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Real hypersurfaces which are contact in a nonflat complex space form

(Dedicated to Professor Shun-ichi Tachibana)

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Abstract. In an n ($\geqq 2$)-dimensional nonflat complex space form $\widetilde{M}_n(c)(= \mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c))$, we classify real hypersurfaces M^{2n-1} which are contact with respect to the almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J and the standard metric g of the ambient space $\widetilde{M}_n(c)$. Our theorems show that this contact manifold M^{2n-1} is congruent to a homogeneous real hypersurface of $\widetilde{M}_n(c)$.

Key words: nonflat complex space forms, real hypersurfaces, totally η-umbilic hypersurfaces, standard real hypersurfaces, homogeneous real hypersurfaces of type (B), almost contact metric structure, contact manifolds, Sasakian manifolds, Sasakian space forms.

1. Introduction

Let $\widetilde{M}_n(c)$ be a complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature c ($\neq 0$). That is, $\widetilde{M}_n(c)$ is holomorphically isometric to either an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c or an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c according as c is positive or negative. It is well-known that every real hypersurface M^{2n-1} of $\widetilde{M}_n(c)$ admits an almost contact metric structure (ϕ, ξ, η, g) induced from this ambient space. Making use of such a structure, many geometers have studied real hypersurfaces in nonflat complex space forms (cf. [6]). On the other hand, contact geometry has been developed also by many geometers (cf. [2]). Contact manifolds, Sasakian manifolds and Sasakian space forms are analogues to Hermitian manifolds, Kähler manifolds and complex space forms, respectively.

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The aim of this paper is to bridge between submanifold theory and contact geometry and show nice examples of contact manifolds from the viewpoint of submanifold theory. We investigate real hypersurfaces in $\widetilde{M}_n(c)$ whose induced structure (ϕ, ξ, η, g) is a contact metric structure (Theorems 1 and 2). Our theorems show that every real hypersurface which is contact in $\widetilde{M}_n(c)$ is an orbit of some subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of the ambient space $\widetilde{M}_n(c)$.

2. Contact metric structures

Let M be an odd dimensional Riemannian manifold. A quartet (ϕ, ξ, η, g) of a $(1,1)$-tensor ϕ, a vector field ξ, a 1-form η and a Riemannian metric g on M is called an almost contact metric structure if

$$\phi^2(X) = -X + \eta(X)\xi, \ \eta(\xi) = 1 \quad \text{and} \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

hold for all vectors $X, Y \in TM$. It is known that these conditions show that $\phi\xi = 0$ and $\eta(\phi(X)) = 0$. We say an odd dimensional manifold to be an almost contact metric manifold if it admits an almost contact metric structure. When the exterior differentiation $d\eta$ of the contact form η on M which is given by $d\eta(X, Y) := (1/2)\{X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])\}$ satisfies

$$d\eta(X, Y) = g(X, \phi Y) \quad \text{for all } X, Y \in TM,$$

the structure (ϕ, ξ, η, g) is said to be a *contact metric structure* on M. An almost contact metric manifold M is said to be a *Sasakian manifold* if the structure tensor ϕ of M satisfies the equation $(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$ with the Riemannian connection ∇ on M associated with g for all $X, Y \in TM$. By easy computation we find that the structure of a Sasakian manifold is a contact metric structure. For a unit tangent vector $u \in TM$ orthogonal to ξ of a Sasakian manifold M we call $K(u, \phi u) := g(R(u, \phi u)\phi u, u)$ its ϕ-sectional curvature, where R is the curvature tensor of M. A *Sasakian space form* is a Sasakian manifold whose ϕ-sectional curvatures do not depend on the choice of unit tangent vectors orthogonal to ξ. For more detail on contact geometry see [2] for example.

3. Standard real hypersurfaces in a complex space form

In order to explain our results we briefly recall some properties on standard real hypersurfaces in a nonflat complex space form. Let M^{2n-1} be a real hypersurface in an n-dimensional Kähler manifold $\widetilde{M}$ with Riemannian metric g and Kähler structure J. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}$ and ∇ of M are related by the following formulas of Gauss and Weingarten with a unit normal local vector field $\mathcal{N}$:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X \mathcal{N} = -AX \tag{3.1}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the ambient space $\widetilde{M}$ and A is the shape operator of M in $\widetilde{M}$ associated with $\mathcal{N}$. On M an almost contact metric structure (ϕ, ξ, η, g) associated with $\mathcal{N}$ is canonically induced from the Kähler structure of the ambient space $\widetilde{M}$. They are defined by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

By the formulas of Gauss and Weingarten and by the property $\widetilde{\nabla} J = 0$ we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \quad \text{and} \quad \nabla_X \xi = \phi AX. \tag{3.2}$$

We now restrict ourselves on real hypersurfaces M in an n $(\geqq 2)$-dimensional nonflat complex space form $\widetilde{M}_n(c)$ which is either $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$. We call eigenvectors and eigenvalues of the shape operator A *principal curvature vectors* and *principal curvatures* of M in $\widetilde{M}_n(c)$, respectively. We usually say that M is a *Hopf hypersurface* if its characteristic vector ξ is a principal curvature vector at each point of M in the ambient space $\widetilde{M}_n(c)$. It is known that every tube of sufficiently small constant radius around each Kähler submanifold of a nonflat complex space form $\widetilde{M}_n(c)$ is a Hopf hypersurface. The following properties on principal curvatures of a Hopf hypersurface M in $\widetilde{M}_n(c)$ are well-known.

Lemma 1

(1) *The principal curvature δ associated with ξ is locally constant.*

(2) *If a vector $v \in TM$ orthogonal to ξ satisfies $Av = \lambda v$, then $(2\lambda - \delta)$ $A\phi v = (\delta\lambda + (c/2))\phi v$ holds. In particular, when $c > 0$, we have $A\phi v =$*

$\big(\delta\lambda + (c/2)\big)/(2\lambda - \delta)\big)\phi v$.

Remark 1 When $c < 0$, in Lemma 1 (2) there exists a case that both of equations $2\lambda - \delta = 0$ and $\delta\lambda + (c/2) = 0$ hold. In fact, for example, we take a horoshere in $\mathbb{C}H^n(c)$. It is known that this real hypersurface has two distinct constant principal curvatures either $\lambda = \sqrt{|c|}/2$, $\delta = \sqrt{|c|}$ or $\lambda = -\sqrt{|c|}/2$, $\delta = -\sqrt{|c|}$. Hence, when $c < 0$, we must consider two cases $2\lambda - \delta = 0$ and $2\lambda - \delta \neq 0$.

Hopf hypersurfaces in a nonflat complex space form all of whose principal curvatures are constant are completely classified. In $\mathbb{C}P^n(c)$ $(n \geqq 2)$, such a Hopf hypersurface is locally congruent to one of the following (cf. [6]):

(A_1) A geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;
(A_2) A tube of radius r around totally geodesic $\mathbb{C}P^{\ell}(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;
(B) A tube of radius r around complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c})$;
(C) A tube of radius r around $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and n $(\geqq 5)$ is odd;
(D) A tube of radius r around complex Grassmann $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 9$;
(E) A tube of radius r around Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. These principal curvatures are given as follows:

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\big)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\big(\frac{\sqrt{c}}{2}r\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\big)$	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\big)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot\big(\frac{\sqrt{c}}{2}r\big)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan\big(\frac{\sqrt{c}}{2}r\big)$
δ	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

In $\mathbb{C}H^n(c)$ ($n \geqq 2$), a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following (c.f. [6]):

(A_0) A horosphere;
($\mathrm{A}_{1,0}$) A geodesic sphere of radius r ($0 < r < \infty$);
($\mathrm{A}_{1,1}$) A tube of radius r around totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;
(A_2) A tube of radius r around totally geodesic $\mathbb{C}H^{\ell}(c)$ ($1 \leqq \ell \leqq n-2$), where $0 < r < \infty$;
(B) A tube of radius r around totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A_0), ($\mathrm{A}_{1,0}$), ($\mathrm{A}_{1,1}$), (A_2) and (B). Summing up real hypersurfaces of types ($\mathrm{A}_{1,0}$) and ($\mathrm{A}_{1,1}$), we call them hypersurfaces of type (A_1). The numbers of distinct principal curvatures of real hypersurfaces of types (A_0), ($\mathrm{A}_{1,0}$), ($\mathrm{A}_{1,1}$), (A_2) are $2, 2, 2, 3$, respectively. A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$ has 2 distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3})$. Except this, a real hypersurface of type (B) has 3 distinct constant principal curvatures. The principal curvatures of these real hypersurfaces are given as follows:

	(A_0)	($\mathrm{A}_{1,0}$)	($\mathrm{A}_{1,1}$)	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}r)$

We denote by $V^0_{\lambda_i}$ the *restricted principal foliation* associated with λ_i, that is $V^0_{\lambda_i} = \{v \in TM \mid Av = \lambda_i v, v \perp \xi\}$. Then we have the following.

(1) For a hypersurface of type (A), that is of type either (A_0), (A_1) or (A_2) in $\widetilde{M}_n(c)$, the restriced foliation $V^0_{\lambda_i}$ is invariant under the action of ϕ. In particular, $A\phi = \phi A$ holds.
(2) For a hypersurface of type (B) in $\widetilde{M}_n(c)$, the restriced foliations satisfy $\phi(V^0_{\lambda_1}) = V^0_{\lambda_2}$, $\phi(V^0_{\lambda_2}) = V^0_{\lambda_1}$.

Real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$ listed above are said to be *standard real hypersurfaces.* It is well-known that they are homogeneous, which means that each of them is an orbit of some subgroup of the full isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of $\widetilde{M}_n(c)$ (see [6]).

For each standard real hypersurface M which is not of type (B) with radius $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$, we see that the restricted principal foliation V_μ^0 coincides with the principal foliation $V_\mu := \{v \in TM | Av = \mu v\}$ for every principal curvature μ of M.

4. Hypersurfaces admitting canonical contact metric structures

We use the following lemma which characterizes real hypersurfaces of types (A_0), (A_1) and (B) by an relationship between their shape operators denoted by A and their structure tensors denoted by ϕ.

Lemma 2 *Let M be a connected real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then M is of type either* (A_0), (A_1) *or* (B) *if and only if $\phi A + A\phi = k\phi$ holds for some nonzero constant k.*

Proof. We shall prove the "only if" part. When M is of type either (A_0) or (A_1), its shape operator satisfies $\phi A = A\phi$ and $A\phi = \alpha\phi$ with some constant α. That is, ϕ preserves subbundles of principal curvature vectors and all vectors orthogonal to ξ are principal. So this real hypersurface M satisfies the relationship. When M is of type (B), the decomposition $TM = \{\xi\}_{\mathbb{R}} \oplus V_{\lambda_1}^0 \oplus V_{\lambda_2}^0$ into subbundles of principal curvature vectors satisfies $\phi V_{\lambda_1}^0 = V_{\lambda_2}^0$ and $\phi V_{\lambda_2}^0 = V_{\lambda_1}^0$. Thus we see that $(\phi A + A\phi)X = (\lambda_1 + \lambda_2)\phi X$ for each $X \in V_{\lambda_i}^0$ $(i = 1, 2)$. Combining $(\phi A + A\phi)\xi = 0 = (\lambda_1 + \lambda_2)\phi\xi$, we find this real hypersurface also satisfies the relationship.

We now prove the "if" part. Suppose that $\phi A + A\phi = k\phi$ holds with some constant k. We then have $\phi A\xi = 0$, which shows that ξ is principal. We denote by δ its principal curvature. We study principal curvatures associated with principal curvature vectors orthogonal to ξ. When such a principal curvature λ satisfies $2\lambda - \delta \neq 0$ at some point of M, the assumption and Lemma 1 (2) show that it satisfies the quadratic equation $4\lambda^2 - 4k\lambda + c + 2k\delta = 0$ on the set $\{p \in M \mid 2\lambda(p) \neq \delta\}$. Since k and δ are constant, this implies that λ is also constant on this set. This also shows that if $2\lambda - \delta = 0$ at some point of M then the continuous function λ satisfies $\lambda \equiv \delta/2$ on some sufficiently small neighborhood of this point. Thus we can see that our real hypersurface is locally congruent to a Hopf hypersurface with at most 4 distinct constant principal curvatures. In view of the list of principal curvatures we find M is of type either (A_0), (A_1), (A_2) or (B). But real hypersurfaces of type (A_2) do not satisfy the condition $\phi A + A\phi = k\phi$.

Therefore we can conclude that M is of type either (A_0), (A_1) or (B). $\square$

We now investigate real hypersurfaces which are contact in a nonflat complex space form. We here clarify the meaning of this condition. On an orientable connected real hypersurface M in a Kähler manifold $\widetilde{M}$, an almost contact metric structure (ϕ, ξ, η, g) associated with a unit normal $\mathcal{N}$ is canonically induced from Kähler structure of the ambient space. Clearly $(\phi, -\xi, -\eta, g)$ is also an almost contact metric structure on M which is associated with a unit nomal $-\mathcal{N}$. We call a real hypersurface M *contact* if one of these induced structures on M is a contact metric structure. In another word, if we fix a unit normal $\mathcal{N}$ of M, this real hypersurface is contact if and only if either $d\eta(X,Y) = g(X, \phi Y)$ holds for arbitrary $X, Y \in TM$ or $d\eta(X,Y) = -g(X, \phi Y)$ holds for arbitrary $X, Y \in TM$.

Theorem 1 *Let M^{2n-1} $(n \geqq 2)$ be a connected real hypersurface of $\mathbb{C}P^n(c)$. If it is contact, then it is locally congruent to one of the following homogeneous real hypersurfaces;*

1) *a geodesic sphere $G(r)$ of radius $r = (2/\sqrt{c})\tan^{-1}(\sqrt{c}/2)$, $0 < r < \pi/\sqrt{c}$,*
2) *a tube of radius $r = (2/\sqrt{c})\tan^{-1}(\sqrt{c+4} - \sqrt{c}\,)/2$ around complex hyperquadric $\mathbb{C}Q^{n-1}$, $0 < r < \pi/(2\sqrt{c}\,)$.*

Theorem 2 *Let M^{2n-1} $(n \geqq 2)$ be a connected real hypersurface of $\mathbb{C}H^n(c)$. If it is contact, then it is locally congruent to one of the following homogeneous real hypersurfaces;*

1) *a horosphere in $\mathbb{C}H^n(c)$ $(c = -4)$,*
2) *either a geodesic sphere $G(r)$ of radius $r = (1/\sqrt{|c|})\{\log(2 + \sqrt{|c|}\,) - \log(2 - \sqrt{|c|}\,)\}$ or a tube of radius $r = \big(1/(2\sqrt{|c|})\big)\{\log(2+\sqrt{|c|}\,) - \log(2 - \sqrt{|c|}\,)\}$ around totally real totally geodesic $\mathbb{R}H^n(c/4)$ $(-4 < c < 0)$,*
3) *a tube of radius $r = (1/\sqrt{|c|})\{\log(\sqrt{|c|} + 2) - \log(\sqrt{|c|} - 2)\}$ around totally geodesic $\mathbb{C}H^{n-1}(c)$ $(c < -4)$.*

Proof of Theorem 1. Since we have $\nabla_X \xi = \phi AX$ (see the second equality in (3.2)), we find that the relation $d\eta(X,Y) = \pm g(X, \phi Y)$ for arbitrary $X, Y \in TM$ means

$$X(g(\xi, Y)) - Y(g(\xi, X)) - g(\nabla_X Y - \nabla_Y X, \xi) \mp 2g(X, \phi Y) = 0,$$

so that

$$0 = g(\phi AX, Y) - g(\phi AY, X) \mp 2g(X, \phi Y) = g\big((\phi A + A\phi \pm 2\phi)X, Y\big).$$

This yields that a real hypersurface M in $\mathbb{C}P^n(c)$ is contact if and only if the following holds:

$$\phi A + A\phi = \mp 2\phi. \tag{4.1}$$

Hence, Lemma 2 tells us that our real hypersurface M is of type either (A_1) or (B).

When M is of type (A_1), as all nonzero vectors orthogonal to ξ are principal associated with the principal curvature $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, the relation (4.1) turns to $\cot(\sqrt{c}\,r/2) = \mp 2/\sqrt{c}$ $(0 < r < \pi/\sqrt{c})$. Thus we find that only the positive sign holds and obtain $r = (2/\sqrt{c})\tan^{-1}(\sqrt{c}/2)$.

When M is of type (B), along the same lines as in the proof of Lemma 2 we find the relation (4.1) turns to $\lambda_1 + \lambda_2 = \mp 2$ with its principal curvatures $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2 - \pi/4)$ and $\lambda_2 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2 + \pi/4)$. Since $0 < r < \pi/(2\sqrt{c}\,)$, we have $\lambda_1 < -\sqrt{c}/2$ and $0 < \lambda_2 < \sqrt{c}/2$. We therefore find that only the negative sign holds. As $\lambda_1 + \lambda_2 = -2$ is equivalent to the equality

$$\frac{\tan(\sqrt{c}\,r/2) + 1}{\tan(\sqrt{c}\,r/2) - 1} - \frac{\tan(\sqrt{c}\,r/2) - 1}{\tan(\sqrt{c}\,r/2) + 1} = -\frac{4}{\sqrt{c}},$$

we obtain $\tan(\sqrt{c}\,r/2) = (\sqrt{c+4} - \sqrt{c}\,)/2$ because $0 < r < \pi/(2\sqrt{c}\,)$. We hence get the conclusion. $\square$

Proof of Theorem 2. By the same discussion as in the proof of Theorem 1, we have $\phi A + A\phi = \mp 2\phi$, hence M is of type either (A_0), (A_1) or (B).

When M is of type (A_0), we find the relation turns to $\sqrt{|c|} = \mp 2$. Hence we find only the positive sign holds and $c = -4$. When M is of type $(\mathrm{A}_{1,0})$, we find the relation turns to $\coth(\sqrt{|c|}\,r/2) = \mp 2/\sqrt{|c|}$. Hence we find only the positive sign holds and $-4 < c < 0$. Solving this, we obtain $r = (1/\sqrt{|c|})\big\{\log(2 + \sqrt{|c|}\,) - \log(2 - \sqrt{|c|}\,)\big\}$. When M is of type $(\mathrm{A}_{1,1})$, we know that the relation turns to $\tanh(\sqrt{|c|}\,r/2) = \mp 2/\sqrt{|c|}$. Hence we see that only the positive sign holds and $c < -4$. Solving this, we obtain $r = (1/\sqrt{|c|})\big\{\log(\sqrt{|c|} + 2) - \log(\sqrt{|c|} - 2)\big\}$.

When M is of type (B), we find the relation (4.1) turns to $\lambda_1 + \lambda_2 = \mp 2$ with its principal curvatures $\lambda_1 = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)$ and

$\lambda_2 = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\,r/2)$. Hence we find only the positive sign holds. Rewritng the relation, we have

$$\frac{\exp(\sqrt{|c|}\,r)+1}{\exp(\sqrt{|c|}\,r)-1}+\frac{\exp(\sqrt{|c|}\,r)-1}{\exp(\sqrt{|c|}\,r)+1}=\frac{4}{\sqrt{|c|}},$$

we therefore obtain $-4 < c < 0$ and $r = \big(1/(2\sqrt{|c|})\big)\big\{\log(2+\sqrt{|c|}\,)-\log(2-\sqrt{|c|}\,)\big\}$. □

As we mentioned before, on an orientable connected real hypersurface M in a Kähler manifold $\widetilde{M}$, if we fix its unit normal $\mathcal{N}$, two almost contact metric structures are canonically induced. We call M *Sasakian* if it is a Sasakian manifold with respect to one of them. As an immediate consequence of Theorems 1 and 2 we obtain the following result which was shown in [1].

Corollary 1 *Let M^{2n-1} $(n \geqq 2)$ be a connected real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$. Then the following three conditions are mutually equivalent to each other*:

(1) *It is Sasakian manifold*;
(2) *It is a Sasakian space form of constant ϕ-sectional curvature $c+1$*;
(3) *It is locally congruent to one of the following homogeneous real hypersurfaces corresponding to c*;
  i) *a geodesic sphere $G((2/\sqrt{c})\tan^{-1}(\sqrt{c}/2))$ when $c > 0$,*
  ii) *a geodesic sphere $G((1/\sqrt{|c|})\{\log(2+\sqrt{|c|}\,)-\log(2-\sqrt{|c|}\,)\})$ when $-4 < c < 0$,*
  iii) *a horosphere when $c = -4$,*
  iv) *a tube of radius $r = (1/\sqrt{|c|})\{\log(\sqrt{|c|}+2)-\log(\sqrt{|c|}-2)\}$ around totally geodesic $\mathbb{C}H^{n-1}(c)$ when $c < -4$.*

A real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$ is called *totally η-umbilic* if its shape operator A is of the form $A = \alpha I + \beta\eta\otimes\xi$ for some smooth functions α and β on M. This definition is equivalent to saying that $Au = \alpha u$ for each vector $u \in TM$ orthogonal to the characteristic vector ξ of M with some smooth function α on M. We remark that this function α is automatically locally constant on M. It is known that a totally η-umbilic hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$ is locally congruent to a homogeneous real hypersurface of type either (A_0)

or (A_1). We hence have the following characterization of real hypersurfaces listed in Corollary 1.

Corollary 2 ([1]) *A connected real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ is Sasakian if and only if it is totally η-umbilic and its shape operator is of the form $A = -I + (c/4)\eta \otimes \xi$.*

In view of the proof of Theorem 1, we find the following by use of Lemma 2.

Proposition 1 *A connected real hypersurface M^{2n-1} $(n \geqq 2)$ in a nonflat complex space form $\widetilde{M}_n(c)$ satisfies $d\eta(X,Y) = kg(X, \phi Y)$ with some constant k for arbitrary $X, Y \in TM$ if and only if it is locally congruent to a homogeneous real hypersurface of type either* $(\mathrm{A}_0), (\mathrm{A}_1)$ *or* (B).

We shall characterize contact hypersurfaces M^{2n-1} in a nonflat complex space form $\widetilde{M}_n(c)$ by investigating geometric properties of these real hypersurfaces. On a Riemannian manifold N with Riemannian connection ∇, a real smooth curve $\gamma = \gamma(s)$ parametrized by its arclenth s is called a *circle* if there exist a nonnegative constant k and a field Y_s of unit vectors along γ satisfying the differential equations $\nabla_{\dot\gamma}\dot\gamma = kY_s$, $\nabla_{\dot\gamma}Y_s = -k\dot\gamma$. The constant k is called the *curvature* of γ. Clearly, a circle of null curvature is a geodesic. It follows from Corollary 2 that

Lemma 3 *Let M^{2n-1} $(n \geqq 2)$ be a Sasakian hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$. Then every geodesic γ whose initial vector $\dot\gamma(0)$ is orthogonal to the characteristic vector $\xi_{\gamma(0)}$ on M is mapped to a circle of the same curvature* 1 *in the ambient space $\widetilde{M}_n(c)$ through the inclusion.*

Considering the converse of Lemma 3, we have

Proposition 2 ([1], [5]) *A connected real hypersurface M^{2n-1} $(n \geqq 2)$ in a nonflat complex space form $\widetilde{M}_n(c)$ is Sasakian if and only if at each fixed point p of M there exist such orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ orthogonal to ξ_p that all geodesics of M through p in the direction $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-2)$ are mapped to circles of the same curvature* 1 *in $\widetilde{M}_n(c)$.*

We next study geometric properties of real hypersurfaces which are contact but not Sasakian in a nonflat complex space form $\widetilde{M}_n(c)$ with the aid of the following (see [3], [4]):

Lemma 4 *Let M^{2n-1} $(n \geqq 2)$ be a connected real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$. Then the following two conditions are mutually equivalent.*

1) *M is of type* (B).
2) *The holomorphic distribution $T^0M = \{X \in TM | X \perp \xi\}$ of M is decomposed into the direct sum of restricted principal foliations $V^0_{\lambda_i} = \{X \in T^0M | AX = \lambda_i X\}$. Moreover, every foliation $V^0_{\lambda_i}$ is integrable and each of its leaves is a totally geodesic submanifold of the real hypersurface M.*

The following proposition follows from Theorems 1, 2 and Lemma 4.

Proposition 3 *A connected real hypersurface M^{2n-1} $(n \geqq 2)$ of a nonflat complex space form $\widetilde{M}_n(c)$ is contact but not Sasakian if and only if M satisfies the following two conditions.*

i) *The holomorphic distribution $T^0M = \{X \in TM | X \perp \xi\}$ of M is decomposed into the direct sum of restricted principal foliations $V^0_{\lambda_i} = \{X \in T^0M | AX = \lambda_i X\}$. Moreover, every foliation $V^0_{\lambda_i}$ is integrable and each of its leaves is a totally geodesic submanifold of the real hypersurface M.*
ii) *There exists an integral curve of ξ on M which is mapped to a circle of positive curvature $|c|/2$ in the ambient space $\widetilde{M}_n(c)$.*

Proof. By virtue of Lemma 4 we only need to show that a homogeneous real hypersurface M of type (B) is contact if and only if M satisfies Condition ii). When $c > 0$, our real hypersurface M has three distinct constant principal curvatures

$$\lambda_1 = \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right),\ \lambda_2 = \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right),\ \delta = \sqrt{c}\cot(\sqrt{c}\,r).$$

The discussion in the proof of Theorem 1 tells us that M is contact if and only if $\lambda_1 + \lambda_2 = -2$. On the other hand, we have

$$\begin{aligned}\lambda_1 + \lambda_2 &= \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right) - \frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right)\\ &= \sqrt{c}\,\cot\left(\sqrt{c}\,r - \frac{\pi}{2}\right)\\ &= -\sqrt{c}\tan(\sqrt{c}\,r).\end{aligned}$$

We hence find that the real hypersurface M is contact if and only if M satisfies $A\xi = (c/2)\xi$ (i.e., $\delta = c/2$). Note that in this case every integral curve of ξ, considered as a curve in the ambient space $\mathbb{C}P^n(c)$, is a circle of curvature $c/2$ (see (3.1) and the second equation in (3.2)). This, together with the constancy of the principal curvature δ, implies that a homogeneous real hypersurface M of type (B) is contact if and only if M satisfies Condition ii).

When $c < 0$, we have

$$\lambda_1 + \lambda_2 = \frac{\sqrt{|c|}}{2}\left\{\coth\left(\frac{\sqrt{|c|}}{2}r\right) + \tanh\left(\frac{\sqrt{|c|}}{2}r\right)\right\} = \sqrt{|c|}\coth\big(\sqrt{|c|}\,r\big).$$

By the same discussion as in the case of $c > 0$, we also obtain the desired conclusion. $\square$

At the end of this paper we pose the following open problem:

Problem Find a geometric condition from the viewpoint of submanifold theory which characterizes all contact hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$.

References

[1] Adachi T., Kameda M. and Maeda S., *Geometric meaning of Sasakian space forms from the viewpoint of Submanifold Theory*. Kodai Math. J. **33** (2010), 383–397.

[2] Blair D. E., *Riemannian geometry of contact and symplectic manifolds*. Progress in Math. **203** (2002), Birkhäuser.

[3] Chen B. Y. and Maeda S., *Hopf hypersurfaces with constant principal curvatures in complex projective or complex hypersbolic spaces*. Tokyo J. Math. **24** (2001), 133–152.

[4] Maeda S., *A characterization of the homogeneous real hypersurfaces of type (B) with two distinct constant principal curvatures in a complex hyperbolic space*. Sci. Math. Jpn. **68** (2008), 1–10.

[5] Maeda S. and Ogiue K., *Charactrerizations of geodesic hyperspheres in a complex projective space by observing the extrinsic shape of geodesics*. Math. Z. **225** (1997), 537–542.

[6] Niebergall R. and Ryan P. J., *Real hypersurfaces in complex space forms*. Tight and Taut Submanifolds, T.E. Cecil and S.S. Chern, eds., Cambridge University Press, 1998, pp. 233–305.

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