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# Null Darboux developable and pseudo-spherical Darboux image of null Cartan curve in Minkowski 3-space

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**Abstract.** Singularities of null Darboux developable, Gaussian surfaces and pseudo-spherical Darboux images associated with a null Cartan curve will be investigated in Minkowski 3-space. The relationships will be revealed between singularities of the above three subjects and differential geometric invariants of null Cartan curves, these invariants are deeply related to the order of contact of null Cartan curves with null helices.

*Key words*: Null Cartan curve, Darboux image, Null Darboux developable, Null helix.

## 1. Introduction

The importance of the study of null curves and its presence in the physical theories are clear [6], [7], [9], [10], [21]. Nersessian and Ramos [18] show us that there exists a geometrical particle model based entirely on the geometry of the null curves in Minkowskian 4-dimensional spacetime which under quantization yields the wave equations corresponding to massive spinning particles of arbitrary spin. They have also studied the simplest geometrical particle model which is associated with null curves in Minkowski 3-space [19].

Many of the classical results from Riemannian geometry have Lorentz counterparts. In fact, spacelike curves or timelike curves can be studied by approaches similar to those taken in positive definite Riemannian geometry. However, null curves have many properties which are very different from spacelike and timelike curves. In other words, null curve theory has many results which have no Riemannian analogues. For general theory of parameterized null curves we refer to [2]. In geometry of null curves difficulties arise because the arc length vanishes, so that it is impossible to normalize the tangent vector in the usual way. Bonner introduces the Cartan frame as the most useful one and he uses this frame to study the behaviors of a null curve [2].

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Nonlightlike (or non-null) curves in Minkowski space and space curves in Euclidean space, regarding singularity, have been studied extensively by S. Izumiya and Pei, et al. [11]–[16]. The classification of singularity of nonlightlike ruled surfaces and their invariants in Minkowski 3-space and ruled surfaces in Euclidean space have been studied systematically in their papers. S. Izumiya [16] has studied rectifying developable and spherical Darboux image in Euclidean 3-space. By constructing volumelike distance functions and volumelike height functions and using unfolding theory [3], he has classified the singularities of spherical Daxboux image and rectifying developable of curve in Euclidean 3-space. M. Che [4] has also studied the singularities of hyperbolic Daxboux image and rectifying Gauss surface of nonlightlike curve in Minkowski 3-space and has given the counterpart. However, to the best of the authors' knowledge, no literature exists regarding the singularities of surfaces and curves as they relate to *null Cartan curves* (see Section 2) in $\mathbb{R}^3_1$. Thus, the current study hopes to serve such a need and is inspired by the reports of [4], [16]. In this paper, we study *null Darboux developable*, *Gaussian surface* and *pseudo-spherical Darboux image* associated with a null Cartan curve in Minkowski 3-space from the standpoint of singularity theory. We adopt the Cartan Frenet frame [5] as the basic tool and use the methods which is similar to [4], [16]. We construct the *volumelike distance function* on null Cartan curves. It is quite useful for the study of generic singularities of null Darboux developable of null Cartan curves. We also introduce the notion of *tangential height function* for the study of singularities of pseudo-spherical Darboux image and Gaussian surface. Our research show that their singularities are deeply related to the geometry of tangential nullcone image of curves.

A brief description of the organization of this paper is as follows. The main results in this paper are stated in Theorem 2.1. In Section 3 we give tangential height functions and volumelike distance functions of a null Cartan curve, by which we can obtain several geometric invariants of the null Cartan curve. The geometric meaning of Theorem 2.1 is described in Section 4. We give the proof of Theorem 2.1 in Section 5.

## 2. Preliminaries

Let $\mathbb{R}^3_1$ denote the 3-dimensional Minkowski space, that is to say, the manifold $\mathbb{R}^3$ with a flat Lorentz metric $\langle , \rangle$ of signature $(-,+,+)$, for any vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, we set $\langle \mathbf{x}, \mathbf{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3$. We also define a vector

$$\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the canonical basis of $\mathbb{R}^3_1$. We say that a vector $\mathbf{x} \in \mathbb{R}^3_1 \backslash \{0\}$ is *spacelike, null* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive, zero or negative, respectively. The *norm* of a vector $\mathbf{x} \in \mathbb{R}^3_1$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. We call $\mathbf{x}$ a *unit vector* if $\|\mathbf{x}\| = 1$ (see [17]). We define the signature of a vector

$$\operatorname{sign}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \text{ is spacelike} \\ 0 & \mathbf{x} \text{ is null} \\ -1 & \mathbf{x} \text{ is timelike.} \end{cases}$$

Let $\gamma : I \to \mathbb{R}^3_1$ be a smooth regular curve in $\mathbb{R}^3_1$ (i.e., $\dot{\gamma}(t) \neq 0$ for any $t \in I$), parametrized by an open interval $I$. For any $t \in I$, the curve $\gamma$ is called a *spacelike curve*, a *null(lightlike) curve* or a *timelike curve* if all its velocity vector satisfy $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ or $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0$, respectively. We call $\gamma$ a *non-null curve* if $\gamma$ is a timelike curve or a spacelike curve.

Let $\gamma : I \to \mathbb{R}^3_1$ be a null curve in $\mathbb{R}^3_1$ (i.e., $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ for any $t \in I$). Now suppose that $\gamma$ is framed by a null frame. A null frame $F = \left\{ \xi = \frac{d\gamma}{dt}, N, B \right\}$ at a point of $\mathbb{R}^3_1$ is a positively oriented 3-tuple of vectors satisfying

$$\langle \xi, \xi \rangle = \langle B, B \rangle = 0, \quad \langle \xi, B \rangle = 1,$$
$$\langle \xi, N \rangle = \langle B, N \rangle = 0, \quad \langle N, N \rangle = 1.$$

The Frenet formula of $\gamma$ with respect to $F$ is given by

$$\begin{cases} \dfrac{d\xi}{dt} = -h\xi + k_1 N \\ \dfrac{dB}{dt} = hB + k_2 N \\ \dfrac{dN}{dt} = -k_2 \xi - k_1 B. \end{cases} \tag{2.1}$$

The functions $h, k_1$ and $k_2$ are called the curvature functions of $\gamma$ (cf. [1]). Null frames of null curves are not uniquely determined. Therefore, the curve and a frame must be given together.

There always exists a parameter $s$ of $\gamma$ such that $h = 0$ in Eqs.(2.1). This parameter is called a *distinguished parameter* of $\gamma$, which is uniquely determined for prescribed screen vector bundle (i.e. a complement in $\left\langle \frac{d\gamma}{dt} \right\rangle^{\perp}$ to $\left\langle \frac{d\gamma}{dt} \right\rangle$) up to affine transformation [5].

Let $\gamma(s)$ be a null curve with a distinguished parameter in $\mathbb{R}^3_1$ (i.e. $h = 0$ in Eqs.(2.1)). Moreover we assume that $\gamma'(s), \gamma''(s), \gamma'''(s)$ are linearly independent for all $s$. Then we consider the basis $E = \{\gamma'(s), \gamma''(s), \gamma'''(s)\}$ such that $\langle \gamma''(s), \gamma''(s) \rangle = k_1(s) = 1$. We choose the $\xi = \frac{d\gamma}{ds}, N = \gamma''(s)$, then there exists only one null frame $F = \{\xi, N, B\}$ for which $\gamma(s)$ is a framed null curve with Frenet equations [5]:

$$\begin{cases} \dfrac{d\xi}{ds} = N \\ \dfrac{dB}{ds} = k_2 N \\ \dfrac{dN}{ds} = -k_2 \xi - B, \end{cases} \tag{2.2}$$

where $\xi = \frac{d\gamma}{ds}, N = \gamma''(s), B = -\gamma''' - k_2\gamma', k_2 = \frac{1}{2}\langle \gamma''', \gamma''' \rangle$. We call Eqs.(2.2) the *Cartan Frenet equations* and $\gamma(s)$ their *null Cartan curve* [5]. We remark that the curvature function $k_2$ is an invariant under Lorentzian transformations.

In case $\gamma$ is a null Cartan curve, labeling $k_2(s) = \tau(s)$, then the Frenet formula of $\gamma(s)$ with respect to $F = \{\xi, N, B\}$ becomes

$$\begin{cases} \dfrac{d\xi}{ds} = N \\ \dfrac{dB}{ds} = \tau N \\ \dfrac{dN}{ds} = -\tau \xi - B. \end{cases} \tag{2.3}$$

This frame satisfies

$$\xi(s) \wedge B(s) = N(s), \quad N(s) \wedge \xi(s) = \xi(s), \quad B(s) \wedge N(s) = B(s).$$

When the Cartan Frenet frame $\{\xi, N, B\}$ of a null Cartan curve makes an instantaneous helix motion at each $s$ time, there exists an axis of frame's rotation. The direction of such axis is given by the *Darboux* (*rotation*) *vector*

$$D(s) = -\tau(s)\xi(s) + B(s).$$

It is evident that $D(s)$ is spacelike, timelike or null is equivalent to $\tau(s) < 0, \tau(s) > 0$ or $\tau(s) = 0$, respectively. One can also see $|\tau(s)| = -\text{sign}(D(s))\tau(s) = -\epsilon\tau(s)$. If $\tau(s) = 0$ at $s_0$, then $\gamma(s)$ is locally a planar curve. In this paper, we only consider the case that $D(s)$ is a non-null vector, i.e., $\tau(s) \neq 0$.

For a null Cartan curve $\gamma(s)$, the Darboux vector $D(s)$ satisfies the equation

$$\begin{cases} \xi'(s) = \xi(s) \wedge D(s) \\ B'(s) = B(s) \wedge D(s) \\ N'(s) = N(s) \wedge D(s). \end{cases}$$

Let $\gamma : I \to \mathbb{R}^3_1$ be a null Cartan curve in Minkowski 3-space. Suppose $D(s)$ is not null for any $s \in I$. Then we define maps

$$\mathcal{DQ}_\epsilon : I \to Q^2_\epsilon, \quad s \mapsto \frac{D(s)}{\|D(s)\|},$$

where $I$ is an open interval, $\epsilon = \pm$ and

$$Q^2_\epsilon = \begin{cases} S^2_1 = \big\{x \in \mathbb{R}^3_1 \mid \langle x, x\rangle = 1\big\} & \text{if } \epsilon = + \\ H^2_0 = \big\{x \in \mathbb{R}^3_1 \mid \langle x, x\rangle = -1\big\} & \text{if } \epsilon = -. \end{cases}$$

If $D(s)$ is spacelike, we take $\epsilon = +$ and if $D(s)$ is timelike, we take $\epsilon = -$. We call $S^2_1$ the *de Sitter space*, $H^2_0$ the *hyperbolic space*, $\mathcal{DQ}_+$ the *de Sitter Darboux image* and $\mathcal{DQ}_-$ the *hyperbolic Darboux image* of null Cartan curve $\gamma$. We simply call every $\mathcal{DQ}_\pm$ the *pseudo-spherical Darboux image* of null Cartan curve $\gamma$.

We also define

$$NC_p = \big\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3_1 \mid -(x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 = 0\big\},$$

where $p = (p_1, p_2, p_3)$, we call $NC_p^* = NC_p \backslash \{p\}$ a *nullcone at the vertex p.*

Now we define surfaces

$$\mathcal{GS}(s, \omega) = \{\omega\xi(s) + B(s) \mid s \in I,\ \omega \in \mathbb{R}\},$$

$$\mathcal{ND}(s, \omega) = \{\gamma(s) + \omega D(s) \mid s \in I,\ \omega \in \mathbb{R}\}.$$

We call $\mathcal{GS}(s, \omega)$ the *Gaussian surface* and $\mathcal{ND}(s, \omega)$ the *null Darboux developable.*

We shall assume throughout the whole paper that all the maps and manifolds are $C^\infty$ and $\tau(s) \neq 0$ unless the contrary is explicitly stated.

The main result in the paper is as follows:

**Theorem 2.1** *Let $\gamma : I \to \mathbb{R}^3_1$ be a regular null Cartan curve, we have the followings*:

(1) *The pseudo-spherical Darboux image $\mathcal{DQ}_\epsilon(s)$ is locally diffeomorphic to the ordinary cusp $C$ at $\mathcal{DQ}_\epsilon(s_0)$ if and only if $\tau'(s_0) = 0$, $\tau''(s_0) \neq 0$.*

(2) (a) *The Gaussian surface $\mathcal{GS}(s, \omega)$ is locally diffeomorphic to the cuspidal edge at $v = \omega_0\xi(s_0) + B(s_0)$ if and only if $\tau'(s_0) \neq 0$ and $\omega_0 = -\tau(s_0)$.*

(b) *The Gaussian surface $\mathcal{GS}(s, \omega)$ is locally diffeomorphic to the swallow tail at $v = \omega_0\xi(s_0) + B(s_0)$ if and only if $\tau'(s_0) = 0, \tau''(s_0) \neq 0$ and $\omega_0 = -\tau(s_0)$.*

(3) (a) *The null Darboux developable $\mathcal{ND}(s, \omega)$ is locally diffeomorphic to the cuspidal edge at $v = \gamma(s_0) + \omega_0(-\tau(s_0)\xi(s_0) + B(s_0))$ if and only if $\tau'(s_0) \neq 0$, $\tau''(s_0) \neq 0$ and $\omega_0 = \frac{1}{\tau'(s_0)}$.*

(b) *The null Darboux developable $\mathcal{ND}(s, \omega)$ is locally diffeomorphic to the swallow tail at $v = \gamma(s_0) + \omega_0(-\tau(s_0)\xi(s_0) + B(s_0))$ if and only if $\tau'(s_0) \neq 0$, $\tau''(s_0) = 0$, $\tau'''(s_0) \neq 0$ and $\omega_0 = \frac{1}{\tau'(s_0)}$.*

## 3. Geometric invariant of null Cartan curves in Minkowski 3-space

In this section we shall introduce two different families of functions on a null Cartan curve that will be useful to the study of geometric invariants of null Cartan curves.

Let $\gamma : I \to \mathbb{R}^3_1$ be a regular null Cartan curve, we define two functions

$$H : I \times Q^2_\epsilon \to \mathbb{R}, \quad H(s, v) = \langle \xi(s), v \rangle.$$

Each of these functions shall be called the *tangential height functions* of null Cartan curve $\gamma(s)$. We denote that $h_v(s) = H(s, v)$ for any fixed vector $v$ in $Q^2_\epsilon$. We have the following proposition.

**Proposition 3.1** *Suppose $\gamma : I \to \mathbb{R}^3_1$ is a regular null Cartan curve. Let $v \in Q^2_\epsilon$, $\epsilon = \pm$. Then*

(1) *$h'_v(s) = 0$ if and only if there exist real numbers $\lambda, \omega$ such that $v = \lambda\xi(s) + \omega B(s)$,$\lambda\omega = \pm\frac{1}{2}$.*
(2) *$h'_v(s) = h''_v(s) = 0$ if and only if $v = \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s) + B(s))$.*
(3) *$h'_v(s) = h''_v(s) = h'''_v(s) = 0$ if and only if $v = \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s) + B(s))$ and $\tau'(s) = 0$.*
(4) *$h'_v(s) = h''_v(s) = h'''_v(s) = h^{(4)}_v(s) = 0$ if and only if $v = \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s) + B(s))$ and $\tau'(s) = \tau''(s) = 0$.*

*Proof.* (1) Let $v = \lambda\xi(s) + \omega B(s) + \mu N(s)$. By the assumption that $v$ is in $Q^2_\epsilon$, we have $2\lambda\omega + \mu^2 = \pm1$. Using Eqs.(2.3), we obtain

$$\begin{aligned} h'_v(s) &= \langle \xi'(s),\ \lambda\xi(s) + \omega B(s) + \mu N(s)\rangle \\ &= \langle N(s),\ \lambda\xi(s) + \omega B(s) + \mu N(s)\rangle \\ &= \mu. \end{aligned}$$

The assertion (1) follows.

(2) By (1), we have $v = \lambda\xi(s) + \omega B(s)$ and $2\lambda\omega = \pm1$. We can also calculate

$$\begin{aligned} h''_v(s) &= \langle N'(s),\ \lambda\xi(s) + \omega B(s) + \mu N(s)\rangle \\ &= \langle -\tau(s)\xi(s) - B(s),\ \lambda\xi(s) + \omega B(s) + \mu N(s)\rangle \\ &= -\omega\tau(s) - \lambda. \end{aligned}$$

Thus $h''_v(s) = 0$ if and only if $\lambda = -\omega\tau(s)$. Therefore, we have $v = \omega(-\tau(s)\xi(s) + B(s))$. By $-2\omega^2\tau(s) = \pm1$, we also have $\omega = \pm\frac{1}{\sqrt{|2\tau(s)|}}$. It follows that $h'(s) = h''(s) = 0$ if and only if $v = \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s)+B(s))$.

(3) Under the assumption that $h'_v(s) = h''_v(s) = 0$, we shall compute $h'''_v(s)$

$$\begin{aligned}
h_v'''(s) &= \Big\langle N''(s),\ \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s)+B(s))\Big\rangle \\
&= \Big\langle -\tau'(s)\xi(s)-2\tau(s)N(s),\ \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s)+B(s))\Big\rangle \\
&= \pm\frac{\tau'(s)}{\sqrt{|2\tau(s)|}}.
\end{aligned}$$

Hence the assertion (3) holds.

(4) Based on the assumption that $h_v'(s) = h_v''(s) = h_v'''(s) = 0$, we can now compute

$$\begin{aligned}
h_v^{(4)}(s) &= \Big\langle N'''(s),\ \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s)+B(s))\Big\rangle \\
&= \Big\langle (-\tau''(s)+2\tau^2(s))\xi(s)-3\tau'(s)N(s)+2\tau(s)B(s), \\
&\qquad\qquad \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s)+B(s))\Big\rangle \\
&= \pm\frac{\tau''(s)}{\sqrt{|2\tau(s)|}}.
\end{aligned}$$

Hence the assertion (4) holds. □

Now let's define a family of smooth functions with three parameters $\widetilde{H} : I \times Q_\epsilon^2 \times \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{H}(s, v, u) = H(s, v) - u,$$

where $v = \big(\pm\sqrt{v_2^2 + v_3^2 \mp 1}, v_2, v_3\big)$ in $Q_\epsilon^2$. We also put $\tilde{h}_{v,u} = \widetilde{H}(s, v, u)$ for any fixed $v$ and $u$. We have the following proposition by Proposition 3.1.

**Proposition 3.2** *Suppose $\gamma : I \to \mathbb{R}_1^3$ is a regular null Cartan curve and $v = \lambda\xi(s) + \omega B(s) + \mu N(s)$ in $Q_\epsilon^2$, where $\lambda, \omega, \mu$ are real numbers. Then*

(1) $\tilde{h}_{v,u}(s) = 0$ *if and only if* $u = \langle \xi(s), v\rangle = \omega$.

(2) $\tilde{h}_{v,u}(s) = \tilde{h}_{v,u}'(s) = 0$ *if and only if there exist real numbers* $\lambda, \omega$ *such that*

$$v = \lambda\xi(s) + \omega B(s), \quad 2\lambda\omega = \pm 1 \quad \textit{and } u = \omega$$

(3) $\tilde{h}_{v,u}(s) = \tilde{h}'_{v,u}(s) = \tilde{h}''_{v,u}(s) = 0$ *if and only if* $v = \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s) + B(s))$ *and* $u = \pm\frac{1}{\sqrt{|2\tau(s)|}}$.

(4) $\tilde{h}_{v,u}(s) = \tilde{h}'_{v,u}(s) = \tilde{h}''_{v,u}(s) = \tilde{h}'''_{v,u}(s) = 0$ *if and only if* $v = \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s) + B(s))$, $u = \pm\frac{1}{\sqrt{|2\tau(s)|}}$ *and* $\tau'(s) = 0$.

(5) $\tilde{h}_{v,u}(s) = \tilde{h}'_{v,u}(s) = \tilde{h}''_{v,u}(s) = \tilde{h}'''_{v,u}(s) = \tilde{h}^{(4)}_{v,u}(s) = 0$ *if and only if* $v = \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s) + B(s))$, $u = \pm\frac{1}{\sqrt{|2\tau(s)|}}$ *and* $\tau'(s) = \tau''(s) = 0$.

We define a three-parameter family of smooth functions

$$G : I \times \mathbb{R}^3_1 \to \mathbb{R}$$

by $G(s, v) = |\xi(s)\ B(s)\ \gamma(s) - v| = \langle \gamma(s) - v, N(s)\rangle$. Here, $|a\ b\ c|$ denotes the determinant of matrix $(a\ b\ c)$. We call $G$ the *volumelike distance function* of null Cartan curve $\gamma$. We denote that $g_v(s) = G(s, v)$ for any fixed $v$ in $\mathbb{R}^3_1$. Then we have the following proposition.

**Proposition 3.3** *Suppose* $\gamma : I \to \mathbb{R}^3_1$ *is a regular null Cartan curve and* $v = \lambda\xi(s) + \omega B(s) + \mu N(s)$ *in* $\mathbb{R}^3_1$, *where* $\lambda, \omega, \mu$ *are real numbers. Then*

(1) $g_v(s) = 0$ *if and only if* $\gamma(s) - v = \lambda\xi(s) + \omega B(s)$.

(2) $g_v(s) = g'_v(s) = 0$ *if and only if* $\gamma(s) - v = \omega(-\tau(s)\xi(s) + B(s))$.

(3) $g_v(s) = g'_v(s) = g''_v(s) = 0$ *if and only if* $\tau'(s) \neq 0$ *and* $\gamma(s) - v = -\frac{1}{\tau'(s)}(-\tau(s)\xi(s) + B(s))$.

(4) $g_v(s) = g'_v(s) = g''_v(s) = g'''_v(s) = 0$ *if and only if* $\tau'(s) \neq 0, \tau''(s) = 0$ *and* $\gamma(s) - v = -\frac{1}{\tau'(s)}(-\tau(s)\xi(s) + B(s))$.

(5) $g_v(s) = g'_v(s) = g''_v(s) = g'''_v(s) = g^{(4)}_v(s) = 0$ *if and only if* $\tau'(s) \neq 0$, $\tau''(s) = \tau'''(s) = 0$ *and* $\gamma(s) - v = -\frac{1}{\tau'(s)}(-\tau(s)\xi(s) + B(s))$.

*Proof.* (1) If $g_v(s) = 0$, then there exist $\lambda, \omega, \mu$ for $\gamma(s) - v = \lambda\xi(s) + \omega B(s) + \mu N(s)$ such that $\langle \gamma(s) - v, N(s)\rangle = 0$, so that we have $\mu = 0$. This shows that $g_v(s) = 0$ if and only if $\gamma(s) - v = \lambda\xi(s) + \omega B(s)$.

(2) On the other hand, when $g_v(s) = 0$ we can calculate

$$\begin{aligned} g_v'(s) &= \big\langle \xi(s), N(s)\big\rangle + \big\langle \gamma(s)-v, N'(s)\big\rangle \\ &= \big\langle \lambda\xi(s)+\omega B(s),\ \ -\tau(s)\xi(s)-B(s)\big\rangle \\ &= -\omega\tau(s)-\lambda, \end{aligned}$$

then $g_v(s)=g_v'(s)=0$ if and only if $\gamma(s)-v=\omega(-\tau(s)\xi(s)+B(s))$.

(3) When $g_v(s)=g_v'(s)=0$, the second derivative

$$\begin{aligned} g_v''(s) &= \big\langle \gamma'(s), N'(s)\big\rangle + \big\langle \gamma(s)-v, N''(s)\big\rangle \\ &= \big\langle \xi(s), -\tau(s)\xi(s)-B(s)\big\rangle \\ &\quad + \omega\big\langle -\tau(s)\xi(s)+B(s),\ \ -\tau'(s)\xi(s)-2\tau(s)N(s)\big\rangle \\ &= -1-\omega\tau'(s), \end{aligned}$$

then $g_v(s)=g_v'(s)=g_v''(s)=0$ if and only if $\tau'(s)\neq 0$ and $\gamma(s)-v=-\frac{1}{\tau'(s)}(-\tau(s)\xi(s)+B(s))$.

(4) When $g_v(s)=g_v'(s)=g_v''(s)=0$, the assertion (4) follows from the fact that

$$\begin{aligned} g_v'''(s) &= \big\langle \gamma''(s), N'(s)\big\rangle + 2\big\langle \gamma'(s), N''(s)\big\rangle + \big\langle \gamma(s)-v, N'''(s)\big\rangle \\ &= \big\langle N(s), -\tau\xi(s)-B(s)\big\rangle + 2\big\langle \xi(s), -\tau'\xi(s)-2\tau N(s)\big\rangle \\ &\quad + \Big\langle -\frac{1}{\tau'(s)}(-\tau(s)\xi(s)+B(s), (-\tau''(s)+2\tau^2(s))\xi(s) \\ &\qquad\qquad -3\tau'(s)N(s)+2\tau(s)B(s)\Big\rangle \\ &= \frac{\tau''(s)}{\tau'(s)}. \end{aligned}$$

(5) Under the condition that $g_v(s)=g_v'(s)=g_v''(s)=g_v'''(s)=0$, this derivative is computed as follows

$$\begin{aligned} g_v^{(4)}(s) &= \big\langle \gamma'(s), N'''(s)\big\rangle + \big\langle \gamma(s)-v, N^{(4)}(s)\big\rangle \\ &= \big\langle \xi(s), (-\tau''(s)+2\tau^2(s))\xi(s)-3\tau'(s)N(s)+2\tau(s)B(s)\big\rangle \\ &\quad + \big\langle \gamma(s)-v, (-\tau'''(s)+7\tau(s)\tau'(s))\xi(s) \\ &\qquad\qquad + (-4\tau''(s)+4\tau^2(s))N(s)+5\tau'(s)B(s)\big\rangle \end{aligned}$$

$$= 2\tau(s) + \frac{1}{\tau'(s)}(\tau'''(s) - 2\tau(s)\tau'(s))$$
$$= \frac{\tau'''(s)}{\tau'(s)}.$$

The assertion (5) follows. □

## 4. Null helices and tangential nullcone image of a null Cartan curve

In this section we study the geometric properties of the null Darboux developable, pseudo-spherical Darboux image and Gassian surface of a null Cartan curve in $\mathbb{R}^3_1$. By the propositions in the last section, we can recognize the functions $\tau(s)$ have special meanings. We have the following proposition.

**Proposition 4.1** *Suppose $\gamma : I \to \mathbb{R}^3_1$ is a regular null Cartan curve. Then*

(1) *If $\tau'(s_0) = 0$, then $\frac{d\mathcal{DQ}_\epsilon}{ds}(s_0) = 0$. If $\tau'(s) \equiv 0$, then the pseudo-spherical Darboux image $\mathcal{DQ}_\epsilon$ of $\gamma$ is constant.*
(2) *The singular set of $\mathcal{ND}$ is $\{(s, u) \mid u\tau'(s) = 1, s \in I\}$.*
(3) *Suppose that $\tau'(s) \neq 0$. If $\mathcal{ND}\big(s, \frac{1}{\tau'(s)}\big) = v_0$ is a constant vector, then $\tau''(s) \equiv 0$.*

*Proof.* (1) Assertion (1) can be verified from the fact that

$$\frac{d\mathcal{DQ}_\epsilon}{ds} = \left(\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s) + B(s))\right)'$$
$$= -\frac{\tau'(s)}{2\tau(s)\sqrt{|2\tau(s)|}}(\tau(s)\xi(s) + B(s)).$$

(2) By the straightforward calculations, we have

$$\frac{\partial\mathcal{ND}}{\partial s} = \gamma'(s) + uD'(s)$$
$$= (1 - u\tau'(s))\xi(s),$$
$$\frac{\partial\mathcal{ND}}{\partial u} = -\tau(s)\xi(s) + B(s).$$

The above two vectors are linearly dependent if and only if $u\tau'(s) = 1$. This completes the proof of the assertion (2).

(3) For a smooth function $\mu : I \to \mathbb{R}$, we define a mapping $f_\mu : I \to \mathbb{R}^3_1$ by

$$f_\mu(s) = \gamma(s) + \mu(s)D(s).$$

Suppose that $f_\mu(s) = v_0$ is a constant. Then we have

$$\frac{df_\mu(s)}{ds} = (1 - \mu'(s)\tau(s) - \mu(s)\tau'(s))\xi(s) + \mu'(s)B(s) = 0.$$

Since the singularities of $\mathcal{ND}$ are $\mu(s) = \frac{1}{\tau'(s)}$, $\mu'(s) = -\frac{\tau''(s)}{\tau'^2(s)}$, substituting this relation into the above equality, we have $\tau''(s) = 0$. $\square$

*Null helix* in the Minkowski space is completely classified in low dimensions [8]. If $\tau'(s) \equiv 0$, then the null Cartan curve $\gamma(s)$ has been classically known as a *null helix* [20]. For a null Cartan curve $\gamma : I \to \mathbb{R}^3_1$, the tangent curve $\xi : I \to NC^*_o$ is called the *tangential nullcone image* of $\gamma$. It can be easily calculated that the geodesic curvature of the tangential nullcone image is equal to the function $\tau(s)$. We have the following proposition.

**Proposition 4.2** *Let $\gamma : I \to \mathbb{R}^3_1$ be a regular null Cartan curve, then $\gamma(s)$ is a null helix if and only if the Darboux vector $D(s)$ is a constant vector. In this case, the tangential nullcone image $\xi(s)$ of $\gamma(s)$ is a circle on the nullcone $NC^*_o$ and the direction of the center of the circle is given by the constant vector $\mathcal{DQ}_\epsilon(s) \equiv c$.*

*Proof.* By the Cartan Frenet formula, we can show that $D'(s) = -\tau'(s)\xi(s)$. Therefore, $\gamma$ is a null helix if and only if $D'(s) = 0$. The condition is equivalent to the condition that $D(s)$ is a constant vector. In this case, since

$$c = \pm\frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s) + B(s)),$$

we have $\langle c, \xi(s)\rangle = \mp\frac{1}{\sqrt{|2\tau(s)|}}$, so $\langle c, \xi(s)\rangle$ is a constant. This means that the tangential nullcone image $\xi(s)$ is a circle on the $NC^*_o$ and the center is directed by $c$, and hence the desired result. $\square$

Let $F : \mathbb{R}^3_1 \to \mathbb{R}$ be a submersion and $\gamma : I \to \mathbb{R}^3_1$ be a null Cartan curve. We say that $\gamma$ and $F^{-1}(0)$ have *k-point contact* for $t = t_0$ if the function $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{k-1}(t_0) = 0$, $g^k(t_0) \neq 0$. We also say that $\gamma$ and $F^{-1}(0)$ have *at least k-point contact* for $t = t_0$ if the function $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{k-1}(t_0) = 0$. We have the following proposition.

**Proposition 4.3** *Let $\gamma : I \to \mathbb{R}^3_1$ be a regular null Cartan curve with $k(s) = 1$ and $\tau(s) \neq 0$. Then there exists an open interval $s_0 \in J \subset I$ and a null helix $\eta : J \to \mathbb{R}^3_1$ such that $\eta(s_0) = \gamma(s_0)$, the curvature of $\eta(s)$ is $k(s) = 1$, the torsion of $\eta$ at $s_0$ is $\tau(s_0)$ and $\gamma$ and $\eta$ have at least 4-point contact at $s_0$.*

*Proof.* We know that there exists an unique solve $\eta(s)$ for the equation $k_\eta(s) = k(s) = 1$, $\tau_\eta(s) = \tau(s)$ under the initial condition $\eta(s_0) = \gamma(s_0)$, $\eta'(s_0) = \gamma'(s_0)$, $\eta''(s_0) = \gamma''(s_0)$, $\eta'''(s_0) = \gamma'''(s_0)$. $\square$

**Remark 4.4** We call the null helix $\eta$ in Proposition 4.4 the *osculating null helix* of $\gamma$ at $s_0$. By Proposition 4.3, the tangential nullcone image $\xi_\eta(s)$ of the null helix $\eta(s)$ is a circle whose center is directed by the pseudo-spherical Darboux image $\frac{-\tau(s_0)\xi(s_0)+B(s_0)}{\sqrt{|2\tau(s_0)|}}$ of $\gamma$ at $s_0$. The singular locus of the Gaussian surface is given by $-\tau(s)\xi(s)+B(s)$, it describes how the shape of the curve $\gamma$ is similar to a null helix. On the other hand, the singular locus of the null Darboux developable is $\gamma(s) + \frac{1}{\tau'(s)}(-\tau(s)\xi(s) + B(s))$, it describes how the shape of the curve $\gamma$ is different from a null helix.

## 5. Unfoldings of functions of one-variables

In this section we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [3].

Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call $F$ an *r-parameter unfolding* of $f$, where $f(s) = F_{x_0}(s, x_0)$. We say that $f(s)$ has $A_k$-singularity at $s_0$ if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$. We also say that $f(s)$ has $A_{\geq k}$-singularity at $s_0$ if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let $F$ be an unfolding of $f$ and $f(s)$ has $A_k$-singularity ($k \geq 1$) at $s_0$. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at $s_0$ by $j^{(k-1)}\big(\frac{\partial F}{\partial x_i}(s, x_0)\big)(s_0) = \sum_{j=1}^{k-1} \alpha_{ji}(s - s_0)^j$, for $i = 1 \ldots r$. Then $F$ is called a $(p)$ *versal unfolding* if the $(k-1) \times r$ matrix of coefficients $(\alpha_{ji})$ has rank $k-1$ $(k-1 \leq r)$. Under the same as the above, $F$ is called a *versal unfolding* if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank $k$ $(k \leq r)$, where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, x_0)$.

We now introduce several important sets concerning the unfolding. The singular set of $F$ is the set

$$\mathfrak{S}_F = \left\{ (s,x) \in \mathbb{R} \times \mathbb{R}^r \;\middle|\; \frac{\partial F}{\partial s}(s,x) = 0 \right\}.$$

The *bifurcation set* $\mathfrak{B}_F$ of $F$ is the critical value set of the restriction to $\mathfrak{S}_F$ of the canonical projection $\pi : \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}^r$ and is given by

$$\mathfrak{B}_F = \left\{ x \in \mathbb{R}^r \;\middle|\; \text{there exists } s \text{ with } \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial s^2} = 0 \text{ at } (s,x) \right\}.$$

The *discriminant set* of $F$ is the set

$$\mathfrak{D}_F = \left\{ x \in \mathbb{R}^r \;\middle|\; \text{there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s,x) \right\}.$$

Then we have the following well-known result [3].

**Theorem 5.1** *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an $r$-parameter unfolding of $f(s)$ which has the $A_k$ singularity at $s_0$.*

(1) *Suppose that $F$ is a (p) versal unfolding.*
   (a) *If $k = 2$, then $(s_0, x_0)$ is the fold point of $\pi \mid \mathfrak{S}_F$ and $\mathfrak{B}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*
   (b) *If $k = 3$, then $\mathfrak{B}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*
(2) *Suppose that $F$ is a versal unfolding.*
   (a) *If $k = 1$, then $\mathfrak{D}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*
   (b) *If $k = 2$, then $\mathfrak{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*
   (c) *If $k = 3$, then $\mathfrak{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

*Here, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v,\ x_2 = 4u^3 + 2uv,\ x_3 = v\}$ is swallow tail and $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$ is the ordinary cusp. We also say that a point $x_0 \in \mathbb{R}^r$ is a fold point of a map germ $f : (\mathbb{R}^r, x_0) \to (\mathbb{R}^r, f(x_0))$ if there exist diffeomorphism germs $\phi : (\mathbb{R}^r, x_0) \to (\mathbb{R}^r, x_0)$ and $\psi : (\mathbb{R}^r, f(x_0)) \to (\mathbb{R}^r, f(x_0))$, $\psi \circ \phi \to \psi^{-1} \circ f \circ \phi$ such that $\psi \circ \phi(x_1, \ldots, x_r) = (x_1, \ldots, x_{r-1}, x_r^2)$.*

For the tangential height function $H$ and the volumelike distance function $G$, we have the following proposition.

**Proposition 5.2** *Let $H : I \times Q^2_\epsilon \to \mathbb{R}$ be the tangential height function on a null Cartan curve $\gamma(s)$. If $h_{v_0}$ has $A_k$-singularity at $s_0$ $(k = 2, 3)$, then $H$ is a (p) versal unfolding of $h_{v_0}$.*

*Proof.* Let

$$\gamma(s) = (x_1(s), x_2(s), x_3(s)) \text{ and } v = \Big(\pm\sqrt{v_2^2 + v_3^2 \mp 1}, v_2, v_3\Big) \text{ in } Q^2_\epsilon.$$

Under this notation the same as above proposition, we obtain

$$H(s, v) = \langle \xi(s), v\rangle = \mp x_1'(s)\sqrt{v_2^2 + v_3^2 \mp 1} + x_2'(s)v_2 + x_3'(s)v_3,$$

$$\frac{\partial H}{\partial v_i}(s, v) = \mp\frac{x_1'(s)v_i}{\sqrt{v_2^2 + v_3^2 \mp 1}} + x_i'(s) \quad (i = 2, 3),$$

$$\frac{\partial}{\partial s}\frac{\partial H}{\partial v_i}(s, v) = \mp\frac{x_1''(s)v_i}{\sqrt{v_2^2 + v_3^2 \mp 1}} + x_i''(s) \quad (i = 2, 3),$$

$$\frac{\partial}{\partial s^2}\frac{\partial H}{\partial v_i}(s, v) = \mp\frac{x_1^{(3)}(s)v_i}{\sqrt{v_2^2 + v_3^2 \mp 1}} + x_i^{(3)}(s) \quad (i = 2, 3).$$

Therefore, the 2-jet of $\frac{\partial H}{\partial v_i}(s, v_0)$ at $s_0 (i = 2, 3)$ is given by

$$\begin{aligned} j^2\Big(\frac{\partial H}{\partial v_i}(s, v_0)\Big)(s_0) &= \frac{\partial}{\partial s}\frac{\partial H}{\partial v_i}(s_0, v_0)(s - s_0) + \frac{1}{2}\frac{\partial}{\partial s^2}\frac{\partial H}{\partial v_i}(s_0, v_0)(s - s_0)^2 \\ &= \alpha_{1,i}(s - s_0) + \frac{1}{2}\alpha_{2,i}(s - s_0)^2. \end{aligned}$$

We have two cases to consider.

Case (1). We consider the rank of the following matrix

$$\begin{aligned} \mathcal{A} &= (\alpha_{1,2} \quad \alpha_{1,3}) \\ &= \left(\mp\frac{x_1''(s_0)v_{0,2}}{\sqrt{v_{0,2}^2 + v_{0,3}^2 \mp 1}} + x_2''(s_0) \quad \mp\frac{x_1''(s_0)v_{0,3}}{\sqrt{v_{0,2}^2 + v_{0,3}^2 \mp 1}} + x_3''(s_0)\right) \end{aligned}$$

When $h_{v_0}$ has the $A_2$-singularity at $s_0$, we have

$$v_0 = \pm\frac{1}{\sqrt{|2\tau(s_0)|}}(-\tau(s_0)\xi(s_0) + B(s_0)) \quad \text{and} \quad \tau'(s_0) \neq 0$$

as shown in Proposition 3.1. Suppose the rank of $\mathcal{A}$ is 0. Then we have

$$\frac{N_1(s_0)v_{0,2}}{v_{0,1}} = N_2(s_0), \quad \frac{N_1(s_0)v_{0,3}}{v_{0,1}} = N_3(s_0).$$

This implies that $v_0 = \pm N(s_0)$. This occurs only if $\epsilon = +$ since $N(s_0)$ belongs to $Q^2_+$. Then $h'_{v_0}(s_0) = \pm\langle N(s_0), N(s_0)\rangle \neq 0$. Therefore $h_{v_0}$ has not the $A_2$-singularity at $s_0$. This leads to a contradiction. Hence we have that the rank of $\mathcal{A}$ is 1.

Case (2). By the assumption that $h_{v_0}$ has the $A_3$-singularity at $s_0$, we know from Proposition 3.1 that

$$v_0 = \pm\frac{1}{\sqrt{|2\tau(s_0)|}}(-\tau(s_0)\xi(s_0) + B(s_0)) \quad \text{and} \quad \tau'(s_0) = 0,\ \tau''(s_0) \neq 0.$$

We require the $2 \times 2$ matrix

$$\mathcal{B} = \begin{pmatrix} \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,2} & \alpha_{3,3} \end{pmatrix}$$

to be nonsingular. By the direct calculation we have

$$\begin{aligned} \det\mathcal{B} &= \pm\frac{1}{\sqrt{v_{0,2}^2 + v_{0,3}^2 \mp 1}}\langle \xi'(s_0) \wedge \xi''(s_0), v\rangle \\ &= \pm\frac{-2\tau(s_0)}{\sqrt{v_{0,2}^2 + v_{0,3}^2 \mp 1}}\langle N(s_0) \wedge \xi(s_0), -B(s_0)\rangle \\ &= \pm\frac{2\tau(s_0)}{\sqrt{v_{0,2}^2 + v_{0,3}^2 \mp 1}} \neq 0. \end{aligned}$$

This means that rank $\mathcal{B} = 2$. $\square$

**Proposition 5.3** *If $\tilde{h}_{v_0,u_0}$ has $A_k$-singularity $(k = 1, 2, 3)$ at $s_0$, then $\widetilde{H}$ is versal unfolding of $\tilde{h}_{v_0,u_0}$.*

*Proof.* Under the same notation as the above proposition, we have

$$\widetilde{H}(s, v, u) = \langle \xi(s), v \rangle - u = \mp x_1'(s)\sqrt{v_2^2 + v_3^2 \mp 1} + x_2'(s)v_2 + x_3'(s)v_3 - u.$$

Let $j^2 \frac{\partial \widetilde{H}}{\partial v_i}(s, v_0, u_0)(s_0)$ and $j^2 \frac{\partial \widetilde{H}}{\partial u}(s, v_0, u_0)(s_0)$ are respectively the 2-jet of $\frac{\partial \widetilde{H}}{\partial v_i}(s, v, u)$ and $\frac{\partial \widetilde{H}}{\partial u}(s, v, u)$ at $s_0$. So

$$\begin{aligned}
&\frac{\partial \widetilde{H}}{\partial v_i}(s_0, v_0, u_0) + j^2\left(\frac{\partial \widetilde{H}}{\partial v_i}(s, v_0, u_0)\right)(s_0) \\
&\quad = \left(\mp \frac{x_1'(s_0)v_{0,i}}{\sqrt{v_{0,2}^2 + v_{0,3}^2 \mp 1}} + x_i'(s_0)\right) \\
&\qquad + \left(\mp \frac{x_1''(s_0)v_{0,i}}{\sqrt{v_{0,2}^2 + v_{0,3}^2 \mp 1}} + x_i''(s_0)\right)(s - s_0) \\
&\qquad + \frac{1}{2}\left(\mp \frac{x_1^{(3)}(s_0)v_{0,i}}{\sqrt{v_{0,2}^2 + v_{0,3}^2 \mp 1}} + x_i^{(3)}(s_0)\right)(s - s_0)^2, \qquad i = 2, 3, \\
&\frac{\partial \widetilde{H}}{\partial u}(s_0, v_0, u_0) + j^2\left(\frac{\partial \widetilde{H}}{\partial u}(s, v_0, u_0)\right)(s_0) = -1,
\end{aligned}$$

We also distinguish three cases.

Case (1). When $\tilde{h}_{v_0,u_0}$ has the $A_1$-singularity at $s_0$, the rank of $1 \times 3$ matrix

$$\begin{aligned}
\mathcal{C} &= (\alpha_{0,2} \quad \alpha_{0,3} \quad \alpha_{0,u}) \\
&= \left(\mp\frac{x_1'(s_0)v_{0,2}}{\sqrt{v_{0,2}^2 + v_{0,3}^2 \mp 1}} + x_2'(s_0) \quad \mp\frac{x_1'(s_0)v_{0,3}}{\sqrt{v_{0,2}^2 + v_{0,3}^2 \mp 1}} + x_3'(s_0) \quad -1\right)
\end{aligned}$$

is clearly 1.

Case (2). When $\tilde{h}_{v_0,u_0}$ has the $A_2$-singularity at $s_0$, we require the matrix

$$\mathcal{D} = \begin{pmatrix} \alpha_{0,2} & \alpha_{0,3} & \alpha_{0,u} \\ \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,u} \end{pmatrix}$$

to have the maximal rank. By the case (1) in Proposition 5.2, the second line of $\mathcal{D}$ does not vanish, so that the rank of $\mathcal{D}$ is 2.

Case (3). When $\tilde{h}_{v_0,u_0}$ has the $A_3$-singularity at $s_0$, we require the matrix

$$\mathcal{E} = \begin{pmatrix} \alpha_{0,2} & \alpha_{0,3} & \alpha_{0,u} \\ \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,u} \\ \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,u} \end{pmatrix}$$

to have the maximal rank. By the case (2) in Proposition 5.2, the rank of $\mathcal{E}$ is 3. □

**Proposition 5.4** *If $g_{v_0}$ has $A_k$-singularity $(k = 1, 2, 3)$ at $s_0$, then $G$ is versal unfolding of $g_{v_0}$.*

*Proof.* For $\gamma(s) = (x_1(s), x_2(s), x_3(s))$, $N(s) = (x_1''(s), x_2''(s), x_3''(s))$, $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3_1$, we have

$$G(s,v) = -(x_1(s) - v_1)x_1''(s) + (x_2(s) - v_2)x_2''(s) + (x_3(s) - v_3)x_3''(s)$$

and

$$\frac{\partial G}{\partial v_1} = x_1''(s), \quad \frac{\partial G}{\partial v_i} = -x_i''(s) \qquad (i = 2, 3).$$

Moreover,

$$\frac{\partial}{\partial s}\frac{\partial G}{\partial v_1} = x_1'''(s), \quad \frac{\partial}{\partial s}\frac{\partial G}{\partial v_i} = -x_i'''(s) \qquad (i = 2, 3)$$

and

$$\frac{\partial^2}{\partial s^2}\frac{\partial G}{\partial v_1} = x_1^{(4)}(s), \quad \frac{\partial^2}{\partial s^2}\frac{\partial G}{\partial v_i} = -x_i^{(4)}(s) \qquad (i = 2, 3).$$

Let $j^2\frac{\partial G}{\partial v_i}(s,v_0)(s_0)$ be the 2-jet of $\frac{\partial G}{\partial v_i}(s,v)$ $(i=1,2,3)$ at $s_0$, then we can show that

$$\begin{aligned}
&\frac{\partial G}{\partial v_i}(s_0,v_0)+j^2\bigg(\frac{\partial G}{\partial v_i}(s,v_0)\bigg)(s_0)\\
&\quad=\frac{\partial G}{\partial v_i}(s_0,v_0)+\frac{\partial}{\partial s}\frac{\partial G}{\partial v_i}(s_0,v_0)(s-s_0)+\frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial G}{\partial v_i}(s_0,v_0)(s-s_0)^2\\
&\quad=\alpha_{0,i}+\alpha_{1,i}(s-s_0)+\frac{1}{2}\alpha_{1,i}(s-s_0)^2.
\end{aligned}$$

We denote that

$$\mathcal{M}=(\alpha_{0,1}\quad \alpha_{0,2}\quad \alpha_{0,3}),\quad \mathcal{Z}=\begin{pmatrix}\alpha_{0,1} & \alpha_{0,2} & \alpha_{0,3}\\ \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3}\end{pmatrix},$$

$$\mathcal{P}=\begin{pmatrix}\alpha_{0,1} & \alpha_{0,2} & \alpha_{0,3}\\ \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3}\\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3}\end{pmatrix}.$$

When $g$ has $A_1$-singularity at $s_0$, there exists a non-zero number $\omega$ such that $\gamma(s_0)-v=\omega(-\tau\xi(s_0)+B(s_0))$ by Proposition 3.3, with this we can see that the rank of $\mathcal{M}=(x_1''(s_0)\ \ -x_2''(s_0)\ \ -x_3''(s_0))$ is 1, since $N(s)\neq 0$.

From Proposition 3.3, we know that $g$ has $A_2$-singularity at $s_0$ if and only if $\gamma(s_0)-v=-\frac{1}{\tau'(s_0)}(-\tau(s_0)\xi(s_0)+B(s_0))$ and $\tau'(s_0)\neq 0$. When $g$ has $A_2$-singularity at $s_0$, we require rank $\mathcal{Z}=2$, which it can be verified from the following proof.

Note the fact showed in Proposition 3.3 that $g$ has $A_3$-singularity at $s_0$ if and only if $\gamma(s_0)-v=-\frac{1}{\tau'(s_0)}(-\tau(s_0)\xi(s_0)+B(s_0))$ and $\tau'(s_0)\neq 0,\tau''(s_0)=0$. When $g$ has $A_3$-singularity at $s_0$, the rank of $\mathcal{P}$ is 3, which is verified from the fact that

$$\begin{aligned}
\det\mathcal{P}&=\det\big((N(s_0)\ N'(s_0)\ N''(s_0))\big)\\
&=\big\langle N(s_0)\wedge(-\tau(s_0)\xi(s_0)-B(s_0)),\ -\tau'(s_0)\xi(s_0)-2\tau(s_0)N(s_0)\big\rangle\\
&=\tau'(s_0)\langle N(s_0)\wedge B(s_0),\xi(s_0)\rangle\\
&=-\tau'(s_0)\neq 0.
\end{aligned}$$

This completes the proof. $\square$

In order to investigate the singularities of Gaussian surface, we introduce the following map

$$\Phi : Q^2_\epsilon \times \mathbb{R} \to \mathbb{R}^3_1, \quad \Phi(v,u) = \frac{1}{u}\Big( \pm \sqrt{v_2^2 + v_3^2 \mp 1}, v_2, v_3\Big).$$

It is clear that $\Phi$ is a diffeomorphism. We can obtain the following Proposition 5.5 by Proposition 3.2.

**Proposition 5.5** *If $\tilde{h}_{v_0,u_0}(s)$ has $A_k (k \geq 2)$-singularity at $s_0$, then the discriminant set $\mathfrak{D}_{\tilde{H}}$ of $\tilde{H}$ is diffeomorphic to $-\tau(s)\xi(s) + B(s)$, i.e., $\Phi(\mathfrak{D}_{\tilde{H}}) = -\tau(s)\xi(s) + B(s)$.*

*Proof of Theorem 2.1.* The bifurcation set $\mathfrak{B}_H$ of $H$ is

$$\mathfrak{B}_H = \Big\{ \pm \frac{1}{\sqrt{|2\tau(s)|}}(-\tau(s)\xi(s) + B(s)) \Big| s \in I \Big\}.$$

the assertion (1) of Theorem 2.1 follows from Proposition 3.1, 5.2 and Theorem 5.1.

We can obtain from Proposition 3.2 that the discriminant set of $\tilde{H}$ is

$$\mathfrak{D}_{\tilde{H}} = \Big\{ \Big( u \Big( \pm \frac{1}{2u^2}\xi(s) + B(s), 1 \Big) \Big| s \in I,\ u \in R \Big\},$$

the assertion (2) of Theorem 2.1 follows from Proposition 3.2, 5.3, 5.5 and Theorem 5.1.

The discriminant set of $G$ is

$$\mathfrak{D}_G = \big\{ \gamma(s) + \omega(-\tau(s)\xi(s) + B(s)) \mid s \in I,\ \omega \in R \big\},$$

the assertion (3) of Theorem 2.1 follows from Proposition 3.3, 5.4 and Theorem 5.1.

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## References


[ 1 ] Bejancu A., *Lightlike curves in Lorentz manifolds*. Publ. Math. Debrecen. **44** (1994), 145–155.

[ 2 ] Bonnor W. B., *Null curves in a Minkowski spacetime*. Tensor (N.S.). **20** (1969), 229–242.

[ 3 ] Bruce J. W. and Giblin P. J., *Curves and singularities (2nd.ed.)*, Cambridge Univ. Press, 1992.

[ 4 ] Che M., Jiang Y. and Pei D., *The hyperbolic Darboux image and rectifying Gaussian surface of nonlightlike curve in Minkowski* 3*-Space*. J. Math. Res. Exposition. **28** (2008), 651–658.

[ 5 ] Duggal K. L., *A Report on canonical null curves and screen distributions for lightlike geometry*. Acta Appl Math. **95** (2007), 135–149.

[ 6 ] Duggal K. L. and Bejancu A., *Lightlike submanifolds of semi-Riemannian manifolds and applications*. **364** of Mathematics and Its Applications, Kluwer, Dordrecht, The Netherlands, 1996.

[ 7 ] Ferrandez A., Gimenez A. and Lucas P., *Geometrical particle models on 3D null curves*. Phys. Lett. B. **543** (2002), 311–317.

[ 8 ] Ferrandez A., Gimenez A. and Lucas P., *Null helices in Lorentzian space forms*. Internat. J. Modern Phys. A. **30** (2001), 4845–4863.

[ 9 ] Hughston L. P. and Shaw W. T., *Classical strings in ten dimensions*. Proc. Roy. Soc. London Ser. A. **414** (1987), 423–431.

[10] Hughston L. P. and Shaw W. T.. *Constraint-free analysis of relativistic strings*. Classical Quantum Gravity. **5** (1988), 69–72.

[11] Izumiya S., Pei D. and Takahashi M., *Curves and surfaces in hyperbolic space*. Banach center publ. **65** (2004), 107–123.

[12] Izumiya S., Pei D. and Sano T., *Horospherical surfaces of curves in hyperbolic space*. Publ. Math. Debrecen. **64** (2004), 1–13.

[13] Izumiya S., Pei D. and Sano T., *The lightlike Gauss map and the lightcone developable of a spacelike curve in Minkowski* 3*-space*. Glasg. Math. J. **42** (2000), 75–89.

[14] Izumiya S., Pei D. and Sano T., *Singularities of hyperbolic Gauss maps*. Proc. London Math. Soc. **86** (2003), 485–512.

[15] Izumiya S., Pei D., Sano T. and Torii E., *Evolutes of hyperbolic plane curves*. Acta Mathematica Sinica,English series. **20** (2004), 543–550.

[16] Izumiya S., Katsumi H. and Yamasaki T., *The rectifying developable and the spherical Darboux image of a space curve*. Banach Center Publ. **50** (1999), 137–149.

[17] O'Neill B., *Semi-Riemannian geometry*. Pure Appl. Math. 103, Academic Press, New York, 1983.

[18] Nersessian A. and Ramos E., *Massive spinning particles and the geometry of null curves.* Phys. Lett. B. **445** (1998), 123–128.

[19] Nersessian A. and Ramos E., *A geometrical particle model for anyons.* Modern Phys. Lett. A. **14** (1999), 2033–2037.

[20] Sahin B., Kilic E. and Gunes R., *Null helices in Minkowski* 3*-space.* Differ. Geom. Dyn. Syst. **3** (2001), 31–36.

[21] Shaw W. T., *Twistors and strings. In Mathematics and General Relativity (Santa Cruz, CA, 1986).* Amer. Math. Soc., RI. (1988), 337–363.

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