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A study on the dimension of global sections of adjoint bundles for polarized manifolds, II

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Abstract. Let X be a smooth complex projective variety of dimension n and let L be an ample line bundle on X. In our previous paper, in order to investigate the dimension of $H^0(K_X + tL)$ more systematically, we introduced the invariant $A_i(X, L)$ for every integer i with $0 \leq i \leq n$. Main purposes of this paper are (1) to study a lower bound of $A_i(X, L)$ for the following two cases: (1.a) the case where L is merely ample and $i \leq 3$, (1.b) the case of $h^0(L) > 0$, and (2) to evaluate a lower bound for the dimension of $H^0(K_X + tL)$ by using $A_i(X, L)$.

Key words: Polarized manifold, adjoint bundles, the i-th sectional H-arithmetic genus, the i-th sectional geometric genus.

1. Introduction

Let X be a projective variety of dimension n defined over the field of complex numbers, and let L be an ample line bundle on X. Then (X, L) is called a *polarized variety*. If X is smooth, then we say that (X, L) is a polarized *manifold*.

This paper is the continuation of [20]. In this paper, we consider the dimension of $H^0(K_X + tL)$. In [3, Conjecture 7.2.7], Beltrametti and Sommese proposed the following conjecture.

Conjecture 1.1 *Let (X, L) be a polarized manifold of dimension n. Assume that $K_X + (n-1)L$ is nef. Then $h^0(K_X + (n-1)L) > 0$.*

At present, there are some answers for Conjecture 1.1. For example, it is known that this conjecture is true if $\dim \mathrm{Bs}|L| \leq 0$ ([3, Corollary 7.2.8], [12, Theorem 3.5]), $\dim X \leq 3$ ([3, Theorem 7.2.6], [18, Theorem 2.4], [6]) or $h^0(L) > 0$ ([21, 1.2 Theorem]). (Here we note that in [21, 1.2 Theorem] Höring proved the following: If X is a normal projective variety of dimension

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$n \geq 2$ with at most rational singularities and L is a nef and big line bundle on X such that $K_X + (n-1)L$ is generically nef, then there exists an integer j with $1 \leq j \leq n-1$ such that $h^0(K_X + jL) > 0$.) But it is unknown whether this conjecture is true or not in general. (Here we note that Conjecture 1.1 does not follow from [21, 1.2 Theorem] in general.) The following conjecture is a generalization of Conjecture 1.1.

Conjecture 1.2 (Ionescu [26, Open problems, p. 321], Ambro [1], Kawamata [23]) *Let (X, L) be a polarized manifold of dimension n. Assume that $K_X + L$ is nef. Then $h^0(K_X + L) > 0$.*

At present, there are some partial answers for this conjecture (for example, [19, Theorem 3.2], [5, Théorème 1.8]). Recently Höring [21, 1.5 Theorem] gave a proof of Conjecture 1.2 for the case of $n = 3$. (More generally, in [21, 1.5 Theorem], Höring proved that $h^0(K_X + L) > 0$ holds if X is a normal projective threefold with at most $\mathbb{Q}$-factorial canonical singularities and L is a nef and big line bundle on X such that $K_X + L$ is nef.) But we don't know whether this conjecture is true or not for the case of $n \geq 4$.

These conjectures motivated the author to begin investigating $h^0(K_X + tL)$ for a positive integer t. Our aim is not only to know the positivity of $h^0(K_X + tL)$ but also to evaluate a lower bound for $h^0(K_X + tL)$. In [20], in order to investigate $h^0(K_X + tL)$ systematically, we introduced an invariant $A_i(X, L)$ for every integer i with $0 \leq i \leq n$, which is called the *i-th Hilbert coefficient of* (X, L) (see Definition 2.2 below). From the following theorem which shows a relationship between $h^0(K_X + tL)$ and $A_i(X, L)$, we see that it is important to study the value of $A_i(X, L)$ in order to know the value of $h^0(K_X + tL)$.

Theorem 1.1 ([20, Corollary 3.1]) *Let (X, L) be a polarized manifold of dimension n, and let t be a positive integer. Then we have $h^0(K_X + tL) = \sum_{j=0}^{n} \binom{t-1}{n-j} A_j(X, L)$.*

In [20] we studied the invariant $A_i(X, L)$ in the case where L is ample and spanned by global sections. In particular we proved that $A_i(X, L) \geq 0$ for every integer i with $0 \leq i \leq n$. And we obtained a lower bound of $h^0(K_X + tL)$ by using some properties of $A_i(X, L)$ (see [20]).

Main purposes of this paper are (i) to investigate $A_i(X, L)$ for the following two cases: (i.1) the case where L is merely ample, (i.2) the case of $h^0(L) > 0$, and (ii) to evaluate a lower bound for $h^0(K_X + tL)$ by using some properties of $A_i(X, L)$.

In [20, Conjecture 5.1] we proposed the following conjecture.

Conjecture 1.3 *Let (X, L) be a polarized manifold of dimension n. Then $A_i(X, L) \geq 0$ holds for every integer i with $0 \leq i \leq n$.*

In this paper, first, in Section 3, we will study a lower bound of $A_i(X, L)$ for the case where L is merely ample and $i \leq 3$ (see Theorem 3.1.1) and the case where $h^0(L) > 0$ (see Theorem 3.2.2). In particular we get a partial answer of this conjecture. As an application of the study of $A_i(X, L)$, we will investigate $h^0(K_X + tL)$. First we will consider the case where $h^0(L) > 0$. Then we can prove that $h^0(K_X + (n-2)L) > 0$ if $\kappa(X) \geq 0$ and $h^0(L) > 0$ (see Theorem 4.1.1). Furthermore we will give a lower bound of $h^0(K_X + tL)$ for (X, L) with $\dim X = 3$ and $h^0(L) \geq 2$ by using Theorem 1.1 above (see Theorem 4.1.2). Next we will investigate the case where $\dim \mathrm{Bs}|L| = 0$ or 1, and we will provide a lower bound of $h^0(K_X + tL)$ (see Theorems 4.2.1 and 4.2.2). Finally we will give a partial answer to a question of Tsuji for $\dim X \leq 4$ (see Theorem 4.3.1).

In this paper, varieties are always assumed to be defined over the field of complex numbers. We use the standard notation from algebraic geometry.

- $\kappa(D)$: the Iitaka dimension of a Cartier divisor D on X.
- $\kappa(X)$: the Kodaira dimension of X.
- $\mathbb{P}^n$: the projective space of dimension n.
- $\mathcal{O}_{\mathbb{P}^n}(1)$: the invertible sheaf defined by a hyperplane of $\mathbb{P}^n$.
- $\mathbb{Q}^n$: a quadric hypersurface in $\mathbb{P}^{n+1}$.
- $\mathcal{O}_{\mathbb{Q}^n}(1)$: the restiction of $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ to a quadric hypersurface $\mathbb{Q}^n$ in $\mathbb{P}^{n+1}$.
- $\mathbb{P}_X(\mathcal{E})$: the projective space bundle associated with a vector bundle $\mathcal{E}$ on X.
- $H(\mathcal{E})$: the tautological line bundle on $\mathbb{P}_X(\mathcal{E})$.

For a real number m and a non-negative integer n, let

$$[m]^n := \begin{cases} m(m+1)\cdots(m+n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0, \end{cases}$$

$$[m]_n := \begin{cases} m(m-1)\cdots(m-n+1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Then for n fixed, $[m]^n$ and $[m]_n$ are polynomials in m whose degree are n.

For any non-negative integer n, we set

$$n! := \begin{cases} [n]_n & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Assume that m and n are integers with $n \geq 0$. Then we put $\binom{m}{n} := \frac{[m]_n}{n!}$. We note that $\binom{m}{n} = 0$ if $0 \leq m < n$, and $\binom{m}{0} = 1$.

2. Preliminaries

Notation 2.1 Let X be a projective variety of dimension n and let L be a line bundle on X. Then $\chi(tL)$ is a polynomial in t of degree at most n, and we can write $\chi(tL)$ as $\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L)\binom{t}{j}$.

Definition 2.1 ([14, Definition 2.1], [17, Definition 2.1]) Let X be a projective variety of dimension n and let L be a line bundle on X. For every integer i with $0 \leq i \leq n$, the *ith sectional geometric genus* $g_i(X, L)$ and the *ith sectional H-arithmetic genus* $\chi_i^H(X, L)$ *of* (X, L) are defined by the following.

$$g_i(X, L) = (-1)^i(\chi_{n-i}(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X),$$

$$\chi_i^H(X, L) = \chi_{n-i}(X, L).$$

Remark 2.1

(1) Since $\chi_{n-i}(X, L) \in \mathbb{Z}$, we see that $\chi_i^H(X, L)$ and $g_i(X, L)$ are integer.
(2) If $i = n$, then $g_n(X, L) = h^n(\mathcal{O}_X)$ and $\chi_n^H(X, L) = \chi(\mathcal{O}_X)$.
(3) If $i = 0$, then $g_0(X, L) = L^n$ and $\chi_0^H(X, L) = L^n$.
(4) If $i = 1$, then $g_1(X, L) = g(L)$, where $g(L)$ is the sectional genus of (X, L). If X is smooth, then the sectional genus $g(L)$ is written as $g(L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1}$.

Definition 2.2 ([20, Definition 3.1 and Definition 3.2]) Let (X, L) be a polarized manifold of dimension n.

(1) Let t be a positive integer. Then set

$$F_0(t) := h^0(K_X + tL)$$
$$F_i(t) := F_{i-1}(t+1) - F_{i-1}(t) \quad \text{for every integer } i \text{ with } 1 \le i \le n.$$

(2) For every integer i with $0 \le i \le n$, the *ith Hilbert coefficient* $A_i(X, L)$ *of* (X, L) is defined by $A_i(X, L) = F_{n-i}(1)$

Remark 2.2

(1) If $1 \le i \le n$, then $A_i(X, L)$ can be written as follows (see [20, Proposition 3.2]).

$$\begin{aligned} A_i(X, L) &= (-1)^i \chi_i^H(X, L) + (-1)^{i-1} \chi_{i-1}^H(X, L) \\ &= g_i(X, L) + g_{i-1}(X, L) - h^{i-1}(\mathcal{O}_X). \end{aligned}$$

(2) By Definition 2.2 and [20, Proposition 3.1 (2)], we have the following:
(2.1) $A_i(X, L) \in \mathbb{Z}$ for every integer i with $0 \le i \le n$.
(2.2) $A_0(X, L) = L^n$.
(2.3) $A_n(X, L) = h^0(K_X + L)$.

Theorem 2.1 *Let* (X, L) *be a polarized manifold of dimension* n *and let* t *be a positive integer. Then for every integer* i *with* $0 \le i \le n$ *we have*

$$F_{n-i}(t) = \sum_{j=0}^{i} \binom{t-1}{i-j} A_j(X, L).$$

Proof. See [20, Theorem 3.1]. Here we note that if $i = n$, then this result is Theorem 1.1 in Introduction. □

Definition 2.3 (1) Let X (resp. Y) be an n-dimensional projective manifold, and L (resp. H) an ample line bundle on X (resp. Y). Then (X, L) is called a *simple blowing up of* (Y, H) if there exists a birational morphism $\pi : X \to Y$ such that π is a blowing up at a point of Y and $L = \pi^*(H) - E$, where E is the π-exceptional effective reduced divisor.

(2) Let X (resp. M) be an n-dimensional projective manifold, and L (resp. A) an ample line bundle on X (resp. M). Then we say that (M, A) is a *reduction of* (X, L) if there exists a birational morphism $\mu : X \to M$ such that μ is a composition of simple blowing ups and (M, A) is not obtained by a simple blowing up of any other polarized manifolds.

Remark 2.3 Let (X, L) be a polarized manifold and let (M, A) be a reduction of (X, L). Let $\mu : X \to M$ be the reduction map, and let γ be the number of simple blowing ups of its reduction. Then by [14, Proposition 2.6] and [17, Remark 2.1 (5)]

$$g_i(X, L) = \begin{cases} g_i(M, A) & \text{if } 1 \le i \le n, \\ A^n - \gamma & \text{if } i = 0, \end{cases}$$

$$\chi_i^H(X, L) = \begin{cases} \chi_i^H(M, A) & \text{if } 1 \le i \le n, \\ A^n - \gamma & \text{if } i = 0. \end{cases}$$

Hence

$$A_i(X, L) = \begin{cases} A_i(M, A) & \text{if } 2 \le i \le n, \\ A_i(M, A) - \gamma & \text{if } i = 0, 1. \end{cases}$$

Definition 2.4 Let (X, L) be a polarized manifold of dimension n. We say that (X, L) is a *scroll* (resp. *quadric fibration*) *over a normal projective variety* Y with $\dim Y = m \ge 1$ if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $n > m$ and $K_X + (n-m+1)L = f^*A$ (resp. $K_X + (n-m)L = f^*A$) for some ample line bundle A on Y.

Definition 2.5 Let (X, L) be a polarized manifold of dimension $n \ge 2$ and let Y be a normal projective variety of dimension $m \ge 1$. Then (X, L) is called a *classical scroll over* Y with $\dim Y = m \ge 1$ if $n > m$ and X is a $\mathbb{P}^{n-m}$-bundle over Y and $L_F = \mathcal{O}_{\mathbb{P}^{n-m}}(1)$ for every fiber F.

Remark 2.4 We note that if $m = 1$, then (X, L) is a classical scroll over Y if and only if (X, L) is either a scroll over Y in the sense of Definition 2.4 or $(\mathbb{P}^1 \times \mathbb{P}^1,\ p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1))$, where $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the ith projection for $i = 1, 2$. (If (X, L) is $(\mathbb{P}^1 \times \mathbb{P}^1,\ p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1))$, then $(X, L) \cong (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1))$.)

We also note that if $m = 2$ and (X, L) is a scroll over Y, then Y is smooth and (X, L) is a classical scroll over Y (see [4, (3.2.1) Theorem] and [8, (11.8.6)]).

Definition 2.6 A polarized manifold (X, L) is called a *hyperquadric fibration over a smooth curve* C if (X, L) is a quadric fibration over C such that every fiber is irreducible and reduced.

Remark 2.5 Assume that (X, L) is a quadric fibration over a smooth curve C with $\dim X = n \geq 3$. Let $f : X \to C$ be its morphism. By [4, (3.2.6) Theorem] and the proof of [22, Lemma (c) in Section 1], we see that (X, L) is one of the following:

(a) f is the contraction morphism of an extremal ray, and every fiber of f is irreducible and reduced. Namely (X, L) is a hyperquadric fibration over C in this case.
(b) X is a $\mathbb{P}^1$-bundle over a smooth surface and $L|_F = \mathcal{O}_{\mathbb{P}^1}(1)$ for every fiber F.

So if (X, L) is not a hyperquadric fibration but a quadric fibration over C, then we may assume that there exists an ample vector bundle $\mathcal{F}$ of rank 2 on a smooth projective surface S such that $(X, L) \cong (\mathbb{P}_S(\mathcal{F}), H(\mathcal{F}))$. In particular $\dim X = 3$ in this case.

Theorem 2.2 *Let (X, L) be a polarized manifold with $n = \dim X \geq 3$. Then (X, L) is one of the following types.*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
(3) *A scroll over a smooth curve.*
(4) $K_X \sim -(n-1)L$, *that is, (X, L) is a Del Pezzo manifold.*
(5) *A hyperquadric fibration over a smooth curve.*
(6) *A classical scroll over a smooth projective surface S. Namely $(X, L) \cong (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S.*
(7) *Let (M, A) be a reduction of (X, L).*
    (7.1) $n = 4$, $(M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.
    (7.2) $n = 3$, $(M, A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.
    (7.3) $n = 3$, $(M, A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.
    (7.4) $n = 3$, *M is a $\mathbb{P}^2$-bundle over a smooth curve C and $(F, A|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for any fiber F of it.*
    (7.5) $K_M + (n-2)A$ *is nef.*

Proof. See [3, Proposition 7.2.2, Theorem 7.2.4, Theorem 7.3.2 and Theorem 7.3.4]. See also [8, (11.2) Theorem, (11.7) Theorem and (11.8) Theorem] and [22, Theorem in Section 1]. □

Remark 2.6 Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Then $\kappa(K_X + (n-2)L) = -\infty$ if and only if (X, L) is one of the types from

(1) to (7.4) in Theorem 2.2.

Proposition 2.1 *Let (X, L) be a polarized surface. Assume that $h^0(L) \geq 2$ and $g_1(X, L) = h^1(\mathcal{O}_X)$. Then (X, L) is one of the following*:

(1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.
(2) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.
(3) *A classical scroll over a smooth curve.*

Proof. By [9, Lemma 1.2], we have $g_1(X, L) - h^1(\mathcal{O}_X) = h^0(K_X + L) - h^0(K_X)$. If $h^0(K_X) > 0$, then by [24, 15.6.2 Lemma] or [14, Lemma 1.12] we have $h^0(K_X + L) - h^0(K_X) \geq h^0(L) - 1 \geq 1$ and this is impossible because $g_1(X, L) = h^1(\mathcal{O}_X)$. Therefore $h^2(\mathcal{O}_X) = h^0(K_X) = 0$. If $\kappa(X) \geq 0$, then $\chi(\mathcal{O}_X) \geq 0$. So we get $h^1(\mathcal{O}_X) \leq 1$ and $g_1(X, L) = h^1(\mathcal{O}_X) \leq 1$. But in this case $K_X L < 0$ and this is impossible because $\kappa(X) \geq 0$ and L is ample. So we have $\kappa(X) = -\infty$. By [9, Theorem 3.1], we get the assertion. □

3. A lower bound of $A_i(X, L)$

3.1. The case where L is merely ample and $i \leq 3$

In this subsection we consider Conjecture 1.3 for the case where L is merely ample and $i \leq 3$.

Theorem 3.1.1 *Let (X, L) be a polarized manifold of dimension $n \geq 2$. Let (M, A) be a reduction of (X, L). Then the following hold*:

(1) $A_1(X, L) \geq 0$ *holds.*
  (1.1) $A_1(X, L) = 0$ *if and only if* $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
  (1.2) $A_1(X, L) = 1$ *if and only if (X, L) is one of the following three types*:
    (1.2.1) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
    (1.2.2) *A scroll over a smooth elliptic curve with* $L^n = 1$.
    (1.2.3) *A Del Pezzo manifold with* $L^n = 1$.
(2) *Assume that $n = 2$ or 3. Then the following hold*:
  (2.1) $A_2(X, L) \geq 0$ *holds.*
  (2.2) *Assume that $n = 2$. Then $A_2(X, L) = 0$ if and only if (X, L) is one of the following*:
    (2.2.1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.
    (2.2.2) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.
    (2.2.3) *A classical scroll over a smooth curve.*

(2.3) *Assume that* $n = 2$. *Then* $A_2(X, L) = 1$ *if and only if* (X, L) *is one of the following*:

(2.3.1) $\kappa(X) = 0, 1$, $\chi(\mathcal{O}_X) = 0$ *and* $g_1(X, L) = 2$.

(2.3.2) $\kappa(X) = -\infty$ *and* $g_1(X, L) = h^1(\mathcal{O}_X) + 1$.

(2.4) *Assume that* $n = 3$. *Then* $A_2(X, L) = 0$ *if and only if* (X, L) *is one of the following*:

(2.4.1) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$.

(2.4.2) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1))$.

(2.4.3) *A scroll over a smooth curve.*

(2.5) *Assume that* $n = 3$. *Then* $A_2(X, L) = 1$ *if and only if* (X, L) *is one of the types in* [18, Theorem 2.4].

(2.6) *Assume that* $n = 3$ *and* $h^0(L) \geq 2$. *Then*

$$A_2(X, L) \geq \begin{cases} h^2(\mathcal{O}_X) & \text{if } \kappa(X) = -\infty, \\ h^1(\mathcal{O}_X) & \text{if } \kappa(X) \geq 0. \end{cases}$$

(3) *If* $n \geq 3$ *and* $\kappa(X) \geq 0$, *then*

$$A_2(X, L) \geq g_1(X, L) - 1 + \frac{(n-2)(n^2-n-1)}{12n} A^n + \frac{(n-2)(n+1)}{12n} K_M A^{n-1}.$$

In particular $A_2(X, L) \geq 2$. *Furthermore, if* $n \geq 4$ *and* $\kappa(X) \geq 0$, *then* $A_2(X, L) \geq 3$.

(4) *If* $n \geq 3$ *and* $\kappa(X) \geq 0$, *then*

$$A_3(X, L) \geq \frac{(n-1)(n-2)(2n-1)}{24n} A^n + \frac{2n-3}{24} K_M A^{n-1} > 0.$$

Proof. (1) First we note that $A_1(X, L) = g_1(X, L) + L^n - 1$ by Remark 2.2 (1). Since $g_1(X, L) \geq 0$ and $L^n \geq 1$, we have $A_1(X, L) \geq 0$. Assume that $A_1(X, L) = 0$. Then we see that $g_1(X, L) = 0$ and $L^n = 1$. By [8, (5.10) Theorem and (12.1) Theorem], [22, Corollary 8] or [3, Proposition 3.1.2 and Theorem 3.1.3], we see that $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Conversely if $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, then we can easily check that $A_1(X, L) = 0$.

Assume that $A_1(X, L) = 1$. Then $(g_1(X, L), L^n) = (0, 2)$ or $(1, 1)$ because $L^n \geq 1$ and $g_1(X, L) \geq 0$. If $(g_1(X, L), L^n) = (0, 2)$ (resp. $(1, 1)$), then (X, L) is the type (1.2.1) (resp. either (1.2.2) or (1.2.3)) by the classification of (X, L) with $g_1(X, L) = 0$ (resp. 1) (see [8, (5.10) Theorem, (12.1) Theorem and (12.3) Theorem]).

If (X, L) is one of the types (1.2.1), (1.2.2) and (1.2.3), then we see $A_1(X, L) = 1$.

(2.1) We are going to investigate the non-negativity of $A_i(X, L)$. First we consider the case where $n = 2$. Since $A_2(X, L) = h^0(K_X + L)$ by Remark 2.2 (2.3), we have $A_2(X, L) \geq 0$.

Next we consider the case where $n = 3$. We note that $A_2(X, L) = \chi_2^H(X, L) - \chi_1^H(X, L) = g_2(X, L) + g_1(X, L) - h^1(\mathcal{O}_X)$ by Remark 2.2 (1).

(A) If $\kappa(K_X + L) \geq 0$, then by [17, Remark 2.1 (3), Theorem 3.2.1 and Theorem 3.3.1 (2)] we have $\chi_2^H(X, L) > 0$, that is, $g_2(X, L) \geq h^1(\mathcal{O}_X)$. Therefore $A_2(X, L) \geq g_1(X, L)$. Since $\kappa(K_X + L) \geq 0$, we have $g_1(X, L) \geq 1 + (1/2)L^3$. Therefore $A_2(X, L) \geq 2$ because $A_2(X, L)$ is an integer.

(B) Assume that $\kappa(K_X + L) = -\infty$. If $h^1(\mathcal{O}_X) = 0$, then $A_2(X, L) \geq 0$ because $g_2(X, L) \geq 0$ by [16, Corollary 2.4] and $g_1(X, L) \geq 0$. So we may assume that $h^1(\mathcal{O}_X) > 0$. Furthermore we may assume that (X, L) is a reduction of iteself by Remark 2.3. Then by Theorem 2.2 and Remark 2.6 (X, L) is one of the following types:

(B.1) A scroll over a smooth curve C.

(B.2) A hyperquadric fibration over a smooth curve C.

(B.3) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank 2 on S.

(B.4) X is a $\mathbb{P}^2$-bundle over a smooth curve C and $L|_F \cong \mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber F.

(B.I) If (X, L) is the type (B.1), (B.2) or (B.4), then $g(C) = h^1(\mathcal{O}_X)$. On the other hand, $g_1(X, L) \geq g(C)$ by [10, Theorem 1.2.1]. Therefore $A_2(X, L) \geq 0$ because $g_2(X, L) \geq 0$ by [16, Corollary 2.4].

(B.II) Assume that (X, L) is the type (B.3). Let $f : X \to S$ be its projection. Then $g_2(X, L) = h^2(\mathcal{O}_X) = h^2(\mathcal{O}_S)$ by [14, Example 2.10 (8)].

(B.II.1) If $\kappa(S) \geq 0$, then $\chi(\mathcal{O}_S) \geq 0$. We note that

$$\begin{aligned} A_2(X,L) &= g_2(X,L) + g_1(X,L) - h^1(\mathcal{O}_X) \\ &= h^2(\mathcal{O}_S) + g_1(X,L) - h^1(\mathcal{O}_S) \\ &= g_1(X,L) - 1 + \chi(\mathcal{O}_S). \end{aligned}$$

On the other hand, $g_1(X,L) \geq 2$ in this case because $g_1(X,L) = g_1(S, c_1(\mathcal{E}))$ and $\kappa(S) \geq 0$. So we get $A_2(X,L) \geq 1$.

(B.II.2) If $\kappa(S) = -\infty$, then there exists the Albanese map $\alpha : S \to B$ such that $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_B)$, where B is a smooth curve. Then $\alpha \circ f : X \to S \to B$ is a fiber space. Therefore $g_1(X,L) \geq h^1(\mathcal{O}_B)$ by [10, Theorem 1.2.1] and $A_2(X,L) \geq g_1(X,L) - h^1(\mathcal{O}_X) = g_1(X,L) - h^1(\mathcal{O}_B) \geq 0$ because $g_2(X,L) \geq 0$ by [16, Corollary 2.4]. Hence we get the assertion of (2.1).

(2.2) Next we consider the assertion of (2.2). Assume that $n = 2$ and $A_2(X,L) = 0$. Then $h^0(K_X + L) = 0$ by Remark 2.2 (2.3). Hence by [25, 3.5 Proposition], (X,L) is either $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ or a classical scroll over a smooth curve. Conversely if (X,L) is one of these types, then we can easily check $A_2(X,L) = 0$.

(2.3) Assume that $n = 2$ and $A_2(X,L) = 1$. Then $A_2(X,L) = g_1(X,L) + h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X) = g_1(X,L) + \chi(\mathcal{O}_X) - 1$ by Remark 2.1 (2) and Remark 2.2 (1).

(2.3.a) If $\kappa(X) \geq 0$, then $\chi(\mathcal{O}_X) \geq 0$ and $g_1(X,L) \geq 2$. Hence $A_2(X,L) \geq 1$. Therefore if $A_2(X,L) = 1$, then $\chi(\mathcal{O}_X) = 0$ and $g_1(X,L) = 2$. Since $\chi(\mathcal{O}_X) = 0$, we have $\kappa(X) = 0$ or 1, and we get the type (2.3.1).

(2.3.b) If $\kappa(X) = -\infty$, then $h^2(\mathcal{O}_X) = 0$ and $A_2(X,L) = g_1(X,L) - h^1(\mathcal{O}_X)$. Hence we get the type (2.3.2).

Next we consider the case of (2.4) and (2.5). Assume that $n = 3$ and $A_2(X,L) \leq 1$. If $\kappa(K_X + L) \geq 0$, then $g_2(X,L) \geq h^1(\mathcal{O}_X)$ by [17, Remark 2.1 (3), Theorem 3.2.1 and Theorem 3.3.1 (2)]. Hence $g_1(X,L) \leq 1$ because $A_2(X,L) = g_1(X,L) + g_2(X,L) - h^1(\mathcal{O}_X)$. But then $\kappa(K_X + L) = -\infty$ and this is a contradiction. Therefore we get $\kappa(K_X + L) = -\infty$. In particular $h^0(K_X + L) = 0$. Hence $A_3(X,L) = 0$ and $A_2(X,L) = h^0(K_X + 2L)$ by Theorem 1.1 and Remark 2.2 (2.3).

(2.4) Assume that $A_2(X,L) = 0$. Then $h^0(K_X + 2L) = 0$ and by [18, Theorem 2.4] $K_X + 2L$ is not nef. Hence (X, L) is one of the types in (2.4) above. If (X, L) is one of the types in (2.4) above, then we see that $A_2(X, L) = 0$.

(2.5) Assume that $A_2(X, L) = 1$. Then $h^0(K_X + 2L) = 1$. Here we note that if $K_X + 2L$ is not nef, then (X, L) is either (1), (2) or (3) in Theorem 2.2 and $h^0(K_X + 2L) = 0$. Hence $K_X + 2L$ is nef in this case. Therefore by [18, Theorem 2.4] we see that (X, L) is one of the types in [18, Theorem 2.4]. Conversely if (X, L) is one of the types in [18, Theorem 2.4], we can easily see that $A_2(X, L) = 1$.

(2.6) By the assumption that $n = 3$ and $h^0(L) \geq 2$ and by [13, Theorem 2.1], we have $g_1(X, L) \geq h^1(\mathcal{O}_X)$. If $\kappa(X) = -\infty$ (resp. ≥ 0), then $g_2(X, L) \geq h^2(\mathcal{O}_X)$ (resp. $\geq h^1(\mathcal{O}_X)$) by [16, Corollary 2.4]. Hence we get the assertion of (2.6).

(3) By the proof of [16, Theorem 2.3.2], we have

$$g_2(X, L) \geq -1 + h^1(\mathcal{O}_X) + \frac{(n-2)(n^2-n-1)}{12n} A^n + \frac{(n-2)(n+1)}{12n} K_M A^{n-1},$$

where (M, A) is a reduction of (X, L). Since $A_2(X, L) = g_2(X, L) + g_1(X, L) - h^1(\mathcal{O}_X)$ by Remark 2.2 (1), we get the first assertion of (3). Here we note that $g_1(X, L) \geq 2$ since $\kappa(X) \geq 0$ and $n \geq 3$. So we have $A_2(X, L) \geq 2$.

Assume that $n \geq 4$. Since $\kappa(X) \geq 0$, we have $K_X L^{n-1} \geq 0$. Hence $g_1(X, L) \geq 3$ because $n \geq 4$ and $g_1(X, L)$ is an integer. Therefore we get the assertion of (3).

(4) By [19, Theorem 3.1], the assertion of (4) holds. □

Remark 3.1.1 By [2, Theorem 1.5 and Theorem 2.7] (resp. [11, Theorem A.1 in Appendix]), we get a classification of (X, L) with the type (2.3.1) (resp. (2.3.2)) in Theorem 3.1.1.

3.2. The case of $h^0(L) > 0$

Notation 3.2.1 Let (X, L) be a polarized manifold of dimension n. Then we put $b(L) := \dim \mathrm{Bs}|L|$. If $\mathrm{Bs}|L| = \emptyset$, then we put $b(L) = -1$.

Proposition 3.2.1 *Let (X, L) be a polarized manifold of dimension $n \geq 2$. Assume that $b(L) \leq n-2$. If $i \geq b(L)+1$, then $g_i(X, L) \geq h^i(\mathcal{O}_X)$.*

Proof. If $i = n$, then by Remark 2.1 (2) this is true. For $b(L)+1 \leq i \leq n-1$, see [15, Corollary 2.8]. □

Proposition 3.2.2 *Let (X, L) be a polarized manifold of dimension $n \geq 2$.*

(1) *If $b(L) \leq n-2$ and $i \geq b(L)+2$, then $A_i(X, L) \geq h^i(\mathcal{O}_X) \geq 0$.*
(2) *Assume that $b(L) \leq 1$. Then $A_i(X, L) \geq h^i(\mathcal{O}_X) \geq 0$ for every integer i with $0 \leq i \leq n$.*

Proof. (1) By Proposition 3.2.1 and Remark 2.2 (1) we get the first assertion.
(2) If $n = 2$, then $g_2(X, L) = h^2(\mathcal{O}_X)$ by Remark 2.1 (2). If $n \geq 3$, then by Proposition 3.2.1, we have $g_i(X, L) \geq h^i(\mathcal{O}_X)$ for every integer i with $2 \leq i \leq n$. Of course, we have $g_0(X, L) \geq 1 = h^0(\mathcal{O}_X)$ because $g_0(X, L) = L^n$. Next we will show that $g_1(X, L) \geq h^1(\mathcal{O}_X)$.

If $b(L) < 0$ (resp. $b(L) = 0$), then by [3, Theorem 7.2.10] (resp. [12, Theorem 3.2]) we have $g_1(X, L) \geq h^1(\mathcal{O}_X)$. So we may assume that $b(L) = 1$.

Claim 3.2.1 *If $b(L) = 1$, then $g_1(X, L) \geq h^1(\mathcal{O}_X)$ holds.*

Proof. If $n = 2$, then this is true by [9, Lemma 1.2] because $h^0(L) > 0$. So we may assume that $n \geq 3$. By [15, Proposition 1.12 (2)] there exists an $(n-3)$-ladder $X \supset X_1 \supset \cdots \supset X_{n-3}$ such that X_j is a normal Gorenstein variety of dimension $n-j$ and $X_j \in |L_{j-1}|$ for every j. Here we set $L_j := L|_{X_j}$ for every integer j with $1 \leq j \leq n-3$ and $L_0 := L$. Then L_{n-3} is ample on X_{n-3} such that $\dim \mathrm{Bs}|L_{n-3}| \leq 1$. Let $\mu : \widetilde{X}_{n-3} \to X_{n-3}$ be a resolution of singularities of X_{n-3}. Then $\mu^* L_{n-3}$ is nef and big on $\widetilde{X}_{n-3}$ and $h^0(\mu^*(L_{n-3})) = h^0(L_{n-3})$. On the other hand since $b(L_{n-3}) \leq 1$, we have $h^0(L_{n-3}) \geq 2$. Hence by [13, Theorem 2.1] we have $g_1(\widetilde{X}_{n-3}, \mu^*(L_{n-3})) \geq h^1(\mathcal{O}_{\widetilde{X}_{n-3}})$. Moreover we see that $g_1(X, L) = g_1(X_{n-3}, L_{n-3}) = g_1(\widetilde{X}_{n-3}, \mu^*(L_{n-3}))$ and $h^1(\mathcal{O}_{\widetilde{X}_{n-3}}) \geq h^1(\mathcal{O}_{X_{n-3}}) = h^1(\mathcal{O}_X)$. Therefore we get $g_1(X, L) \geq h^1(\mathcal{O}_X)$. □

Therefore by Remark 2.2 (1) we can show that $A_i(X, L) \geq h^i(\mathcal{O}_X) \geq 0$ for every integer i with $0 \leq i \leq n$. □

Here we consider [20, Conjecture 5.1 (2)]. By Propositions 3.2.1 and 3.2.2 (1) and [20, Remark 5.1 (1), (2), (3) and (4)] we get the following.

Theorem 3.2.1 *Let (X, L) be a polarized manifold of dimension $n \geq 2$. Assume that $b(L) \leq n-2$. If i is an integer with $i \geq b(L)+1$, then Conjecture* 5.1 (2) *in* [20] *is true.*

Proposition 3.2.3 *Let (X, L) be a polarized manifold of dimension $n \geq 2$. Assume that $b(L) \leq n-2$. If $b(L)+1 \leq i \leq n-1$ and $h^0(L) \geq n+s_i+1-i$, then $g_{i+1}(X, L) = 0$. (Here we set $s_i := g_i(X, L) - h^i(\mathcal{O}_X)$.)*

Proof. By [15, Proposition 1.12 (2)], there exists an $(n-b(L)-2)$-ladder $X \supset X_1 \supset \cdots \supset X_{n-b(L)-2}$ such that X_j is a normal and Gorenstein variety of dimension $n-j$ and $h^0(L_{n-b(L)-2}) > 0$. Here we set $L_j := L|_{X_j}$ for every integer j with $1 \leq j \leq n-b(L)-2$. By [15, Propositions 2.1 and 2.3], we have $s_i = g_i(X, L) - h^i(\mathcal{O}_X) = g_i(X_{n-i-1}, L_{n-i-1}) - h^i(\mathcal{O}_{X_{n-i-1}})$ for $b(L)+1 \leq i \leq n-1$. From [15, Claim 2.1.1 and Theorem 1.3 (1)] and the Serre duality we also see that $g_i(X_{n-i-1}, L_{n-i-1}) - h^i(\mathcal{O}_{X_{n-i-1}}) = h^0(K_{X_{n-i-1}} + L_{n-i-1}) - h^0(K_{X_{n-i-1}})$.

Assume that $h^0(K_{X_{n-i-1}}) > 0$. Then

$$h^0\big(K_{X_{n-i-1}} + L_{n-i-1}\big) - h^0\big(K_{X_{n-i-1}}\big) \geq h^0(L_{n-i-1}) - 1$$

by [24, 15.6.2 Lemma] or [14, Lemma 1.12]. On the other hand, by assumption, we see that $h^0(L_{n-i-1}) \geq h^0(L_{n-i-2}) - 1 \geq \cdots \geq h^0(L) - (n-i-1) \geq s_i + 2$. Hence $g_i(X, L) - h^i(\mathcal{O}_X) \geq s_i + 1$. But this is impossible because of the definition of s_i. Therefore we get $h^0(K_{X_{n-i-1}}) = 0$. Since $g_{i+1}(X, L) = h^{i+1}(\mathcal{O}_{X_{n-i-1}}) = h^0(K_{X_{n-i-1}})$ by [15, Remark 1.2.1 (2) and Propositions 2.1 and 2.3] and the Serre duality, we get the assertion. □

Corollary 3.2.1 *Let (X, L) be a polarized manifold of dimension $n \geq 2$. Assume that $b(L) \leq n-2$. If $b(L)+1 \leq i \leq n-1$ and $g_i(X, L) - h^i(\mathcal{O}_X) \leq i-1-b(L)$, then $g_{i+1}(X, L) = 0$. In particular if $b(L)+1 \leq i \leq n-1$ and $g_i(X, L) = 0$, then $g_{i+1}(X, L) = 0$.*

Proof. We note that $h^0(L) \geq n - b(L)$ in this case (see e.g. [7, (1.7) Lemma]). If $g_i(X, L) - h^i(\mathcal{O}_X) \leq i-1-b(L)$, then $n-b(L) \geq n+s_i+1-i$, where $s_i := g_i(X, L) - h^i(\mathcal{O}_X)$. Hence we have $h^0(L) \geq n+s_i+1-i$ and we get the assertion by Proposition 3.2.3. □

Notation 3.2.2 Let (X, L) be a polarized manifold of dimension n and let

$$p(X, L) := \min\{t > 0 \mid t \in \mathbb{Z},\ h^0(K_X + tL) \neq 0\}.$$

Theorem 3.2.2 *Let (X, L) be a polarized manifold of dimension $n \geq 2$. Assume that $b(L) \leq n - 2$. Then we get the following*:

(1) $A_j(X, L) = 0$ *if* $j \geq n - p(X, L) + 2$.
(2) $A_{n-p(X,L)+1}(X, L) \geq h^{n-p(X,L)+1}(\mathcal{O}_X) + 1$ *if* $n - p(X, L) + 1 \geq b(L) + 2$.
(3) $A_{n-p(X,L)}(X, L) \geq h^{n-p(X,L)}(\mathcal{O}_X) + n - p(X, L) - b(L)$ *if* $n - p(X, L) \geq b(L) + 2$.
(4) $A_k(X, L) \geq h^k(\mathcal{O}_X) + 2k - 2b(L) - 1$ *if* $b(L) + 2 \leq k \leq n - p(X, L) - 1$.

Proof. First we are going to consider (1). In this case we may assume that $p(X, L) \geq 2$ because we study $A_j(X, L)$ with $j \geq n - p(X, L) + 2$. If $1 \leq t < p(X, L)$, then $\binom{t-1}{n-j} = 0$ for every j with $0 \leq j \leq n - p(X, L) + 1$. Hence by Theorem 1.1, we have

$$h^0(K_X + tL) = \sum_{j=n-p(X,L)+2}^{n} \binom{t-1}{n-j} A_j(X, L).$$

Moreover by the definition of $p(X, L)$ we have $h^0(K_X + tL) = 0$ if $1 \leq t < p(X, L)$. Hence we get the first assertion (1). Here we note that if $n - p(X, L) + 2 \geq b(L) + 2$, then by Proposition 3.2.1 and Remark 2.2 (1) we have

$$g_j(X, L) = 0 \quad \text{if} \ \ j \geq n - p(X, L) + 2 \geq b(L) + 2 \tag{3.1}$$

and

$$g_{n-p(X,L)+1}(X, L) = h^{n-p(X,L)+1}(\mathcal{O}_X) \quad \text{if} \ \ n - p(X, L) + 2 \geq b(L) + 2. \tag{3.2}$$

Next we consider the value of $A_{n-p(X,L)+1}(X, L)$ if $n - p(X, L) + 1 \geq b(L) + 2$. Since $h^0(K_X + p(X, L)L) > 0$, we have $A_{n-p(X,L)+1}(X, L) \geq 1$ by Theorem 1.1 and by (1) above. Here we note that the following:

Claim 3.2.2

$$g_{n-p(X,L)}(X,L) - h^{n-p(X,L)}(\mathcal{O}_X) \geq 1 \qquad \text{if} \ \ b(L)+2 \leq n-p(X,L)+1. \tag{3.3}$$

Proof. Assume that $g_{n-p(X,L)}(X,L) = h^{n-p(X,L)}(\mathcal{O}_X)$. Then since $b(L)+1 \leq n-p(X,L) \leq n-1$ by assumption, we have $g_{n-p(X,L)+1}(X,L) = 0$ by Corollary 3.2.1. But since $A_{n-p(X,L)+1}(X,L) = g_{n-p(X,L)+1}(X,L) + g_{n-p(X,L)}(X,L) - h^{n-p(X,L)}(\mathcal{O}_X)$, we get $A_{n-p(X,L)+1}(X,L) = 0$ and this is a contradiction. □

Hence by (3.2), (3.3) and Remark 2.2 (1), we get the assertion of (2).

Finally we consider the value of $A_k(X,L)$ if $b(L)+2 \leq k \leq n-p(X,L)$. By Claim 3.2.2 and Corollary 3.2.1 we have

$$g_j(X,L) - h^j(\mathcal{O}_X) \geq j - b(L) \quad \text{if} \ \ b(L)+1 \leq j \leq n-p(X,L)-1. \tag{3.4}$$

Hence by Remark 2.2 (1) and Claim 3.2.2 we get $A_{n-p(X,L)}(X,L) \geq h^{n-p(X,L)}(\mathcal{O}_X) + 1 + n - p(X,L) - 1 - b(L) = h^{n-p(X,L)}(\mathcal{O}_X) + n - p(X,L) - b(L)$ and $A_k(X,L) \geq h^k(\mathcal{O}_X) + k - b(L) + (k-1) - b(L) = h^k(\mathcal{O}_X) + 2k - 2b(L) - 1$. Therefore we get the assertion of (3) and (4). □

Remark 3.2.1 Assume that $b(L) \leq n-2$. Then by (3.1), (3.2), (3.3), (3.4) in the proof of Theorem 3.2.2, we get the following.

(1) $g_j(X,L) = 0$ if $j \geq n-p(X,L)+2 \geq b(L)+2$.
(2) $g_{n-p(X,L)+1}(X,L) = h^{n-p(X,L)+1}(\mathcal{O}_X)$ if $b(L)+1 \leq n-p(X,L)+1$.
(3) $g_{n-p(X,L)}(X,L) \geq h^{n-p(X,L)}(\mathcal{O}_X) + 1$ if $b(L)+1 \leq n-p(X,L)$.
(4) $g_j(X,L) \geq h^j(\mathcal{O}_X) + j - b(L)$ if $b(L)+1 \leq j \leq n-p(X,L)-1$.

4. On the dimension of global sections of $K_X + tL$

Here we will give some results about the dimension of global sections of adjoint bundles, which are obtained by using Theorems 1.1, 3.1.1 and 3.2.2, and Remark 3.2.1.

4.1. The case where $h^0(L) > 0$.

In this subsection, we consider a lower bound of the global sections of adjoint bundles under the assumption that $h^0(L) > 0$.

First we are going to investigate the positivity of $h^0(K_X + (n-2)L)$ under the assumption that $\kappa(X) \geq 0$ and $h^0(L) > 0$.

Theorem 4.1.1 *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that $\kappa(X) \geq 0$ and $h^0(L) > 0$. Then $h^0(K_X + (n-2)L) > 0$.*

Proof. (1) Assume that $n \geq 4$. If $h^0(K_X + tL) \neq 0$ for some integer t with $1 \leq t \leq n-3$, then by [24, 15.6.2 Lemma] or [14, Lemma 1.12] we obtain $h^0(K_X + (n-2)L) > 0$ since $h^0(L) > 0$. So we may assume that $h^0(K_X + tL) = 0$ for any integer t with $1 \leq t \leq n-3$.

(1.1) Assume that $n = 4$. Since $h^0(K_X + L) = 0$, we have $F_1(1) = h^0(K_X + 2L)$. (Here we use notation in Definition 2.2 (1).) But then $F_1(1) = A_3(X, L)$ by Theorem 2.1 and we see that $h^0(K_X + 2L) > 0$ because $A_3(X, L) > 0$ by Theorem 3.1.1 (4).

(1.2) Assume that $n \geq 5$. Since $F_1(t) = 0$ for every integer t with $1 \leq t \leq n-4$, by Theorem 2.1 we see that $A_{n-1}(X, L) = 0, \ldots, A_4(X, L) = 0$ and $F_1(n-3) = \sum_{j=0}^{n-1} \binom{n-4}{n-1-j} A_j(X, L) = A_3(X, L)$. Therefore $F_1(n-3) = A_3(X, L) > 0$ by Theorem 3.1.1 (4) and we get $h^0(K_X + (n-2)L) > 0$.

(2) Assume that $n = 3$. Then by [19, Theorem 3.2] we have already obtained $h^0(K_X + L) > 0$. (In this case we don't need the assumption that $h^0(L) > 0$.) Therefore we get the assertion. $\square$

Remark 4.1.1 We note that [21, 1.2 Theorem] does not imply Theorem 4.1.1 above.

Next we are going to study $h^0(K_X + tL)$ under the assumption that $\dim X = 3$ and $h^0(L) \geq 2$.

Theorem 4.1.2 *Let (X, L) be a polarized manifold of dimension 3. Assume that $h^0(L) \geq 2$. Then for every positive integer t we have the following inequality*:

$$h^0(K_X + tL) \geq \begin{cases} (t-1)h^2(\mathcal{O}_X) + \binom{t-1}{2} h^1(\mathcal{O}_X) + \binom{t-1}{3} & \text{if } \kappa(X) = -\infty, \\ \binom{t}{2} \max\{2, h^1(\mathcal{O}_X)\} + \binom{t-1}{3} & \text{if } \kappa(X) \geq 0. \end{cases}$$

Proof. First we note that by Remark 2.2 (2.2) and (2.3)

$$A_3(X,L) = h^0(K_X + L) \geq 0, \tag{4.1}$$

$$A_0(X,L) = L^3 \geq 1. \tag{4.2}$$

Next we consider a lower bound for $A_1(X,L)$. Since $L^3 \geq 1$, we have

$$A_1(X,L) = g_1(X,L) + L^3 - 1 \geq g_1(X,L). \tag{4.3}$$

By assumption and [13, Theorem 2.1], we have

$$g_1(X,L) \geq h^1(\mathcal{O}_X). \tag{4.4}$$

On the other hand if $\kappa(X) \geq 0$, then $g_1(X,L) \geq 1 + L^3 \geq 2$. Therefore if $\kappa(X) \geq 0$, then by (4.4)

$$g_1(X,L) \geq \max\{2, h^1(\mathcal{O}_X)\}. \tag{4.5}$$

Hence by (4.3), (4.4) and (4.5)

$$A_1(X,L) \geq \begin{cases} h^1(\mathcal{O}_X) & \text{if } \kappa(X) = -\infty, \\ \max\{2, h^1(\mathcal{O}_X)\} & \text{if } \kappa(X) \geq 0. \end{cases} \tag{4.6}$$

Finally we consider a lower bound for $A_2(X,L)$. If $\kappa(X) = -\infty$, then by [16, Corollary 2.4] we have $g_2(X,L) \geq h^2(\mathcal{O}_X)$. Hence by (4.4)

$$A_2(X,L) = g_2(X,L) + g_1(X,L) - h^1(\mathcal{O}_X) \geq h^2(\mathcal{O}_X). \tag{4.7}$$

If $\kappa(X) \geq 0$, then by [16, Corollary 2.4] we get $g_2(X,L) \geq h^1(\mathcal{O}_X)$. Hence by (4.5)

$$\begin{aligned} A_2(X,L) &= g_2(X,L) + g_1(X,L) - h^1(\mathcal{O}_X) \\ &\geq g_1(X,L) \geq \max\{2, h^1(\mathcal{O}_X)\}. \end{aligned} \tag{4.8}$$

On the other hand by Theorem 1.1 or Theorem 2.1

$$\begin{aligned} h^0(K_X+tL) &= A_3(X,L)+(t-1)A_2(X,L) \\ &\quad + \binom{t-1}{2}A_1(X,L)+\binom{t-1}{3}A_0(X,L). \end{aligned}$$

Therefore we get the assertion by (4.1), (4.2), (4.6), (4.7) and (4.8). □

4.2. The case of $\dim \mathrm{Bs}|L| = 0$ or 1

Here we use Notation 3.2.1 and Notation 3.2.2.

In [20], we studied a lower bound of $h^0(K_X + tL)$ for the case where $\mathrm{Bs}|L| = \emptyset$. In this subsection, we consider the case where $\dim \mathrm{Bs}|L| = 0$ or 1. First we prove the following.

Proposition 4.2.1 *Let (X, L) be a polarized manifold of dimension $n \geq 2$. Assume that $0 \leq b(L) \leq 1$. Then $p(X, L) \leq n$. Moreover if $b(L) = 0$, then $p(X, L) = n$ if and only if (X, L) is a scroll over a smooth curve.*

Proof. First we note that by Proposition 3.2.2 (2) we have $A_i(X, L) \geq 0$ for every i with $1 \leq i \leq n$ and $A_0(X, L) > 0$ in this case.

Assume that $p(X, L) \geq n + 1$. Then $h^0(K_X + nL) = 0$. Hence we see that $A_i(X, L) = 0$ for every integer i with $1 \leq i \leq n$ by Theorem 1.1. In particular, $A_1(X, L) = 0$ implies $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ by Theorem 3.1.1 (1.1). But this is impossible because here we assume that $b(L) = 0$ or 1. Hence $p(X, L) \leq n$.

Assume that $b(L) = 0$ and $p(X, L) = n$. Then $h^0(L) \geq n \geq 2$ and $h^0(K_X + (n-1)L) = 0$. So by Theorem 1.1 and Proposition 3.2.2 (2), we see that $A_i(X, L) = 0$ for every integer i with $2 \leq i \leq n$. In particular, $A_2(X, L) = 0$ implies that $g_2(X, L) = 0$ and $g_1(X, L) = h^1(\mathcal{O}_X)$. If $n = 2$ and $g_1(X, L) = h^1(\mathcal{O}_X)$, then by Proposition 2.1 we see that (X, L) is either $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ or a classical scroll over a smooth curve. If $n \geq 3$ and $g_1(X, L) = h^1(\mathcal{O}_X)$, then by [12, Theorem 3.2], we see that (X, L) is either $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ or a scroll over a smooth curve. But since $b(L) = 0$, we see that (X, L) is a scroll over a smooth curve. Conversely if (X, L) is a scroll over a smooth curve, then we can easily check that $p(X, L) = n$. This completes the proof. □

(I) The case of $b(L) = 0$.

By Proposition 4.2.1, we see that if $p(X, L) = n$, then (X, L) is a scroll over a smooth curve, and in this case we can compute $h^0(K_X + tL)$. In this case by [14, Example 2.10 (8)] we have $A_0(X, L) \geq 1$, $A_1(X, L) \geq h^1(\mathcal{O}_X)$

and $A_j(X,L)=0$ for every j with $j\geq 2$. Hence

$$h^0(K_X+tL)\geq\binom{t-1}{n}+\binom{t-1}{n-1}h^1(\mathcal{O}_X).$$

So, as the next step, we consider a lower bound of $h^0(K_X+tL)$ for the case where $p(X,L)\leq n-1$.

Theorem 4.2.1 *Let (X,L) be a polarized manifold of dimension $n\geq 2$. Assume that $b(L)=0$ and $p(X,L)\leq n-1$. Then*

$$h^0(K_X+tL)$$
$$\geq\begin{cases}\binom{t-1}{n}+\binom{t-1}{n-1}(h^1(\mathcal{O}_X)+1)+\binom{t-1}{n-2}(h^2(\mathcal{O}_X)+1) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } p(X,L)=n-1, \\ \binom{t-1}{n}+\binom{t-1}{n-1}\max\{h^1(\mathcal{O}_X)+1,2\}+\binom{t-1}{n-2}(h^2(\mathcal{O}_X)+2) & \\ \quad+\binom{t-1}{n-3}(h^3(\mathcal{O}_X)+1) \qquad\qquad \textit{if } p(X,L)=n-2, \\ \binom{t-1}{n}+\binom{t-1}{n-1}\max\{h^1(\mathcal{O}_X)+1,2\} & \\ \quad+\binom{t-1}{p(X,L)-1}(h^{n-p(X,L)+1}(\mathcal{O}_X)+1) & \\ \quad+\binom{t-1}{p(X,L)}(h^{n-p(X,L)}(\mathcal{O}_X)+n-p(X,L)) & \\ \quad+\sum_{j=2}^{n-p(X,L)-1}\binom{t-1}{n-j}(h^j(\mathcal{O}_X)+2j-1) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } 1\leq p(X,L)\leq n-3. \end{cases}$$

Proof. First we note that by Theorem 3.2.2 (1) we have

$$h^0(K_X+tL)=\sum_{j=0}^{n}\binom{t-1}{n-j}A_j(X,L)=\sum_{j=0}^{n-p(X,L)+1}\binom{t-1}{n-j}A_j(X,L).$$

First we note that $A_0(X,L)=L^n\geq 1$. If $p(X,L)=n-1$, then by Theorem 3.2.2 (2) we see that $A_2(X,L)\geq h^2(\mathcal{O}_X)+1$. Here we note that by Remark 3.2.1 (3) we have $A_1(X,L)=g_1(X,L)+L^n-1\geq h^1(\mathcal{O}_X)+1$.

Next we consider the case where $p(X,L)\leq n-2$. Then we note that $n\geq 3$ in this case.

If $p(X,L)=n-2$, then we have $A_2(X,L)\geq h^2(\mathcal{O}_X)+2$, and $A_3(X,L)\geq h^3(\mathcal{O}_X)+1$ by Theorem 3.2.2 (2) and (3). Moreover by Remark 3.2.1 (4) we have $A_1(X,L)\geq h^1(\mathcal{O}_X)+1$. Moreover by Theorem 3.1.1 (1) we have $A_1(X,L)\geq 2$ because we assume $p(X,L)=n-2$. Hence $A_1(X,L)\geq \max\{h^1(\mathcal{O}_X)+1,2\}$.

Assume that $p(X,L)\leq n-3$. In this case by the same reason as above we have $A_1(X,L)\geq \max\{h^1(\mathcal{O}_X)+1,2\}$. By Theorem 3.2.2 we have $A_{n-p(X,L)+1}(X,L)\geq h^{n-p(X,L)+1}(\mathcal{O}_X)+1$, $A_{n-p(X,L)}(X,L)\geq h^{n-p(X,L)}(\mathcal{O}_X)+n-p(X,L)$ and $A_k(X,L)\geq h^k(\mathcal{O}_X)+2k-1$ if $2\leq k\leq n-p(X,L)-1$.

From Theorem 1.1 and the above argument we obtain the inequalities in Theorem 4.2.1. $\square$

(II) The case of $b(L)=1$.

Next we consider the case where $b(L)=1$. In this case we assume that $n\geq 3$ and $p(X,L)\leq n$.

Theorem 4.2.2 *Let (X,L) be a polarized manifold of dimension $n\geq 3$. Assume that $b(L)=1$ and $p(X,L)\leq n$. Then the following inequalities hold.*

$$h^0(K_X+tL)$$
$$\geq \begin{cases} \binom{t-1}{n}+\binom{t-1}{n-1}\max\{h^1(\mathcal{O}_X),2\} & \text{if } p(X,L)=n,\\ \binom{t-1}{n}+\binom{t-1}{n-1}\max\{h^1(\mathcal{O}_X),2\}+\binom{t-1}{n-2}h^2(\mathcal{O}_X) & \text{if } p(X,L)=n-1,\\ \binom{t-1}{n}+\binom{t-1}{n-1}\max\{h^1(\mathcal{O}_X),2\}+\binom{t-1}{n-2}(h^2(\mathcal{O}_X)+1) & \\ \quad +\binom{t-1}{n-3}(h^3(\mathcal{O}_X)+1) & \text{if } p(X,L)=n-2,\\ \binom{t-1}{n}+\binom{t-1}{n-1}\max\{h^1(\mathcal{O}_X),2\}+\binom{t-1}{n-2}(h^2(\mathcal{O}_X)+1) & \\ \quad +\binom{t-1}{n-3}(h^3(\mathcal{O}_X)+2)+\binom{t-1}{n-4}(h^4(\mathcal{O}_X)+1) & \text{if } p(X,L)=n-3,\\ \binom{t-1}{n}+\binom{t-1}{n-1}\max\{h^1(\mathcal{O}_X),2\}+\binom{t-1}{p(X,L)-1}(h^{n-p(X,L)+1}(\mathcal{O}_X)+1) & \\ \quad +\binom{t-1}{p(X,L)}(h^{n-p(X,L)}(\mathcal{O}_X)+n-p(X,L)-1) & \\ \quad +\sum_{j=2}^{n-p(X,L)-1}\binom{t-1}{n-j}(h^j(\mathcal{O}_X)+2j-3) & \text{if } 1\leq p(X,L)\leq n-4. \end{cases}$$

Proof. As we said in Theorem 4.2.1, by Theorem 3.2.2 (1) we have

$$h^0(K_X+tL)=\sum_{j=0}^{n}\binom{t-1}{n-j}A_j(X,L)=\sum_{j=0}^{n-p(X,L)+1}\binom{t-1}{n-j}A_j(X,L).$$

First we note that $A_0(X,L)=L^n\geq 1$, and $A_1(X,L)\geq h^1(\mathcal{O}_X)$ by Claim 3.2.1. Here we note that if $A_1(X,L)\leq 1$, then by Theorem 3.1.1 (1) we see that (X,L) is a scroll over a smooth elliptic curve C with $L^n=1$ because $b(L)=1$. Then there exists an ample vector bundle $\mathcal{E}$ on C such that $X=\mathbb{P}_C(\mathcal{E})$ and $L=H(\mathcal{E})$. Since $c_1(\mathcal{E})=L^n=1$, by [3, Lemma 3.2.5] we have $h^0(L)=h^0(\mathcal{E})=1$. Since $b(L)=1$, we have $h^0(L)\geq n-1$. Hence $n\leq 2$ holds. But this contradicts the assumption. Therefore we have $A_1(X,L)\geq 2$. Hence

$$A_1(X,L)\geq\max\{h^1(\mathcal{O}_X),2\}.$$

Hence if $p(X,L)=n$, then we get

$$h^0(K_X+tL)\geq\binom{t-1}{n}+\binom{t-1}{n-1}\max\{h^1(\mathcal{O}_X),2\}.$$

Next we assume that $p(X,L)\leq n-1$. Then we consider the value of $A_j(X,L)$ for $j\geq 2$.

If $p(X,L)=n-1$, then by Remark 3.2.1 (2) and Claim 3.2.1 we see that $A_2(X,L)\geq h^2(\mathcal{O}_X)$.

Assume that $p(X,L)\leq n-2$. We note that $n\geq 3$. If $p(X,L)=n-2$, then by Theorem 3.2.2 (2), Remark 3.2.1 (3) and Claim 3.2.1 we have $A_2(X,L)\geq h^2(\mathcal{O}_X)+1$, and $A_3(X,L)\geq h^3(\mathcal{O}_X)+1$.

Assume that $p(X,L)=n-3$. By Theorem 3.2.2 (2) and (3) we have $A_3(X,L)\geq h^3(\mathcal{O}_X)+2$ and $A_4(X,L)\geq h^4(\mathcal{O}_X)+1$. By Remark 3.2.1 (4) we have $g_2(X,L)-h^2(\mathcal{O}_X)\geq 1$. So we have $A_2(X,L)=g_2(X,L)+g_1(X,L)-h^1(\mathcal{O}_X)\geq h^2(\mathcal{O}_X)+1$ by Claim 3.2.1.

Assume that $p(X,L)\leq n-4$. By Remark 3.2.1 (4) and Claim 3.2.1 we have $A_2(X,L)\geq h^2(\mathcal{O}_X)+1$. Moreover by Theorem 3.2.2 (2), (3) and (4) we have $A_{n-p(X,L)+1}(X,L)\geq h^{n-p(X,L)+1}(\mathcal{O}_X)+1$, $A_{n-p(X,L)}(X,L)\geq h^{n-p(X,L)}(\mathcal{O}_X)+n-p(X,L)-1$, and $A_k(X,L)\geq h^k(\mathcal{O}_X)+2k-3$ if $3\leq k\leq n-p(X,L)-1$.

Therefore by using Theorem 1.1 we obtain the inequalities in Theorem 4.2.2. □

4.3. On the difference between $h^0(K_X + mL)$ and $h^0(K_X + (m-1)L)$

The following Theorem 4.3.1 is a partial answer of the following problem proposed by H. Tsuji [27, Problem 1] for $\dim X \leq 4$.

Problem 4.3.1 *Let (X, L) be a polarized manifold of dimension n. Then is it true that $h^0(K_X + mL) \geq h^0(K_X + (m-1)L)$ for every integer m with $m \geq 2$?*

Theorem 4.3.1 *Let (X, L) be a polarized manifold of dimension n.*

(1) *If $n = 2$, then for every integer m with $m \geq 2$ we have*

$$h^0(K_X + mL) - h^0(K_X + (m-1)L) \geq m - 2.$$

Moreover this equality holds if and only if $(X, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

(2) *If $n = 3$, then for every integer m with $m \geq 2$ we have*

$$h^0(K_X + mL) - h^0(K_X + (m-1)L) \geq \binom{m-2}{2}.$$

Moreover the following hold.

(2.1) *$h^0(K_X + 2L) - h^0(K_X + L) = 0$ if and only if (X, L) is one of the following:*

(2.1.1) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$.

(2.1.2) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1))$.

(2.1.3) *A scroll over a smooth curve.*

(2.2) *For $m \geq 3$, $h^0(K_X + mL) - h^0(K_X + (m-1)L) = \binom{m-2}{2}$ if and only if $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$.*

(3) *If $n = 4$ and $\kappa(X) \geq 0$, then for every integer m with $m \geq 2$ we have*

$$h^0(K_X + mL) - h^0(K_X + (m-1)L) \geq \binom{m+1}{3} > 0.$$

Proof. We consider (1) (resp. (2)). Then $\dim X = 2$ (resp. 3). So by Theorem 2.1 we have

$$F_1(m-1) = (m-2)A_0(X,L) + A_1(X,L)$$
$$\left(\text{resp. } F_1(m-1) = \binom{m-2}{2}A_0(X,L) + (m-2)A_1(X,L) + A_2(X,L)\right).$$

Here we note that $F_1(m-1) = h^0(K_X + mL) - h^0(K_X + (m-1)L)$ and $A_0(X,L) = L^n \geq 1$. Hence by Theorem 3.1.1 (1) (resp. Theorem 3.1.1 (1) and (2.1)), we have

$$h^0(K_X + mL) - h^0(K_X + (m-1)L) \geq m - 2$$
$$\left(\text{resp. } h^0(K_X + mL) - h^0(K_X + (m-1)L) \geq \binom{m-2}{2}\right).$$

Next we consider the case where $\dim X = 2$ and $h^0(K_X + mL) - h^0(K_X + (m-1)L) = m-2$. Then by the above proof, we see that $A_1(X,L) = 0$. By Theorem 3.1.1 (1.1) we see that $(X,L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Conversely if $(X,L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, then we can easily check that $h^0(K_X+mL)-h^0(K_X+(m-1)L) = m-2$.

Assume that $n = 3$, $m = 2$ and $h^0(K_X + 2L) - h^0(K_X + L) = 0$. Then by the above proof we see that $A_2(X,L) = 0$. By Theorem 3.1.1 (2.4), (X,L) is either (2.1.1), (2.1.2) or (2.1.3) in the statement of Theorem 4.3.1. Conversely if (X,L) is one of these types, then we see that $h^0(K_X + 2L) - h^0(K_X + L) = 0$.

Assume that $n = 3$, $m \geq 3$ and $h^0(K_X + mL) - h^0(K_X + (m-1)L) = \binom{m-2}{2}$. Then $A_1(X,L) = 0$ and $A_2(X,L) = 0$ hold. Hence by Theorem 3.1.1 (1.1) and (2.4) we get $(X,L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. Conversely if $(X,L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, then we can easily check that $h^0(K_X + mL) - h^0(K_X + (m-1)L) = \binom{m-2}{2}$.

Next we consider (3). Then we assume that $\dim X = 4$ and $\kappa(X) \geq 0$. By Theorem 2.1,

$$\begin{aligned} h^0(K_X &+ mL) - h^0(K_X + (m-1)L) \\ &= \binom{m-2}{3}A_0(X,L) + \binom{m-2}{2}A_1(X,L) \\ &\quad + (m-2)A_2(X,L) + A_3(X,L). \end{aligned} \tag{4.9}$$

Since $\dim X = 4$ and $\kappa(X) \geq 0$, we have $g_1(X,L) = 1+(1/2)(K_X+3L)L^3 \geq$

3 and $L^4 \geq 1$. So we get $A_1(X,L) = g_1(X,L) + L^4 - 1 \geq 3$. Hence by (4.9), Theorem 3.1.1 (1), (3) and (4), we have

$$\begin{aligned} &h^0(K_X + mL) - h^0(K_X + (m-1)L) \\ &\quad \geq \binom{m-2}{3} + 3\binom{m-2}{2} + 3(m-2) + 1 = \binom{m+1}{3}. \end{aligned}$$

This completes the proof. □

Remark 4.1 In [19, Theorem 3.5] we proved Conjecture 1.1 for the case of $\dim X = 4$ and $\kappa(X) \geq 0$. We note that also by using Theorem 4.3.1 (3) and [21, 1.2 Theorem] we can prove that Conjecture 1.1 is true if $\dim X = 4$ and $\kappa(X) \geq 0$.

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