{9},$$

which means that $m(z_1, z_2) \in D$ for any $(z_1, z_2) \in D$. $\square$

For a given $(a, b) \in D$, we define a pair of sequences with initial $(a_0, b_0) = (a, b)$ by the recurrence relation

$$(a_{n+1}, b_{n+1}) = (m_1(a_n, b_n), m_2(a_n, b_n)) = m(a_n, b_n) = m \circ m^n(a, b) \qquad (2)$$

for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$.

Proposition 2 *The pair of sequences* (2) *converges uniformly on any compact set in D and it satisfies*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \neq 0;$$

this common limit is denoted by $m_*^\infty(a, b)$. If a and b are positive real numbers, then this pair of sequences satisfies

(i) $\frac{1}{2}(a_n + b_n)$ *is a positive real number for any $n \in \mathbb{N}$,*
(ii) a_{2n}, b_{2n} *are positive real numbers, and*

$$b_{2n} < a_{2n} \Rightarrow b_{2n} < a_{2n+2} < b_{2n+2} < a_{2n},$$
$$a_{2n} < b_{2n} \Rightarrow a_{2n} < b_{2n+2} < a_{2n+2} < b_{2n},$$

(iii) *the common limit $m_*^\infty(a, b)$ is a positive real number.*

In the rest of this subsection, we prove Proposition 2. We put

$$c_n = \frac{a_n + b_n}{2}, \quad d_n = \frac{a_n - b_n}{2},$$

$$a'_n = \frac{a_n + b_n}{2} - \frac{1}{\sqrt{-3}}\frac{a_n - b_n}{2} = c_n - \frac{1}{\sqrt{-3}}d_n,$$

$$b'_n = \frac{a_n + b_n}{2} + \frac{1}{\sqrt{-3}}\frac{a_n - b_n}{2} = c_n + \frac{1}{\sqrt{-3}}d_n,$$

for any $n \in \mathbb{N}$. We give some lemmas.

Lemma 2 *If $a_0 \neq b_0$ then*

$$|d_{n+1}| < \frac{1}{\sqrt{3}}|d_n|.$$

Proof. Note that

$$d_n = \frac{1}{2}(a_n - b_n) = \frac{-\sqrt{-3}}{2}\left(a'_n - b'_n\right) = \frac{-\sqrt{-3}}{2}a'_n(1-\xi)(1+\xi+\xi^2),$$

$$d_{n+1} = \frac{a_{n+1} - b_{n+1}}{2} = \frac{a'_n\left(\sqrt[3]{b'_n/a'_n} - (\sqrt[3]{b'_n/a'_n})^2\right)}{2} = \frac{a'_n\xi(1-\xi)}{2},$$

where $\xi = \sqrt[3]{b'_n/a'_n}$. We have

$$\left|\frac{d_{n+1}}{d_n}\right| = \frac{1}{\sqrt{3}}\frac{1}{|1+\xi+\xi^{-1}|} < \frac{1}{\sqrt{3}},$$

since the real part of $(\xi + \xi^{-1})$ is positive for $|\arg(\xi)| < \frac{\pi}{3}$. $\square$

Lemma 3 *We have*

$$c_{n+1}^2 = c_n^2 + \frac{1}{3}d_n^2 + d_{n+1}^2.$$

Proof. We have

$$a_{n+1}b_{n+1} = a'_n b'_n,$$

$$a_{n+1}b_{n+1} = \left(\frac{a_{n+1} + b_{n+1}}{2}\right)^2 - \left(\frac{a_{n+1} - b_{n+1}}{2}\right)^2 = c_{n+1}^2 - d_{n+1}^2,$$

$$a'_n b'_n = \left(c_n - \frac{1}{\sqrt{-3}}d_n\right)\left(c_n + \frac{1}{\sqrt{-3}}d_n\right) = c_n^2 + \frac{1}{3}d_n^2;$$

these imply this lemma. □

Proof of Proposition 2. By Lemma 2, the sequence $\{d_n\}_{n\in\mathbb{N}}$ converges to 0 uniformly on any compact set in D. Lemma 3 implies

$$\begin{aligned}
c_n^2 &= c_{n-1}^2 + \frac{1}{3}d_{n-1}^2 + d_n^2,\\
c_{n-1}^2 &= c_{n-2}^2 + \frac{1}{3}d_{n-2}^2 + d_{n-1}^2,\\
&\ \vdots\\
c_{k+1}^2 &= c_k^2 + \frac{1}{3}d_k^2 + d_{k+1}^2;\\
c_n^2 - c_k^2 &= \frac{1}{3}\sum_{j=k}^{n-1} d_j^2 + \sum_{j=k+1}^{n} d_j^2
\end{aligned}$$

for $n > k \geq 0$. Thus we have

$$\begin{aligned}
\left|c_n^2 - c_k^2\right| &\leq \frac{1}{3}\sum_{j=k}^{n-1} |d_j|^2 + \sum_{j=k+1}^{n} |d_j|^2\\
&< \left(\frac{1}{3}\sum_{j=k}^{n-1}\frac{1}{3^j} + \sum_{j=k+1}^{n}\frac{1}{3^j}\right)|d_0|^2\\
&< \frac{2}{3}\sum_{j=k}^{\infty}\frac{1}{3^j}|d_0|^2 = \frac{1}{3^k}|d_0|^2.
\end{aligned}$$

Hence the sequence $\{c_n^2\}_{n\in\mathbb{N}}$ converges uniformly on any compact set in D. By putting $k = 0$, we obtain

$$\left|c_0^2\right| - \left|c_n^2\right| \leq \left|c_n^2 - c_0^2\right| < |d_0|^2.$$

Thus

$$\left|c_n^2\right| > \left|c_0^2\right| - \left|d_0^2\right| = \frac{1}{4}\left(|a_0 + b_0|^2 - |a_0 - b_0|^2\right) \geq \frac{1}{2}|a_0||b_0| > 0,$$

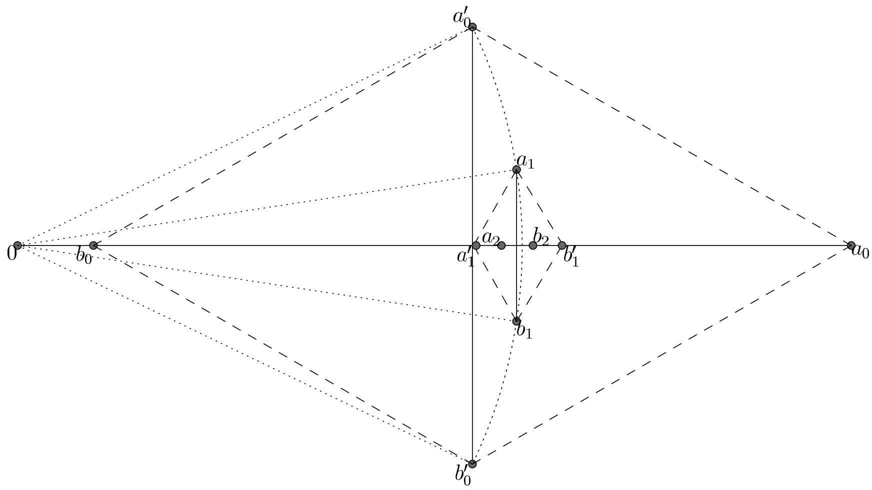

Figure 2. a_1, a_2, b_1, b_2 for positive real a_0, b_0.

which implies $\lim_{n\to\infty} c_n^2 \neq 0$. By the continuity of m, c_{n+1} is very near to c_n for a sufficiently large n. Thus the original sequence $\{c_n\}_{n\in N}$ also converges uniformly on any compact set in D. By $\lim_{n\to\infty} d_n = 0$, the sequences $\{a_n\}$ and $\{b_n\}$ converge and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n \neq 0$.

Let a_0 and b_0 be positive real numbers with $b_0 < a_0$. Then $(a_0 + b_0)/2$ is positive real, and a'_0 and b'_0 are complex conjugate each other. Thus a_1 and b_1 are complex conjugate each other and they satisfy

$$\frac{a_0+b_0}{2} < \frac{a_1+b_1}{2} < \sqrt{\left(\frac{a_0+b_0}{2}\right)^2 + \left(\frac{a_0-b_0}{2\sqrt{3}}\right)^2} < \frac{a_0+b_0}{2} + \frac{a_0-b_0}{2\sqrt{3}},$$

i.e.,

$$c_0 < c_1 < \sqrt{c_0^2 + \frac{1}{3}|d_0|^2} < c_0 + \frac{|d_0|}{\sqrt{3}};$$

see Figure 2. It is clear that $c_1 = (a_1 + b_1)/2$ is positive real and $d_1 = (a_1 - b_1)/2$ is pure imaginary with $|d_1| < |d_0|/\sqrt{3}$. Since a'_1 and b'_1 are given by $c_1 \mp d_1/\sqrt{-3}$ and a_2 and b_2 are given by $\sqrt[3]{(a'_1)^2 b'_1}$ and $\sqrt[3]{a'_1 (b'_1)^2}$, we have

$$b_0 < c_0 - \frac{|d_0|}{3} < c_1 - \frac{|d_0|}{3} < c_1 - \frac{|d_1|}{\sqrt{3}} = a_1' < a_2 < c_1,$$

$$c_1 < b_2 < b_1' = c_1 + \frac{|d_1|}{\sqrt{3}} < c_1 + \frac{|d_0|}{3} < c_0 + \frac{|d_0|}{\sqrt{3}} + \frac{|d_0|}{3} < a_0.$$

For $a_0 > b_0$, exchange the role of them. Therefore, we have inductively (i) and (ii), and we have (iii) as a consequence of them. □

3.2. Expression of the limit of the pair of sequences

The Gauss hypergeometric series $F(\alpha, \beta, \gamma; z)$ of a variable z with parameter α, β, γ is defined by

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha, n)(\beta, n)}{(\gamma, n)(1, n)} z^n,$$

where $|z| < 1$, $\gamma \neq 0, -1, -2, \ldots$ and $(\alpha, n) = \alpha(\alpha+1)\cdots(\alpha+n-1)$.

This series satisfies the Gauss hypergeometric differential equation

$$E(\alpha, \beta, \gamma) : z(1-z)\frac{d^2 f}{dz^2} + (\gamma - (\alpha+\beta+1)z)\frac{df}{dz} - \alpha\beta f = 0.$$

By this differential equation, we can make the analytic continuation of the Gauss hypergeometric series $F(\alpha, \beta, \gamma, z)$ to a single valued holomorphic function on the simply connected domain $\mathbb{C} - [1, \infty)$.

The Gauss hypergeometric series $F(\alpha, \beta, \gamma; z)$ satisfies the following functional equation due to Vidūnas.

Fact 1 ((23) in [V]) *For z sufficiently near to 0, we have*

$$F\left(\alpha, \frac{1+\alpha}{3}, \frac{2+2\alpha}{3}; z\right)$$
$$= (1+\bar{\omega}z)^{-\alpha} F\left(\frac{\alpha}{3}, \frac{1+\alpha}{3}, \frac{2\alpha+2}{3}; \frac{3(2\omega+1)z(z-1)}{(z+\omega)^3}\right),$$

where $\alpha \neq -1, -\frac{5}{2}, -4, -\frac{11}{2}, -7, \ldots$, $\bar{\omega}$ is the complex conjugate of ω and $(1+\bar{\omega}z)^{-\alpha} = 1$ at $z = 0$.

Our first theorem is as follows.

Theorem 1 *The common limit $m_*^\infty(a,b)$ of the pair of sequences* (2) *can be expressed by the Gauss hypergeometric function*:

$$m_*^\infty(a,b) = \frac{a}{F\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};1-\left(\frac{b}{a}\right)^3\right)}.$$

Remark 1

(1) Since $1-b^3/a^3 \in \mathbb{C}-[1,\infty)$ for any $(a,b)\in D$, Theorem 1 is effective for any $(a,b)\in D$.

(2) The monodromy group of the Gauss hypergeometric differential equation $E(1/3,2/3,4/3)$ is isomorphic to the triangle group $(3,3,3)$. In [HKM], several analogs of the arithmetic-geometric mean are studied and the triangle groups (r_1,r_2,r_3) with $1/r_1+1/r_2+1/r_3=1$ except $(3,3,3)$ appear. Theorem 1 completes the correspondence between analogs of the arithmetic-geometric mean and the triangle groups acting on Euclidean space $\mathbb{C}$.

Proof of Theorem 1. We apply Proposition 1 to $m(a,b)=(m_1(a,b), m_2(a,b))$ in this section and $\mu(a,b)=a/F(1/3,2/3,4/3;1-b^3/a^3)$. We have shown in Section 3.1 that the map m satisfies the conditions for Proposition 1. We have only to show the condition (i) and (ii) in Proposition 1 for $\mu(a,b)$. Since $F(\alpha,\beta,\gamma;0)=1$ for any parameters α,β,γ, we have $\mu(a,a)=a$ for any $a\in\mathbb{C}^*$; (i) is satisfied. We remark that we need the assumption $m_*^\infty(a,b)\neq 0$.

Let us show that $\mu(a,b)$ satisfies the condition (ii). By using Fact 1 with $\alpha=1$ and a well-known formula

$$F(\alpha,\beta,\gamma;z)=(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma;z),$$

we have

$$F\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};z\right)=\frac{\sqrt[3]{1-z}}{1+\bar{\omega}z}F\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};\frac{3(2\omega+1)z(z-1)}{(z+\omega)^3}\right),$$

which is equivalent to

$$F\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};1-x\right)=\frac{\sqrt[3]{x}}{1+\bar{\omega}(1-x)}F\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};1-\left(\frac{x+\omega}{\omega x+1}\right)^3\right)$$

under the variable change $z = 1-x$. By substituting $(x+\omega)/(\omega x+1) = b/a$, we can transform the above equality into

$$\frac{a}{F\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};1-\frac{b^3}{a^3}\right)} = \frac{m_1(a,b)}{F\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};1-\frac{m_2(a,b)^3}{m_1(a,b)^3}\right)},$$

which implies that (ii) is satisfied. □

4. Limit of a triple of sequences

In this section, we consider a map

$$m : z = (z_1, z_2, z_3) \mapsto (m_1(z), m_2(z), m_3(z)),$$
$$m_1(z) = \frac{z_1 + z_2 + z_3}{3}, \quad m_\omega(z) = \frac{z_1 + \omega z_2 + \bar{\omega} z_3}{3},$$
$$m_{\bar{\omega}}(z) = \frac{z_1 + \bar{\omega} z_2 + \omega z_3}{3},$$
$$m_2(z) = \sqrt[3]{m_1(z)^3 - m_\omega(z)^3}, \quad m_3(z) = \sqrt[3]{m_1(z)^3 - m_{\bar{\omega}}(z)^3},$$

on a domain

$$D = \left\{ z \in (\mathbb{C}^*)^3 \;\middle|\; |\arg(z_2/z_1)|, |\arg(z_3/z_1)| < \frac{\pi}{3} \right\},$$

which satisfies $(1,1,1) \in D$ and $t \cdot (z_1, z_2, z_3) \in D$ for any $t \in \mathbb{C}^*$ and $(z_1, z_2, z_3) \in D$. By assigning branches of the functions m_2 and m_3, we study the iteration of the map m.

Remark 2 The limit of the iteration of m is studied in [KS1] when z_1, z_2 and z_3 are positive real numbers.

4.1. Iteration of three terms

We begin with the following elementary lemma.

Lemma 4 *Let z_1 and z_2 be different elements of $\mathbb{C}^*$ and let θ be $\arg(z_2/z_1)$. If $|\theta| < \pi/3$ then*

$$|z_1 - z_2| < \sqrt[3]{|z_1^3 - z_2^3|}.$$

Proof. We have $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos|\theta|$. If $|z_1| = |z_1'|$ and $|z_2| = |z_2'|$, then

$$|\theta| > |\theta'| = \left|\arg(z_2'/z_1')\right| \Rightarrow |z_1 - z_2| > |z_1' - z_2'|$$

for $-\pi < \theta, \theta' < \pi$. Since $|\theta| < \pi/3$, we have $|\arg(\omega z_2/z_1)|, |\arg(\bar{\omega} z_2/z_1)| > |\theta|$. Thus inequalities $|z_1 - z_2| < |z_1 - \omega z_2|$ and $|z_1 - z_2| < |z_1 - \bar{\omega} z_2|$ hold. Hence if $z_1 \neq z_2$ then

$$\frac{|z_1 - z_2|^3}{|z_1^3 - z_2^3|} = \frac{|z_1 - z_2|^2}{|z_1 - \omega z_2||z_1 - \bar{\omega} z_2|} < 1,$$

which implies this Lemma. □

Lemma 5

(1) *If $z = (z_1, z_2, z_3)$ belongs to D then the functions $(m_\omega(z)/m_1(z))^3$ and $(m_{\bar{\omega}}(z)/m_1(z))^3$ do not take values in the interval $[1, \infty)$.*
(2) *If $z = (z_1, z_2, z_3) \in D$ satisfies $|z_2|, |z_3| < 2|z_1|$ then*

$$|m_\omega(z)|,\ |m_{\bar{\omega}}(z)| < |m_1(z)|.$$

Proof. (1) Put $z_2/z_1 = a + b\sqrt{-1}$ and $z_3/z_1 = c + d\sqrt{-1}$, where $a, b, c, d \in \mathbb{R}$. By the assumption, we have

$$a, c > 0, \quad |b| < \sqrt{3}a, \quad |d| < \sqrt{3}c. \tag{3}$$

Suppose that $(m_\omega(z)/m_1(z))^3 \in \mathbb{R}$. Then one of

$$\mathrm{Im}\left(\frac{m_\omega(z)}{m_1(z)}\right) = 0, \quad \mathrm{Im}\left(\omega\frac{m_\omega(z)}{m_1(z)}\right) = 0, \quad \mathrm{Im}\left(\omega^2\frac{m_\omega(z)}{m_1(z)}\right) = 0$$

is satisfied. The condition $\mathrm{Im}(m_\omega(z)/m_1(z)) = 0$ is equivalent to

$$a^2 + b^2 - c^2 - d^2 + a - \sqrt{3}b - c - \sqrt{3}d = 0.$$

By solving this quadratic equation with respect to d, we have

$$d = -\frac{\sqrt{3}}{2} \pm \frac{1}{2}\sqrt{4a^2 + 4b^2 - 4c^2 + 4a - 4b\sqrt{3} - 4c + 3}.$$

Substitute this into

$$\begin{aligned}&\mathrm{Re}(1-m_\omega(z)/m_1(z))\\&\quad=\frac{(3a^2+b^2+c^2+d^2+a+c+2ac+2bd)+\sqrt{3}(b-d-2ad+2bc)}{2((1+a+c)^2+(b+d)^2)}.\end{aligned}$$

Then we have

$$\mathrm{Re}(1-m_\omega(z)/m_1(z))=\frac{\sqrt{3}\big(\sqrt{3}(a+c)+(b-d)\big)}{2(a+c+1)},$$

which is positive under the assumption (3).

The conditions $\mathrm{Im}(\omega m_\omega(z)/m_1(z))=0$ and $\mathrm{Im}(\omega^2 m_\omega(z)/m_1(z))=0$ are equivalent to

$$\begin{aligned}&a^2+b^2+bd+ac-c-1+\sqrt{3}(bc-ad-d)=0,\\&c^2+d^2+ac+bd-a-1+\sqrt{3}(-ad+bc+b)=0,\end{aligned}$$

respectively. By these relations, we have

$$\begin{aligned}d&=\frac{a^2+b^2+ac+\sqrt{3}bc-c-1}{\sqrt{3}a-b+\sqrt{3}},\\b&=-\frac{c^2+d^2+ac-\sqrt{3}ad-a-1}{\sqrt{3}c+d+\sqrt{3}}.\end{aligned}$$

We can transform $\mathrm{Re}(1-\omega m_\omega(z)/m_1(z))$ and $\mathrm{Re}(1-\omega^2 m_\omega(z)/m_1(z))$ into

$$\frac{\sqrt{3}}{2}\frac{\sqrt{3}+\sqrt{3}a-b}{1+a+c},\quad \frac{\sqrt{3}}{2}\frac{\sqrt{3}+\sqrt{3}c+d}{1+a+c},$$

respectively. They are positive under the assumption (3). Use a similar argument for $(m_{\bar\omega}(z)/m_1(z))^3$.

(2) By the assumption, we have the additional condition $|b|,|d|<\sqrt{3}$. A straightforward calculation implies

$$\begin{aligned}
&\frac{3\sqrt{3}}{|z_1|^2}(|m_1(z)|^2 - |m_\omega(z)|^2) \\
&\quad = \big(\sqrt{3}a + b\big) + \big(\sqrt{3}c - d\big) + \big(\sqrt{3}ac + \sqrt{3}bd + ad - bc\big). \qquad (4)
\end{aligned}$$

It is clear that the first term $(\sqrt{3}a + b)$ and the second term $(\sqrt{3}c - d)$ of (4) are positive. The last term $(\sqrt{3}ac + \sqrt{3}bd + ad - bc)$ of (4) is positive for the following cases

$$\begin{aligned}
b, d \geq 0 &\Rightarrow \big(\sqrt{3}a - b\big)c + \sqrt{3}bd + ad > 0, \\
b, d \leq 0 &\Rightarrow \big(\sqrt{3}c + d\big)a + \sqrt{3}bd - bc > 0, \\
b \leq 0, d \geq 0 &\Rightarrow \frac{1}{\sqrt{3}}\big((3ac + bd) + \big(\sqrt{3}a + b\big)d - b\big(\sqrt{3}c - d\big)\big) > 0.
\end{aligned}$$

If $b \geq 0, d \leq 0$, then the positivity of (4) is shown as follows:

$$\begin{aligned}
(4) &= \big(\sqrt{3}c + b - d + \sqrt{3}bd\big) + \big(\sqrt{3} + d\big)a + \big(\sqrt{3}a - b\big)c \\
&> \big(\sqrt{3}c + b - d + 3d\big) + \big(\sqrt{3} + d\big)a + \big(\sqrt{3}a - b\big)c \\
&= \big(\sqrt{3}c + d\big) + (b + d) + \big(\sqrt{3} + d\big)a + \big(\sqrt{3}a - b\big)c > 0
\end{aligned}$$

for $|b| \geq |d|$;

$$\begin{aligned}
(4) &= \big(\sqrt{3}a + b - d + \sqrt{3}bd\big) + \big(\sqrt{3} - b\big)c + \big(\sqrt{3}c + d\big)a \\
&> \big(\sqrt{3}a + b - d - 3b\big) + \big(\sqrt{3} - b\big)c + \big(\sqrt{3}c + d\big)a \\
&= \big(\sqrt{3}a - b\big) + (-d - b) + \big(\sqrt{3} - b\big)c + \big(\sqrt{3}c + d\big)a > 0
\end{aligned}$$

for $|b| \leq |d|$. Similarly we have $|m_1(z)|^2 > |m_{\bar{\omega}}(z)|^2$. $\square$

The function $f(y) = \sqrt[3]{1 - y^3}$ on the unit disk $U = \{y \in \mathbb{C} \mid |y| < 1\}$ is defined by the power series

$$f(y) = \sum_{n=0}^{\infty} \frac{(-1/3, n)}{n!} y^{3n}, \qquad (5)$$

It satisfies

$$|f(y)|^3 = |1 - y^3|, \quad 3\arg(f(y)) = \arg(1 - y^3) \in (-\pi/2, \pi/2)$$

for any $y \in U$, and admits the analytic continuation along any path in $\mathbb{C} - \{1\}$. Now we strictly define two functions m_2 and m_3 on D. When $(z_1, z_2, z_3) \in D$ and $|z_2|, |z_3| < 2|z_1|$, $m_\omega(z)/m_1(z)$ and $m_{\bar{\omega}}(z)/m_1(z)$ belong to the unit disk U by Lemma 5. Thus we define m_2 and m_3 by the convergent power series $f(y)$:

$$m_2(z) = m_1(z) f(m_\omega(z)/m_1(z)) = m_1(z)\sqrt[3]{1 - m_\omega(z)^3/m_1(z)^3},$$
$$m_3(z) = m_1(z) f(m_{\bar{\omega}}(z)/m_1(z)) = m_1(z)\sqrt[3]{1 - m_{\bar{\omega}}(z)^3/m_1(z)^3}.$$

We can make their analytic continuations to the whole domain D; the extended functions $m_2(z)$ and $m_3(z)$ are single valued on D by Lemma 5. Note that

$$m_j(1,1,1) = 1, \quad m_j(t \cdot z) = t \cdot m_j(z) \quad (j = 1, 2, 3)$$

for any $t \in \mathbb{C}^*$ and $z \in D$.

Lemma 6 *We have $m(D) \subset D$.*

Proof. By Lemma 5, we have inequalities

$$-\pi < \arg(m_2(z)^3/m_1(z)^3), \ \arg(m_3(z)^3/m_1(z)^3) < \pi,$$

which imply this lemma. □

For any $(a, b, c) = (a_0, b_0, c_0) \in D$, we define a triple $(a_n, b_n, c_n)_{n \in \mathbb{N}}$ of sequences by the recurrence relation

$$\begin{aligned}(a_{n+1}, b_{n+1}, c_{n+1}) &= \big(m_1(a_n, b_n, c_n), m_2(a_n, b_n, c_n), m_3(a_n, b_n, c_n)\big) \\ &= m(a_n, b_n, c_n) = m \circ m^n(a, b, c). \end{aligned} \tag{6}$$

Note that b_n and c_n are uniquely determined by the previous terms and that $(a_n, b_n, c_n) \in D$ for any $n \in \mathbb{N}$ by Lemmas 5 and 6.

Proposition 3 *The triple* (6) *of sequences converges uniformly on any compact set in D, and it has a common limit*:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n,$$

which is denoted by $m_*^\infty(a,b,c)$. *If* (a,b,c) *satisfies*

$$|a-b| < \frac{1}{2}|a|, \quad |a-c| < \frac{1}{2}|a|, \tag{7}$$

then $\lim_{n\to\infty} a_n \neq 0$.

Proof. Lemma 4 implies

$$\begin{aligned} |a_n - b_n| &\le \sqrt[3]{|a_n^3 - b_n^3|} = \frac{1}{3}\big|a_{n-1} + \omega b_{n-1} + \bar{\omega} c_{n-1}\big| \\ &= \frac{1}{3}\big|\omega(b_{n-1} - a_{n-1}) + \bar{\omega}(c_{n-1} - a_{n-1})\big| \\ &\le \frac{1}{3}(|b_{n-1} - a_{n-1}| + |c_{n-1} - a_{n-1}|), \\ |a_n - c_n| &\le \frac{1}{3}(|b_{n-1} - a_{n-1}| + |c_{n-1} - a_{n-1}|). \end{aligned}$$

Thus we have

$$\begin{aligned} |a_n - b_n| + |a_n - c_n| &\le \frac{2}{3}(|a_{n-1} - b_{n-1}| + |a_{n-1} - c_{n-1}|) \\ &\le \left(\frac{2}{3}\right)^2 (|a_{n-2} - b_{n-2}| + |a_{n-2} - c_{n-2}|) \\ &\le \cdots \le \left(\frac{2}{3}\right)^n (|a_0 - b_0| + |a_0 - c_0|), \end{aligned}$$

which implies

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty}(a_n - c_n) = 0. \tag{8}$$

Since

$$\begin{aligned} a_{n+1} - a_n &= \frac{a_n + b_n + c_n}{3} - a_n = \frac{(b_n - a_n) + (c_n - a_n)}{3}, \\ a_{n+2} - a_{n+1} &= \frac{(b_{n+1} - a_{n+1}) + (c_{n+1} - a_{n+1})}{3}, \end{aligned}$$

$$\vdots$$

$$a_{n+k} - a_{n+k-1} = \frac{(b_{n+k-1} - a_{n+k-1}) + (c_{n+k-1} - a_{n+k-1})}{3},$$

we have

$$\begin{aligned}
a_{n+k} - a_n &= \frac{1}{3} \sum_{i=n}^{n+k-1} ((b_i - a_i) + (c_i - a_i)), \\
|a_{n+k} - a_n| &\leq \frac{1}{3} \sum_{i=n}^{n+k-1} (|b_i - a_i| + |c_i - a_i|) \\
&\leq \frac{|b_n - a_n| + |c_n - a_n|}{3} \sum_{i=0}^{k-1} \left(\frac{2}{3}\right)^i \leq |b_n - a_n| + |c_n - a_n|. \quad (9)
\end{aligned}$$

By (8), the sequence $(a_n)_{n\in\mathbb{N}}$ is fundamental. Thus it converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Note that this convergence is uniformly on any compact set in D by the inequalities

$$|a_{n+k} - a_n| \leq |a_n - b_n| + |a_n - c_n| \leq \left(\frac{2}{3}\right)^n (|a_0 - b_0| + |a_0 - c_0|).$$

If (a, b, c) satisfies (7), then there exists a small positive real number ε such that

$$|a - b| + |a - c| < |a| - \varepsilon.$$

By putting $n = 0$ in (9), we have

$$|a| - |a_k| \leq |a - a_k| \leq |a - b| + |a - c| < |a| - \varepsilon.$$

Let $k \to \infty$ for the above inequality, then we obtain $\lim_{k\to\infty} |a_k| \geq \varepsilon > 0$.

$\square$

4.2. Expression of the limit of the triple of sequences

The Appell hypergeometric series F_1 of two variables z_1, z_2 with parameters $\alpha, \beta_1, \beta_2, \gamma$ is defined as

$$F_1(\alpha, \beta_1, \beta_2, \gamma; z_1, z_2) = \sum_{n_1, n_2 \geq 0}^{\infty} \frac{(\alpha, n_1+n_2)(\beta_1, n_1)(\beta_2, n_2)}{(\gamma, n_1+n_2)(1, n_1)(1, n_2)} z_1^{n_1} z_2^{n_2},$$

where $\gamma \neq 0, -1, -2, \dots$ and z_j satisfies $|z_j| < 1$ $(j = 1, 2)$. It is known that we can make the analytic continuation of the series $F_1(\alpha, \beta_1, \beta_2, \gamma; z_1, z_2)$ along any path in

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 (z_1 - 1)(z_2 - 1)(z_1 - z_2) \neq 0\}$$

by the Appell hypergeometric system $E_1(\alpha, \beta_1, \beta_2, \gamma)$ of differential equations. In particular, $F_1(\alpha, \beta_1, \beta_2, \gamma; z_1, z_2)$ can be regarded as a single valued holomorphic function on the simply connected domain $(\mathbb{C} - [1, \infty))^2$.

Fact 2 ([KS1], [MO]) *We have a transformation formula*

$$\begin{aligned}&\left(\frac{1+z_1+z_2}{3}\right)^p F_1\left(\frac{p}{3}, \frac{p+1}{6}, \frac{p+1}{6}, \frac{p+1}{2}; 1 - z_1^3, 1 - z_2^3\right) \\ &\quad = F_1\left(\frac{p}{3}, \frac{p+1}{6}, \frac{p+1}{6}, \frac{p+5}{6}; z_1', z_2'\right),\end{aligned}$$

where $p \neq -1, -3, -5, \dots$, $z = (z_1, z_2)$ is in a small neighborhood of $(1, 1)$, the value of $(\frac{1+z_1+z_2}{3})^p$ at $(z_1, z_2) = (1, 1)$ is 1, and

$$(z_1', z_2') = \left(\left(\frac{1 + \omega z_1 + \bar{\omega} z_2}{1 + z_1 + z_2}\right)^3, \left(\frac{1 + \bar{\omega} z_1 + \omega z_2}{1 + z_1 + z_2}\right)^3\right).$$

In particular,

$$\left(\frac{1+z_1+z_2}{3}\right) F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1 - z_1^3, 1 - z_2^3\right) = F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; z_1', z_2'\right). \quad (10)$$

Our second theorem is as follows.

Theorem 2 *Let (a, b, c) be any element of D. Then the common limit $m_*^\infty(a, b, c)$ of the triple (6) of sequences can be expressed by*

$$m_*^\infty(a,b,c) = \frac{a}{F_1\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},1;1-\frac{b^3}{a^3},1-\frac{c^3}{a^3}\right)}.$$

Remark 3

(1) The right hand side of Theorem 2 is a single valued holomorphic function. The common limit $m_*^\infty(a,b,c)$ never vanishes for any $(a,b,c)\in D$.
(2) This theorem is an extension of Theorem 2.2 in [KS1].

Proof of Theorem 2. We apply Proposition 1 to the map m in this section and

$$\mu(a,b,c) = \frac{a}{F_1\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},1;1-\frac{b^3}{a^3},1-\frac{c^3}{a^3}\right)}.$$

Suppose that $(a,b,c)\in D$ satisfies (7). Then $m_*^\infty(a,b,c)\neq 0$ and the map m satisfies the conditions for Proposition 1. We show the condition (i) and (ii) in Proposition 1 for $\mu(a,b,c)$. Since $F(\alpha,\beta_1,\beta_2,\gamma;0,0)=1$, we have $\mu(a,a,a)=a$ for any $a\in\mathbb{C}^*$; (i) is satisfied. By substituting $(z_1,z_2)=(b/a,c/a)$ in (10), we have

$$\frac{a}{F\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},1;1-\frac{b^3}{a^3},1-\frac{b^3}{a^3}\right)} = \frac{m_1(a,b,c)}{F\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},1;1-\frac{m_2(a,b,c)^3}{m_1(a,b,c)^3},1-\frac{m_3(a,b,c)^3}{m_1(a,b,c)^3}\right)},$$

which implies that (ii) is satisfied. By the analytic continuation, this theorem is effective and $m_*^\infty(a,b,c)\neq 0$ for any $(a,b,c)\in D$. $\square$

References

[BB1] Borwein J. M. and Borwein P. B., *A cubic counterpart of Jacobi's identity and the AGM.* Trans. Amer. Math. Soc. **323** (1991), 691–701.

[BB2] Borwein J. M. and Borwein P. B., *Pi and the AGM.* volume 4, A Wiley-Interscience Publ., New York, 1998.

[C] Carlson B. C., *Algorithms involving arithmetic and geometric means.* MAA Monthly. **78** (1971), 496–505.

[HKM] Hattori R., Kato T. and Matsumoto K., *Mean iterations derived from transformation formulas for the hypergeometric function.* Hokkaido Math. J. **38** (2009), 563–586.

[KM] Kato T. and Matsumoto K., *The common limit of a quadruple sequence and the hypergeometric function F_D of three variables.* Nagoya Math. J. **195** (2009), 113–124.

[KS1] Koike K. and Shiga H., *Isogeny formulas for the Picard modular form and a three terms arithmetic geometric mean.* J. Number Theory. **124** (2007), 123–141.

[KS2] Koike K. and Shiga H., *Extended Gauss AGM and corresponding Picard modular forms.* J. Number Theory. **128** (2008), 2097–2126.

[M] Matsumoto K., *A transformation formula for Appell's hypergeometric function F_1 and common limits of triple sequences by mean iterations.* Tôhoku Math. J. **62** (2010), 263–268.

[MO] Matsumoto K. and Ohara K., *Some transformation formulas for Lauricella's hypergeometric functions F_D.* Funkcial. Ekvac. **52** (2009), 203–212.

[V] Vidūnas R., *Algebraic Transformations of Gauss Hypergeometric Functions.* Funkcial. Ekvac. **52** (2009), 139–180.

Keiji Matsumoto
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: matsu@math.sci.hokudai.ac.jp

Takashi Oikawa
Hokkaido Kushiro Higashi High School
Tomihara 3-1, Kushiro-Cho
Kushiro-Gun, 088-0618, Japan
E-mail: T.Oikawa.0727.agm@gmail.com