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Bases for the derivation modules of two-dimensional multi-Coxeter arrangements and universal derivations

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Abstract. Let $\mathcal{A}$ be an irreducible Coxeter arrangement and $\mathbf{k}$ be a multiplicity of $\mathcal{A}$. We study the derivation module $D(\mathcal{A}, \mathbf{k})$. Any two-dimensional irreducible Coxeter arrangement with even number of lines is decomposed into two orbits under the action of the Coxeter group. In this paper, we will explicitly construct a basis for $D(\mathcal{A}, \mathbf{k})$ assuming $\mathbf{k}$ is constant on each orbit. Consequently we will determine the exponents of $(\mathcal{A}, \mathbf{k})$ under this assumption. For this purpose we develop a theory of universal derivations and introduce a map to deal with our exceptional cases.

Key words: Coxeter arrangement, Coxeter group, multi-arrangement, primitive derivation, multi-derivation module, logarithmic differential form

1. Introduction

Let V be an ℓ-dimensional Euclidean space with inner product I. Let S denote the symmetric algebra of the dual space V^* over $\mathbb{R}$. Denote the S-module of $\mathbb{R}$-linear derivations of S by Der_S. Let F be the field of quotients of S and Der_F be the F-vector space of $\mathbb{R}$-linear derivations of F. Let $W \subseteq O(V, I)$ be a finite irreducible reflection group (a Coxeter group) and $\mathcal{A}$ be the corresponding **Coxeter arrangement**, i.e., $\mathcal{A}$ is the set of all reflecting hyperplanes of W. An arbitrary map $\mathbf{k}\colon \mathcal{A} \to \mathbb{Z}$ is called a **multiplicity** of $\mathcal{A}$. We say that the pair $(\mathcal{A}, \mathbf{k})$ is a **multi-Coxeter arrangement**. The S-module $D(\mathcal{A}, \mathbf{k})$, defined in Section 2, of derivations associated with $(\mathcal{A}, \mathbf{k})$ was introduced by Ziegler [13] when $\operatorname{im} \mathbf{k} \subseteq \mathbb{Z}_{\geq 0}$ and in [1], [2] for any multiplicity $\mathbf{k}$. We say that $(\mathcal{A}, \mathbf{k})$ is **free** if $D(\mathcal{A}, \mathbf{k})$ is a free S-module. The polynomial degrees (= pdeg) [7] of a homogeneous S-basis for $D(\mathcal{A}, \mathbf{k})$ are called the **exponents** of $(\mathcal{A}, \mathbf{k})$. If $\mathbf{k} \equiv 1$, then $D(\mathcal{A}, \mathbf{k})$ coincides with the S-module $D(\mathcal{A})$ of logarithmic derivations and $(\mathcal{A}, \mathbf{k})$ is free (e.g., [8], [7]). More in general, when $\mathbf{k}$ is a constant function, $(\mathcal{A}, \mathbf{k})$ is free and we can explicitly construct a basis using basic invariants and a primitive derivation as in [2], [11]. In the case that $\mathbf{k}$ is not constant, however, we do not know how we can construct a basis for $D(\mathcal{A}, \mathbf{k})$ even when $\ell = 2$. The main result

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of this paper gives an explicit construction of a basis for the module $D(\mathcal{A}, \mathbf{k})$ when $\ell = 2$ and the multiplicity $\mathbf{k}$ is W-equivariant: $\mathbf{k}(H) = \mathbf{k}(wH)$ for any $w \in W$ and $H \in \mathcal{A}$.

The structure of this paper is as follows: In Section 2, we define and discuss the **universal derivations** which will be used in the subsequent sections. Theorem 2.8 is the key result there. In Sections 3 and 4, we assume that $\ell = 2$. Then $W = I_2(h)$ is isomorphic to the dihedral group of order $2h$. When h is odd, $\mathcal{A}$ itself is the unique W-orbit. Thus $\mathbf{k}$ is constant and we can construct a basis (e.g., see [11], [5], [1], [2]). So we may assume that h is even with $h \geq 4$. In this case, we have the W-orbit decomposition: $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Then both $\mathcal{A}_1$ and $\mathcal{A}_2$ are again irreducible arrangements if $h \geq 6$ (or equivalently if $W \neq B_2$). The corresponding irreducible Coxeter groups W_1 and W_2 are both isomorphic to $I_2(\frac{h}{2})$. For $a_1, a_2 \in \mathbb{Z}$, let (a_1, a_2) denote the multiplicity $\mathbf{k} : \mathcal{A} \to \mathbb{Z}$ with $\mathbf{k}(H) = a_1$ $(H \in \mathcal{A}_1)$ and $\mathbf{k}(H) = a_2$ $(H \in \mathcal{A}_2)$. We classify the set $\{(a_1, a_2) \mid a_1, a_2 \in \mathbb{Z}\}$ into sixteen cases. The first fourteen cases are listed in Table 1. We call the fourteen cases **ordinary**. The remaining two cases, which are when either $(a_1, a_2) = (4p, 4q + 2)$ or $(4p + 2, 4q)$, are called to be **exceptional**

(a_1, a_2)	ζ	θ_1, θ_2	basis for $D(\mathcal{A}, (a_1, a_2))$
$(4p+1, 4q+1)$	$E^{(2p,2q)}$	$E, I^*(dP_2)$	$\nabla_{\theta_1}\zeta, \nabla_{\theta_2}\zeta$
$(4p-1, 4q-1)$	$E^{(2p,2q)}$	$D, I^*(dQ/Q)$	
$(4p-1, 4q+1)$	$E^{(2p,2q)}$	$I^*(dQ_1/Q_1), E$	
$(4p+1, 4q-1)$	$E^{(2p,2q)}$	$I^*(dQ_2/Q_2), E$	
$(4p+1, 4q)$	$E^{(2p,2q)}$	$E, I^*(dQ_2)$	
$(4p+3, 4q+2)$	$E^{(2p+1,2q+1)}$		
$(4p-1, 4q)$	$E^{(2p,2q)}$	$D_1, I^*(dQ_1/Q_1)$	
$(4p+1, 4q+2)$	$E^{(2p+1,2q+1)}$		
$(4p, 4q+1)$	$E^{(2p,2q)}$	$E, I^*(dQ_1)$	
$(4p+2, 4q+3)$	$E^{(2p+1,2q+1)}$		
$(4p, 4q-1)$	$E^{(2p,2q)}$	$D_2, I^*(dQ_2/Q_2)$	
$(4p+2, 4q+1)$	$E^{(2p+1,2q+1)}$		
$(4p, 4q)$	$E^{(2p,2q)}$	$\partial_{x_1}, \partial_{x_2}$	
$(4p+2, 4q+2)$	$E^{(2p+1,2q+1)}$		

Table 1. Bases for $D(\mathcal{A}, (a_1, a_2))$ (ordinary cases) $(p \geq 0$ or $q \geq 0)$

because our basis construction method in the ordinary cases does not work for the exceptional ones. The exceptional cases are listed in Table 2. The derivations $\zeta = E^{(s,t)}$ are universal. We will explain how to read the two Tables in Sections 3 and 4. Section 3 is devoted to the ordinary cases where the main tool is the **Levi-Civita connection**

$$\nabla : \mathrm{Der}_F \times \mathrm{Der}_F \to \mathrm{Der}_F$$

with respect to I together with **primitive derivations** D and D_i corresponding to W and W_i $(i = 1, 2)$ respectively. The recipe here is Abe-Yoshinaga's theory developed in [5] and [1]. The main ingredient in Section 4 is the maps

$$\Phi_\zeta^{(1)} : \mathrm{Der}_S \to D(\mathcal{A}, (4p+2, 4q)),$$

$$\Phi_\zeta^{(2)} : \mathrm{Der}_S \to D(\mathcal{A}, (4p, 4q+2)),$$

defined by

$$\Phi_\zeta^{(1)}(\theta) := Q_1(\nabla_\theta\, \zeta) - (4p+1)\theta(Q_1)\zeta,$$

$$\Phi_\zeta^{(2)}(\theta) := Q_2(\nabla_\theta\, \zeta) - (4q+1)\theta(Q_2)\zeta,$$

where Q_i is a defining polynomial for $\mathcal{A}_i$ $(i = 1, 2)$ and ζ is $(2p, 2q)$-universal. Actually in Sections 3 and 4, we will construct bases only when either $p \geq 0$ or $q \geq 0$ in Tables 1 and 2. Lastly we cover the remaining cases using the duality: the existence of a non-degenerate S-bilinear pairing

$$\Omega(\mathcal{A}, \mathbf{k}) \times D(\mathcal{A}, \mathbf{k}) \longrightarrow S,$$

where $\Omega(\mathcal{A}, \mathbf{k})$ is the S-module of logarithmic differential 1-forms associated with the multi-Coxeter arrangement $(\mathcal{A}, \mathbf{k})$ defined in [13], [1] and [3]. We

(a_1, a_2)	ζ	θ_1, θ_2	basis for $D(\mathcal{A}, (a_1, a_2))$
$(4p+2, 4q)$	$E^{(2p,2q)}$	$\partial_{x_1}, \partial_{x_2}$	$\Phi_\zeta^{(1)}(\theta_1), \Phi_\zeta^{(1)}(\theta_2)$
$(4p, 4q+2)$	$E^{(2p,2q)}$	$\partial_{x_1}, \partial_{x_2}$	$\Phi_\zeta^{(2)}(\theta_1), \Phi_\zeta^{(2)}(\theta_2)$

Table 2. Bases for $D(\mathcal{A}, (a_1, a_2))$ (exceptional cases) ($p \geq 0$ or $q \geq 0$)

conclude this paper with Section 5 in which we present Table 4 showing the exponents of $(\mathcal{A}, \mathbf{k})$.

Remark In addition to $I_2(h)$ with $h \geq 4$ even, there exist two kinds of irreducible Coxeter arrangements which have two W-orbits: B_ℓ $(\ell \geq 2)$ and F_4. For each of these two cases, when $\mathbf{k}$ is an equivariant multiplicity, a basis for $D(\mathcal{A}, \mathbf{k})$ is constructed with a method similar to the one applied to the ordinary cases here. Details are found in [4].

2. Universal derivations

Let $\mathcal{A}$ be an irreducible Coxeter arrangement. For each hyperplane $H \in \mathcal{A}$, choose a linear form $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. The product $Q := \prod_{H \in \mathcal{A}} \alpha_H$ lies in S. Let Ω_S be the S-module of regular 1-forms and Ω_F be the F-vector space of rational 1-forms on V. Let I^* denote the inner product on V^* induced from the inner product I on V. Then I^* naturally induces an S-bilinear map $I^* : \Omega_F \times \Omega_F \to F$. Thus we have an F-linear isomorphism

$$I^* : \Omega_F \to \mathrm{Der}_F$$

by $[I^*(\omega)](f) = I^*(\omega, df)$ where $\omega \in \Omega_F$, $f \in F$. Recall the S-module

$$\Omega(\mathcal{A}, \infty) := \big\{\omega \in \Omega_F \mid Q^N \omega \text{ and } (Q/\alpha_H)^N \omega \wedge d\alpha_H \\ \text{are both regular for any } H \in \mathcal{A} \text{ and } N \gg 0\big\}$$

of logarithmic 1-forms [2]. We also have the S-module

$$\begin{aligned} D(\mathcal{A}, -\infty) &:= I^*(\Omega(\mathcal{A}, \infty)) \\ &= \big\{\theta \in \mathrm{Der}_F \mid Q^N \theta \in \mathrm{Der}_S \text{ and } (Q/\alpha_H)^N \theta(\beta) \text{ is regular for } \\ &\qquad \beta \in V^* \text{ whenever } I^*(\beta, \alpha_H) = 0 \text{ for any } H \in \\ &\qquad \mathcal{A} \text{ and } N \gg 0\big\} \end{aligned}$$

of logarithmic derivations [2]. Let

$$\begin{aligned} \nabla \colon \mathrm{Der}_F \times \mathrm{Der}_F &\longrightarrow \mathrm{Der}_F \\ (\theta, \delta) &\longmapsto \nabla_\theta\, \delta \end{aligned}$$

be the Levi-Civita connection with respect to I. The derivation $\nabla_\theta \delta \in \mathrm{Der}_F$ is characterized by the equality $(\nabla_\theta \delta)(\alpha) = \theta(\delta(\alpha))$ for any $\alpha \in V^*$.

For $\alpha \in V^*$ let $S_{(\alpha)}$ denote the localization of S at the prime ideal (α) of S. For an arbitrary multiplicity $\mathbf{k} : \mathcal{A} \to \mathbb{Z}$, define an S-submodule $D(\mathcal{A}, \mathbf{k})$ of $D(\mathcal{A}, -\infty)$ by

$$D(\mathcal{A}, \mathbf{k}) := \left\{\theta \in D(\mathcal{A}, -\infty) \mid \theta(\alpha_H) \in \alpha_H^{\mathbf{k}(H)} S_{(\alpha_H)} \text{ for any } H \in \mathcal{A}\right\}$$

from [3]. The module $D(\mathcal{A}, \mathbf{k})$ was introduced by Ziegler [13] when $\operatorname{im} \mathbf{k} \subseteq \mathbb{Z}_{\geq 0}$. Note $D(\mathcal{A}, \mathbf{0}) = \mathrm{Der}_S$ where $\mathbf{0}$ is the zero multiplicity. For each $\mathbf{k} \colon \mathcal{A} \to \mathbb{Z}$, define $Q^{\mathbf{k}} := \prod_{H \in \mathcal{A}} \alpha_H^{\mathbf{k}(H)} \in F$. Recall the following generalization of Saito's criterion [9]:

Theorem 2.1 (Abe [1, Theorem 1.4]) *Let $\mathbf{k} \colon \mathcal{A} \to \mathbb{Z}$ and $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A}, \mathbf{k})$. Then $\theta_1, \ldots, \theta_\ell$ form an S-basis for $D(\mathcal{A}, \mathbf{k})$ if and only if $\det[\theta_j(x_i)] \doteq Q^{\mathbf{k}}$. Here $\doteq$ implies the equality up to a non-zero constant multiple.*

Definition 2.2 Let $\mathbf{k} : \mathcal{A} \to \mathbb{Z}$ and $\zeta \in D(\mathcal{A}, -\infty)^W$, where the superscript W stands for the W-invariant part. We say that ζ is **k-universal** when ζ is homogeneous and the S-linear map

$$\begin{aligned} \Psi_\zeta : \mathrm{Der}_S &\longrightarrow D(\mathcal{A}, 2\mathbf{k}) \\ \theta &\longmapsto \nabla_\theta \zeta \end{aligned}$$

is bijective.

Example 2.3 The **Euler derivation** E, which is the derivation characterized by $E(\alpha) = \alpha$ for any $\alpha \in V^*$, is $\mathbf{0}$-universal because $\Psi_E(\delta) = \nabla_\delta E = \delta$.

For an irreducible Coxeter group W, there exist algebraically independent homogeneous polynomials $P_1, P_2, \ldots, P_\ell$ with $\deg P_1 < \deg P_2 \leq \cdots \leq \deg P_{\ell-1} < \deg P_\ell$ by Chevalley's Theorem [6], which are called **basic invariants**. When $D \in \mathrm{Der}_F$ satisfies

$$D(P_j) = \begin{cases} 0 & \text{if } 1 \leq j < \ell, \\ 1 & \text{if } j = \ell, \end{cases}$$

we say that D is a **primitive derivation**. It is unique up to a nonzero constant multiple. Let $R := S^W$ be the W-invariant subring of S and

$$T := \{f \in R \mid D(f) = 0\}.$$

Theorem 2.4 ([2, Theorem 3.9 (1)], [3, Theorem 4.4])

(1) *We have a T-linear automorphism*

$$\begin{aligned} \nabla_D : D(\mathcal{A}, -\infty)^W &\longrightarrow D(\mathcal{A}, -\infty)^W, \\ \theta &\longmapsto \nabla_D \theta \end{aligned}$$

(2) $\nabla_D(D(\mathcal{A}, 2\mathbf{k}+\mathbf{1})^W) = D(\mathcal{A}, 2\mathbf{k}-\mathbf{1})^W$ *for any multiplicity* $\mathbf{k} : \mathcal{A} \to \mathbb{Z}$.

Note that ∇_D^{-1} and ∇_D^k ($k \in \mathbb{Z}$) are also T-linear automorphisms.

Let $x_1, \ldots, x_\ell$ be a basis for V^*. Put $A := [I^*(x_i, x_j)]_{ij}$ which is a nonsingular real symmetric matrix. For simplicity let ∂_{x_j} and ∂_{P_j} denote $\partial/\partial x_j$ and $\partial/\partial P_j$ respectively. Note that $D = \partial_{P_\ell}$.

Proposition 2.5 *Let $k \in \mathbb{Z}$. Here $\mathbf{k}$ is a constant multiplicity: $\mathbf{k} \equiv k$. Then the derivation $\nabla_D^k E$ is $(-\mathbf{k})$-universal.*

Proof. When $k \le 0$, the result was first proved by Yoshinaga in [12]. Assume $k > 0$. Recall a basis $\eta_1^{(-2k)}, \ldots, \eta_\ell^{(-2k)}$ for $D(\mathcal{A}, -2k)$ introduced in [2, Definition 3.1]. Then we have

$$\left[\nabla_{\partial_{x_1}} \nabla_D^k E, \ldots, \nabla_{\partial_{x_\ell}} \nabla_D^k E\right] = \left[\eta_1^{(-2k)}, \ldots, \eta_\ell^{(-2k)}\right] A^{-1},$$

which is the second equality of [2, Proposition 4.3] (in the differential-form version). □

Proposition 2.6 *Let $\zeta \in D(\mathcal{A}, -\infty)^W$ be $\mathbf{k}$-universal. Then*

(1) *the S-linear map*

$$\begin{aligned} \Psi_\zeta : D(\mathcal{A}, -\mathbf{1}) &\longrightarrow D(\mathcal{A}, 2\mathbf{k}-\mathbf{1}) \\ \theta &\longmapsto \nabla_\theta \zeta \end{aligned}$$

is bijective,

(2) $\zeta \in D(\mathcal{A}, 2\mathbf{k}+\mathbf{1})^W$, *and*

(3) $\alpha_H^{-2\mathbf{k}(H)-1}\zeta(\alpha_H)$ *is a unit in* $S_{(\alpha_H)}$ *for any* $H \in \mathcal{A}$.

Proof. (1) Note that $\partial_{P_1}, \ldots, \partial_{P_\ell}$ form an S-basis for $D(\mathcal{A}, -\mathbf{1})$ [2, p. 823]. Let $1 \leq j \leq \ell$. Then

$$Q\nabla_{\partial_{P_j}}\zeta = \nabla_{Q\partial_{P_j}}\zeta \in D(\mathcal{A}, 2\mathbf{k})$$

because $Q\partial_{P_j} \in \mathrm{Der}_S$. Thus

$$\big(\nabla_{\partial_{P_j}}\zeta\big)(\alpha_H) \in \alpha_H^{2\mathbf{k}(H)-1}S_{(\alpha_H)} \quad (H \in \mathcal{A}).$$

Pick $H \in \mathcal{A}$ arbitrarily and choose an orthonormal basis $x_1, \ldots, x_\ell$ for V^* so that $H = \ker(x_1)$. For $i = 2, \ldots, \ell$ define $g_i := (Q/x_1)^N Q(\nabla_{\partial_{P_j}}\zeta)(x_i) \in S$ for a sufficiently large positive integer N. Let $s = s_H$ denote the orthogonal reflection through H. Then $s(g_i) = -g_i$. Thus $g_i \in x_1 S$ and

$$\big(\nabla_{\partial_{P_j}}\zeta\big)(x_i) = (Q/x_1)^{-N} g_i/Q \in S_{(x_1)}.$$

This implies $\nabla_{\partial_{P_j}}\zeta \in D(\mathcal{A}, -\infty)$ and thus $\nabla_{\partial_{P_j}}\zeta \in D(\mathcal{A}, 2\mathbf{k}-\mathbf{1})$. One has

$$\begin{aligned}
&\det\left[\big(\nabla_{\partial_{P_j}}\zeta\big)(x_i)\right] \\
&\quad = \det\left(\left[\big(\nabla_{\partial_{x_j}}\zeta\big)(x_i)\right]\left[\partial P_i/\partial x_j\right]^{-1}\right) \doteq Q^{-1}\det\left[\big(\nabla_{\partial_{x_j}}\zeta\big)(x_i)\right] \\
&\quad \doteq Q^{2\mathbf{k}-\mathbf{1}}
\end{aligned}$$

by the chain rule $\partial_{x_j} = \sum_{s=1}^{\ell}(\partial P_s/\partial x_j)\partial_{P_s}$ and the equality $\det[\partial P_i/\partial x_j] \doteq Q$. Applying Theorem 2.1 we conclude that $\nabla_{\partial_{P_1}}\zeta, \ldots, \nabla_{\partial_{P_\ell}}\zeta$ form an S-basis for $D(\mathcal{A}, 2\mathbf{k}-\mathbf{1})$.

(2) By (1), $\nabla_D\,\zeta \in D(\mathcal{A}, 2\mathbf{k}-\mathbf{1})^W$. Thanks to Theorem 2.4, we have $\zeta \in D(\mathcal{A}, 2\mathbf{k}+\mathbf{1})^W$.

(3) By (2), $\zeta(\alpha_H) \in \alpha_H^{2\mathbf{k}(H)+1}S_{(\alpha_H)}$ for any $H \in \mathcal{A}$. Assume that $\alpha_H^{-2\mathbf{k}(H)-1}\zeta(\alpha_H)$ is not a unit in $S_{(\alpha_H)}$ for some $H \in \mathcal{A}$. Choose an orthonormal basis $x_1, x_2, \ldots, x_\ell$ for V^* so that $H = \ker(x_1)$. Then $\zeta(x_1) \in x_1^{2\mathbf{k}(H)+2}S_{(x_1)}$. Thus $(\nabla_{\partial_{x_j}}\zeta)(x_1) \in x_1^{2\mathbf{k}(H)+1}S_{(x_1)}$ for each j with $1 \leq j \leq \ell$ and $Q^{2\mathbf{k}} \doteq \det[(\nabla_{\partial_{x_j}}\zeta)(x_i)] \in x_1^{2\mathbf{k}(H)+1}S_{(x_1)}$, which is a contradiction. □

Proposition 2.7 (cf. [5, Theorem 10], [1, Theorem 2.1]) *If* $\zeta \in D(\mathcal{A},$

$-\infty)^W$ is $\mathbf{k}$-universal and $\mathbf{m} : \mathcal{A} \to \{-1, 0, 1\}$ is a multiplicity, then the S-linear map

$$\begin{aligned}\Psi_\zeta \colon D(\mathcal{A}, \mathbf{m}) &\longrightarrow D(\mathcal{A}, 2\mathbf{k} + \mathbf{m})\\ \theta &\longmapsto \nabla_\theta\, \zeta\end{aligned}$$

is bijective.

Proof. Note that $D(\mathcal{A}, \mathbf{m}) \subseteq D(\mathcal{A}, -\mathbf{1})$ and $D(\mathcal{A}, 2\mathbf{k}+\mathbf{m}) \subseteq D(\mathcal{A}, 2\mathbf{k}-\mathbf{1})$. By Proposition 2.6 (1), the restriction of

$$\Psi_\zeta \colon D(\mathcal{A}, -\mathbf{1}) \longrightarrow D(\mathcal{A}, 2\mathbf{k}-\mathbf{1})$$

to $D(\mathcal{A}, \mathbf{m})$ is injective. Thus it is enough to prove $\Psi_\zeta(D(\mathcal{A}, \mathbf{m})) = D(\mathcal{A}, 2\mathbf{k}+\mathbf{m})$. Let $\theta \in D(\mathcal{A}, -\mathbf{1})$. Pick $H \in \mathcal{A}$ arbitrarily and fix it. Choose an orthonormal basis $x_1, x_2, \ldots, x_\ell$ with $H = \ker(x_1)$. Let $k := \mathbf{k}(H)$ and $m := \mathbf{m}(H)$. Then, by Proposition 2.6 (3), $g := x_1^{-2k-1}\zeta(x_1)$ is a unit in $S_{(x_1)}$. Compute

$$\begin{aligned}(\Psi_\zeta(\theta))(x_1) &= (\nabla_\theta\, \zeta)(x_1) = \theta(\zeta(x_1)) = \theta(x_1^{2k+1} g)\\ &= x_1^{2k+1}\theta(g) + (2k+1)x_1^{2k}\theta(x_1)g\\ &= x_1^{2k+1}\sum_{j=1}^{\ell}\theta(x_j)(\partial g/\partial x_j) + (2k+1)x_1^{2k}\theta(x_1)g\\ &= x_1^{2k}\theta(x_1)\{x_1(\partial g/\partial x_1) + (2k+1)g\} + x_1^{2k+1}\sum_{j=2}^{\ell}\theta(x_j)(\partial g/\partial x_j)\\ &= x_1^{2k}\theta(x_1)U + x_1^{2k+1}C,\end{aligned}$$

where $U := x_1(\partial g/\partial x_1) + (2k+1)g$ is a unit in $S_{(x_1)}$ and $C := \sum_{j=2}^{\ell}\theta(x_j)(\partial g/\partial x_j)$. Dividing the both sides by x_1^{2k+m}, we get

$$x_1^{-2k-m}(\Psi_\zeta(\theta))(x_1) = x_1^{-m}\theta(x_1)U + x_1^{1-m}C.$$

Note that $\partial g/\partial x_j \in S_{(x_1)}$ and $\theta(x_j) \in S_{(x_1)}$ $(j \geq 2)$ because $\theta \in D(\mathcal{A}, -\infty)$. So one has $C \in S_{(x_1)}$ and $x_1^{1-m}C \in S_{(x_1)}$ for $m \in \{\pm 1, 0\}$. Thus we conclude that

$$x_1^{-2k-m}(\Psi_\zeta(\theta))(x_1) \in S_{(x_1)} \Longleftrightarrow x_1^{-m}\theta(x_1) \in S_{(x_1)}.$$

This implies that

$$\Psi_\zeta(\theta) \in D(\mathcal{A}, 2\mathbf{k}+\mathbf{m}) \Longleftrightarrow \theta \in D(\mathcal{A}, \mathbf{m})$$

because $H \in \mathcal{A}$ was arbitrarily chosen. This completes the proof. $\square$

The following is the main result in this section.

Theorem 2.8 *Let $\mathbf{k}\colon \mathcal{A} \to \mathbb{Z}$ be a multiplicity of $\mathcal{A}$. Let $\zeta \in D(\mathcal{A}, -\infty)^W$ be $\mathbf{k}$-universal. Then $\nabla_D^{-1}\zeta$ is $(\mathbf{k}+\mathbf{1})$-universal.*

Proof. It is classically known [8] that $\xi_j := I^*(dP_j) \in D(\mathcal{A}, \mathbf{1})^W$ ($j = 1, \ldots, \ell$) form an S-basis for $D(\mathcal{A}, \mathbf{1})$. By Proposition 2.7, $\nabla_{\xi_j}\zeta \in D(\mathcal{A}, 2\mathbf{k}+\mathbf{1})^W$ ($j = 1, \ldots, \ell$) form an S-basis for $D(\mathcal{A}, 2\mathbf{k}+\mathbf{1})$. Since $\nabla_D\nabla_{\xi_j}\zeta \in D(\mathcal{A}, 2\mathbf{k}-\mathbf{1})^W$ ($j = 1, \ldots, \ell$) by Theorem 2.4, we can write

$$\nabla_D\nabla_{\xi_j}\zeta = \sum_{i=1}^{\ell} f_{ij}\nabla_{\partial_{P_i}}\zeta$$

with W-invariant polynomials $f_{ij} \in R$ because of Proposition 2.6 (1). Then f_{ij} is a homogeneous element with degree $m_i + m_j - h < h$, where h is the Coxeter number, and f_{ij} belongs to $T = \{f \in R \mid Df = 0\}$. Since $m_i + m_{\ell+1-i} - h = 0$, $\det[f_{ij}] \in \mathbb{R}$. Apply ∇_D^{-1} to the both sides to get

$$\nabla_{\xi_j}\zeta = \nabla_D^{-1}\sum_{i=1}^{\ell} f_{ij}\nabla_{\partial_{P_i}}\zeta = \sum_{i=1}^{\ell} f_{ij}\nabla_{\partial_{P_i}}\nabla_D^{-1}\zeta.$$

Since $\nabla_{\xi_j}\zeta \in D(\mathcal{A}, 2\mathbf{k}+\mathbf{1})^W$ ($j = 1, \ldots, \ell$) form an S-basis for $D(\mathcal{A}, 2\mathbf{k}+\mathbf{1})$, we have $\det[f_{ij}] \in \mathbb{R}^\times$. This implies that $\nabla_{\partial_{P_j}}\nabla_D^{-1}\zeta$ ($j = 1, \ldots, \ell$) form an S-basis for $D(\mathcal{A}, 2\mathbf{k}+\mathbf{1})$. Since $\nabla_D^{-1}\zeta \in D(\mathcal{A}, 2\mathbf{k}+\mathbf{3})$ by Proposition 2.6 (2) and Theorem 2.4, we conclude that

$$\nabla_{\partial_{x_j}}\nabla_D^{-1}\zeta = \sum_{i=1}^{\ell}(\partial_{x_j}P_i)\nabla_{\partial_{P_i}}\nabla_D^{-1}\zeta \quad (j = 1, \ldots, \ell)$$

form an S-basis for $D(\mathcal{A}, 2\mathbf{k}+\mathbf{2})$ by Theorem 2.1. $\square$

3. The ordinary cases

In the rest of this paper we assume $\dim V = \ell = 2$ and $W = I_2(h)$ such that $h \geq 4$ is an even number. The orbit decomposition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ satisfies $|\mathcal{A}_1| = |\mathcal{A}_2| = h/2$. Recall the equivariant multiplicities $\mathbf{k} = (a_1, a_2)$, $a_1, a_2 \in \mathbb{Z}$, defined by

$$\mathbf{k}(H) = \begin{cases} a_1 & \text{if } H \in \mathcal{A}_1, \\ a_2 & \text{if } H \in \mathcal{A}_2. \end{cases}$$

Let x_1, x_2 be an orthonormal basis for V^*. Suppose that $P_1 := (x_1^2 + x_2^2)/2$ and P_2 are basic invariants of W. Then $\deg P_2 = h$ and $R = S^W = \mathbb{R}[P_1, P_2]$. Let W_i be the (normal) subgroup of W generated by all reflections through $H \in \mathcal{A}_i$ $(i = 1, 2)$. Let $Q_i = \prod_{H \in \mathcal{A}_i} \alpha_H$ and $R_i := S^{W_i}$ $(i = 1, 2)$. Let D be a primitive derivation corresponding to the whole group W. Then it is known [10, (5.1)] that

$$D \doteq \frac{1}{Q}(-x_2 \partial_{x_1} + x_1 \partial_{x_2}).$$

Lemma 3.1 *Define*

$$D_1 := Q_2 D \doteq \frac{1}{Q_1}(-x_2 \partial_{x_1} + x_1 \partial_{x_2}), \quad D_2 := Q_1 D \doteq \frac{1}{Q_2}(-x_2 \partial_{x_1} + x_1 \partial_{x_2}).$$

Then

(1) $R_1 = \mathbb{R}[P_1, Q_2]$, $R_2 = \mathbb{R}[P_1, Q_1]$ *and* $R = \mathbb{R}[P_1, Q_1^2] = \mathbb{R}[P_1, Q_2^2]$,
(2) $-x_2(\partial Q_2/\partial x_1) + x_1(\partial Q_2/\partial x_2) \doteq Q_1$ *and* $-x_2(\partial Q_1/\partial x_1) + x_1(\partial Q_1/\partial x_2) \doteq Q_2$,
(3) $D_1(P_1) = D_2(P_1) = 0$, $D_1(Q_2) \in \mathbb{R}^\times$ *and* $D_2(Q_1) \in \mathbb{R}^\times$.

Proof. Thanks to the symmetry we only have to prove a half of the statement. Since Q and Q_1 are both W_1-antiinvariant, $Q_2 = Q/Q_1$ is W_1-invariant and Q_2^2 is W-invariant. Note that Q_2 is a product of real linear forms. So Q_2 and P_1 are algebraically independent. Since

$$|\mathcal{A}_1| = h/2 = (\deg Q_2 - 1) + (\deg P_1 - 1),$$

we have $R_1 = \mathbb{R}[P_1, Q_2]$. Similarly we obtain $R = \mathbb{R}[P_1, Q_2^2]$. This proves

(1). The Jacobian

$$-x_2(\partial Q_2/\partial x_1) + x_1(\partial Q_2/\partial x_2) = \det \begin{pmatrix} \partial P_1/\partial x_1 & \partial Q_2/\partial x_1 \\ \partial P_1/\partial x_2 & \partial Q_2/\partial x_2 \end{pmatrix} \neq 0$$

is equal to Q_1 up to a nonzero constant multiple, which is (2). Compute

$$D_1(P_1) = Q_2 D(P_1) = 0, \quad 2D_1(Q_2) = 2Q_2 D(Q_2) = D(Q_2^2) \in \mathbb{R}^\times.$$

This proves (3). □

The Euler derivation $E = I^*(dP_1) = I^*(x_1 dx_1 + x_2 dx_2) = x_1\partial_{x_1} + x_2\partial_{x_2}$ satisfies $E(\alpha) = \alpha$ for all $\alpha \in V^*$ and belongs to $D(\mathcal{A}, (1,1))$.

Proposition 3.2 *A basis for $D(\mathcal{A}, (a_1, a_2))$ is given in Table* 3 *for $-1 \le a_1 \le 1$, $-1 \le a_2 \le 1$.*

(a_1, a_2)	basis for $D(\mathcal{A}, (a_1, a_2))$	exponents of $(\mathcal{A}, (a_1, a_2))$	their difference
$(1,1)$	$E, I^*(dP_2)$	$1, h-1$	$h-2$
$(1,0)$	$E, I^*(dQ_2)$	$1, (h/2)-1$	$(h/2)-2$
$(0,1)$	$E, I^*(dQ_1)$	$1, (h/2)-1$	$(h/2)-2$
$(1,-1)$	$I^*(dQ_2/Q_2), E$	$-1, 1$	2
$(0,0)$	$\partial_{x_1}, \partial_{x_2}$	$0, 0$	0
$(-1,1)$	$I^*(dQ_1/Q_1), E$	$-1, 1$	2
$(0,-1)$	$D_2, I^*(dQ_2/Q_2)$	$1-(h/2), -1$	$(h/2)-2$
$(-1,0)$	$D_1, I^*(dQ_1/Q_1)$	$1-(h/2), -1$	$(h/2)-2$
$(-1,-1)$	$D, I^*(dQ/Q)$	$1-h, -1$	$h-2$

Table 3. The exponents of $(\mathcal{A}, (a_1, a_2))$ $(-1 \le a_1 \le 1, -1 \le a_2 \le 1)$

Proof. Let $\omega_0 = -x_2 dx_1 + x_1 dx_2$. Note that $\omega_0 \wedge d\alpha = -\alpha(dx_1 \wedge dx_2)$ for any $\alpha \in V^*$. It is easy to see that each of $dP_1, dP_2, dQ_1, dQ_2, dQ_1/Q_1, dQ_2/Q_2, \omega_0/Q, \omega_0/Q_1$ and ω_0/Q_2 belongs to $\Omega(\mathcal{A}, \infty)$ defined in Section 2. Note that $D = I^*(\omega_0)/Q$ and $D_i = I^*(\omega_0)/Q_i$ $(i = 1, 2)$. Thus all of the derivations in the table lie in $D(\mathcal{A}, -\infty) = I^*(\Omega(\mathcal{A}, \infty))$.

If P is W-invariant, then $I^*(dP) \in D(\mathcal{A}, (1,1))$. Therefore $I^*(dQ_1) \in D(\mathcal{A}, (0,1))$ and $I^*(dQ_2) \in D(\mathcal{A}, (1,0))$ because of Lemma 3.1 (1). We thus have $I^*(dQ_1/Q_1) \in D(\mathcal{A}, (-1,1))$ and $I^*(dQ_2/Q_2) \in D(\mathcal{A}, (1,-1))$. Since $QD = Q_1 D_1 = Q_2 D_2$ lies in Der_S, we get $D \in D(\mathcal{A}, (-1,-1))$,

$D_1 \in D(\mathcal{A}, (-1, 0))$ and $D_2 \in D(\mathcal{A}, (0, -1))$. Now apply Theorem 2.1 noting Lemma 3.1 (2). □

Lemma 3.3 *When $h \geq 6$ is even, D_i is a primitive derivation of the irreducible Coxeter arrangement $\mathcal{A}_i$ $(i = 1, 2)$.*

Proof. By Lemma 3.1 (3). □

For $s, t \in \mathbb{Z}$ with $t - s \in 2\mathbb{Z}$, define

$$E_1^{(s,t)} := \nabla_D^{-t} \nabla_{D_1}^{t-s} E, \quad E_2^{(s,t)} := \nabla_D^{-s} \nabla_{D_2}^{s-t} E.$$

Proposition 3.4
(1) *If $t \in \mathbb{Z}_{\geq 0}$ and $t - s \in 2\mathbb{Z}$, then $E_1^{(s,t)}$ is (s, t)-universal,*
(2) *If $s \in \mathbb{Z}_{\geq 0}$ and $s - t \in 2\mathbb{Z}$, then $E_2^{(s,t)}$ is (s, t)-universal.*

Proof. It is enough to show (1) because of the symmetry of the statement.

Case 1. When $h \geq 6$ is even, $\mathcal{A}_1$ is an irreducible Coxeter arrangement of $h/2$ lines. By Lemma 3.3, D_1 is a primitive derivation of $\mathcal{A}_1$. Thus

$$\nabla_{\partial_{x_1}} \nabla_{D_1}^{t-s} E, \ldots, \nabla_{\partial_{x_\ell}} \nabla_{D_1}^{t-s} E$$

form an S-basis for $D(\mathcal{A}, (2(s-t), 0))$. Note that $D_1 = Q_2 D$ satisfies

$$w_1 D_1 = D_1, \quad w_2 D_1 = \det(w_2) D_1$$

for any $w_1 \in W_1$, $w_2 \in W_2$. Since W_1 is a normal subgroup of W, $D(\mathcal{A}_1, -\infty)^{W_1}$ is naturally a W-module and the map $\nabla_{D_1}^n : D(\mathcal{A}_1, -\infty)^{W_1} \to D(\mathcal{A}_1, -\infty)^{W_1}$ is a W-equivariant bijection when n is even. Thus $\nabla_{D_1}^{t-s} E \in D(\mathcal{A}, -\infty)^W$. This implies that $\nabla_{D_1}^{t-s} E$ is $(s-t, 0)$-universal when $t - s \in 2\mathbb{Z}$. Apply Theorem 2.8.

Case 2. Let $h = 4$. Then W is of type B_2. We may choose an orthonormal basis for V^* with $Q_1 = x_1 x_2$ and $Q_2 = (x_1 + x_2)(x_1 - x_2)$. Then

$$D_1 = -\frac{1}{x_1} \partial_{x_1} + \frac{1}{x_2} \partial_{x_2}$$

and

$$\nabla_{D_1}^{2n} E = -(4n-3)!!\big(x_1^{1-4n}\partial_{x_1} + x_2^{1-4n}\partial_{x_2}\big) \in D(\mathcal{A}, -\infty)^W \quad (n > 0),$$

$$\nabla_{D_1}^{-2n} E = \frac{1}{(4n+1)!!}\big(x_1^{4n+1}\partial_{x_1} + x_2^{4n+1}\partial_{x_2}\big) \in D(\mathcal{A}, -\infty)^W \quad (n \geq 0),$$

where $(2m-1)!! = \prod_{i=1}^{m}(2i-1)$. Thus

$$\nabla_{\partial_{x_1}} \nabla_{D_1}^{2n} E \doteq x_1^{-4n}\partial_{x_1}, \quad \nabla_{\partial_{x_2}} \nabla_{D_1}^{2n} E \doteq x_2^{-4n}\partial_{x_2} \quad (n \in \mathbb{Z}).$$

This implies that $\nabla_{D_1}^{t-s} E$ is $(s-t, 0)$-universal when $s - t \in 2\mathbb{Z}$. Apply Theorem 2.8. $\square$

We say that a pair (a_1, a_2) is **exceptional** if

$$a_1 \in 2\mathbb{Z} \text{ and } a_1 - a_2 \equiv 2 \pmod 4.$$

If (a_1, a_2) is not exceptional, then we call (a_1, a_2) **ordinary**. We may apply Propositions 3.2 and 2.7 to get the following proposition:

Proposition 3.5 *Suppose that (a_1, a_2) is ordinary and that either $p \geq 0$ or $q \geq 0$ in Table 1. Then $\nabla_{\theta_1}\zeta$, $\nabla_{\theta_2}\zeta$ form an S-basis for $D(\mathcal{A}, (a_1, a_2))$ as in Table 1, where $E^{(s,t)}$ stands for $E_1^{(s,t)}$ if $t \geq 0$ or it stands for $E_2^{(s,t)}$ if $s \geq 0$.*

4. The exceptional cases

Suppose that $(a_1, a_2) \in \mathbb{Z}^2$ is exceptional. Write

$$(a_1, a_2) = (4p+2, 4q) \ \text{ or } \ (a_1, a_2) = (4p, 4q+2) \quad (p, q \in \mathbb{Z}).$$

Proposition 4.1 *Suppose that ζ is $(2p, 2q)$-universal. Then the map*

$$\begin{aligned} \Phi_\zeta^{(1)} : \mathrm{Der}_S &\longrightarrow D(\mathcal{A}, (4p+2, 4q)) \\ \theta &\longmapsto Q_1(\nabla_\theta \zeta) - (4p+1)\theta(Q_1)\zeta \end{aligned}$$

is an S-linear bijection. Similarly the map

$$\begin{aligned} \Phi_\zeta^{(2)} : \mathrm{Der}_S &\longrightarrow D(\mathcal{A}, (4p, 4q+2)) \\ \theta &\longmapsto Q_2(\nabla_\theta \zeta) - (4q+1)\theta(Q_2)\zeta \end{aligned}$$

is an S-linear bijection.

Proof. It is enough to show the first half because of the symmetry. Let $\theta \in \mathrm{Der}_S$. We first prove that $\Phi_\zeta^{(1)}(\theta) \in D(\mathcal{A}, (4p+2, 4q))$. Let $H_i \in \mathcal{A}_i$ and $\alpha_i := \alpha_{H_i}$ $(i = 1, 2)$. Since $\zeta \in D(\mathcal{A}, (4p+1, 4q+1))$ by Proposition 2.6 (2), write

$$\zeta(\alpha_1) = \alpha_1^{4p+1} f_1, \quad \zeta(\alpha_2) = \alpha_2^{4q+1} f_2 \quad (f_1 \in S_{(\alpha_1)},\ f_2 \in S_{(\alpha_2)}).$$

Compute

$$\begin{aligned}
&\big[\Phi_\zeta^{(1)}(\theta)\big](\alpha_1) \\
&= Q_1(\nabla_\theta \zeta)(\alpha_1) - (4p+1)\theta(Q_1)\zeta(\alpha_1) \\
&= Q_1\big(\theta(\alpha_1^{4p+1} f_1)\big) - (4p+1)\theta(Q_1)\alpha_1^{4p+1} f_1 \\
&= Q_1 \alpha_1^{4p+1}\theta(f_1) + (4p+1) f_1 \alpha_1^{4p} Q_1 \theta(\alpha_1) - (4p+1) f_1 \alpha_1^{4p+1}\theta(Q_1) \\
&= Q_1 \alpha_1^{4p+1}\theta(f_1) - (4p+1) f_1 \alpha_1^{4p+2}\big\{(1/\alpha_1)\theta(Q_1) - (Q_1/\alpha_1^2)\theta(\alpha_1)\big\} \\
&= Q_1 \alpha_1^{4p+1}\theta(f_1) - (4p+1) f_1 \alpha_1^{4p+2}\theta(Q_1/\alpha_1) \in \alpha_1^{4p+2} S_{(\alpha_1)}.
\end{aligned}$$

Also

$$\begin{aligned}
&\big[\Phi_\zeta^{(1)}(\theta)\big](\alpha_2) \\
&= Q_1(\nabla_\theta \zeta)(\alpha_2) - (4p+1)\theta(Q_1)\zeta(\alpha_2) \\
&= Q_1\big(\theta(\alpha_2^{4q+1} f_2)\big) - (4p+1)\theta(Q_1)\alpha_2^{4q+1} f_2 \\
&= Q_1 \alpha_2^{4q+1}\theta(f_2) + (4q+1) f_2 \alpha_2^{4q} Q_1 \theta(\alpha_2) - (4p+1) f_2 \alpha_2^{4q+1}\theta(Q_1) \\
&\in \alpha_2^{4q} S_{(\alpha_2)}.
\end{aligned}$$

This shows $\Phi_\zeta^{(1)}(\theta) \in D(\mathcal{A}, (4p+2, 4q))$. Next we will prove that $\Phi_\zeta^{(1)}(\partial_{x_1})$ and $\Phi_\zeta^{(1)}(\partial_{x_2})$ form an S-basis for $D(\mathcal{A}, (4p+2, 4q))$. Define $M(\theta_1, \theta_2) := [\theta_i(x_j)]_{1 \le i,j \le 2}$. Then

$$\begin{aligned}\det M\big(\Phi_\zeta^{(1)}(\partial_{x_1}),\Phi_\zeta^{(1)}(\partial_{x_2})\big) &= \det M\big(Q_1\nabla_{\partial_{x_1}}\zeta, Q_1\nabla_{\partial_{x_2}}\zeta\big)\\ &\quad -(4p+1)\det M\big(Q_1\nabla_{\partial_{x_1}}\zeta,(\partial_{x_2}Q_1)\zeta\big)\\ &\quad -(4p+1)\det M\big((\partial_{x_1}Q_1)\zeta, Q_1\nabla_{\partial_{x_2}}\zeta\big).\end{aligned}$$

Note

$$x_1\big(\nabla_{\partial_{x_1}}\zeta\big)+x_2\big(\nabla_{\partial_{x_2}}\zeta\big)=\nabla_E\zeta=\{1+h(p+q)\}\zeta$$

because $\nabla_{\partial_{x_1}}\zeta$, $\nabla_{\partial_{x_2}}\zeta$ are a basis for $D(\mathcal{A},(4p,4q))$ and $\operatorname{pdeg}\zeta=1+h(p+q)$. Thus

$$\begin{aligned}&\det M\big(\Phi_\zeta^{(1)}(\partial_{x_1}),\Phi_\zeta^{(1)}(\partial_{x_2})\big)\\ &= Q_1^2\det M\big(\nabla_{\partial_{x_1}}\zeta,\nabla_{\partial_{x_2}}\zeta\big)-\frac{(4p+1)Q_1x_2(\partial_{x_2}Q_1)}{1+h(p+q)}\det M\big(\nabla_{\partial_{x_1}}\zeta,\nabla_{\partial_{x_2}}\zeta\big)\\ &\quad -\frac{(4p+1)Q_1x_1(\partial_{x_1}Q_1)}{1+h(p+q)}\det M\big(\nabla_{\partial_{x_1}}\zeta,\nabla_{\partial_{x_2}}\zeta\big)\\ &=\left\{Q_1^2-\frac{(4p+1)Q_1(x_1(\partial_{x_1}Q_1)+x_2(\partial_{x_2}Q_1))}{1+h(p+q)}\right\}\det M\big(\nabla_{\partial_{x_1}}\zeta,\nabla_{\partial_{x_2}}\zeta\big)\\ &\doteq\left\{1-\frac{(4p+1)h}{2(1+h(p+q))}\right\}Q_1^2Q_1^{4p}Q_2^{4q}=\frac{2-h(2p-2q+1)}{2(1+h(p+q))}Q_1^{4p+2}Q_2^{4q}.\end{aligned}$$

Note that $2-h(2p-2q+1)\neq 0$ and $1+h(p+q)\neq 0$ because $h\geq 4$. Therefore $\Phi_\zeta^{(1)}(\partial_{x_1})$ and $\Phi_\zeta^{(1)}(\partial_{x_2})$ form an S-basis for $D(\mathcal{A},(4p+2,4q))$ thanks to Theorem 2.1. Thus $\Phi_\zeta^{(1)}$ is an S-linear bijection. □

We may apply Proposition 4.1 to get the following proposition:

Proposition 4.2 *Suppose that (a_1,a_2) is exceptional and that either $p\geq 0$ or $q\geq 0$ in Table 2. Then, for $i=1,2$, $\Phi_\zeta^{(i)}(\theta_1)$ and $\Phi_\zeta^{(i)}(\theta_2)$ form an S-basis for $D(\mathcal{A},(a_1,a_2))$ as in Table 2.*

Proposition 3.4 asserts that $E_1^{(s,t)}$ is (s,t)-universal when $s-t\in 2\mathbb{Z}$, $t\geq 0$ and that $E_2^{(s,t)}$ is (s,t)-universal when $t-s\in 2\mathbb{Z}$, $s\geq 0$. So Tables 1 and 2 show how to construct a basis for $D(\mathcal{A},(a_1,a_2))$ when $a_1\geq 0$ or $a_2\geq 0$. We will construct a basis for $D(\mathcal{A},(a_1,a_2))$ in the remaining case that $a_1 < 0$ and $a_2 < 0$. Let

$$\begin{aligned}\Omega(\mathcal{A},\mathbf{k}) &:= (I^*)^{-1}(D(\mathcal{A},-\mathbf{k}))\\ &= \left\{\omega \in \Omega(\mathcal{A},\infty) \mid I^*(\omega, d\alpha_H) \in \alpha_H^{-\mathbf{k}(H)} S_{(\alpha_H)} \text{ for all } H \in \mathcal{A}\right\}.\end{aligned}$$

Theorem 4.3 (Ziegler [13], Abe [1, Theorem 1.7]) *The natural S-bilinear coupling*

$$D(\mathcal{A},\mathbf{k}) \times \Omega(\mathcal{A},\mathbf{k}) \longrightarrow S$$

is non-degenerate and provides S-linear isomorphisms:

$$\alpha : D(\mathcal{A},\mathbf{k}) \to \Omega(\mathcal{A},\mathbf{k})^*, \quad \beta : \Omega(\mathcal{A},\mathbf{k}) \to D(\mathcal{A},\mathbf{k})^*.$$

Thus we have the following proposition:

Proposition 4.4 *Let $(a_1, a_2) \in (\mathbb{Z}_{<0})^2$ and x_1, x_2 be an orthonormal basis. Let θ_1, θ_2 be an S-basis for $D(\mathcal{A},(-a_1,-a_2))$. Then*

$$\eta_1 := g_{11}\partial_{x_1} + g_{21}\partial_{x_2}, \quad \eta_2 := g_{12}\partial_{x_1} + g_{22}\partial_{x_2},$$

form an S-basis for $D(\mathcal{A},(a_1,a_2))$. Here

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} = \begin{pmatrix} \theta_1(x_1) & \theta_1(x_2) \\ \theta_2(x_1) & \theta_2(x_2)\end{pmatrix}^{-1} = Q_1^{a_1} Q_2^{a_2} \begin{pmatrix} \theta_2(x_2) & -\theta_1(x_2) \\ -\theta_2(x_1) & \theta_1(x_1)\end{pmatrix}.$$

5. Conclusion

Let $\mathcal{A}$ be a two-dimensional irreducible Coxeter arrangement such that $|\mathcal{A}|$ is even with $|\mathcal{A}| \geq 4$. We have constructed an explicit basis for $D(\mathcal{A},(a_1,a_2))$ for an arbitrary equivariant multiplicity $\mathbf{k} = (a_1, a_2)$ with a_1, $a_2 \in \mathbb{Z}$. Our recipes are presented in the Tables 1, 2, Propositions 3.5, 4.2 and 4.4. Lastly we show Table 4 for the exponents.

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a_1	a_2	$a_1 - a_2$	exponents of $(\mathcal{A},(a_1,a_2))$	their difference
odd	odd	$\equiv 0\,(\mathrm{mod}\ 4)$	$\frac{(a_1+a_2-2)h}{4}+1,\ \frac{(a_1+a_2+2)h}{4}-1$	$h-2$
odd	odd	$\equiv 2\,(\mathrm{mod}\ 4)$	$\frac{(a_1+a_2)h}{4}+1,\ \frac{(a_1+a_2)h}{4}-1$	2
odd	even		$\frac{(a_1+a_2-1)h}{4}+1,\ \frac{(a_1+a_2+1)h}{4}-1$	$(h/2)-2$
even	odd		$\frac{(a_1+a_2-1)h}{4}+1,\ \frac{(a_1+a_2+1)h}{4}-1$	$(h/2)-2$
even	even		$\frac{(a_1+a_2)h}{4},\ \frac{(a_1+a_2)h}{4}$	0

Table 4. The exponents of $(\mathcal{A},(a_1,a_2))$ $(a_1,a_2 \in \mathbb{Z})$

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