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Analysis of strongly commuting self-adjoint operators with applications to a spin-$\frac{1}{2}$ neutral particle with anomalous magnetic moment

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Abstract. Using strong commuting self-adjoint operators in the Minkowski space, we showed that the operator concerning a neutral particle with an anomalous magnetic moment is related to that of a free particle by a non-unitary transformation.

Key words: strongly commuting self-adjoint operators, Dirac operator, quantum field theory, external field problem, anomalous magnetic moment

1. Introduction

The Green's functions of the Klein-Gordon equation and the Dirac equation in an external electromagnetic field were computed algebraically by Vaidya et al. [V-F-H] and Vaidya and Hott [V-H]. In the preceding papers ([A-T] and [T]), we found that some ideas in [V-F-H] and [V-H] could be justified from the view point of operator theory. And we have developed an operator theory concerning a family of strongly commuting self-adjoint operators in $L^2(\mathbf{R}^d)$ and $\bigoplus_{k=1}^{m} L^2(\mathbf{R}^d)$ $(m \geq 2)$ and applied the theory to the external field problem of a charged particle. Vaidya and Silva Filho [V-S] also algebraically computed Green's functions for a neutral particle with an anomalous magnetic moment in an external plane-wave electromagnetic field. The Green's function $G(x, x')$ for a spin-$\frac{1}{2}$ neutral particle with an anomalous magnetic moment in an external plane-wave field $F_{\mu\nu}$ satisfies the following equation,

$$\big(\gamma_\mu p^\mu - a\sigma \cdot F - m\big) G(x, x') = \delta(x - x'), \tag{1}$$

where $\delta(x-x')$ is the Dirac's delta-distribution on $\mathbf{R}^4 \times \mathbf{R}^4$, γ_μ $(\mu = 0, 1, 2, 3)$ are the gamma matrices, i.e., γ_0 is a 4×4 Hermitian matrix, γ_j $(j = 1, 2, 3)$ is a 4×4 anti-Hermitian matrix such that $\gamma_0^2 = E$, $\gamma_j^2 = -E$ (E is the 4×4

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unit matrix) and $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$, $\mu \neq \nu$, $\mu, \nu = 0, 1, 2, 3$, and $\sigma \cdot F = \sigma_{\mu\nu}F^{\mu\nu}$ with $\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ $(\mu, \nu = 0, 1, 2, 3)$. Note that, in what follows, we obey the Einstein's rule as to summation with Greek indices. In [V-S], they found algebraically an operator W satisfying

$$W\gamma_\mu p^\mu W^{-1} = \gamma_\mu p^\mu - a\sigma \cdot F, \tag{2}$$

and showed that the Green function for a particle with an anomalous magnetic moment is also related to that for a free particle.

In this paper, we apply the operator theory developed in [A-T] and [T] and justify the result of [V-S] and compute a Green's function for a spin-$\frac{1}{2}$ neutral particle with an anomalous magnetic moment. Since some γ_μ are symmetric and some are anti-symmetric as matrices, we found that $\gamma^\mu p_\mu - a\sigma \cdot F$ is not unitarily equaivarent to $\gamma_\mu p^\mu$. However, we obtained W in (2) as an operator on a dense domain as shown in Theorem 2.3 in Section 2. In Section 3, we discuss the corresponding integral kernels and then, in Section 4, we apply these results to the external field problem.

2. Operator calculus in the Minkowski space

We first introduce some basic symbols. Let $d \geq 2$ be a natural number and $(g_{\mu\nu})_{\mu,\nu=0,\ldots,d-1}$ be the metric tensor of the d-dimensional Minkowski space $\mathbf{M}^d$ with

$$g_{\mu\nu} = \begin{cases} 1 & (\mu = \nu = 0) \\ -1 & (\mu = \nu \neq 0) \\ 0 & (\text{otherwise}) \end{cases} \qquad (\mu,\, \nu = 0, 1, \ldots, d-1). \tag{3}$$

We denote a vector in $\mathbf{M}^d$ (or the Euclidean space $\mathbf{R}^d$) as $x = (x^0, x^1, \ldots, x^{d-1})$. For x^μ and $(g_{\mu\nu})_{\mu,\nu=0,1,\ldots,d-1}$, the metric tensor of $\mathbf{M}^d$ (or $d \times d$ matrix) defined in (3), we define x_μ by

$$x_\mu = g_{\mu\nu}x^\nu. \tag{4}$$

The indefinite inner product of $\mathbf{M}^d$ is given by

$$xy = x_\mu y^\mu = g_{\mu\nu}x^\nu y^\mu = x^0y^0 - \sum_{j=1}^{d-1} x^j y^j. \tag{5}$$

We can also write $xy = x^\mu y_\mu$. The inverse of the matrix $g = (g_{\mu\nu})_{\mu,\nu=0,1,\ldots,d-1}$ is given by $g^{-1} = (g^{\mu\nu})_{\mu,\nu=0,1,\ldots,d-1}$ with $g^{\mu\nu} = g_{\mu\nu}$ ($\mu,\, \nu = 0,1,\ldots,d-1$) so that we can write $x^\mu = g^{\mu\nu}x_\nu$.

For each natural number m, we denote the m direct sum of the Hilbert space $L^2(\mathbf{R}^d)$ by $\bigoplus^m L^2(\mathbf{R}^d)$. For a linear operator A on a Hilbert space, we denote its domain by $D(A)$. For linear operators A_k ($k = 1,\ldots,m$) on $L^2(\mathbf{R}^d)$, we denote by $\bigoplus^m A_k$ the direct sum of A_k on $\bigoplus^m L^2(\mathbf{R}^d)$, that is

$$D\Big(\bigoplus^m A_k\Big) := \Big\{\boldsymbol{\psi} = \{\psi_k\}_{k=1}^m \in \bigoplus^m L^2(\mathbf{R}^d) \,\Big|\, \psi_k \in D(A_k),\, k = 1,\ldots,m\Big\},$$

$$\bigoplus^m A_k\boldsymbol{\psi} := \{A_k\psi_k\}_{k=1}^m, \quad \boldsymbol{\psi} = \{\psi_k\}_{k=1}^m \in D\Big(\bigoplus^m A_k\Big) \tag{6}$$

where $D(\,\cdot\,)$ is operator domain.

For each $a \in \mathbf{M}^d$, the function ax defines a self-adjoint multiplication operator on $L^2(\mathbf{R}^d)$ with domain

$$D(ax) = \big\{\psi \in L^2(\mathbf{R}^d) \mid ax\psi \in L^2(\mathbf{R}^d)\big\}. \tag{7}$$

Let ∂_μ be the generalized partial differential operator in x^μ acting in $L^2(\mathbf{R}^d)$. Then, the operator $p_\mu := i\partial_\mu$ ($\mu = 0,1,\ldots,d-1$) is self-adjoint on $L^2(\mathbf{R}^d)$.

For each $b \in \mathbf{M}^d$, we define a self-adjoint operator on $L^2(\mathbf{R}^d)$, denoted by bp as follows:

$$D(bp) := \Big\{\psi \in L^2(\mathbf{R}^d) \,\Big|\, \int_{\mathbf{R}^d} \big|b\xi\widetilde{\psi}(\xi)\big|^2\, d\xi < \infty\Big\},$$

$$\big(\widetilde{bp\psi}\big)(\xi) := b\xi\widetilde{\psi}(\xi), \quad \psi \in D(bp),\ \text{a.e. } \xi \in \mathbf{R}^d \tag{8}$$

where $\widetilde{\psi}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbf{R}^d} \psi(x)e^{i\xi x}\, dx$ ($\xi \in \mathbf{R}^d$) is the Fourier transform of ψ with ξx being the Minkowski inner product of ξ and x as in (5).

In what follows, for simplicity, we denote the m direct sum $\bigoplus^m A$ of a linear operator A on a Hilbert space by A. With this convention, both ax and bp are self-adjoint on $\bigoplus^m L^2(\mathbf{R}^d)$.

For a complex $m \times m$ matrix $A = (c_{ij})_{i,j=0,1,\ldots,d-1}$ ($c_{ij} \in \mathbf{C}$, $i,j = 0,1,\ldots,d-1$), we define a linear operator $\hat{A}$ on $\bigoplus^m L^2(\mathbf{R}^d)$ as follows:

$$D(\hat{A}) := \bigoplus^m L^2(\mathbf{R}^d),$$

$$(\hat{A}\boldsymbol{\psi})_k := \sum_{j=1}^m c_{kj}\psi_j, \quad \boldsymbol{\psi} = \{\psi_k\}_{k=1}^m \in \bigoplus^m L^2(\mathbf{R}^d),\ k = 1, \ldots, m \tag{9}$$

It is easy to see that $\hat{A}$ is bounded. Moreover, if A is Hermitian, then $\hat{A}$ is self-adjoint.

We introduce a subset $\mathbf{M}_0$ of $\mathbf{M}^d \times \mathbf{M}^d$ as follows:

$$\mathbf{M}_0 = \{(a, b) \in \mathbf{M}^d \times \mathbf{M}^d \mid a \neq 0,\ b \neq 0,\ ab = 0\}.$$

We denote by $M_d^{\mathrm{as}}(\mathbf{R})$ the set of $d \times d$ real anti-symmetric matrices, that is,

$$M_d^{\mathrm{as}}(\mathbf{R}) = \{f = (f_{\mu\nu}) \mid f_{\mu\nu} \in \mathbf{R},\ f_{\mu\nu} = -f_{\nu\mu},\ \mu,\ \nu = 0, 1, \ldots, d-1\}. \tag{10}$$

For each $f_{\mu\nu}$, we have $f^{\mu}{}_{\nu} = g^{\mu\lambda} f_{\lambda\nu}$, $f_{\mu}{}^{\nu} = f_{\mu\lambda} g^{\lambda\nu}$ and $f^{\mu\nu} = g^{\mu\lambda} f_{\lambda}{}^{\nu}$.

For each $a \in \mathbf{M}^d$, we define $\mathcal{F}_a$ as follows:

$$\mathcal{F}_a := \{f \in M_d^{\mathrm{as}}(\mathbf{R}) \mid a^{\mu} f_{\mu\nu} = 0,\ \nu = 0, 1, \ldots, d-1\}. \tag{11}$$

If $f \in \mathcal{F}_a$, then we can see that $a^{\mu} f_{\mu}{}^{\nu} = a_{\mu} f^{\mu\nu} = a_{\mu} f^{\mu}{}_{\nu} = 0$.

For $f \in M_d^{\mathrm{as}}(\mathbf{R})$ and $\mu = 0, 1, \ldots, d-1$, we denote the direct sum of operators $\bigoplus^m f^{\mu\nu} p_{\nu}$ by $f^{\mu\nu} p_{\nu}$ for simplicity. Since $f^{\mu\nu} p_{\nu}$ is self-adjoint on $L^2(\mathbf{R}^d)$, $f^{\mu\nu} p_{\nu}$ is self-adjoint on $\bigoplus^m L^2(\mathbf{R}^d)$.

We define $\mathcal{G}_a$, a subset of $\mathcal{F}_a$, as follows:

$$\mathcal{G}_a := \{f \in \mathcal{F}_a \mid f^{\mu\lambda} f_{\lambda\nu} = a^{\mu} a_{\nu},\ \mu, \nu = 0, \ldots, d-1\}. \tag{12}$$

We say that two self-adjoint operators on a Hilbert space strongly commutes if their spectral measures commute. The following fact is well known (e.g., Theorem VIII. 13 in [R-S]).

Lemma 2.1 *Let A and B be self-adjoint operators on a Hilbert space. Then the following two statements are equivalent.*

(1) *A and B strongly commute.*
(2) *For all s, $t \in \mathbf{R}$, $e^{isA} e^{itB} = e^{itB} e^{isA}$*

Using this lemma, we can prove the following statement.

Lemma 2.2 *Let $(a,b) \in \mathbf{M}_0$. If $f \in \mathcal{F}_a$, then ax, bp and $f^{\mu\nu}p_\nu$ strongly commute for all $\mu = 0, \dots, d-1$.*

Proof. See Lemma 2.2 in [A-T]. □

Let $\mathbf{B}_{\text{real}}(\mathbf{R}^d)$ be the set of real-valued Borel measurable functions on $\mathbf{R}^d$ which are almost everywhere finite with respect to the d-dimensional Lebesgue measure. Let $E_{ax}(\,\cdot\,)$ and $E_{bp}(\,\cdot\,)$ be the spectral measures of ax and bp, respectively. Then there exists a unique joint spectral measure $E(\,\cdot\,)$ such that $E(B_1 \times B_2) = E_{ax}(B_1)E_{bp}(B_2)$ (where B_1, B_2 are Borel sets in $\mathbf{R}$). Let u be a Borel measurable function on $\mathbf{R}^2$. Then, by functional calculus, we can define a linear operator $u(ax, bp)$ on $L^2(\mathbf{R}^d)$ as follows:

$$u(ax, bp) = \int_{\mathbf{R}^2} u(\lambda_1, \lambda_2)\, dE(\lambda_1, \lambda_2). \tag{13}$$

We denote by $L^\infty(\mathbf{R}^d)$ the set of essentially bounded Borel measurable functions on $\mathbf{R}^d$ and denote by $\|\psi\|_\infty$ the essential supremum of $\psi \in L^\infty(\mathbf{R}^d)$. The subset of real-valued functions in $L^\infty(\mathbf{R}^d)$ is denoted $L^\infty_{\text{real}}(\mathbf{R}^d)$. In what follows, for simplicity, we mean by a bounded function on $\mathbf{R}^2$ an element of $L^\infty(\mathbf{R}^d)$. If u is real-valued then $u(ax, bp)$ is self-adjoint. If $u \in L^\infty(\mathbf{R}^2)$, then $u(ax, bp)$ is bounded.

Let $\mathbf{N} = \{1, 2, 3, \dots\}$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. For $r \in \mathbf{N}_0$, we denote by $C^r_{\text{real}}(\mathbf{R}^n)$ the set of r times continuously differentiable real-valued functions on $\mathbf{R}^n$ and by $\mathfrak{B}^r(\mathbf{R}^n)$ the set of bounded functions u in $C^r_{\text{real}}(\mathbf{R}^n)$ such that the partial derivatives of u of order j ($j = 0, 1, \dots, r$) is bounded on $\mathbf{R}^n$.

We say that a real-valued function $v = v(x_1, x_2)$ on $\mathbf{R}^2$ is in the set $\mathfrak{B}^{r,\infty}(\mathbf{R}^2)$ ($r \in \mathbf{N}_0$) if $v(\cdot, x_2) \in \mathfrak{B}^r(\mathbf{R})$ for a.e. $x_2 \in \mathbf{R}$ and the function $\partial_1^j v := \frac{\partial^j v}{\partial x_1^j}$ is bounded on $\mathbf{R}^2$ for $j = 0, 1, \dots, r$.

Let $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. For $f \in M_d^{\text{as}}(\mathbf{R})$ and $\mu = 0, 1, \dots, d-1$, we denote $(fp)^\mu := f^{\lambda\mu}p_\lambda$. Since, by Lemma 2.2, ax, bp and $(fp)^\mu$ strongly commute for all $\mu = 0, \dots, d-1$, $u(ax, bp)$ and $(fp)^\mu$ strongly commute. We set

$$D^\infty_{u,fp} := \bigcap_{n,k \in \mathbf{N}_0} \bigcap_{\mu_1, \dots, \mu_n = 0}^{d-1} D\big(u(ax, bp)^n (fp)^{\mu_1} \dots (fp)^{\mu_k}\big). \tag{14}$$

Let $\mathcal{S}(\mathbf{R}^d)$ be the set of rapidly decreasing functions on $\mathbf{R}^d$. $D^\infty_{u,fp}$ is dense in $\bigoplus^m L^2(\mathbf{R}^d)$ since $\bigoplus^m \mathcal{S}(\mathbf{R}^d) \subset D^\infty_{u,fp}$. For each $\mu = 0, 1, \ldots, d-1$, we have a self-adjoint operator

$$M^\mu(u, fp) := \overline{u(ax, bp)(fp)^\mu \restriction_{D^\infty_{u,fp}}}. \tag{15}$$

We see that $M^0(u, fp)$, $M^1(u, fp), \ldots, M^{d-2}(u, fp)$ and $M^{d-1}(u, fp)$ strongly commute by the strong commutativity of $u(ax, bp)$ and $(fp)^\mu$.

Let $G_0, G_1, \ldots, G_{d-1}$ and $\varGamma$ be $m \times m$ regular matrices satisfying

(i) Each $iG_1, \ldots, iG_{d-1}$ and G_0 is Hermitian and $\{G_\mu, G_\nu\} = 2g_{\mu\nu}E \quad (\mu, \nu = 0, 1, \ldots, d-1)$, (16)

(ii) $\varGamma$ is Hermitian and $\varGamma^2 = E$, (17)

(iii) $G_\mu \varGamma = -\varGamma G_\mu \quad (\mu = 0, 1, \ldots, d-1)$, (18)

where E is the unit matrix and, for linear operators (or matrices) A and B, we denote

$$[A, B] := AB - BA, \quad \{A, B\} := AB + BA. \tag{19}$$

There exists such a set of matrices. For example, let $d = 4$ and $\{\gamma^\mu\}_{\mu=0}^3$ be the gamma matrices as explained in Introduction. Then, putting

$$G_0 = \gamma_0, \quad G_j = \gamma_j \ \ (j = 1, 2, 3), \quad \varGamma = \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \tag{20}$$

we see that G_0, iG_1, iG_2, iG_3 and $\varGamma$ are Hermitian and satisfying (16), (18) and (17). By (16) and (18), for each $k = 1, \ldots, d-1$, $G_k\varGamma$ is Hermitian. On the other hand, $G_0\varGamma$ is anti-Hermitian.

Let $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$, $(a, b) \in \mathbf{M}_0$ and $f \in \mathcal{F}_a$. We define a closed operator $M(u, f, G, \varGamma)$ on $\bigoplus^m L^2(\mathbf{R}^d)$ as follows:

$$\begin{aligned} M(u, f, G, \varGamma) &:= \overline{M^\mu(u, fp) G_\mu \varGamma} \\ &= \overline{M^0(u, fp) G_0 \varGamma + \sum_{j=1}^{d-1} M^j(u, fp) G_j \varGamma}. \end{aligned} \tag{21}$$

Since $G_0\varGamma$ is anti-symmetric, $M(u, f, G, \varGamma)$ may not be symmetric. (However, since $G_\mu\varGamma$ is Hermitian for $\mu \neq 0$, $\sum_{j=1}^{d-1} M^j(u, fp) G_j \varGamma$ is essentially self-adjoint on $\bigoplus^m L^2(\mathbf{R}^d)$.)

Lemma 2.3 *Let $f \in M_d^{as}(\mathbf{R})$, $f \in \mathcal{F}_a$ and $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$.*

(1) $\{M(u,f,G,\Gamma)\}^2 = \{u(ax,bp)\}^2 f^{\lambda\mu} p_\lambda f_\mu{}^\nu p_\nu$ *on* $D^\infty_{u,fp}$.
(2) *If* $f \in \mathcal{F}_a \cap \mathcal{G}_b$, $\{M(u,f,G,\Gamma)\}^2 = \{u(ax,bp)bp\}^2$ *on* $D^\infty_{u,\,fp}$.

Proof. (1) Let $\boldsymbol{\psi} \in D^\infty_{u,fp}$. Since $u(ax,\, bp)$ strongly commute with $(fp)^\mu$,

$$\begin{aligned}\{M(u,f,G,\Gamma)\}^2\boldsymbol{\psi} &= \{u(ax,bp)\}^2\Big(- f^{\lambda 0}p_\lambda f^{\nu 0}p_\nu + \sum_{k=1}^{d-1} f^{\lambda k}p_\lambda f^{\nu k}p_\nu\Big)\boldsymbol{\psi}\\ &\quad - \{u(ax,bp)\}^2\sum_{k=1}^{d-1} f^{\nu k}p_\nu f^{\lambda 0}p_\lambda (G_0G_k + G_kG_0)\boldsymbol{\psi}\\ &\quad - \{u(ax,bp)\}^2 \sum_{\substack{i,j=1\\ i\neq j}}^{d-1} f^{\nu i}p_\nu f^{\lambda j}p_\lambda G_iG_j\boldsymbol{\psi}.\end{aligned}$$

Using $G_0G_k + G_kG_0 = O$ and

$$\begin{aligned}\sum_{\substack{i,j=1\\ i\neq j}}^{d-1} f^{i\nu}p_\nu f^{j\lambda}p_\lambda G_iG_j\boldsymbol{\psi} &= \sum_{\substack{i,j=1\\ i\neq j}}^{d-1} f^{j\nu}p_\nu f^{i\lambda}p_\lambda G_jG_i\boldsymbol{\psi}\\ &= -\sum_{\substack{i,j=1\\ i\neq j}}^{d-1} f^{i\nu}p_\nu f^{j\lambda}p_\lambda G_iG_j\boldsymbol{\psi},\end{aligned}$$

we see that

$$\begin{aligned}&\sum_{k=1}^{d-1} f^{k\nu}p_\nu f^{0\lambda}p_\lambda (G_0G_k + G_kG_0)\boldsymbol{\psi} = 0 \quad \text{and}\\ &\sum_{\substack{i,j=1\\ i\neq j}}^{d-1} f^{i\nu}p_\nu f^{j\lambda}p_\lambda G_iG_j\boldsymbol{\psi} = 0.\end{aligned}$$

Thus for $\boldsymbol{\psi} \in D^\infty_{u,fp}$

$$\begin{aligned}\{M(u,f,G,\mathit{\Gamma})\}^2\boldsymbol{\psi} &= \{u(ax,bp)\}^2\Big(-f^{\lambda 0}p_\lambda f^{\nu 0}p_\nu + \sum_{k=1}^{d-1} f^{\lambda k}p_\lambda f^{\nu k}p_\nu\Big)\boldsymbol{\psi}\\ &= \{u(ax,bp)\}^2\Big(-f^{\lambda 0}p_\lambda f^{0\nu}p_\nu + \sum_{k=1}^{d-1} f^{\lambda k}p_\lambda f^{k\nu}p_\nu\Big)\boldsymbol{\psi}\\ &= \{u(ax,bp)\}^2 f^{\lambda\mu}p_\lambda f_\mu{}^\nu p_\nu\boldsymbol{\psi}.\end{aligned}$$

(2) If $f \in \mathcal{G}_b$, using strong commutativity, we see that

$$\{u(ax,bp)\}^2 f^{\lambda\mu}p_\lambda f_\mu{}^\nu p_\nu\boldsymbol{\psi} = \{u(ax,bp)\}^2(bp)^2\boldsymbol{\psi} = \{u(ax,bp)bp\}^2\boldsymbol{\psi}$$

for $\boldsymbol{\psi} \in D^\infty_{u,fp}$. □

Using Lemma 2.3, we can prove the following statement.

Theorem 2.1 *Let $f \in \mathcal{F}_a \cap \mathcal{G}_b$ and $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then*

$$\{M(u,f,G,\mathit{\Gamma})\}^n\boldsymbol{\psi} = \begin{cases}\{u(ax,bp)bp\}^n\boldsymbol{\psi} & (\textit{if } n \textit{ is even})\\ M(u,f,G,\mathit{\Gamma})\{u(ax,bp)bp\}^{n-1}\boldsymbol{\psi} & (\textit{if } n \textit{ is odd})\end{cases}, \tag{22}$$

for all $\boldsymbol{\psi} \in D^\infty_{u,fp}$.

Proof. Let $\boldsymbol{\psi} \in D^\infty_{u,fp}$ and $\boldsymbol{\Psi}_n = \{M(u,f,G,\mathit{\Gamma})\}^n\boldsymbol{\psi}$. We can easily show (22) by induction. Equation (22) is clear for $n = 1, 2$ from Lemma 2.3. Suppose that $\boldsymbol{\Psi}_{2k} = \{u(ax,bp)bp\}^{2k}\boldsymbol{\psi}$ and $\boldsymbol{\Psi}_{2k-1} = M(u,f,G,\mathit{\Gamma}) \cdot\{u(ax,bp)bp\}^{2k-2}\boldsymbol{\psi}$ for some $k \in \mathbf{N}$. Then, by Lemma 2.3, $\boldsymbol{\Psi}_{2k}$, $\boldsymbol{\Psi}_{2k-1} \in D^\infty_{u,fp}$ and

$$\begin{aligned}\{M(u,f,G,\mathit{\Gamma})\}^{2(k+1)}\boldsymbol{\psi} &= \{M(u,f,G,\mathit{\Gamma})\}^2\boldsymbol{\Psi}_{2k}\\ &= \{u(ax,bp)bp\}^2\boldsymbol{\Psi}_{2k} = \{u(ax,bp)bp\}^{2(k+1)}\boldsymbol{\psi}\end{aligned}$$

and, by the strong commutativity of $u(ax,bp)$, bp and $(fp)^\mu$,

$$\begin{aligned}&\{M(u,f,G,\mathit{\Gamma})\}^{2(k+1)-1}\boldsymbol{\psi}\\ &\quad = \{M(u,f,G,\mathit{\Gamma})\}^2\boldsymbol{\Psi}_{2k-1} = \{u(ax,bp)bp\}^2\boldsymbol{\Psi}_{2k-1}\end{aligned}$$

$$= \{u(ax,bp)bp\}^2 M(u,f,G,\Gamma)\{u(ax,bp)bp\}^{2k-2}\boldsymbol{\psi}$$
$$= M(u,f,G,\Gamma)\{u(ax,bp)bp\}^{2(k+1)-2}\boldsymbol{\psi}.$$

This implies (22) for arbitrary $n \in \mathbf{N}$. □

Now, we suppose that the real-valued function $u(\lambda_1,\lambda_2)\lambda_2$ is in $\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then, the self-adjoint operator $u(ax,bp)bp$ is bounded on $\bigoplus^m L^2(\mathbf{R}^d)$. Then, by Theorem 2.1, $\{M(u,f,G,\Gamma)\}^{2k}$ is bounded on $\bigoplus^m L^2(\mathbf{R}^d)$ for all $k \in \mathbf{N}$.

Using (22), for all $f \in \mathcal{F}_a \cap \mathcal{G}_b$, $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ and $\boldsymbol{\psi} \in D^\infty_{u,fp}$, we have

$$\begin{aligned}\sum_{n=0}^{N} &\frac{(it)^n}{n!}\{M(u,f,G,\Gamma)\}^n\boldsymbol{\psi} \\ &= \sum_{k=0}^{[N/2]} \frac{(-1)^k}{(2k)!}\{t\,u(ax,bp)bp\}^{2k}\boldsymbol{\psi} \\ &\quad + it\,G_\mu\Gamma(fp)^\mu u(ax,bp)\sum_{k=0}^{[(N-1)/2]}\frac{(-1)^k}{(2k+1)!}\{t\,u(ax,bp)bp\}^{2k}\boldsymbol{\psi}, \\ &\qquad (t \in \mathbf{R}). \quad (23)\end{aligned}$$

$\sum_{k=0}^{[N/2]}\frac{(-1)^k}{(2k)!}\{t\,u(ax,bp)bp\}^{2k}$ and $\sum_{k=0}^{[(N-1)/2]}\frac{(-1)^k}{(2k+1)!}\{t\,u(ax,bp)bp\}^{2k}$ converge in norm for all $t \in \mathbf{R}$ as $N \to \infty$ and $(fp)^\mu u(ax,bp)\sum_{k=0}^{[(N-1)/2]} \cdot\frac{(-1)^k}{(2k+1)!}\{t\,u(ax,bp)bp\}^{2k}\boldsymbol{\psi}$ converges for all $t \in \mathbf{R}$ as $N \to \infty$. And using the closedness of $(fp)^\mu$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{(it)^n}{n!}\{M(u,f,G,\Gamma)\}^n\boldsymbol{\psi} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\{t\,u(ax,bp)bp\}^{2k}\boldsymbol{\psi} \\ &\quad + it\,G_\mu\Gamma(fp)^\mu u(ax,bp)\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\{t\,u(ax,bp)bp\}^{2k}\boldsymbol{\psi}. \qquad (24)\end{aligned}$$

We denote the operator $e^{itM(u,f,G,\Gamma)}$ by

$$e^{itM(u,f,G,\mathit{\Gamma})} := \overline{\sum_{n=0}^{\infty} \frac{(it)^n}{n!} \{M(u,f,G,\mathit{\Gamma})\}^n \restriction_{D^{\infty}_{u,fp}}}.$$

Since $M(u,f,G,\mathit{\Gamma})$ may not be self-adjoint, $e^{itM(u,f,G,\mathit{\Gamma})}$ may be non-unitary and unbounded in general.

Lemma 2.4 *Let $f \in \mathcal{F}_a \cap \mathcal{G}_b$, $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ and $u(\lambda_1,\lambda_2)\lambda_2$ be in $\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then, for all $\boldsymbol{\psi} \in D^{\infty}_{u,fp}$,*

$$e^{i(s+t)M(u,f,G,\mathit{\Gamma})}\boldsymbol{\psi} = e^{isM(u,f,G,\mathit{\Gamma})}e^{itM(u,f,G,\mathit{\Gamma})}\boldsymbol{\psi}. \tag{25}$$

Proof. For all $\boldsymbol{\psi} \in D^{\infty}_{u,fp}$, $\lim_{N\to\infty}\sum_{n=0}^{N} \frac{\{i(s+t)\}^n}{n!}\{M(u,f,G,\mathit{\Gamma})\}^n\boldsymbol{\psi} = e^{i(s+t)M(u,f,G,\mathit{\Gamma})}\boldsymbol{\psi}$ and

$$\begin{aligned}
&\sum_{n=0}^{N} \frac{\{i(s+t)\}^n}{n!}\{M(u,f,G,\mathit{\Gamma})\}^n\boldsymbol{\psi} \\
&\quad= \sum_{j=0}^{N} \frac{(is)^j}{j!}\{M(u,f,G,\mathit{\Gamma})\}^j \sum_{m=0}^{N-j} \frac{(it)^m}{m!}\{M(u,f,G,\mathit{\Gamma})\}^m\boldsymbol{\psi} \\
&\quad= \sum_{k=0}^{[N/2]} \frac{(-1)^k}{(2k)!}\{s\,u(ax,bp)bp\}^{2k} \sum_{m=0}^{N-2k} \frac{(it)^m}{m!}\{M(u,f,G,\mathit{\Gamma})\}^m\boldsymbol{\psi} \\
&\qquad+ it\,G_\mu \mathit{\Gamma}(fp)^\mu u(ax,bp) \sum_{k=0}^{[(N-1)/2]} \frac{(-1)^k}{(2k+1)!}\{s\,u(ax,bp)bp\}^{2k} \\
&\qquad\times \sum_{m=0}^{N-2k+1} \frac{(it)^m}{m!}\{M(u,f,G,\mathit{\Gamma})\}^m\boldsymbol{\psi}.
\end{aligned}$$

Note that, for each $j \in \mathbf{N}_0$, $\lim_{N\to\infty}\sum_{m=0}^{N-j} \frac{(it)^m}{m!}\{M(u,f,G,\mathit{\Gamma})\}^m\boldsymbol{\psi} = e^{itM(u,f,G,\mathit{\Gamma})}\boldsymbol{\psi}$ and $\sum_{k=0}^{[N/2]} \frac{(-1)^k}{(2k)!}\{s\,u(ax,bp)bp\}^{2k}$ and $\sum_{k=0}^{[(N-1)/2]} \frac{(-1)^k}{(2k+1)!}$ $\cdot\{s\,u(ax,bp)bp\}^{2k}$ converge in norm. By the closedness of $(fp)^\mu$, we have

$$\lim_{N\to\infty}\sum_{n=0}^{N}\frac{\{i(s+t)\}^n}{n!}\{M(u,f,G,\Gamma)\}^n\boldsymbol{\psi}$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\{s\,u(ax,bp)bp\}^{2k}e^{itM(u,f,G,\Gamma)}\boldsymbol{\psi}$$
$$+\,it\,G_\mu\Gamma(fp)^\mu u(ax,bp)\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\{s\,u(ax,bp)bp\}^{2k}e^{itM(u,f,G,\Gamma)}\boldsymbol{\psi}$$
$$=e^{isM(u,f,G,\Gamma)}e^{itM(u,f,G,\Gamma)}\boldsymbol{\psi}.$$

Hence, for all $\boldsymbol{\psi}\in D^\infty_{u,fp}$, $e^{i(s+t)M(u,f,G,\Gamma)}\boldsymbol{\psi}=e^{isM(u,f,G,\Gamma)}e^{itM(u,f,G,\Gamma)}\boldsymbol{\psi}$.
□

We apply Lemma 2.4 with $t=-s$, then, for $\boldsymbol{\psi}\in D^\infty_{u,fp}$,

$$e^{isM(u,f,G,\Gamma)}e^{-isM(u,f,G,\Gamma)}\boldsymbol{\psi}=e^{-isM(u,f,G,\Gamma)}e^{isM(u,f,G,\Gamma)}\boldsymbol{\psi}=\boldsymbol{\psi}.$$

For an operator A bounded on $D(A)$ which is a dense subset of a Hilbert space $\mathscr{H}$, we also denote by A the extension of A to $\mathscr{H}$. From Lemma 2.4 and Lemma 5.7 in [A-T], if $u(\lambda_1,\lambda_2)\lambda_2$ is in $\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$, $u(ax,bp)bp$ leaves $D(p_\mu)$ invariant and

$$\big[p_\mu,\,u(ax,bp)bp\big]=ia_\mu\partial_1u(ax,bp)bp \tag{26}$$
$$\left(\text{we denote }\ \partial_1u(ax,bp):=\bigoplus_{k=1}^{m}\partial_1u(ax,bp)\ \text{ on }\ \bigoplus^{m}L^2(\mathbf{R}^d)\right)$$

on $D(p_\mu)$. If $u(\lambda_1,\lambda_2)\lambda_2$ is in $\mathfrak{B}^{2,\infty}(\mathbf{R}^2)$, then $u(ax,bp)bp$ leaves $D(p_\mu p_\nu)\cap D(p_\nu p_\mu)$ invariant and

$$\begin{aligned}p_\mu p_\nu u(ax,bp)bp&=i\,u(ax,bp)bp\,p_\mu p_\nu+i\partial_1u(ax,bp)bp(a_\nu p_\mu+a_\mu p_\nu)\\&\quad-a_\nu a_\mu\partial_1^2u(ax,bp)bp\end{aligned} \tag{27}$$
$$\left(\text{we denote }\ \partial_1^2u(ax,bp):=\bigoplus_{k=1}^{m}\partial_1^2u(ax,bp)\ \text{ on }\ \bigoplus^{m}L^2(\mathbf{R}^d)\right)$$

on $D(p_\mu p_\nu)\cap D(p_\nu p_\mu)$.

Let the operator

$$\square = -p^2 = \partial_0^2 - \sum_{j=1}^{d-1} \partial_j^2 \tag{28}$$

be the free d'Alembertian with $D(\square) := \bigcap_{\mu=0}^{d-1} D(p_\mu^2)$. $\square$ is essentially self-adjoint on $\mathcal{S}(\mathbf{R}^d)$. We denote the closure of $\square$ by H_0. We also denote by $\square$ the direct sum of operators $\bigoplus^m \square$ on $\bigoplus^m L^2(\mathbf{R}^d)$.

A vector $x \in \mathbf{M}^d$ satisfying $x^2 = 0$ is called a null vector. We denote by $\mathcal{N}_d$ the set of null vectors in $\mathbf{M}^d$. From (27), we have the following lemma.

Lemma 2.5 *Let $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$, $(a,b) \in \mathbf{M}_0$, $f \in \mathcal{F}_a$ and $a \in \mathcal{N}_d$. And let $u(\lambda_1,\lambda_2)\lambda_2$ be in $\mathfrak{B}^{2,\infty}(\mathbf{R}^2)$. Then, for each $\lambda = 0,1,\ldots,d-1$, $u(ax,bp)bp$ leaves $D(\square)$ invariant and*

$$\square\, u(ax,bp)bp\boldsymbol{\psi} = i\, u(ax,bp)bp\,\square\,\boldsymbol{\psi} - 2i\partial_1 u(ax,bp)bpap\boldsymbol{\psi}, \tag{29}$$

for all $\boldsymbol{\psi} \in D(\square)$.

Theorem 2.2 *Let $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$, $(a,b) \in \mathbf{M}_0$, $f \in \mathcal{F}_a \cap \mathcal{G}_b$. And let $u(\lambda_1,\lambda_2)\lambda_2$ be in $\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then, for all $\boldsymbol{\psi} \in D^\infty_{u,fp}$, $e^{-itM(u,f,G,\mathit{\Gamma})}\boldsymbol{\psi}$ is in $D(p_\mu)$ and*

$$\begin{aligned} &p_\mu e^{-itM(u,f,G,\mathit{\Gamma})}\boldsymbol{\psi} \\ &\quad = e^{-itM(u,f,G,\mathit{\Gamma})}p_\mu\boldsymbol{\psi} + ta_\mu e^{-itM(u,f,G,\mathit{\Gamma})}\partial_1 u(ax,bp)(fp)^\lambda G_\lambda \mathit{\Gamma}\boldsymbol{\psi}, \end{aligned} \tag{30}$$

$$\begin{aligned} &G_\mu p^\mu e^{-itM(u,f,G,\mathit{\Gamma})}\boldsymbol{\psi} \\ &\quad = e^{-itM(u,f,G,\mathit{\Gamma})}G_\mu p^\mu\boldsymbol{\psi} + tG_\mu a^\mu e^{-itM(u,f,G,\mathit{\Gamma})}\partial_1 u(ax,bp)(fp)^\lambda G_\lambda \mathit{\Gamma}\boldsymbol{\psi}. \end{aligned} \tag{31}$$

Proof. For each $\boldsymbol{\psi} \in D^\infty_{u,fp}$, let

$$\begin{aligned} W_N\boldsymbol{\psi} &= \sum_{n=0}^{N} \frac{(-it)^n}{n!}\{M(u,f,G,\mathit{\Gamma})\}^n\boldsymbol{\psi} \\ &= \sum_{k=0}^{[N/2]} \frac{(-1)^k}{(2k)!}\{t\,u(ax,bp)bp\}^{2k}\boldsymbol{\psi} \\ &\quad - it\,G_\mu \mathit{\Gamma} u(ax,bp)(fp)^\mu \sum_{k=0}^{[(N-1)/2]} \frac{(-1)^k}{(2k+1)!}\{t\,u(ax,bp)bp\}^{2k}\boldsymbol{\psi}, \\ &\qquad (t \in \mathbf{R}). \end{aligned} \tag{32}$$

Since $u(ax,bp)bp$ and $u(ax,bp)$ leave $D(p_\mu)$ invariant, we see that $W_N\boldsymbol{\psi} \in D(p_\mu)$ and

$$p_\mu W_N\boldsymbol{\psi} = W_N p_\mu\boldsymbol{\psi} + ta_\mu W_{N-1}\partial_1 u(ax,bp)(fp)^\lambda G_\lambda \Gamma\boldsymbol{\psi}.$$

Since $\lim_{N\to\infty} W_N\boldsymbol{\psi} = e^{-itM(u,f,G,\Gamma)}\boldsymbol{\psi}$ and $p_\mu W_N\boldsymbol{\psi}$ converges, by the closedness of p_μ, $e^{-itM(u,f,G,\Gamma)}\boldsymbol{\psi}$ is in $D(p_\mu)$ and we obtain (30).

To prove (31), we use

$$G_\mu\{u(ax,bp)(fp)^\lambda G_\lambda\Gamma\}^n\boldsymbol{\psi}$$
$$= \begin{cases} u(ax,bp)(fp)^\lambda G_\lambda\Gamma\{u(ax,bp)bp\}^{n-1}G_\mu\boldsymbol{\psi} \\ \quad +2\,u(ax,bp)(fp)_\mu\{u(ax,bp)bp\}^{n-1}\Gamma\boldsymbol{\psi} & (n \text{ is odd}) \\ \{u(ax,bp)bp\}^n G_\mu\boldsymbol{\psi} & (n \text{ is even}) \end{cases}.$$

Thus, by (18),

$$\begin{aligned} G_\mu p^\mu W_N\boldsymbol{\psi} &= G_\mu W_N p^\mu\boldsymbol{\psi} + tG_\mu a^\mu W_{N-1}\partial_1 u(ax,bp)(fp)^\lambda G_\lambda\Gamma\boldsymbol{\psi} \\ &= W_N G_\mu p^\mu\boldsymbol{\psi} + 2\sum_{n:\text{odd}} \frac{(-it)^n}{n!}\{u(ax,bp)(fp)^\lambda G_\lambda\Gamma\}^{n-1} \\ &\qquad\qquad \cdot\Gamma u(ax,bp)(fp)_\mu p^\mu\boldsymbol{\psi} \\ &\quad + tG_\mu a^\mu W_{N-1}\partial_1 u(ax,bp)(fp)^\lambda G_\lambda\Gamma\boldsymbol{\psi}. \end{aligned}$$

Since $(fp)_\mu p^\mu\boldsymbol{\psi} = 0$ by $f \in M_d^{\rm as}(\mathbf{R}^d)$, we have

$$G_\mu p^\mu W_N\boldsymbol{\psi} = W_N G_\mu p^\mu\boldsymbol{\psi} + tG_\mu a^\mu W_{N-1}\partial_1 u(ax,bp)(fp)^\lambda G_\lambda\Gamma\boldsymbol{\psi}.$$

We conclude (31) as $N \to \infty$. □

For all $\boldsymbol{\psi} \in D^\infty_{u,fp}$, we can see that the right hand side of (30) is in $D^\infty_{u,fp}$. Using Lemma 2.4 and Theorem 2.2, we obtain the following theorem.

Theorem 2.3 *Let $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$, $(a,b) \in \mathbf{M}_0$, $f \in \mathcal{F}_a \cap \mathcal{G}_b$. Suppose that $u(\lambda_1,\lambda_2)\lambda_2$ be in $\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then:*

(1) *For all $\boldsymbol{\psi} \in D^\infty_{u,fp}$,*

$$\begin{aligned} &e^{itM(u,f,G,\Gamma)}p_\mu e^{-itM(u,f,G,\Gamma)}\boldsymbol{\psi} \\ &\qquad = \{p_\mu + ta_\mu\partial_1 u(ax,bp)(fp)^\lambda G_\lambda\Gamma\}\boldsymbol{\psi}. \end{aligned} \tag{33}$$

(2) *For all* $\boldsymbol{\psi} \in D^{\infty}_{u,fp}$,

$$\begin{aligned} & e^{itM(u,f,G,\mathit{\Gamma})} G_\mu p^\mu e^{-itM(u,f,G,\mathit{\Gamma})} \boldsymbol{\psi} \\ & \quad = \{ G_\mu p^\mu + t G_\mu a^\mu \partial_1 u(ax, bp)(fp)^\lambda G_\lambda \mathit{\Gamma} \} \boldsymbol{\psi}. \end{aligned} \tag{34}$$

Let $a = b$ and $f \in \mathcal{F}_a \cap \mathcal{G}_a$. Then $a^2 = 0$ and $f^{\mu\lambda} f_{\lambda\mu} = a^\mu a_\nu$. Hence we have the following theorem.

Theorem 2.4 *Let* $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$, $a \in \mathcal{N}_d$ *and* $f \in \mathcal{F}_a \cap \mathcal{G}_a$. *And let* $u(\lambda_1, \lambda_2)\lambda_2$ *be in* $\mathfrak{B}^{2,\infty}(\mathbf{R}^2)$. *Then*:

(1) *For all* $\boldsymbol{\psi} \in D^{\infty}_{u,fp}$,

$$\begin{aligned} & p_\mu p^\mu e^{-itM(u,f,G,\mathit{\Gamma})} \boldsymbol{\psi} \\ & \quad = e^{-itM(u,f,G,\mathit{\Gamma})} \{ p_\mu p^\mu \boldsymbol{\psi} + 2t \partial_1 u(ax, ap) ap (fp)^\lambda G_\lambda \mathit{\Gamma} \} \boldsymbol{\psi} \\ & \quad = e^{-itM(u,f,G,\mathit{\Gamma})} \{ G_\mu p^\mu + t G_\mu a^\mu \partial_1 u(ax, ap)(fp)^\lambda G_\lambda \mathit{\Gamma} \}^2 \boldsymbol{\psi}. \end{aligned} \tag{35}$$

That is, for all $\boldsymbol{\psi} \in D^{\infty}_{u,fp}$,

$$\begin{aligned} & e^{itM(u,f,G,\mathit{\Gamma})} \Box e^{-itM(u,f,G,\mathit{\Gamma})} \boldsymbol{\psi} \\ & \quad = - \{ G_\mu p^\mu + t G_\mu a^\mu \partial_1 u(ax, ap)(fp)^\lambda G_\lambda \mathit{\Gamma} \}^2 \boldsymbol{\psi}. \end{aligned} \tag{36}$$

(2) *The following operator equality holds*:

$$\begin{aligned} & e^{itM(u,f,G,\mathit{\Gamma})} H_0 e^{-itM(u,f,G,\mathit{\Gamma})} \\ & \quad = - \overline{\{ G_\mu p^\mu + t G_\mu a^\mu \partial_1 u(ax, ap)(fp)^\lambda G_\lambda \mathit{\Gamma} \}^2 \restriction_{D^{\infty}_{u,fp}}}. \end{aligned} \tag{37}$$

Proof. (1) Let $\boldsymbol{\psi} \in D^{\infty}_{u,fp}$. Since, for all $\boldsymbol{\psi} \in D^{\infty}_{u,fp}$, the right hand side of (30) is in $D^{\infty}_{u,fp}$, we can define $p_\mu p^\mu e^{-itM(u,f,G,\mathit{\Gamma})} \boldsymbol{\psi}$ and

$$\begin{aligned} & p_\mu p^\mu e^{-itM(u,f,G,\mathit{\Gamma})} \boldsymbol{\psi} \\ & \quad = p_\mu e^{-itM(u,f,G,\mathit{\Gamma})} p^\mu \boldsymbol{\psi} + t a^\mu p_\mu e^{-itM(u,f,G,\mathit{\Gamma})} \partial_1 u(ax, ap)(fp)^\lambda G_\lambda \mathit{\Gamma} \boldsymbol{\psi} \\ & \quad = e^{-itM(u,f,G,\mathit{\Gamma})} \{ p_\mu p^\mu + 2t \partial_1 u(ax, ap) ap (fp)^\lambda G_\lambda \mathit{\Gamma} \} \boldsymbol{\psi}. \end{aligned} \tag{38}$$

For all $\boldsymbol{\psi} \in D^{\infty}_{u,fp}$, we have

$$\begin{aligned}
&\big\{G_\mu p^\mu + tG_\mu a^\mu \partial_1 u(ax,ap)(fp)^\lambda G_\lambda \varGamma\big\}^2 \boldsymbol{\psi} \\
&\quad = p_\mu p^\mu \boldsymbol{\psi} + 2t\partial_1 u(ax,ap)ap(fp)^\lambda G_\lambda \varGamma \boldsymbol{\psi}. \qquad (39)
\end{aligned}$$

Since $\{p_\mu p^\mu + 2t\partial_1 u(ax,ap)ap(fp)^\lambda G_\lambda \varGamma\}\boldsymbol{\psi} \in D^{\infty}_{u,fp}$, by Lemma 2.5,

$$\begin{aligned}
&e^{itM(u,f,G,\varGamma)}\Box e^{-itM(u,f,G,\varGamma)}\boldsymbol{\psi} \\
&\quad = -e^{itM(u,f,G,\varGamma)}e^{-itM(u,f,G,\varGamma)}\big\{p_\mu p^\mu + 2t\partial_1 p(ax,ap)ap(fp)^\lambda G_\lambda \varGamma\big\}\boldsymbol{\psi} \\
&\quad = -\big\{p_\mu p^\mu + 2t\partial_1 u(ax,ap)ap(fp)^\lambda G_\lambda \varGamma\big\}\boldsymbol{\psi} \\
&\quad = -\big\{G_\mu p^\mu + tG_\mu a^\mu \partial_1 u(ax,ap)(fp)^\lambda G_\lambda \varGamma\big\}^2 \boldsymbol{\psi}. \qquad (40)
\end{aligned}$$

(2) Since $\Box$ is essentially self-adjoint on $\bigoplus^m \mathcal{S}(\mathbf{R}^d)$ and $\bigoplus^m \mathcal{S}(\mathbf{R}^d) \subset D^{\infty}_{u,fp}$, $\Box$ is essentially self-adjoint on $D^{\infty}_{u,fp}$. It gives (37). $\Box$

3. Calculation of integral kernels

For $H_0 = \overline{\Box}$, e^{isH_0} $(s \in \mathbf{R}\setminus\{0\})$ is an integral operator in the sense that

$$(e^{isH_0}\psi)(x) = \int_{\mathbf{R}^d} \Delta_s(x,y)\psi(y)\,dy, \quad \psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d), \qquad (41)$$

with

$$\Delta_s(x,y) = \frac{e^{i\varepsilon(s)\pi(d-2)/4}}{2^d \pi^{d/2}|s|^{d/2}} e^{i(x-y)^2/4s}, \qquad (42)$$

where $\varepsilon(s)$ is the sign function, that is, $\varepsilon(s) = 1$ if $s > 0$ and $\varepsilon(s) = -1$ if $s < 0$.

For e^{isH_0}, we can write

$$\big(e^{isH_0}\boldsymbol{\psi}\big)(x) = \bigg\{\int_{\mathbf{R}^d} \Delta_s(x,y)\psi_k(y)\,dy\bigg\}_{k=1}^m, \quad \boldsymbol{\psi} \in \bigoplus^m \big(L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)\big). \qquad (43)$$

We denote $\big\{\int_{\mathbf{R}^d} \Delta_s(x,y)\psi_k(y)\,dy\big\}_{k=1}^m$ by $\int_{\mathbf{R}^d} \Delta_s(x,y)\boldsymbol{\psi}(y)\,dy$.

We use the follwing lemma in [A-T] (Lemma 6.3).

Lemma 3.1 *Let $F \in L^\infty(\mathbf{R}^{r+1})$, $a \in \mathbf{M}^d$ and (a, b_j), $(b_j, b_k) \in \mathbf{M}_0$, for $j, k = 1, \dots, r$. Then*

$$\begin{aligned}&\big(F(ax, b_1p, \dots, b_rp)e^{isH_0}\psi\big)(x)\\&\quad = \int_{\mathbf{R}^d} F\biggl(ax, \frac{b_1y - b_1x}{2s}, \dots, \frac{b_ry - b_rx}{2s}\biggr)\Delta_s(x, y)\psi(y)\, dy,\end{aligned} \tag{44}$$

for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$.

In this section, we suppose that $(a, b) \in \mathbf{M}_0$, $f \in \mathcal{F}_a \cap \mathcal{F}_b \cap \mathcal{G}_b$ and a, $b \in \mathcal{N}_d$. Moreover, suppose that $u(\lambda_1, \lambda_2)$ and $u(\lambda_1, \lambda_2)\lambda_2$ are in $L^\infty_{\mathrm{real}}(\mathbf{R}^2)$. Then, for all $\boldsymbol{\psi} \in D^\infty_{u,fp}$, the closed operators $e^{\pm iM(u,f,G,\Gamma)}$ can be written

$$\begin{aligned}e^{\pm iM(u,f,G,\Gamma)}\boldsymbol{\psi} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\{u(ax, bp)bp\}^{2k}\boldsymbol{\psi}\\&\qquad \pm i\, G_\mu \Gamma (fp)^\mu\, \boldsymbol{u}(ax, bp) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\{u(ax, bp)bp\}^{2k}\boldsymbol{\psi}.\end{aligned} \tag{45}$$

Let

$$H(u, f, G, \Gamma) = e^{iM(u,f,G,\Gamma)} H_0 e^{-iM(u,f,G,\Gamma)}. \tag{46}$$

For all $\boldsymbol{\psi} \in D^\infty_{u,fp}$, we have

$$\begin{aligned}e^{isH(u,f,G,\Gamma)}\boldsymbol{\psi} &= e^{iM(u,f,G,\Gamma)} e^{isH_0} e^{-iM(u,f,G,\Gamma)}\boldsymbol{\psi}\\&= e^{iM(u,f,G,\Gamma)} e^{-iM(s)} e^{isH_0}\boldsymbol{\psi},\end{aligned}$$

where $M(s) = e^{isH_0} M(u, f, G, \Gamma) e^{-isH_0}$ and, for $\boldsymbol{\phi} = e^{isH_0}\boldsymbol{\psi}$ ($\boldsymbol{\psi} \in D^\infty_{u,fp}$),

$$\begin{aligned}e^{-iM(s)}\boldsymbol{\phi} &= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-i)^n}{n!}\{M(s)\}^n\boldsymbol{\phi}\\&= \lim_{N\to\infty} e^{isH_0} \sum_{n=0}^{N} \frac{(-i)^n}{n!}\{M(u, f, G, \Gamma)\}^n e^{-isH_0}\boldsymbol{\phi}\\&= e^{isH_0} e^{-iM(u,f,G,\Gamma)} e^{-isH_0}\boldsymbol{\phi}.\end{aligned}$$

Let $x^\mu(s) = e^{isH_0} x^\mu e^{-isH_0}$ and $X(s) = e^{isH_0} ax e^{-isH_0}$. Then $x^\mu(s) = x^\mu + 2sp^\mu$ and $X(s) = ax + 2sap$ on $\mathcal{S}(\mathbf{R}^d)$. Since $ax + 2sap$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$, we can see that $X(s) = \overline{ax + 2sap}$.

The operator $X(s)$ strongly commutes with ax, ap, bp and $(fp)^\mu$. Also note that p^μ strongly commutes with H_0. Hence, by functional calculus, we have

$$e^{isH_0} u(ax, bp) bp e^{-isH_0} = u\big(\overline{ax + 2sap}, bp\big) bp, \tag{47}$$

$$e^{isH_0} (fp)^\mu u(ax, bp) e^{-isH_0} = (fp)^\mu u\big(\overline{ax + 2sap}, bp\big), \tag{48}$$

and for $\boldsymbol{\phi} = e^{isH_0} \boldsymbol{\psi}$ ($\boldsymbol{\psi} \in D^\infty_{u,fp}$),

$$\begin{aligned}
&e^{isH_0} \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!} \{u(ax, bp)bp\}^{2k} e^{-isH_0} \boldsymbol{\phi} \\
&\quad = \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!} \big\{u(\overline{ax + 2sap}, bp)bp\big\}^{2k} \boldsymbol{\phi} \\
&e^{isH_0} (fp)^\mu u(ax, bp) \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!} \{u(ax, bp)bp\}^{2k} e^{-isH_0} \boldsymbol{\phi} \\
&\quad = (fp)^\mu u\big(\overline{ax + 2sap}, bp\big) \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!} \big\{u(\overline{ax + 2sap}, bp)bp\big\}^{2k} \boldsymbol{\phi}.
\end{aligned}$$

Hence

$$\begin{aligned}
e^{-iM(s)} \boldsymbol{\phi} &= \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!} \big\{u(\overline{ax + 2sap}, bp)bp\big\}^{2k} \boldsymbol{\phi} \\
&\quad - i\, G_\mu \Gamma (fp)^\mu u\big(\overline{ax + 2sap}, bp\big) \\
&\quad \times \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!} \big\{u(\overline{ax + 2sap}, bp)bp\big\}^{2k} \boldsymbol{\phi} \\
&= e^{-iM(u_s, f, G, \Gamma)} \boldsymbol{\phi},
\end{aligned}$$

where $u_s = u(\overline{ax + 2sap}, bp)$ and, for all $\boldsymbol{\psi} \in D^\infty_{u,fp}$,

$$
\begin{aligned}
e^{isH(u,f,G,\Gamma)}\boldsymbol{\psi} &= e^{i\{M(u,f,G,\Gamma)-M(u_s,f,G,\Gamma)\}}e^{isH_0}\boldsymbol{\psi} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\big[\big\{u(ax,bp)-u(\overline{ax+2sap},bp)\big\}bp\big]^{2k}e^{isH_0}\boldsymbol{\psi} \\
&\quad + i\,G_\mu\Gamma(fp)^\mu\big\{u(ax,bp)-u(\overline{ax+2sap},bp)\big\} \\
&\quad \times \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\big[\big\{u(ax,bp)-u(\overline{ax+2sap},bp)\big\}bp\big]^{2k}e^{isH_0}\boldsymbol{\psi}.
\end{aligned}
$$

Now we define

$$
\begin{aligned}
F_1(x_1,x_2,x_3) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\big[\{u(x_1,x_3)-u(x_1+2sx_2,x_3)\}x_3\big]^{2k} \\
&= \cos\big[\{u(x_1,x_3)-u(x_1+2sx_2,x_3)\}x_3\big], \qquad (49)
\end{aligned}
$$

and

$$
\begin{aligned}
F_2(x_1,x_2,x_3) &= \{u(x_1,x_3)-u(x_1+2sx_2,x_3)\} \\
&\quad \times \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\big[\{u(x_1,x_3)-u(x_1+2sx_2,x_3)\}x_3\big]^{2k}.
\end{aligned}
$$

If $x_3 \neq 0$, then we have

$$
F_2(x_1,x_2,x_3) = \frac{1}{x_3}\sin\big[\{u(x_1,x_3)-u(x_1+2sx_2,x_3)\}x_3\big]. \qquad (50)
$$

Note that

$$
\begin{aligned}
F_2(x_1,x_2,0) &= \lim_{x_3\to 0}\frac{1}{x_3}\sin\big[\{u(x_1,x_3)-u(x_1+2sx_2,x_3)\}x_3\big] \\
&= u(x_1,0)-u(x_1+2sx_2,0).
\end{aligned}
$$

By the above calculation, we have

$$
\begin{aligned}
&e^{-isH(u,f,G,\Gamma)}\boldsymbol{\psi} \\
&\quad = F_1(ax,ap,bp)e^{isH_0}\boldsymbol{\psi} + iG\Gamma(fp)^\mu F_2(ax,ap,bp)e^{isH_0}\boldsymbol{\psi} \qquad (51)
\end{aligned}
$$

for all $\boldsymbol{\psi} \in D^{\infty}_{u,fp}$. It is obvious that F_1, $F_2 \in L^{\infty}(\mathbf{R}^3)$. Hence, by Lemma 3.1, for all $\boldsymbol{\psi} \in \bigoplus^m \{L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)\}$ and $s \in \mathbf{R} \setminus \{0\}$,

$$\begin{aligned}&\big(F_1(ax, ap, bp)e^{isH_0}\boldsymbol{\psi}\big)(x)\\&= \int_{\mathbf{R}^d} \cos\left[\left\{u\left(ax, \frac{by-bx}{2s}\right) - u\left(ay, \frac{by-bx}{2s}\right)\right\}\frac{by-bx}{2s}\right]\\&\quad\times \Delta_s(x,y)\boldsymbol{\psi}(y)dy, \end{aligned}\tag{52}$$

and

$$\begin{aligned}&\big(F_2(ax, ap, bp)e^{isH_0}\boldsymbol{\psi}\big)(x)\\&= \int_{\mathbf{R}^d} \frac{2s}{by-bx}\sin\left[\left\{u\left(ax, \frac{by-bx}{2s}\right) - u\left(ay, \frac{by-bx}{2s}\right)\right\}\frac{by-bx}{2s}\right]\\&\quad\times \Delta_s(x,y)\boldsymbol{\psi}(y)\, dy. \end{aligned}\tag{53}$$

We say that $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ is in the set D_x if, for all $j = 0, \ldots, d-1$, $x^j\psi(x)$ ($x = (x^0, \ldots, x^{d-1})$) is in $L^1(\mathbf{R}^d)$. Since $\mathcal{S}(\mathbf{R}^d) \subset D_x$, D_x is dense in $L^2(\mathbf{R}^d)$.

Lemma 3.2 *Let $u \in L^{\infty}_{real}(\mathbf{R}^2) \cap C^1(\mathbf{R}^2)$, $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $a, b \in \mathcal{N}_d$. Then, forall $\psi \in D_x$,*

$$\begin{aligned}&f^{\mu\lambda}p_\lambda \int_{\mathbf{R}^d} \frac{2s}{by-bx}\sin\left[\left\{u\left(ax, \frac{by-bx}{2s}\right) - u\left(ay, \frac{by-bx}{2s}\right)\right\}\frac{by-bx}{2s}\right]\\&\quad\times \Delta_s(x,y)\psi(y)\, dy\\&= \int_{\mathbf{R}^d} \frac{f^{\mu\lambda}y_\lambda - f^{\mu\lambda}x_\lambda}{by-bx}\sin\left[\left\{u\left(ax, \frac{by-bx}{2s}\right) - u\left(ay, \frac{by-bx}{2s}\right)\right\}\frac{by-bx}{2s}\right]\\&\quad\times \Delta_s(x,y)\psi(y)\, dy. \end{aligned}\tag{54}$$

Proof. Let

$$S_s(x,y) = \frac{2s}{by-bx}\sin\left[\left\{u\left(ax, \frac{by-bx}{2s}\right) - u\left(ay, \frac{by-bx}{2s}\right)\right\}\frac{by-bx}{2s}\right]$$

and $\Psi(x) = \int_{\mathbf{R}^d} S_s(x,y)\Delta_s(x,y)\psi(y)\, dy$. Let $x + h^j := (x^0, \ldots, x^{j-1}, x^j +$

$h^j, x^{j+1}, \ldots, x^{d-1})$ $(h^j \in \mathbf{R})$ for $j = 0, 1, \ldots, d-1$. Then

$$\sum_{j=0}^{d-1} f^{\mu j} \frac{\Psi(x+h^j) - \Psi(x)}{h^j}$$
$$= \int_{\mathbf{R}^d} \sum_{j=0}^{d-1} f^{\mu j} \left\{ \frac{S_s(x+h^j, y) - S_s(x,y)}{h^j} \Delta_s(x+h^j, y) \right.$$
$$\left. + S_s(x,y) \frac{\Delta_s(x+h^j, y) - \Delta_s(x,y)}{h^j} \right\} \psi(y)\, dy. \tag{55}$$

Since $f \in \mathcal{F}_a \cap \mathcal{F}_b$, we have

$$\lim_{(h^0,\ldots,h^{d-1}) \to 0} \sum_{j=0}^{d-1} f^{\mu j} \frac{S_s(x+h^j, y) - S_s(x,y)}{h^j} = 0, \tag{56}$$

and

$$\lim_{(h^0,\ldots,h^{d-1}) \to 0} \sum_{j=0}^{d-1} f^{\mu j} S_s(x,y) \frac{\Delta_s(x+h^j, y) - \Delta_s(x,y)}{h^j}$$
$$= i \frac{f^{\mu\lambda} x_\lambda - f^{\mu\lambda} y_\lambda}{2s} S_s(x,y) \Delta_s(x,y), \tag{57}$$

for almost everywhere y. We set $|h^j| < 1$ for $j = 0, \ldots, d-1$. Since $\psi \in L^1(\mathbf{R}^d)$ and $y^j \psi \in L^1(\mathbf{R}^d)$ for all $j = 0, \ldots, d-1$, there exists a function $G(x,y)$ independent on $(h^0, \ldots, h^{d-1})$ such that $G(x, \cdot) \in L^1(\mathbf{R}^d)$ and

$$\left| \sum_{j=0}^{d-1} f^{\mu j} \left\{ \frac{S_s(x+h^j, y) - S_s(x,y)}{h^j} \Delta_s(x+h^j, y) \right. \right.$$
$$\left. \left. + S_s(x,y) \frac{\Delta_s(x+h^j, y) - \Delta_s(x,y)}{h^j} \right\} \psi(y) \right| \leq G(x,y).$$

Hence, by the dominated convergence theorem, we have

$$
\begin{aligned}
&f^{\mu\lambda}p_\lambda \Psi(x)\\
&= i \lim_{(h^0,\dots,h^{d-1})\to 0} \sum_{j=0}^{d-1} f^{\mu j}\,\frac{\Psi(x+h^j)-\Psi(x)}{h^j}\\
&= i \int_{\mathbf{R}^d} \lim_{(h^0,\dots,h^{d-1})\to 0} \sum_{j=0}^{d-1} f^{\mu j}\left\{ \frac{S_s(x+h^j,y)-S_s(x,y)}{h^j}\Delta_s(x+h^j,y)\right.\\
&\qquad\qquad \left. + S_s(x+h^j,y)\,\frac{\Delta_s(x+h^j,y)-\Delta_s(x,y)}{h^j}\right\}\psi(y)\,dy\\
&= i \int_{\mathbf{R}^d} i\,\frac{f^{\mu\lambda}x_\lambda - f^{\mu\lambda}y_\lambda}{2s} S_s(x,y)\Delta_s(x,y)\psi_k(y)\,dy\\
&= \int_{\mathbf{R}^d} \frac{f^{\mu\lambda}y_\lambda - f^{\mu\lambda}x_\lambda}{by-bx}\\
&\qquad \times \sin\left[\left\{u\left(ax,\frac{by-bx}{2s}\right)-u\left(ay,\frac{by-bx}{2s}\right)\right\}\frac{by-bx}{2s}\right]\Delta_s(x,y)\psi(y)\,dy.
\end{aligned}
$$

Hence, we obtain (54). □

For $\boldsymbol{\psi} = \{\psi_k\}_{k=1}^m$, we also denote $(\boldsymbol{\psi})_k := \psi_k$. And for a matrix M, we denote the (i,j)-th component of M by $(M)_{ij}$. By (52), (53) and Lemma 3.2, we obtain the following theorem.

Theorem 3.1 *Let $u \in L^\infty_{real}(\mathbf{R}^2)\cap C^1(\mathbf{R}^2)$, $f \in \mathcal{F}_a\cap\mathcal{F}_b\cap\mathcal{G}_b$ and a, $b \in \mathcal{N}_d$. Then, for all $\boldsymbol{\psi} \in \bigoplus^m D_x$ and $s \in \mathbf{R}\setminus\{0\}$,*

$$
\begin{aligned}
&\left(e^{isH(u,f,G,\Gamma)}\boldsymbol{\psi}\right)_k(x)\\
&= \int_{\mathbf{R}^d} \cos\left[\left\{u\left(ax,\frac{by-bx}{2s}\right)-u\left(ay,\frac{by-bx}{2s}\right)\right\}\frac{by-bx}{2s}\right]\Delta_s(x,y)\psi_k(y)\,dy\\
&\quad + i\sum_{j=1}^m (G_\mu\Gamma)_{kj}\int_{\mathbf{R}^d}\frac{f^{\mu\lambda}y_\lambda - f^{\mu\lambda}x_\lambda}{by-bx}\\
&\quad \times \sin\left[\left\{u\left(ax,\frac{by-bx}{2s}\right)-u\left(ay,\frac{by-bx}{2s}\right)\right\}\frac{by-bx}{2s}\right]\Delta_s(x,y)\psi_j(y)\,dy.
\end{aligned}
\tag{58}
$$

We denote $\{(e^{isH(u,f,G,\Gamma)}\boldsymbol{\psi})_k\}_{k=1}^m$ by

$$
\begin{aligned}
&\big(e^{isH(u,f,G,\Gamma)}\boldsymbol{\psi}\big)(x)\\
&= \int_{\mathbf{R}^d} \cos\left[\left\{u\left(ax, \frac{by-bx}{2s}\right) - u\left(ay, \frac{by-bx}{2s}\right)\right\}\frac{by-bx}{2s}\right]\Delta_s(x,y)\boldsymbol{\psi}(y)\,dy\\
&\quad + i\,G_\mu \Gamma \int_{\mathbf{R}^d} \frac{f^{\mu\lambda}y_\lambda - f^{\mu\lambda}x_\lambda}{by-bx}\\
&\quad \times \sin\left[\left\{u\left(ax, \frac{by-bx}{2s}\right) - u\left(ay, \frac{by-bx}{2s}\right)\right\}\frac{by-bx}{2s}\right]\Delta_s(x,y)\boldsymbol{\psi}(y)\,dy.
\end{aligned}
\tag{59}
$$

4. Application to the external field problem with anomalous magnetic moment

In this section, we apply the operator theory developed in the preceding sections to the plane-wave external electromagnetic field mentioned in Introduction and calculate the Green's function for a spin-$\frac{1}{2}$ neutral particle with anomalous magnetic moment.

We consider a quantum system of such a particle moving in the Minkowski spsce $\mathbf{M}^d$ under the influence of an electromagnetic field $F = (F_{\mu\nu})_{\mu,\nu=0,\ldots,d-1}$, a tensor field on $\mathbf{M}^d$.

A plane wave is characterized by the field strength tensor

$$F_{\mu\nu} = f_{\mu\nu}\frac{dA}{d\xi} = f_{\mu\nu}F(\xi), \quad \mu,\nu = 0,1,\ldots,d-1, \tag{60}$$

where $\xi = ax$ with a null vector $a \in \mathbf{M}^d$, $A \in C^1(\mathbf{R})$, $F := A'$ and $f_{\mu\nu}$ $(\mu,\nu = 0,1,\ldots,d-1)$ are constants satisfying

$$f_{\mu\nu} = -f_{\nu\mu}, \quad a_\lambda f^{\lambda\nu} = 0, \quad \mu,\nu = 0,1,\ldots,d-1, \tag{61}$$

and the normalization condition

$$f_{\mu\lambda}f^{\lambda}{}_{\nu} = a_\mu a_\nu, \quad \lambda,\mu,\nu = 0,1,\ldots,d-1. \tag{62}$$

Let $\varepsilon > 0$ be a parameter and $u_\varepsilon = u_\varepsilon(t)$ be a function in $C^1_{\mathrm{real}}(\mathbf{R})$, depending on ε with the folloiwing properties:

$$\text{(i)}\quad tu_\varepsilon \in \mathfrak{B}^1(\mathbf{R}) \tag{63}$$

$$\text{(ii)}\quad \sup_{t\in\mathbf{R}} |tu_\varepsilon| \le C \quad \text{with } C \text{ a constant independent of } \varepsilon. \tag{64}$$

$$\text{(iii)}\quad \lim_{\varepsilon\to 0} u_\varepsilon(t) = \frac{1}{t}, \quad (t \in \mathbf{R}\setminus\{0\}). \tag{65}$$

A simple example of u_ε is $u_\varepsilon(t) = \frac{t}{t^2+\varepsilon^2}$. In what follows, we assume that $a \in \mathcal{N}_d$.

Let $A \in \mathfrak{B}^1(\mathbf{R})$ and we set

$$w_\varepsilon(\lambda_1, \lambda_2) := A(\lambda_1)u_\varepsilon(\lambda_2). \tag{66}$$

Then, for all $\boldsymbol{\psi} \in D^\infty_{u_\varepsilon, fp}$,

$$\begin{aligned} &e^{itM(w_\varepsilon,f,G,\Gamma)}\boldsymbol{\psi} \\ &\quad = \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!}\{t\,A(ax)u_\varepsilon(bp)bp\}^{2k}\boldsymbol{\psi} \\ &\qquad + t\,G_\mu \Gamma (fp)^\mu A(ax)u_\varepsilon(bp)\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!}\{t\,A(ax)u_\varepsilon(bp)\}^{2k}\boldsymbol{\psi}. \end{aligned} \tag{67}$$

We apply Theorem 2.3 and Theorem 2.4 with $M(u,f,G,\Gamma) = M(w_\varepsilon, f, G, \Gamma)$. We denote the operator $M(w_\varepsilon, f, G, \Gamma)$ with $b = a$ by $M_a(w_\varepsilon, f, G, \Gamma)$.

Theorem 4.1

(1) *For all $A \in \mathfrak{B}^1(\mathbf{R})$ and $\boldsymbol{\psi} \in D^\infty_{u_\varepsilon, fp}$,*

$$\begin{aligned} &e^{itM_a(w_\varepsilon,f,G,\Gamma)}p^\mu e^{-itM_a(w_\varepsilon,f,G,\Gamma)}\boldsymbol{\psi} \\ &\quad = \left\{p^\mu + ta^\mu F(ax)u_\varepsilon(ap)f^{\nu\lambda}p_\nu G_\lambda \Gamma\right\}\boldsymbol{\psi}, \end{aligned} \tag{68}$$

and

$$\begin{aligned} &e^{itM_a(w_\varepsilon,f,G,\Gamma)}G_\mu p^\mu e^{-itM_a(w_\varepsilon,f,G,\Gamma)}\boldsymbol{\psi} \\ &\quad = \left\{G_\mu p^\mu + tG_\mu a^\mu F(ax)u_\varepsilon(ap)f^{\nu\lambda}p_\nu G_\lambda \Gamma\right\}\boldsymbol{\psi}. \end{aligned} \tag{69}$$

(2) *For all* $A \in \mathfrak{B}^2(\mathbf{R})$ *and* $\boldsymbol{\psi} \in D^\infty_{u_\varepsilon, fp}$,

$$\begin{aligned} &e^{itM_a(w_\varepsilon,f,G,\Gamma)} \Box e^{-itM_a(w_\varepsilon,f,G,\Gamma)} \boldsymbol{\psi} \\ &\quad = -\big\{p_\mu p^\mu \boldsymbol{\psi} + 2tF(ax)u_\varepsilon(ap)apf^{\nu\lambda}p_\nu G_\lambda \Gamma\big\}\boldsymbol{\psi} \\ &\quad = -\big\{G_\mu p^\mu + tG_\mu a^\mu F(ax)u_\varepsilon(ap)f^{\nu\lambda}p_\nu G_\lambda \Gamma\big\}^2 \boldsymbol{\psi}. \end{aligned} \tag{70}$$

Next, we apply Theorem 3.1 with $u = w_\varepsilon$. Let

$$H_\varepsilon := H(w_\varepsilon, f, G, \Gamma). \tag{71}$$

Theorem 4.2 *For all* $\boldsymbol{\psi} \in \bigoplus^m D_x$ *and* $s \in \mathbf{R} \setminus \{0\}$,

$$\begin{aligned} &\big(e^{isH_\varepsilon}\boldsymbol{\psi}\big)_k(x) \\ &= \int_{\mathbf{R}^d} \cos\left[\{A(ay) - A(ax)\}u_\varepsilon\left(\frac{ay-ax}{2s}\right)\frac{ay-ax}{2s}\right]\Delta_s(x,y)\psi_k(y)\,dy \\ &\quad + i\sum_{k=1}^m (G_\mu\Gamma)_{kj}\int_{\mathbf{R}^d}\frac{f^{\mu\lambda}y_\lambda - f^{\mu\lambda}x_\lambda}{ay-ax} \\ &\quad \times \sin\left[\{A(ay) - A(ax)\}u_\varepsilon\left(\frac{ay-ax}{2s}\right)\frac{ay-ax}{2s}\right]\Delta_s(x,y)\psi_k(y)\,dy, \end{aligned} \tag{72}$$

where $(G_\mu\Gamma)_{kj}$ *is the* (k,j)*-th component of the matrix* $G_\mu\Gamma$.

We next consider the limit $\varepsilon \to 0$. Let

$$u_{-1}(t) = \frac{1}{t}, \quad t \in \mathbf{R} \setminus \{0\}, \tag{73}$$

and $w_{-1}(\lambda_1, \lambda_2) = A(\lambda_1)u_{-1}(\lambda_2)$ with $A \in \mathfrak{B}^1(\mathbf{R})$.

Lemma 4.1 *Let* $D = \bigcap_{j,k\in\mathbf{N}_0}\bigcap_{\mu_1,\ldots,\mu_k=0}^{d-1} D((ap)^{-j}(fp)^{\mu_1}\ldots(fp)^{\mu_k})$. *Then for all* $\boldsymbol{\psi} \in D$, *we have*

$$\begin{aligned} &\lim_{\varepsilon\to 0} e^{\pm itM_a(w_\varepsilon,f,G,\Gamma)}\boldsymbol{\psi} \\ &\quad = e^{\pm itM_a(w_{-1},f,G,\Gamma)}\boldsymbol{\psi} \\ &\quad = \big\{\cos[tA(ax)] \pm i\,G_\mu\Gamma(fp)^\mu(ap)^{-1}\sin[tA(ax)]\big\}\boldsymbol{\psi}. \end{aligned} \tag{74}$$

Proof. Since, by (24), for all $\boldsymbol{\psi} \in D$,

$$
\begin{aligned}
&e^{\pm itM_a(w_\varepsilon,f,G,\Gamma)}\boldsymbol{\psi} \\
&\quad = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\{t\,A(ax)u_\varepsilon(ap)ap\}^{2k}\boldsymbol{\psi} \\
&\qquad \pm it\,G_\mu \Gamma(fp)^\mu A(ax)u_\varepsilon(ap) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\{t\,A(ax)u_\varepsilon(ap)ap\}^{2k}\boldsymbol{\psi}, \quad (75)
\end{aligned}
$$

by the functional calculus, we can show that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} e^{\pm itM_a(w_\varepsilon,f,G,\Gamma)}\boldsymbol{\psi} \\
&\quad = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\{t\,A(ax)\}^{2k}\boldsymbol{\psi} \\
&\qquad \pm i\,G_\mu \Gamma(fp)^\mu (ap)^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\{t\,A(ax)\}^{2k+1}\boldsymbol{\psi} \\
&\quad = \cos[tA(ax)]\boldsymbol{\psi} \pm i\,G_\lambda \Gamma(fp)^\lambda (ap)^{-1}\sin[tA(ax)]\boldsymbol{\psi}. \qquad \square
\end{aligned}
$$

It follows from Theorem 2.3 that, for all $\boldsymbol{\psi} \in D$,

$$
\begin{aligned}
p_\mu e^{-itM_a(w_\varepsilon,f,G,\Gamma)}\boldsymbol{\psi} &= e^{-itM_a(w_\varepsilon,f,G,\Gamma)}p_\mu\boldsymbol{\psi} \\
&\quad + ta_\mu e^{-itM_a(w_\varepsilon,f,G,\Gamma)}G_\lambda\Gamma(fp)^\lambda F(ax)u_\varepsilon(ap)\boldsymbol{\psi} \quad (76)
\end{aligned}
$$

converges as $\varepsilon \to 0$. Hence, by the closedness of p_μ, $e^{-itM(u_{-1},f,G,\Gamma)}\boldsymbol{\psi} \in D(p_\mu)$ and

$$
\begin{aligned}
p_\mu e^{-itM_a(w_{-1},f,G,\Gamma)}\boldsymbol{\psi} &= e^{-itM_a(w_{-1},f,G,\Gamma)}p_\mu\boldsymbol{\psi} \\
&\quad + ta_\mu e^{-itM_a(w_{-1},f,G,\Gamma)}G_\lambda\Gamma(fp)^\lambda F(ax)(ap)^{-1}\boldsymbol{\psi}. \quad (77)
\end{aligned}
$$

Since, for all $\boldsymbol{\psi} \in D$, $e^{itM(w_{-1},f,G,\Gamma)}e^{-itM(w_{-1},f,G,\Gamma)}\boldsymbol{\psi} = \boldsymbol{\psi}$, we have

$$
\begin{aligned}
&e^{itM_a(w_{-1},f,G,\Gamma)}p_\mu e^{-itM_a(w_{-1},f,G,\Gamma)}\boldsymbol{\psi} \\
&\quad = p_\mu\boldsymbol{\psi} + ta_\mu G_\lambda \Gamma(fp)^\lambda F(ax)(ap)^{-1}\boldsymbol{\psi} \quad (78)
\end{aligned}
$$

and

$$e^{itM_a(w_{-1},f,G,\Gamma)}G_\mu p^\mu e^{-itM_a(w_{-1},f,G,\Gamma)}\boldsymbol{\psi}$$
$$= G_\mu p^\mu \boldsymbol{\psi} + tG_\mu a^\mu G_\lambda \Gamma (fp)^\lambda F(ax)(ap)^{-1}\boldsymbol{\psi}. \tag{79}$$

Note that (76) converges to (77) and

$$H_0 e^{-itM_a(w_\varepsilon,f,G,\Gamma)}\boldsymbol{\psi}$$
$$= e^{-itM_a(w_\varepsilon,f,G,\Gamma)}H_0\boldsymbol{\psi} - 2te^{-itM_a(w_\varepsilon,f,G,\Gamma)}G_\mu \Gamma (fp)^\mu F(ax)u_\varepsilon(ap)ap\boldsymbol{\psi}. \tag{80}$$

Taking $\varepsilon \to 0$, we obtain

$$H_0 e^{-itM_a(w_{-1},f,G,\Gamma)}\boldsymbol{\psi}$$
$$= e^{-itM_a(w_{-1},f,G,\Gamma)}H_0\boldsymbol{\psi} - 2te^{-itM_a(w_{-1},f,G,\Gamma)}G_\mu \Gamma (fp)^\mu F(ax)\boldsymbol{\psi} \tag{81}$$

and

$$e^{itM_a(w_{-1},f,G,\Gamma)}H_0 e^{-itM_a(w_{-1},f,G,\Gamma)}\boldsymbol{\psi}$$
$$= H_0\boldsymbol{\psi} - 2tG_\mu \Gamma (fp)^\mu F(ax)\boldsymbol{\psi}$$
$$= -\left\{G_\mu p^\mu + tG_\mu a^\mu F(ax)(ap)^{-1}(fp)^\lambda G_\lambda \Gamma\right\}^2\boldsymbol{\psi}. \tag{82}$$

Let $d = 4$ and $\varepsilon_{\mu\nu\alpha\beta}$ be 1 or -1, if $(\mu\nu\alpha\beta)$ forms an even, or odd permutation of (0123), and be zero otherwise. For the numerical tensor $f_{\mu\nu}$, its dual ${}^*f_{\mu\nu}$ is defined as follows:

$${}^*f_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}f^{\alpha\beta} \quad \alpha,\beta,\mu,\nu = 0,1,2,3. \tag{83}$$

${}^*f_{\mu\nu}$ are restricted by Maxwell equations,

$$a_\mu {}^*f^{\mu\nu} = 0, \quad \mu,\nu = 0,1,2,3, \tag{84}$$

$${}^*f_{\mu\lambda}f^\lambda{}_\nu = 0, \tag{85}$$

and the normalization condition

$$ {}^*f_{\mu\lambda}\, {}^*f^{\lambda}{}_{\nu} = a_\mu a_\nu, \quad \lambda, \mu, \nu = 0, 1, 2, 3. \tag{86} $$

For the regular matrices $G_0, \dots, G_3$ satisfying (16),

$$ \Gamma := i\, G_0 G_1 G_2 G_3 \tag{87} $$

satisfies (17) and (18). We define

$$ \sigma_{\alpha\beta} := \frac{i}{2}[G_\alpha, G_\beta]. \tag{88} $$

Using ${}^*f^{\nu\lambda} p_\nu G_\lambda \Gamma = i\, f^{\nu\lambda} p_\nu G_\lambda - \frac{1}{2} G^\nu p_\nu \sigma_{\alpha\beta} f^{\alpha\beta}$, we have

$$ a^\mu G_\mu ({}^*fp)^\lambda G_\lambda \Gamma = -a^\mu G_\mu ({}^*fp)^\lambda G_\lambda \Gamma - ap\sigma_{\alpha\beta} f^{\alpha\beta}. $$

Hence,

$$ a^\mu G_\mu ({}^*fp)^\lambda G_\lambda \Gamma = -\frac{1}{2} ap\sigma_{\alpha\beta} f^{\alpha\beta}. \tag{89} $$

Thus, for all $\boldsymbol{\psi} \in D$, we have from (79) and (89),

$$ \begin{aligned} & e^{itM_a(w_{-1},\, {}^*f, G, \Gamma)} G_\mu p^\mu e^{-itM_a(w_{-1},\, {}^*f, G, \Gamma)} \boldsymbol{\psi} \\ & \quad = \left\{ G_\mu p^\mu - \frac{1}{2} t F(ax) \sigma_{\alpha\beta} f^{\alpha\beta} \right\} \boldsymbol{\psi}. \end{aligned} \tag{90} $$

By (46) and (71), we have

$$ H_\varepsilon \boldsymbol{\psi} = e^{tM_a(w_\varepsilon, f, G, \Gamma)} H_0 e^{-tM_a(w_\varepsilon, f, G, \Gamma)} \boldsymbol{\psi}. \tag{91} $$

By the dominated convergence theorem, we have, for $\boldsymbol{\psi} \in \bigoplus^m D_x$ and $s \in \mathbf{R} \setminus \{0\}$,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}(e^{isH_\varepsilon}\boldsymbol{\psi})_k(x)\\
&\quad=\int_{\mathbf{R}^d}\lim_{\varepsilon\to 0}\cos\left[\{A(ay)-A(ax)\}u_\varepsilon\left(\frac{ay-ax}{2s}\right)\frac{ay-ax}{2s}\right]\Delta_s(x,y)\psi_k(y)\,dy\\
&\qquad+i\sum_{k=1}^{m}(G_\mu\Gamma)_{kj}\int_{\mathbf{R}^d}\lim_{\varepsilon\to 0}\frac{f^{\mu\lambda}y_\lambda-f^{\mu\lambda}x_\lambda}{ay-ax}\\
&\qquad\times\sin\left[\{A(ay)-A(ax)\}u_\varepsilon\left(\frac{ay-ax}{2s}\right)\frac{ay-ax}{2s}\right]\Delta_s(x,y)\psi_k(y)\,dy\\
&\quad=\int_{\mathbf{R}^d}\cos\{A(ay)-A(ax)\}\Delta_s(x,y)\psi_k(y)\,dy\\
&\qquad-i\sum_{j=1}^{m}(G_\mu\Gamma)_{kj}\int_{\mathbf{R}^d}\frac{f^{\mu\lambda}y_\lambda-f^{\mu\lambda}x_\lambda}{ay-ax}\\
&\qquad\times\sin\{A(ay)-A(ax)\}\Delta_s(x,y)\psi_j(y)\,dy.
\end{aligned}
\tag{92}
$$

We denote $e^{isH}\boldsymbol{\psi}:=\lim_{\varepsilon\to 0}e^{isH_\varepsilon}\boldsymbol{\psi}$.

Finally we consider the implications of the preceeding results for approximate Green's functions of neutral particle with an anomalous electromagnetic moment in an external plane-wave electromagnetic field. The Green's functions of $H+m^2$ may be defined as the limit of $\varepsilon\to 0$ of $G_{\pm,\varepsilon}:=\mp i\int_0^\infty e^{isH}e^{\pm ism^2-s\varepsilon}ds$ in a suitable sense, where $\varepsilon>0$ is a constant parameter.

Let $\rho>0$ be a constant and $\boldsymbol{\psi}:=\{\psi_k\}_{k=1}^m$, $\boldsymbol{\phi}:=\{\phi_k\}_{k=1}^m$ be in $\bigoplus^m\mathcal{S}(\mathbf{R}^d)$. We define

$$
\begin{aligned}
&\left(G^\rho_{\pm,\varepsilon}\boldsymbol{\psi}\right)_k(x)\\
&\quad=\mp i\int_\rho^\infty\int_{\mathbf{R}^d}e^{-s\varepsilon\pm ism^2}\cos\{A(ay)-A(ax)\}\Delta_{\pm s}(x,y)\psi_k(y)\,dyds\\
&\qquad\mp\sum_{j=1}^{m}(G_\mu\Gamma)_{kj}\int_\rho^\infty\int_{\mathbf{R}^d}e^{-s\varepsilon\pm ism^2}\frac{f^{\mu\lambda}y_\lambda-f^{\mu\lambda}x_\lambda}{ay-ax}\\
&\qquad\times\sin\{A(ay)-A(ax)\}\Delta_{\pm s}(x,y)\psi_j(y)\,dyds,
\end{aligned}
\tag{93}
$$

and

$$\begin{aligned}\left\langle \boldsymbol{\phi}, G^{\rho}_{\pm,\varepsilon}\boldsymbol{\psi}\right\rangle = &\mp i \sum_{k=1}^{m}\int_{\rho}^{\infty}\int_{\mathbf{R}^d\times\mathbf{R}^d} e^{-s\varepsilon\pm ism^2} \\ &\times \cos\{A(ay)-A(ax)\}\Delta_{\pm s}(x,y)\overline{\phi_k(x)}\psi_k(y)\,dydxds \\ &\mp \sum_{k=1}^{m}\sum_{j=1}^{m}(G_\mu\Gamma)_{kj}\int_{\rho}^{\infty}\int_{\mathbf{R}^d\times\mathbf{R}^d} e^{-s\varepsilon\pm ism^2}\frac{f^{\mu\lambda}y_\lambda - f^{\mu\lambda}x_\lambda}{ay-ax} \\ &\times \sin\{A(ay)-A(ax)\}\Delta_{\pm s}(x,y)\overline{\phi_k(x)}\psi_j(y)\,dydxds. \end{aligned} \tag{94}$$

For each ρ and ε, (93) and (94) are absolutely convergent. Since $\Delta_s(x,y)$ as a function s has singularity of order $\frac{d}{2}$ at $s=0$, we introduce the cutoff parameter ρ in the above integral. In particular, for $d\geq 3$, then $\int_\rho^\infty e^{\pm ism^2}\Delta_{\pm s}(x,y)ds$ is absolutely convergent and, by the dominated convergence theorem, we see that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left\langle \boldsymbol{\phi}, G^{\rho}_{\pm,\varepsilon}\boldsymbol{\psi}\right\rangle \\ &= \mp i \sum_{k=1}^{m}\int_{\rho}^{\infty}\int_{\mathbf{R}^d\times\mathbf{R}^d} e^{ism^2} \\ &\quad\times \cos\{A(ay)-A(ax)\}\Delta_{\pm s}(x,y)\overline{\phi_k(x)}\psi_k(y)\,dydxds \\ &\quad\mp \sum_{k=1}^{m}\sum_{j=1}^{m}(G_\mu\Gamma)_{kj}\int_{\rho}^{\infty}\int_{\mathbf{R}^d\times\mathbf{R}^d} e^{-ism^2}\frac{f^{\mu\lambda}y_\lambda - f^{\mu\lambda}x_\lambda}{ay-ax} \\ &\quad\times \sin\{A(ay)-A(ax)\}\Delta_{\pm s}(x,y)\overline{\phi_k(x)}\psi_j(y)\,dydxds. \end{aligned} \tag{95}$$

We set $\boldsymbol{\phi}^* := \{\overline{\phi_k}\}_{k=1}^m$ for $\phi_k\in\mathcal{S}(\mathbf{R}^d)$, $k=1,\ldots,m$, where $\overline{\phi_k}$ is the complex conjugate of ϕ_k.

Theorem 4.3 *Let $d\geq 3$. Then, there are unique tempered distributions $G^\rho_\pm$ satisfying*

$$G^\rho_\pm(\boldsymbol{\phi}\otimes\boldsymbol{\psi}) = \lim_{\varepsilon\to 0}\left\langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon}\boldsymbol{\psi}\right\rangle. \tag{96}$$

Proof. Let $B^\rho_\pm(\boldsymbol{\phi},\boldsymbol{\psi}) := \lim_{\varepsilon\to 0}\langle\boldsymbol{\phi}^*, G^\rho_{\pm,\varepsilon}\boldsymbol{\psi}\rangle$ for the cutoff parameter $\rho>0$. Since

$$\begin{aligned}\left|B^\rho_\pm(\boldsymbol{\phi},\boldsymbol{\psi})\right| \le &\sum_{k=1}^m \int_\rho^\infty \int_{\mathbf{R}^d\times\mathbf{R}^d} |\Delta_{\pm s}(x,y)||\phi_k(x)||\psi_k(y)|dydxds \\ &+\sum_{k=1}^m\sum_{j=1}^m \left|(G_\mu \mathit{\Gamma})_{kj}\right| \int_\rho^\infty \int_{\mathbf{R}^d\times\mathbf{R}^d} \left|f^{\mu\lambda}y_\lambda - f^{\mu\lambda}x_\lambda\right| \\ &\times \left|\frac{\sin\{A(ay)-A(ax)\}}{ay-ax}\right| |\Delta_{\pm s}(x,y)||\phi_k(x)||\psi_k(y)|dydxds \end{aligned} \tag{97}$$

and there exist some C_1 and C_2 such that

$$\begin{aligned}&\int_\rho^\infty \int_{\mathbf{R}^d\times\mathbf{R}^d} |\Delta_{\pm s}(x,y)||\phi_k(x)||\psi_k(y)|dydxds \\ &\le C_1\|\phi_k\|_{L^1(\mathbf{R}^d)}\|\psi_k\|_{L^1(\mathbf{R}^d)} \int_\rho^\infty \int_{\mathbf{R}^d\times\mathbf{R}^d} \left|f^{\mu\lambda}y_\lambda - f^{\mu\lambda}x_\lambda\right| \\ &\quad\times \left|\frac{\sin\{A(ay)-A(ax)\}}{ay-ax}\right| |\Delta_{\pm s}(x,y)||\phi_k(x)||\psi_k(y)|dydxds \\ &\le C_2\|\phi_k\|_{\mathcal{S}(\mathbf{R}^d)}\|\psi_k\|_{\mathcal{S}(\mathbf{R}^d)},\end{aligned}$$

we see that $B^\rho_\pm(\boldsymbol{\phi},\boldsymbol{\psi})$ are separately continuous bilinear functionals on $\bigoplus^m \mathcal{S}(\mathbf{R}^d)\times\bigoplus^m \mathcal{S}(\mathbf{R}^d)$. Hence, by the nuclear theorem, there are unique tempered distributions $G^\rho_\pm$ satisfying $G^\rho_\pm(\boldsymbol{\phi}\otimes\boldsymbol{\psi}) = B^\rho_\pm(\boldsymbol{\phi},\boldsymbol{\psi}) = \lim_{\varepsilon\to 0}\langle\boldsymbol{\phi}^*, G^\rho_{\pm,\varepsilon}\boldsymbol{\psi}\rangle$. □

Let $d=4$ and we take ${}^*f^{\mu\nu}$ defined (83) as $f^{\mu\nu}$. Suppose that $A(t)$ is slowly increasing C^∞-function. Let

$$C^\rho_\pm(\boldsymbol{\phi},\boldsymbol{\psi}) = B^\rho_\pm\left(\left({}^tG_\mu p^\mu - \frac{1}{2}tF(ax)\,{}^t\sigma_{\alpha\beta}f^{\alpha\beta} - m\right)\boldsymbol{\phi},\boldsymbol{\psi}\right),$$

where ${}^tG$ and ${}^t\sigma_{\alpha\beta}$ are transposed matrices of G and $\sigma_{\alpha\beta} := \frac{i}{2}[G_\alpha, G_\beta]$, respectively. We see that $C^\rho_\pm(\boldsymbol{\phi},\boldsymbol{\psi})$ are separately continuous bilinear functionals on $\bigoplus^m \mathcal{S}(\mathbf{R}^d)\times\bigoplus^m \mathcal{S}(\mathbf{R}^d)$. Hence, by the nuclear theorem, there are unique tempered distributions $H^\rho_\pm$ satisfying

$$\begin{aligned} H^{\rho}_{\pm}(\boldsymbol{\phi}\otimes\boldsymbol{\psi}) &= C^{\rho}_{\pm}(\boldsymbol{\phi},\boldsymbol{\psi}) \\ &= B^{\rho}_{\pm}\left(\left({}^{t}G_{\mu}p^{\mu} - \frac{1}{2}tF(ax)\,{}^{t}\sigma_{\alpha\beta}f^{\alpha\beta} - m\right)\boldsymbol{\phi},\boldsymbol{\psi}\right). \end{aligned} \tag{98}$$

$H^{\rho}_{\pm}(\boldsymbol{\phi}\otimes\boldsymbol{\psi})$ are not exactly Green's functions of ${}^{t}G_{\mu}p^{\mu} - \frac{1}{2}tF(ax){}^{t}\sigma_{\alpha\beta}f^{\alpha\beta} - m$, since $\rho \neq 0$. However, it suggests that it gives some approximate Green's functions in distribution sense.

Remarks For $\rho > 0$ and $\boldsymbol{\psi} := \{\psi_k\}_{k=1}^{m}$, $\boldsymbol{\phi} := \{\phi_k\}_{k=1}^{m}$ in $\bigoplus^{m}\mathcal{S}(\mathbf{R}^d)$, let

$$\langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon}\boldsymbol{\psi}\rangle = \langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon,1}\boldsymbol{\psi}\rangle + \langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon,2}\boldsymbol{\psi}\rangle \tag{99}$$

with

$$\begin{aligned} \langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon,1}\boldsymbol{\psi}\rangle := \mp i\sum_{k=1}^{m}\int_{\rho}^{\infty}\int_{\mathbf{R}^d\times\mathbf{R}^d} & e^{-s\varepsilon\pm ism^2}\cos\{A(ay)-A(ax)\} \\ & \times \Delta_{\pm s}(x,y)\phi_k(x)\psi_k(y)\,dydxds \end{aligned} \tag{100}$$

and

$$\begin{aligned} \langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon,2}\boldsymbol{\psi}\rangle := \mp\sum_{k=1}^{m}\sum_{j=1}^{m}(G_{\mu}\Gamma)_{kj}\int_{\rho}^{\infty}\int_{\mathbf{R}^d\times\mathbf{R}^d} & e^{-s\varepsilon\pm ism^2}\frac{f^{\mu\lambda}y_{\lambda} - f^{\mu\lambda}x_{\lambda}}{ay-ax} \\ & \times \sin\{A(ay)-A(ax)\}\Delta_{\pm s}(x,y)\phi_k(x)\psi_j(y)\,dydxds. \end{aligned} \tag{101}$$

Moreover, as for the term $\langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon,1}\boldsymbol{\psi}\rangle$, the following statement holds.

Theorem 4.4 *Let $d \geq 3$. Then, there exist unique tempered distributions $G_{\pm,1}$ satisfying*

$$G_{\pm,1}(\boldsymbol{\phi}\otimes\boldsymbol{\psi}) = \lim_{\varepsilon\to 0}\lim_{\rho\to 0}\langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon,1}\boldsymbol{\psi}\rangle. \tag{102}$$

Proof. Let $0 < \rho < 1$ and

$$\langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon,1}\boldsymbol{\psi}\rangle = T_{\pm,\varepsilon,1}(\boldsymbol{\phi},\boldsymbol{\psi}) + S^{\rho}_{\pm,\varepsilon,1}(\boldsymbol{\phi},\boldsymbol{\psi}) \tag{103}$$

with

$$T_{\pm,\varepsilon,1}(\boldsymbol{\phi},\boldsymbol{\psi}) = \mp i \sum_{k=1}^{m} \int_1^{\infty} \int_{\mathbf{R}^d\times\mathbf{R}^d} e^{-s\varepsilon \pm ism^2} \cos\{A(ay)-A(ax)\} \times \Delta_{\pm s}(x,y)\overline{\phi_k(x)}\psi_k(y)\, dydxds$$

$$S^{\rho}_{\pm,\varepsilon,1}(\boldsymbol{\phi},\boldsymbol{\psi}) = \mp i \sum_{k=1}^{m} \int_{\rho}^{1} \int_{\mathbf{R}^d\times\mathbf{R}^d} e^{-s\varepsilon \pm ism^2} \cos\{A(ay)-A(ax)\} \times \Delta_{\pm s}(x,y)\overline{\phi_k(x)}\psi_k(y)\, dydxds$$

Since

$$S^{\rho}_{\pm,\varepsilon,1}(\boldsymbol{\phi},\boldsymbol{\psi}) = \mp i \sum_{k=1}^{m} \int_{\rho}^{1} \frac{e^{-s\varepsilon \pm ism^2}}{2} \big(e^{iA(ax)}\phi_k,\, e^{\pm isH_0} e^{iA(ay)}\psi_k\big)\, ds \mp i \sum_{k=1}^{m} \int_{\rho}^{1} \frac{e^{-s\varepsilon \pm ism^2}}{2} \big(e^{-iA(ax)}\phi_k,\, e^{\pm isH_0} e^{-iA(ay)}\psi_k\big)\, ds,$$

there exist $\lim_{\varepsilon\to 0}\lim_{\rho\to 0} S^{\rho}_{\pm,\varepsilon,1}(\boldsymbol{\phi},\boldsymbol{\psi})$ and $\lim_{\varepsilon\to 0}\lim_{\rho\to 0}\langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon,1}\boldsymbol{\psi}\rangle$. Let

$$S_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi}) := \lim_{\varepsilon\to 0}\lim_{\rho\to 0} S^{\rho}_{\pm,\varepsilon,1}(\boldsymbol{\phi},\boldsymbol{\psi}), \quad B_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi}) := \lim_{\varepsilon\to 0}\lim_{\rho\to 0} \big\langle \boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon,1}\boldsymbol{\psi}\big\rangle$$
$$\text{and} \quad T_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi}) := \lim_{\varepsilon\to 0} T_{\pm,\varepsilon,1}(\boldsymbol{\phi},\boldsymbol{\psi}).$$

Then $B_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi}) = T_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi}) + S_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi})$. Since

$$T_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi}) = \mp i \sum_{k=1}^{m} \int_1^{\infty} \int_{\mathbf{R}^d\times\mathbf{R}^d} \cos\{A(ay)-A(ax)\} \times \Delta_{\pm s}(x,y)\overline{\phi_k(x)}\psi_k(y)\, dydxds,$$

and

$$S_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi}) = \mp i \sum_{k=1}^{m} \int_0^{1} \frac{1}{2} \big(e^{iA(ax)}\phi_k,\, e^{\pm isH_0} e^{iA(ay)}\psi_k\big)\, ds \mp i \sum_{k=1}^{m} \int_{\sigma}^{1} \frac{1}{2} \big(e^{-iA(ax)}\phi_k,\, e^{\pm isH_0} e^{-iA(ay)}\psi_k\big)\, ds,$$

we can see that $|T_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi})| \leq C_1\|\boldsymbol{\phi}\|_{L^1}\|\boldsymbol{\psi}\|_{L^1}$ and $|S_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi})| \leq C_2\|\boldsymbol{\phi}\|_{L^2}\|\boldsymbol{\psi}\|_{L^2}$, where C_1 and C_2 are positive constants. Hence $B_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi})$ are separately continuous bilinear functionals on $\bigoplus^m \mathcal{S}(\mathbf{R}^d) \times \bigoplus^m \mathcal{S}(\mathbf{R}^d)$ and, by the nuclear theorem, there are unique tempered distributions satisfying $G_{\pm,1}(\boldsymbol{\phi}\otimes\boldsymbol{\psi}) = B_{\pm,1}(\boldsymbol{\phi},\boldsymbol{\psi}) = \lim_{\varepsilon\to 0}\lim_{\rho\to 0}\langle\boldsymbol{\phi}^*, G^{\rho}_{\pm,\varepsilon,1}\boldsymbol{\psi}\rangle$. □

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