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Orthogonal almost complex structures of hypersurfaces of purely imaginary octonions

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Abstract. First we give the new elementary proof of the structure equations of G_2 and the congruence theorem of hypersurfaces of the purely imaginary octonions $\mathrm{Im}\,\mathfrak{C}$ under the action of G_2. Next, we classify almost complex structures of homogeneous hypersurfaces of $\mathrm{Im}\,\mathfrak{C}$ into 4-types.

Key words: octonions, almost complex structure, G_2-congruent, G_2-orbits decomposition.

1. Introduction

It is well known that the octonions $\mathfrak{C}$ is a non-commutative, non-associative, alternative division normed algebra ([5]). The automorphism group of the octonions is an exceptional simple Lie group G_2.

One of the purposes of this paper is to give the new elementary proof of the structure equations of G_2 which are obtained by E. Calabi ([2]) and R. L. Bryant ([1]). Our method is basically the analogy of calculations of the formula of Frenet-Serre about a curve in a 3-dimensional Euclidean space.

Let $\varphi : M^6 \to \mathrm{Im}\,\mathfrak{C}$ be an immersion from a 6-dimensional orientable manifold M^6 into the purely imaginary octonions $\mathrm{Im}\,\mathfrak{C} = \{x \in \mathfrak{C} \mid \langle x, 1\rangle = 0\} \cong \mathbf{R}^7$, where 1 is a unit element of $\mathfrak{C}$. Then we define the metric of M^6 induced from the canonical metric of $\mathrm{Im}\,\mathfrak{C}(\cong \mathbf{R}^7)$.

Next we define the canonical orientation of the hypersurface M^6. The octonions is considered as a pair of the quaternions $\mathbf{H} \oplus \mathbf{H}$. We define the oriented basis (the orientation) of $\mathrm{Im}\,\mathfrak{C}$ as

$$\mathrm{Im}\,\mathfrak{C} = span_{\mathbf{R}}\{i, j, k, \varepsilon, i\varepsilon, j\varepsilon, k\varepsilon\},$$

where $\{i, j, k\}$ is the basis of pure imaginary part of quaternions and $\varepsilon = (0, 1) \in \mathbf{H} \oplus \mathbf{H}$. Then M^6 admits the orientation which is compatible with the above orientation of $\mathrm{Im}\,\mathfrak{C}$ such that

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$$\xi \wedge T_p(M) = \operatorname{Im}\mathfrak{C},$$

where ξ is a unit normal vector field whole on M^6. By algebraic properties of $\mathfrak{C}$, we define the (induced) almost complex structure J of M^6 by

$$\varphi_*(JX) = \varphi_*(X)\xi \ (= \varphi_*(X) \times \xi),$$

for any $X \in T_pM^6$, $(p \in M^6)$, which is compatible with the induced metric, where $\times$ is the exterior product of $\mathfrak{C}$ (see Section 2). Then the orientation of M^6 is compatible with the one which comes from the almost complex structure J.

Let $\varphi : M^6 \to \operatorname{Im}\mathfrak{C}$ and $\varphi' : N^6 \to \operatorname{Im}\mathfrak{C}$ be two isometric immersions. We call φ and φ' are G_2 (resp. $SO(7)$)-congruent if there exist a $g \in G_2$ (resp. $\in SO(7)$) and an orientation preserving diffeomorphism $\psi : M^6 \to N^6$ satisfying

$$g \circ \varphi = \varphi' \circ \psi$$

up to a parallel displacement. We can easily see that, if φ and φ' are G_2-congruent, then the two induced almost complex structures coincide.

In Section 3, we give the congruence theorem of hypersurfaces of $\operatorname{Im}\mathfrak{C}$ under the action of G_2. We note that this theorem is also related to the orbit decomposition (under the action of G_2), of the Grassmann manifold $G_k^+(\operatorname{Im}\mathfrak{C})$ of oriented $k-$planes in $\operatorname{Im}\mathfrak{C}$. This decomposition is also related to the double coset decomposition with respect to $G_2\backslash(SO(7)/SO(3)\times SO(4))$.

Let $\varphi : M^6 \to \mathbf{R}^7$ be an orientable hypersurface of a 7-dimensional Euclidean space. The main purpose of this paper is to describe the set of all induced almost complex structures of $g \circ \varphi$ for any $g \in SO(7)$. We restrict our attention to the Riemannian homogeneous hypersurfaces $S^k \times \mathbf{R}^{6-k}$ (generalized cylinders) for any $k \in \{0, \ldots, 6\}$. We will classify almost complex structures of $S^k \times \mathbf{R}^{6-k}$ into 4-types. In particular, we can show that (for general $g \in SO(7)$) the induced almost complex structures of $g \circ \varphi$ are different from that of φ, in the case $S^2 \times \mathbf{R}^4$ and $S^3 \times \mathbf{R}^3$. We also describe the moduli space of imbeddings from $S^2 \times \mathbf{R}^4$ and $S^3 \times \mathbf{R}^3$, to $\operatorname{Im}\mathfrak{C}$ up to the action of G_2.

In the present paper, all manifolds and tensor fields are always assumed to be of class C^∞, unless otherwise specified.

2. Preliminaries

Let $\mathbf{H}$ be the skew field of all quaternions with canonical basis $\{1, i, j, k\}$, which satisfies

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The octonions (or Cayley algebra) $\mathfrak{C}$ over $\mathbf{R}$ can be considered as a direct sum $\mathbf{H} \oplus \mathbf{H} = \mathfrak{C}$ with the following multiplication

$$(a + b\varepsilon)(c + d\varepsilon) = ac - \bar{d}b + (da + b\bar{c})\varepsilon,$$

where $\varepsilon = (0, 1) \in \mathbf{H} \oplus \mathbf{H}$ and $a, b, c, d \in \mathbf{H}$, where the symbol "$^-$" denotes the conjugation of the quaternions. For any $x, y \in \mathfrak{C}$, we have

$$\langle xy, xy \rangle = \langle x, x \rangle \langle y, y \rangle,$$

which is called "normed algebra" in ([5]). The octonions is a non-commutative, non-associative alternative division algebra. The group of automorphisms of the octonions is the exceptional simple Lie group

$$G_2 = \{g \in SO(8) \mid g(uv) = g(u)g(v) \text{ for any } u, v \in \mathfrak{C}\}.$$

The "exterior product" of $\mathfrak{C}$ is defined by

$$u \times v = (1/2)(\bar{v}u - \bar{u}v),$$

where $\bar{v} = 2\langle v, 1 \rangle - v$ is the conjugation of $v \in \mathfrak{C}$. We note that $u \times v \in \operatorname{Im} \mathfrak{C}$, where

$$\operatorname{Im} \mathfrak{C} = \{u \in \mathfrak{C} \mid \langle u, 1 \rangle = 0\}.$$

2.1. G_2-structure equations

In this section, we shall recall the structure equation of G_2 which was established by R. Bryant ([1]). To do this, we fix a basis of the complexification of the octonions $\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{C}$ over $\mathbf{C}$ given by

$$N = (1/2)\big(1 - \sqrt{-1}\varepsilon\big), \quad \bar{N} = (1/2)\big(1 + \sqrt{-1}\varepsilon\big),$$

$$E_1 = iN, \quad E_2 = jN, \quad E_3 = -kN, \quad \bar{E}_1 = i\bar{N}, \quad \bar{E}_2 = j\bar{N}, \quad \bar{E}_3 = -k\bar{N},$$

where $^-$ denote the complex conjugation of $\mathbf{C}\otimes_{\mathbf{R}}\mathfrak{C}$. We use the same symbol of the conjugation in the three ways, but it is possible to distinguish the conjugation, if the element included in $\mathbf{H}$ or $\mathfrak{C}$ or $\mathbf{C}\otimes_{\mathbf{R}}\mathfrak{C}$. We extend the multiplication of the octonions complex linearly on $\mathbf{C}\otimes_{\mathbf{R}}\mathfrak{C}$ and denote by AB. Then we have the following multiplication table;

$A\backslash B$	ε	E_1	E_2	E_3	$\bar{E}_1$	$\bar{E}_2$	$\bar{E}_3$
ε	-1	$-\sqrt{-1}E_1$	$-\sqrt{-1}E_2$	$-\sqrt{-1}E_3$	$\sqrt{-1}\bar{E}_1$	$\sqrt{-1}\bar{E}_2$	$\sqrt{-1}\bar{E}_3$
E_1	$\sqrt{-1}E_1$	0	$-\bar{E}_3$	$\bar{E}_2$	$-\bar{N}$	0	0
E_2	$\sqrt{-1}E_2$	$\bar{E}_3$	0	$-\bar{E}_1$	0	$-\bar{N}$	0
E_3	$\sqrt{-1}E_3$	$-\bar{E}_2$	$\bar{E}_1$	0	0	0	$-\bar{N}$
$\bar{E}_1$	$-\sqrt{-1}\bar{E}_1$	$-N$	0	0	0	$-E_3$	E_2
$\bar{E}_2$	$-\sqrt{-1}\bar{E}_2$	0	$-N$	0	E_3	0	$-E_1$
$\bar{E}_3$	$-\sqrt{-1}\bar{E}_3$	0	0	$-N$	$-E_2$	E_1	0

The multiplication table of the exterior product $A\times B$ is given by

$A\backslash B$	ε	E_1	E_2	E_3	$\bar{E}_1$	$\bar{E}_2$	$\bar{E}_3$
ε	0	$-\sqrt{-1}E_1$	$-\sqrt{-1}E_2$	$-\sqrt{-1}E_3$	$\sqrt{-1}\bar{E}_1$	$\sqrt{-1}\bar{E}_2$	$\sqrt{-1}\bar{E}_3$
E_1	$\sqrt{-1}E_1$	0	$-\bar{E}_3$	$\bar{E}_2$	$-\sqrt{-1}\varepsilon/2$	0	0
E_2	$\sqrt{-1}E_2$	$\bar{E}_3$	0	$-\bar{E}_1$	0	$-\sqrt{-1}\varepsilon/2$	0
E_3	$\sqrt{-1}E_3$	$-\bar{E}_2$	$\bar{E}_1$	0	0	0	$-\sqrt{-1}\varepsilon/2$
$\bar{E}_1$	$-\sqrt{-1}\bar{E}_1$	$\sqrt{-1}\varepsilon/2$	0	0	0	$-E_3$	E_2
$\bar{E}_2$	$-\sqrt{-1}\bar{E}_2$	0	$\sqrt{-1}\varepsilon/2$	0	E_3	0	$-E_1$
$\bar{E}_3$	$-\sqrt{-1}\bar{E}_3$	0	0	$\sqrt{-1}\varepsilon/2$	$-E_2$	E_1	0

To calculate the Maurer-Cartan form of G_2, we define the representation $\rho : G_2 \hookrightarrow End_{\boldsymbol{R}}(\mathrm{Im}\,\mathfrak{C})$ of G_2 by

$$\rho(g)(u) = g(u), \tag{2.1}$$

for any $u \in \mathrm{Im}\,\mathfrak{C}$, where $End_{\boldsymbol{R}}(\mathrm{Im}\,\mathfrak{C})$ is the set of all linear endomorphisms of $\mathrm{Im}\,\mathfrak{C}$. Extending the representation $\rho(g)$ complex linearly on $\mathbf{C}\otimes_{\boldsymbol{R}}\mathrm{Im}\,\mathfrak{C}$, we set

$$\begin{pmatrix} u & f & \bar{f} \end{pmatrix} = \begin{pmatrix} \rho(g)(\varepsilon) & \rho(g)(E) & \rho(g)(\bar{E}) \end{pmatrix} = \begin{pmatrix} \varepsilon & E & \bar{E} \end{pmatrix} M,$$

where

$$f = (f_1, f_2, f_3), \quad E = (E_1, E_2, E_3), \quad \bar{E} = (\bar{E}_1, \bar{E}_2, \bar{E}_3),$$

and $M = M(g)$ is a $M_{7\times 7}(\boldsymbol{C})$-valued function on G_2. Each components of $(u, f, \bar{f})$ can be considered as a vector valued function on G_2, that is, $u : G_2 \to \mathrm{Im}\,\mathfrak{C}$, $f_i : G_2 \to \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathrm{Im}\,\mathfrak{C}$, $\bar{f}_i : G_2 \to \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathrm{Im}\,\mathfrak{C}$. The (local) section $(u, f, \bar{f})$ on G_2 is called the G_2-frame field. It satisfies

$$\langle u, f_i \rangle = 0, \quad \langle f_i, f_j \rangle = \langle \bar{f}_i, \bar{f}_j \rangle = 0, \quad \langle f_i, \bar{f}_j \rangle = \delta_{ij}/2.$$

Also we extend the exterior product $\times$ complex linearly, we have the following relations.

$$f_i \times u = \sqrt{-1} f_i, \quad \langle f_1 \times f_2, f_3 \rangle = -1/2,$$

for any $i \in \{1, 2, 3\}$.

Proposition 2.1 ([1]) *Let $\begin{pmatrix} u & f & \bar{f} \end{pmatrix}$ be the G_2-frame field. Then we have*

$$d\begin{pmatrix} u & f & \bar{f} \end{pmatrix} = \begin{pmatrix} u & f & \bar{f} \end{pmatrix} \left(\begin{array}{c|c|c} 0 & -\sqrt{-1}\,{}^t\bar{\theta} & \sqrt{-1}\,{}^t\theta \\ \hline -2\sqrt{-1}\theta & \kappa & [\bar{\theta}] \\ \hline 2\sqrt{-1}\bar{\theta} & [\theta] & \bar{\kappa} \end{array}\right) = \begin{pmatrix} u & f & \bar{f} \end{pmatrix} \Phi, \tag{2.2}$$

where, $\theta = {}^t\begin{pmatrix} \theta^1 & \theta^2 & \theta^3 \end{pmatrix}$ is an $M_{3\times 1}(\boldsymbol{C})$ valued 1-form, κ is an $\mathfrak{su}(3)$ valued 1-form, which satisfies

$$\kappa + {}^t\bar{\kappa} = 0_{3\times 3}, \quad tr\kappa = 0,$$

and

$$[\theta] = \begin{pmatrix} 0 & \theta^3 & -\theta^2 \\ -\theta^3 & 0 & \theta^1 \\ \theta^2 & -\theta^1 & 0 \end{pmatrix}.$$

The integrability condition $d \circ d = 0$ implies that

$$d\theta = -\kappa \wedge \theta + [\bar{\theta}] \wedge \bar{\theta}, \tag{2.3}$$

$$d\kappa = -\kappa \wedge \kappa + 3\theta \wedge {}^t\bar{\theta} - ({}^t\theta \wedge \bar{\theta}) I_3, \tag{2.4}$$

where I_3 denote the 3×3 identity matrix.

We will give the direct proof of Proposition 2.1.

Proof. Taking a exterior derivative of G_2-frame field $\begin{pmatrix} u & f & \bar{f} \end{pmatrix}$, then we get

$$d\begin{pmatrix} u & f & \bar{f} \end{pmatrix} = \begin{pmatrix} \varepsilon & E & \bar{E} \end{pmatrix} dM = \begin{pmatrix} u & f & \bar{f} \end{pmatrix} M^{-1} dM.$$

where $M^{-1}dM$ is a $\mathfrak{g}_2$-valued left invariant 1-from on G_2, that is, the Maurer-Cartan form of G_2, where $\mathfrak{g}_2(= \rho_*(T_e G_2))$ is the Lie algebra of G_2. By (2.2), we will prove the following equality

$$M^{-1}dM = \Phi. \tag{2.5}$$

To do this, we set

$$M^{-1}dM = \left(\begin{array}{c|c|c} \psi_{00} & \psi_{01} & \psi_{02} \\ \hline \psi_{10} & \psi_{11} & \psi_{12} \\ \hline \psi_{20} & \psi_{21} & \psi_{22} \end{array}\right), \tag{2.6}$$

where ψ_{00} is a $\boldsymbol{R}$-valued 1-form, ψ_{01}, ψ_{02} are $M_{1\times 3}(\boldsymbol{C})$-valued 1-forms, ψ_{10}, ψ_{20} are $M_{3\times 1}(\boldsymbol{C})$-valued 1-forms, $\psi_{11}, \psi_{22}, \psi_{12}, \psi_{21}$ are $M_{3\times 3}(\boldsymbol{C})$-valued 1-forms, respectively.

(1) Since $\langle u, u \rangle = 1$, we get $\psi_{00} = 0$.
(2) We show that $\psi_{20} = \overline{\psi_{10}}$. Since $du = \overline{du}$, we have

$$\sum_{i=1}^{3} f_i (\psi_{10})^i + \sum_{i=1}^{3} \bar{f}_i (\psi_{20})^i = \sum_{i=1}^{3} \bar{f}_i \overline{(\psi_{10})^i} + \sum_{i=1}^{3} f_i \overline{(\psi_{20})^i}.$$

From which, we obtain

$$\psi_{20} = \overline{\psi_{10}}.$$

(3) We show that $\psi_{01} = -\frac{1}{2} {}^t\overline{\psi_{10}}$. Since $\langle u, f_i \rangle = 0$, we have

$$0 = \langle du, f_i\rangle + \langle u, df_i\rangle = \frac{1}{2}\overline{(\psi_{10})^i} + (\psi_{01})^i.$$

Hence we obtain the desired result.

(4) We show that $\psi_{02} = \frac{1}{2}{}^t\psi_{10}$, $\psi_{12} = \overline{\psi_{21}}$, $\psi_{22} = \overline{\psi_{11}}$. In fact,

$$df_i = u(\psi_{01})^i + \sum_{j=1}^{3} f_j(\psi_{11})^j_i + \sum_{j=1}^{3} \bar{f}_j(\psi_{21})^j_i.$$

Since $\overline{df_i} = d\bar{f}_i$, we see that

$$(\psi_{02})^i = \overline{(\psi_{01})^i} = -\frac{1}{2}(\psi_{10})^i, \quad (\psi_{12})^j_i = \overline{(\psi_{21})^j_i}, \quad (\psi_{22})^j_i = \overline{(\psi_{11})^j_i},$$

for any $i, j \in \{1, 2, 3\}$. We get the desired result.

(5) We will prove that $\psi_{21} = \frac{\sqrt{-1}}{2}\begin{pmatrix} 0 & (\psi_{10})^3 & -(\psi_{10})^2 \\ -(\psi_{10})^3 & 0 & (\psi_{10})^1 \\ (\psi_{10})^2 & -(\psi_{10})^1 & 0 \end{pmatrix}$. Since, $f_1 \times u = \sqrt{-1} f_1$, we get

$$df_1 \times u + f_1 \times du = \sqrt{-1} df_1. \tag{2.7}$$

By (2.6), we get

$$\begin{aligned}
&\text{l. h. s. of (2.7)}\\
&= df_1 \times u + f_1 \times du\\
&= \Big\{ u(\psi_{01})^1 + \sum_{i=1}^{3} f_i(\psi_{11})^i_1 + \sum_{i=1}^{3} \bar{f}_i(\psi_{21})^i_1 \Big\} \times u\\
&\quad + f_1 \times \Big\{ \sum_{i=1}^{3} f_i(\psi_{10})^i + \sum_{i=1}^{3} \bar{f}_i\overline{(\psi_{10})^i} \Big\}\\
&= \sqrt{-1}\Big\{ u\Big(-\frac{1}{2}\overline{(\psi_{10})^1} \Big) + \sum_{i=1}^{3} f_i(\psi_{11})^i_1 + \bar{f}_1\big(-(\psi_{21})^1_1 \big)\\
&\qquad + \bar{f}_2\big(-(\psi_{21})^2_1 - \sqrt{-1}(\psi_{10})^3 \big) + \bar{f}_3\big(-(\psi_{21})^3_1 + \sqrt{-1}(\psi_{10})^2 \big) \Big\}.
\end{aligned} \tag{2.8}$$

On the other hand,

r. h. s. of (2.7)

$$= \sqrt{-1}\Big\{u(\psi_{01})^1 + \sum_{i=1}^{3} f_i(\psi_{11})^i_1 + \bar{f}_1(\psi_{21})^1_1 + \bar{f}_2(\psi_{21})^2_1 + \bar{f}_3(\psi_{21})^3_1\Big\}. \tag{2.9}$$

Therefore, by (2.8), (2.9), we have

$$(\psi_{21})^1_1 = -(\psi_{21})^1_1,$$
$$(\psi_{21})^2_1 = -(\psi_{21})^2_1 - \sqrt{-1}(\psi_{10})^3, \quad (\psi_{21})^3_1 = -(\psi_{21})^3_1 + \sqrt{-1}(\psi_{10})^2.$$

Hence, we obtain

$$(\psi_{21})^1_1 = 0, \quad (\psi_{21})^2_1 = -\frac{\sqrt{-1}}{2}(\psi_{10})^3, \quad (\psi_{21})^3_1 = \frac{\sqrt{-1}}{2}(\psi_{10})^2.$$

In the same way, since $f_2 \times u = \sqrt{-1}f_2$, $f_3 \times u = \sqrt{-1}f_3$. We get the desired result.

(6) Since $\langle f_i, \bar{f}_j\rangle = \frac{1}{2}\delta_{ij}$, we see that $\psi_{11} + \overline{{}^t\psi_{11}} = 0_{3\times 3}$.

(7) We will show that $tr(\psi_{11}) = 0$. Since $\langle f_1 \times f_2, f_3\rangle = -\frac{1}{2}$, we get

$$\begin{aligned} 0 &= \langle df_1 \times f_2, f_3\rangle + \langle f_1 \times df_2, f_3\rangle + \langle f_1 \times f_2, df_3\rangle \\ &= \big\langle (f_1 \times f_2)(\psi_{11})^1_1, f_3\big\rangle + \big\langle (f_1 \times f_2)(\psi_{11})^2_2, f_3\big\rangle + \big\langle -\bar{f}_3,\ f_3(\psi_{11})^3_3\big\rangle \\ &= -\frac{1}{2}\big((\psi_{11})^1_1 + (\psi_{11})^2_2 + (\psi_{11})^3_3\big). \end{aligned}$$

Therefore, we obtain

$$tr(\psi_{11}) = 0.$$

Summing up the above arguments, if we set $\frac{\sqrt{-1}}{2}\psi_{10} = {}^t\big(\theta^1 \ \ \theta^2 \ \ \theta^3\big)$, we have $[\theta] = \begin{pmatrix} 0 & \theta^3 & -\theta^2 \\ -\theta^3 & 0 & \theta^1 \\ \theta^2 & -\theta^1 & 0 \end{pmatrix}$. Furthermore, if we put $\kappa = \psi_{11}$, then we obtain (2.5).

(8) Since $d \circ d = 0$, we can easily see that

$$d\psi = -\psi \wedge \psi.$$

From which we obtain (2.3), (2.4). □

2.2. $\mathrm{Im}\,\mathfrak{C} \rtimes G_2$-structure equations

We obtain $\mathrm{Im}\,\mathfrak{C} \rtimes G_2$-structure equations from those of G_2. For $(x, g) \in \mathrm{Im}\,\mathfrak{C} \rtimes G_2$, by (2.1), we define

$$\tilde{\rho} : \mathrm{Im}\,\mathfrak{C} \rtimes G_2 \hookrightarrow End_{\boldsymbol{R}}(\mathrm{Im}\,\mathfrak{C}).$$

such that

$$\tilde{\rho}(x, g)(v) = \rho(g)(v) + x = g(v) + x,$$

for any $v \in \mathrm{Im}\,\mathfrak{C}$. Since $g(0) = 0$, we can easily see that

$$\tilde{\rho}(x, g)(0) = g(0) + x = x.$$

Extending the representation $\tilde{\rho}$ complex linearly on $\mathbf{C} \otimes_{\boldsymbol{R}} \mathrm{Im}\,\mathfrak{C}$, we set

$$\big(x \ ; \ u \ \ f \ \ \bar{f}\big) = \big(\tilde{\rho}(x, g)(0) \ ; \ \rho(g)(\varepsilon) \ \ \rho(g)(E) \ \ \rho(g)(\bar{E})\big).$$

Then we obtain

Proposition 2.2 *Let* $\big(x \ ; \ u \ \ f \ \ \bar{f}\big)$ *be the* $\mathrm{Im}\,\mathfrak{C} \rtimes G_2$*-frame field. Then we have*

$$d\big(x \ ; \ u \ \ f \ \ \bar{f}\big) = \big(x \ ; \ u \ \ f \ \ \bar{f}\big) \left(\begin{array}{c|ccc} 0 & & 0_{1\times 7} & \\ \hline \mu & 0 & -\sqrt{-1}\,{}^t\bar{\theta} & \sqrt{-1}\,{}^t\theta \\ \omega & -2\sqrt{-1}\theta & \kappa & [\bar{\theta}] \\ \bar{\omega} & 2\sqrt{-1}\bar{\theta} & [\theta] & \bar{\kappa} \end{array}\right)$$

$$= \big(x \ ; \ u \ \ f \ \ \bar{f}\big)\Psi,$$

where μ *is a* $\boldsymbol{R}$*-valued* 1*-form, and* ω *is a* $M_{3\times 1}(\boldsymbol{C})$ *valued* 1*-form, respectively. The integrability condition implies that*

$$d\mu - \sqrt{-1}\,{}^t\bar{\theta} \wedge \omega + \sqrt{-1}\,{}^t\theta \wedge \bar{\omega} = 0,$$

$$d\omega - 2\sqrt{-1}\theta \wedge \mu + \kappa \wedge \omega + [\theta] \wedge \bar{\omega} = 0,$$

$$d\theta = -\kappa \wedge \theta + [\bar{\theta}] \wedge \bar{\theta},$$

$$d\kappa = -\kappa \wedge \kappa + 3\theta \wedge {}^t\bar{\theta} - ({}^t\theta \wedge \bar{\theta})I_3.$$

3. Almost complex structures of hypersurfaces of $\mathrm{Im}\,\mathfrak{C}$

In this section we define the almost complex (Hermitian) structures on hypersurafces of $\mathrm{Im}\,\mathfrak{C}$, and give some its fundamaenatl properties.

Let M be a connected orientable 6-dimensional manifold and $\varphi : M \to \mathrm{Im}\,\mathfrak{C}$ be an immersion from M to $\mathrm{Im}\,\mathfrak{C}$. Then M admits the induced metric g and the global unit normal vector field ξ. For any $X \in T_pM$ ($^\forall p \in M$), we define the linear transformation J_p

$$J_p : T_pM \to T_pM, \quad (\varphi_*(J_pX) = \varphi_*(X)\xi),$$

For any $X, Y \in T_pM$ the linear transformation J_p satisfies $J_p(J_pX) = -X$, $g(J_pX, J_pY) = g(X, Y)$. Let TM, T^*M be the tangent bundle, cotangent bundle of M, respectively. We denote $\Gamma(M, T^*M \otimes TM)$ the space of $T^*M \otimes TM$-valued global C^∞ sections on M. We define the almost complex structure $J \in \Gamma(M, T^*M \otimes TM)$ as $J(p) = J_p$ for any $p \in M$.

3.1. G_2-congruence class of hypersurfaces

Let M, N be two 6-dimensional orientable manifolds and $\varphi : M \hookrightarrow \mathrm{Im}\,\mathfrak{C}$, $\varphi' : N \hookrightarrow \mathrm{Im}\,\mathfrak{C}$ be two isometric immersions. The two hypersurfaces (M, φ) and (N, φ') are said to be G_2-congruent if there exist an element $(g, a) \in G_2 \times \mathrm{Im}\,\mathfrak{C}$ and an orientation preserving isometry $\psi : M \to N$ satisfying

$$\varphi'(\psi(p)) = g(\varphi(p)) + a$$

for any $p \in M$, that is, the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\ \varphi\ } & \mathrm{Im}\,\mathfrak{C} \\ \psi \big\downarrow & & \big\downarrow h_{(g,a)} \\ N & \xrightarrow{\ \varphi'\ } & \mathrm{Im}\,\mathfrak{C} \end{array}$$

where $h_{(g,a)}(u) = g(u) + a$ for any $u \in \mathrm{Im}\,\mathfrak{C}$. We can easily see that the G_2-congruency of hypersurfaces in $\mathrm{Im}\,\mathfrak{C}$ is an equivalent relation. We will show that the almost complex structure J is an invariant up to the action of G_2 in the following sense.

Lemma 3.1 *Let $\varphi : M \hookrightarrow \mathrm{Im}\,\mathfrak{C}$, $\varphi' : N \hookrightarrow \mathrm{Im}\,\mathfrak{C}$ be two isometric immersions with same orientation. Suppose that they are G_2-congruent. Then we have*

$$J = (\psi_*)^{-1} \circ J' \circ \psi_*,$$

where J and J' are almost complex structures on M and N, respectively.

Proof. Since $g \in G_2$ and a 6-sphere $S^6 = G_2/SU(3)$, we have $\xi' = g(\xi)$. Therefore we obtain

$$\begin{aligned}\varphi'_*(J'\psi_*(X)) &= \varphi'_*(\psi_*(X))\xi' = g(\varphi_*(X))g(\xi) = g(\varphi_*(X)\xi) \\ &= g(\varphi_*(JX)) = \varphi'_*(\psi_*(JX)),\end{aligned}$$

and φ'_* is injective, we obtain

$$J'\psi_*(X) = \psi_*(JX),$$

for any $X \in T_pM$. We get the desired result. □

We note that $\psi_*(T^{1,0}M) = T^{1,0}N$. If $g \in SO(7)$, then the induced almost complex structures do not necessarily coincide.

3.2. Construction of G_2-frame field on a hypersurface

Let $\varphi : M \hookrightarrow \mathrm{Im}\,\mathfrak{C}$ be an oriented hypersurface of $\mathrm{Im}\,\mathfrak{C}$ and ξ be the unit normal vector field on M. We construct the (local $\mathfrak{C}$-valued) G_2−frame field $(e_1, \dots, e_7)$ on M, from ξ. For any $p \in M$, we set $e_4(p) = \xi(p)$. Next, we put $e_1(p) \in T_{\varphi(p)}\varphi(M)$, $(|e_1(p)| = 1)$. We define $e_5(p)$ as $e_5(p) = e_1(p)e_4(p)$. We take $e_2(p)$ satisfying $e_2(p) \in (span_{\boldsymbol{R}}\{e_1(p), e_4(p), e_5(p)\})^\perp$, $(|e_2(p)| = 1)$. Lastly, we set $e_3(p), e_6(p), e_7(p)$ as

$$e_3(p) = e_1(p)e_2(p), \quad e_6(p) = e_2(p)e_4(p), \quad e_7(p) = e_3(p)e_4(p).$$

Then the multiplication table of the product of $(e_1(p), \dots, e_7(p))$ coincides with that of $(i, j, k, \varepsilon, i\varepsilon, j\varepsilon\, k\varepsilon)$. Therefore, there exists an $A_p \in G_2 \subset M_{7\times 7}$ such that

$$\begin{pmatrix} e_1(p) & \cdots & e_7(p) \end{pmatrix} = \begin{pmatrix} i & j & k & \varepsilon & i\varepsilon & j\varepsilon & k\varepsilon \end{pmatrix} A_p.$$

Let U $(\subset M)$ be a neighborhood of p. Then we can define the C^∞ map A from U to $M_{7\times 7}$ by $A(q) = A_q$ for any $q \in U$. Also, we obtain the local G_2-frame field $\begin{pmatrix} e_1 & \cdots & e_7 \end{pmatrix}$ on U.

3.3. Invariants of G_2-congruence class

The purpose of this section, we define the geometrical invariants of hypersurfaces under the action of G_2. Let T_pM be the tangent space at $p \in M$ and $T_pM \otimes \boldsymbol{C}$ be the complexification of T_pM. We define the eigen-space of the almost complex structure J as

$$T_p^{1,0}M = \big\{X \in T_pM \otimes \boldsymbol{C} \mid J_pX = \sqrt{-1}X\big\},$$
$$T_p^{0,1}M = \big\{X \in T_pM \otimes \boldsymbol{C} \mid J_pX = -\sqrt{-1}X\big\}.$$

Then, we have

$$T_pM \otimes \boldsymbol{C} = T_p^{1,0}M \oplus T_p^{0,1}M.$$

We represent the above spaces by using the G_2 frame field. Let $(e_1, \dots, e_7)$ be a local G_2 frame field as above. We set

$$f_1 = \big(e_1 - \sqrt{-1}e_5\big)/2, \quad f_2 = \big(e_2 - \sqrt{-1}e_6\big)/2, \quad f_3 = -\big(e_3 - \sqrt{-1}e_7\big)/2.$$

Then we can easily see that $Jf_i = \sqrt{-1}f_i$, for any $i \in \{1, 2, 3\}$. Therefore, we obtain

$$T_p^{1,0}M = span_{\boldsymbol{C}}\{f_1, f_2, f_3\}, \quad T_p^{0,1}M = span_{\boldsymbol{C}}\big\{\bar{f}_1, \bar{f}_2, \bar{f}_3\big\}.$$

We also note that $\begin{pmatrix} \xi & f & \bar{f} \end{pmatrix}$ is a local G_2-frame field on M. Next, we define the map $\tilde{\varphi} : U \to \operatorname{Im}\mathfrak{C} \rtimes G_2$ $(\subset \operatorname{Im}\mathfrak{C} \rtimes M_{7\times 7}(\boldsymbol{C}))$, (which is called the local lift of φ) by

$$\tilde{\varphi} = \begin{pmatrix} \varphi & \xi & f & \bar{f} \end{pmatrix}.$$

By Proposition 2.2, we have

$$d\tilde{\varphi} = \tilde{\varphi} \cdot \tilde{\varphi}^*\Psi.$$

In this case, we see that $\tilde{\varphi}^*\mu = 0$, and that

$$d\varphi = \sum_{i=1}^{3}(f_i\omega^i + \bar{f}_i\bar{\omega^i}) = \begin{pmatrix} f & \bar{f} \end{pmatrix}\begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix},$$

where ω^i $(i \in \{1,2,3\})$ are $\boldsymbol{C}$-valued 1-forms, $\omega = {}^t\begin{pmatrix}\omega^1 & \omega^2 & \omega^3\end{pmatrix}$. Also, we have

$$d\xi = \sum_{j=1}^{3} f_j\big(-2\sqrt{-1}\theta^j\big) + \bar{f}_j\big(2\sqrt{-1}\bar{\theta}^j\big) = \begin{pmatrix} f & \bar{f} \end{pmatrix}\begin{pmatrix} -2\sqrt{-1}\theta \\ 2\sqrt{-1}\bar{\theta} \end{pmatrix},$$

where θ^j is a $\boldsymbol{C}$-valued 1-form and $\theta = {}^t\begin{pmatrix}\theta^1 & \theta^2 & \theta^3\end{pmatrix}$. By Cartan's Lemma, there exist $M_{3\times 3}(\boldsymbol{C})$-valued (local) functions $\mathfrak{A}$, $\mathfrak{B}$ such that

$$\sqrt{-1}\theta = \begin{pmatrix} {}^t\mathfrak{B} & \bar{\mathfrak{A}} \end{pmatrix}\begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}, \tag{3.1}$$

where the each component of $\mathfrak{B}$, $\mathfrak{A}$ is given by

$$\mathfrak{B}_{ij} = \big\langle \mathrm{II}(f_i, \bar{f}_j), \xi\big\rangle, \quad \mathfrak{A}_{ij} = \big\langle \mathrm{II}(f_i, f_j), \xi\big\rangle.$$

We can easily see that

Lemma 3.2 *The functions on a hypersurface of* $\mathrm{Im}\,\mathfrak{C}$

$$tr\mathfrak{B}, \quad tr({}^t\bar{\mathfrak{B}}\mathfrak{B}), \quad \det\mathfrak{B}, \quad tr({}^t\bar{\mathfrak{A}}\mathfrak{A}), \quad tr\{({}^t\bar{\mathfrak{A}}\mathfrak{A})^2\}, \quad \det({}^t\bar{\mathfrak{A}}\mathfrak{A}),$$

are invariants up to the action of G_2.

We note that $tr\mathfrak{B}$ is independent of the almost complex structure, which corresponds to the norm of a mean curvature vector field.

3.4. G_2-congruence theorem of hypersurfaces of $\mathrm{Im}\,\mathfrak{C}$

The purpose of this section is to prove the following

Theorem 3.1 *Let* M, N *be two* 6*-dimensional orientable manifolds and* $\varphi : M \hookrightarrow \mathrm{Im}\,\mathfrak{C}$, $\varphi' : N \hookrightarrow \mathrm{Im}\,\mathfrak{C}$ *be two isometric immersions. Suppose that there exists an orientation preserving diffeomorphism* $\psi : M \to N$ *which satisfies*

$$d\psi \circ J_M = J_N \circ d\psi, \quad \psi^* g_N = g_M, \quad \psi^*\big(\omega_N^1 \wedge \omega_N^2 \wedge \omega_N^3\big) = \omega_M^1 \wedge \omega_M^2 \wedge \omega_M^3,$$

where g_M, g_N (resp. J_M, J_N, and ω^i_M, ω^i_N) are the induced metrics (resp. induced almost complex structures and the dual 1-forms) on M, N, rspecyively. Then there exits an $(a, g) \in \mathrm{Im}\,\mathfrak{C} \times G_2$ satisfying

$$g \circ \varphi + a = \varphi' \circ \psi,$$

that is, φ, φ' are G_2-congruent.

Proof. Let $(\xi, f, \bar{f})$ be the (local) G_2-frame fields on $\varphi(M)$. We set the (complexified) vector field v_i on M such that

$$d\varphi(v_i) = f_i,$$

for any $i \in \{1, 2, 3\}$. From the assumption, we see that $d\psi(v_i)$, is also the local $(1, 0)$ vector fields on N, for any $i \in \{1, 2, 3\}$, and $(d\psi(v_1), d\psi(v_2), d\psi(v_3))$ is an $SU(3)$-frame field on N. If we identify $d(\varphi' \circ \psi)(v_i)$ with f'_i, then the corresponding dual 1-forms ω_N, ω_M satisfy

$$\psi^* \omega_N = \omega_M. \tag{3.2}$$

Since ψ is an isometry from M to N, the corresponding Levi-Civita connections ∇^M, ∇^N satisfy

$$d\psi\big({\nabla^M}_X(Y)\big) = {\nabla^N}_{d\psi(X)}(d\psi(Y)), \tag{3.3}$$

for any vector fields X, Y on M. From which, we show that

$$\psi^* \kappa^N = \kappa^M, \quad \psi^* \theta^N = \theta^M, \tag{3.4}$$

where κ^M, κ^N, (resp. θ^M, θ^N) are the $\mathfrak{su}(3)$ (resp. $M_{3\times 1}(\boldsymbol{C})$)-valued 1-forms of M, N, respectively. In fact,

$$\begin{aligned}
2(\psi^* \kappa^N)^j_i &= g_N\big(\nabla^N(d\psi(v_i)), d\psi(\bar{v_j})\big) \\
&= g_N\big(d\psi(\nabla^M(v_i)), d\psi(\bar{v_j})\big) \\
&= g_N\bigg(d\psi\bigg(\sum_{k=1}^{3}\big(v_k(\kappa^M)^k_i + \bar{v_k}([\theta^M])^k_i\big)\bigg), d\psi(\bar{v_j})\bigg) \\
&= 2(\kappa^M)^j_i,
\end{aligned} \tag{3.5}$$

therefore, we get the first equality of (3.4). Similary, we have the second equality of (3.4).

Since $\mathrm{Im}\,\mathfrak{C} \rtimes G_2$ is a Lie group, there exists a $\mathrm{Im}\,\mathfrak{C} \rtimes G_2$-valued function $\tilde{g}$ on M such that

$$\tilde{\varphi'} \circ \psi(p) = \tilde{g}(p) \cdot \tilde{\varphi}(p), \tag{3.6}$$

for any $p \in M$, where $\tilde{\varphi'}, \tilde{\varphi}$ are $\mathrm{Im}\,\mathfrak{C} \rtimes G_2$-valued functions (the lift of φ', φ) on N, M, respectively. To prove Theorem 3.1, we will show that the function $\tilde{g}$ is constant on M. Hence we may show that

$$d\tilde{g} = d\big(\tilde{\varphi'} \circ \psi \cdot (\tilde{\varphi})^{-1}\big) = 0. \tag{3.7}$$

In fact, by Proposition 2.2, we have

$$\begin{aligned} d\big(\tilde{\varphi'} \circ \psi \cdot (\tilde{\varphi})^{-1}\big) &= d(\tilde{\varphi'} \circ \psi) \cdot (\tilde{\varphi})^{-1} + (\tilde{\varphi'} \circ \psi) \cdot d(\tilde{\varphi})^{-1} \\ &= (\tilde{\varphi'} \circ \psi) \cdot \big((\tilde{\varphi'} \circ \psi)^* \Psi - \tilde{\varphi}^* \Psi\big) \cdot (\tilde{\varphi})^{-1}. \end{aligned} \tag{3.8}$$

By Proposition 2.2, (3.2) and (3.4), we see that

$$(\tilde{\varphi'} \circ \psi)^* \Psi = \tilde{\varphi}^* \Psi.$$

Therefore, we get the desired result. □

3.5. G_2-orbits

3.5.1 $S^6, S^5, V_2^+(\mathrm{Im}\,\mathfrak{C})$ and $G_2^+(\mathrm{Im}\,\mathfrak{C})$

Let S^6 and S^5 be a 6-dimensional unit sphere in $\mathrm{Im}\,\mathfrak{C}$ and a 5-dimensional unit sphere in $\mathbf{R}^6 \subset \mathrm{Im}\,\mathfrak{C}$ where $\mathbf{R}^6 = \{u \in \mathrm{Im}\,\mathfrak{C} \mid \langle u, \varepsilon \rangle = 0\}$, respectively. It is well known that

$$S^6 \cong G_2/SU(3), \quad S^5 \cong SU(3)/SU(2). \tag{3.9}$$

Let $V_2^+(\mathrm{Im}\,\mathfrak{C})$ be a Stiefel manifold of oriented 2-frames in $\mathrm{Im}\,\mathfrak{C}$. It is well known that

$$V_2^+(\mathrm{Im}\,\mathfrak{C}) = \big\{(u, v) \in S^6 \times S^6 \mid \langle u, v \rangle = 0\big\}.$$

We shall prove the following

Proposition 3.1

$$V_2^+(\mathrm{Im}\,\mathfrak{C}) \cong G_2/SU(2).$$

Proof. First, we prove that G_2 acts transitively on $V_2^+(\mathrm{Im}\,\mathfrak{C})$. For any $(u,v) \in V_2^+(\mathrm{Im}\,\mathfrak{C})$, by (3.9), there exists a $g \in G_2$ such that $u = g(\varepsilon)$. Then we get $\langle u, g(i)\rangle = \langle g(\varepsilon), g(i)\rangle = \langle \varepsilon, i\rangle = 0$, and, since $\langle u, v\rangle = 0$

$$g(i), \quad v \in T_u^1 S^6,$$

where $T_u^1 S^6 = \{X \in T_u S^6 \mid |X| = 1\}$. Here, we will identify $T_u S^6$ with $\boldsymbol{R}^6$, then we have

$$i, \quad g^{-1}(v) \in T_\varepsilon^1 S^6 \cong S^5.$$

Since $S^5 \cong SU(3)/SU(2)$, there exists an $h \in SU(3) \subset G_2$ such that

$$g^{-1}(v) = h(i),$$

where, $SU(3) = \{g \in G_2 \mid g(\varepsilon) = \varepsilon\}$. Therefore

$$g(h(i)) = v. \tag{3.10}$$

Moreover, since $h(\varepsilon) = \varepsilon$, we get

$$g(h(\varepsilon)) = g(\varepsilon) = u. \tag{3.11}$$

By (3.10), (3.11), we have

$$\big(g(h(i)), g(h(\varepsilon))\big) = (u, v).$$

Hence the G_2 acts on $V_2^+(\mathrm{Im}\,\mathfrak{C})$ transitively, and its isotropy subgroup is $SU(2)$. □

By Proposition 3.1, we can see that

Corollary 3.1

$$G_2^+(\mathrm{Im}\,\mathfrak{C}) \cong G_2/U(2),$$

where $G_2^+(\mathrm{Im}\,\mathfrak{C})$ *be a Grassmann manifold of oriented* 2*-planes in* $\mathrm{Im}\,\mathfrak{C}$*.*

3.6. $V_3^+(\boldsymbol{R}^7)$ and $G_3^+(\boldsymbol{R}^7)$ (G_2-orbit decomposition)

Let $V_3^+(\boldsymbol{R}^7)$ and $G_3^+(\boldsymbol{R}^7)$ be a Stiefel manifold of oriented 3-frames in $\boldsymbol{R}^7$ and a Grassmann manifold of oriented 3-planes in $\boldsymbol{R}^7$, respectively. For any $(e_1, e_2, e_3) \in V_3^+(\boldsymbol{R}^7)$, by Proposition 3.1, there exits a $g \in G_2$ such that $g(i) = e_1$, $g(j) = e_2$. Since $g \in G_2$ we have $g(i)g(j) = g(k)$. In general, the following equality does not hold $e_1 e_2 = e_3$. Therefore we see that two manifolds $V_3^+(\boldsymbol{R}^7)$, $G_3^+(\boldsymbol{R}^7)$ can not be represented as orbits of G_2.

Next we consider the canonical form of the each element of $G_3^+(\boldsymbol{R}^7) \ni V$ by G_2. Let $V = span_{\boldsymbol{R}}\{e_1, e_2, e_3\} \in G_3^+(\boldsymbol{R}^7)$.

(1) If we assume that $e_1 e_2 = e_3$, then there exists a $g \in G_2$ satisfying

$$V = span_{\boldsymbol{R}}\{g(i), g(j), g(k)\}.$$

In this case V is called an associative 3-plane.

(2) Suppose that $e_1 e_2 \neq e_3$. We note that there exists a $g \in G_2$ such that $g(i) = e_1$, $g(j) = e_2$. By the assumption, we may assume that $g(k) \neq e_3$, then we have

$$\dim(span_{\boldsymbol{R}}\{g(k), e_3\}) = 2.$$

We can take $u \in span_{\boldsymbol{R}}\{g(k), e_3\}$ so that

$$|u| = 1, \quad \langle u, g(k)\rangle = 0.$$

If we put $\langle e_3, g(k)\rangle = \cos\theta (0 \leq \theta \leq \pi)$, then

$$e_3 = \cos\theta g(k) + \sin\theta u.$$

Since $u \in (span_{\boldsymbol{R}}\{g(i), g(j), g(k)\})^{\perp}$, we may put $u = g(\varepsilon)$. Hence we have

$$V = span_{\boldsymbol{R}}\{g(i), g(j), g(\cos\theta k + \sin\theta\varepsilon)\}.$$

Summing up the above arguments, we obtain

Proposition 3.2 *For any $V \in G_3^+(\boldsymbol{R}^7)$, there exist a $g \in G_2$ and a $\theta \in \boldsymbol{R}$ $(0 \leq \theta \leq \pi)$ satisfying*

$$V = span_{\boldsymbol{R}}\{g(i), g(j), g(\cos\theta k + \sin\theta\varepsilon)\}.$$

A 3-dimensional vector space V in $\mathrm{Im}\mathfrak{C}$ is called *associative* if $span_{\mathbf{R}}\{u, v, uv\} = V$, where $\{u, v\}$ is an oriented orthonormal pair of V. We also note that the Grassmann manifold $G_{ass}(\mathrm{Im}\mathfrak{C})$ of associative 3-planes are given by

$$G_{ass}(\mathrm{Im}\mathfrak{C}) \simeq G_2/SO(4).$$

We note that the representation

$$\rho_{SO(4)} : SO(4)(\simeq Sp(1) \times Sp(1)/Z_2) \to G_2$$

is given by

$$\rho_{SO(4)}(q_1, q_2)(a + b\varepsilon) = q_1 a\overline{q_1} + (q_2 b\overline{q_1})\varepsilon,$$

where $(q_1, q_2) \in Sp(1) \times Sp(1)$ and $a + b\varepsilon \in \mathrm{Im}\mathfrak{C}$.

4. Second fundamental forms of the generalized cylinder of Im $\mathfrak{C}$

4.1. Homogeneous hypersurfaces of Im $\mathfrak{C}$ with unique homogeneous almost complex structure

In this section, we shall give the invariants of $\boldsymbol{R}^6$, $S^1 \times \boldsymbol{R}^5$, $\boldsymbol{R} \times S^5$, S^6 and proof of the uniqueness of the induced almost complex structure up to the action of G_2.

4.1.1 $\boldsymbol{R}^6$

Proposition 4.1 *Let $\psi_0 : \boldsymbol{R}^6 \hookrightarrow \mathrm{Im}\,\mathfrak{C}$ be an isometric imbedding defined by*

$$\psi_0(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 i + x_2 j + x_3 k + x_4 i\varepsilon + x_5 j\varepsilon + x_6 k\varepsilon,$$

where $(x_1, x_2, x_3, x_4, x_5, x_6) \in \boldsymbol{R}^6$. Then we have

$$tr\,{}^t\bar{\mathfrak{B}}\mathfrak{B} = 0, \quad tr\,{}^t\bar{\mathfrak{A}}\mathfrak{A} = 0.$$

The automorphism group of the induced almost Hermitian structure coincides with $\boldsymbol{R}^6 \rtimes SU(3)(\subset \boldsymbol{R}^6 \rtimes SO(6))$ and it acts transitively on $\boldsymbol{R}^6$. The induced almost Hermitian structure is also unique under the action of G_2.

Proof. Since the isotropy subgroup of G_2 at ε is $SU(3)$, we observe that the automorphism group of $\boldsymbol{R}^6$ coincides with $\boldsymbol{R}^6 \rtimes SU(3)$. □

4.1.2 $\boldsymbol{S^1 \times R^5}$

Proposition 4.2 *Let $\psi_1 : S^1 \times \boldsymbol{R}^5 \hookrightarrow \operatorname{Im}\mathfrak{C}$ be the mapping defined by*

$$\psi_1(\theta, x_0, q) = e^{i\theta} j e^{-i\theta} + x_0 i + q\varepsilon,$$

where $[\theta] \in S^1$, $(x_0, q) \in \boldsymbol{R} \times \boldsymbol{H}(\cong \boldsymbol{R}^5)$. Then we obtain

$$tr\,{}^t\bar{\mathfrak{B}}\mathfrak{B} = \frac{1}{16}, \quad tr\,{}^t\bar{\mathfrak{A}}\mathfrak{A} = \frac{1}{16}.$$

The automorphism group of the induced almost Hermitian structure coincides with $U(2) \ltimes \boldsymbol{R}^5$ $(\subset SO(2) \times (SO(5) \ltimes \boldsymbol{R}^5))$, and it acts transitively on $S^1 \times \boldsymbol{R}^5$. The representation $\rho_{U(2)} : U(2)(\simeq S^1 \times S^3) \to \operatorname{Im}\mathfrak{C}$, is given by

$$\rho_{U(2)}(\theta, q')(a + b\varepsilon) = e^{i\theta} a e^{-i\theta} + (q' b e^{-i\theta})\varepsilon,$$

where $a + b\varepsilon \in \operatorname{Im}\mathfrak{C}$, and $([\theta], q') \in S^1 \times S^3$.

Proof. First, we construct the G_2-frame field along the map ψ_1. Let ξ be the unit normal vector field, given by $e_4 = \xi = e^{i\theta} j e^{-i\theta} = e^{2i\theta} j = \cos 2\theta j + \sin 2\theta k$. Next, we take a tangent vector $e_1 = i$ of $S^1 \times \boldsymbol{R}^5$, then we have $\langle e_1, e_4 \rangle = 0$. We set e_5 by

$$e_5 = e_1 e_4 = i(\cos 2\theta j + \sin 2\theta k) = -\sin 2\theta j + \cos 2\theta k.$$

Also we take the vector field $e_2 = \varepsilon$ on $S^1 \times \boldsymbol{R}^5$, then e_2 is orthogonal to the associative 3-plane $span_R\{e_1, e_4, e_5\}$. Lastly we put $\{e_3, e_6, e_7\}$ as $e_3 = e_1 e_2 = i\varepsilon$, $e_6 = e_2 e_4 = -\cos 2\theta j\varepsilon - \sin 2\theta k\varepsilon$, $e_7 = e_3 e_4 = \sin 2\theta j\varepsilon - \cos 2\theta k\varepsilon$. Then the frame field $(e_1, \ldots, e_7)$ is a G_2-valued function on $S^1 \times \boldsymbol{R}^5$. Therefore we have

$$\begin{cases} f_1 = \dfrac{1}{2}\big(i - \sqrt{-1}(-\sin 2\theta j + \cos 2\theta k)\big), \\ f_2 = \dfrac{1}{2}\big(\varepsilon + \sqrt{-1}(\cos 2\theta j\varepsilon + \sin 2\theta k\varepsilon)\big), \\ f_3 = -\dfrac{1}{2}\big(i\varepsilon - \sqrt{-1}(\sin 2\theta j\varepsilon - \cos 2\theta k\varepsilon)\big). \end{cases}$$

We note that $Jf_i = \sqrt{-1}f_i$. Therefore

$$d\psi_1 = idx_0 - 2(\sin 2\theta j - \cos 2\theta k)d\theta + (dq)\varepsilon,$$

From which, we have

$$\begin{aligned}\omega^1 &= dx_0 + \sqrt{-1}d\theta,\\ \omega^2 &= \langle dq, 1\rangle - \sqrt{-1}(\cos 2\theta\langle dq, j\rangle + \sin 2\theta\langle dq, k\rangle),\\ \omega^3 &= -\langle dq, i\rangle - \sqrt{-1}(\sin 2\theta\langle dq, j\rangle - \cos 2\theta\langle dq, k\rangle).\end{aligned}$$

In the same way, since

$$d\xi = -2(\sin 2\theta j - \cos 2\theta k)d\theta,$$

we have

$$\sqrt{-1}\theta^1 = -\frac{\sqrt{-1}}{2}d\theta, \quad \sqrt{-1}\theta^2 = \sqrt{-1}\theta^3 = 0.$$

Hence, we obtain

$$\sqrt{-1}\theta = -\frac{1}{4}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right)\begin{pmatrix}\omega\\ \bar{\omega}\end{pmatrix}.$$

The 2nd fundamental form of ψ_1 is given by

$$\mathfrak{B} = -\frac{1}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \mathfrak{A} = \frac{1}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

From which, we get the desired result. From the definition of ψ_1 and the representation $\rho_{u(2)}$, we see that the automorphism group of $S^1 \times \boldsymbol{R}^5$ is $U(2) \ltimes \boldsymbol{R}^5$. □

Proposition 4.3 *The almost complex structure on* $S^1 \times \boldsymbol{R}^5$ *induced from* $\mathrm{Im}\,\mathfrak{C}$ *is unique up to the action of* G_2.

Proof. Let φ_0 be a fixed imbedding from $S^1 \times \boldsymbol{R}^5$ to $\mathrm{Im}\,\mathfrak{C}$ by

$$\varphi_0(u_0, u_1, v_1, \dots, v_5) = iu_0 + ju_1 + kv_1 + \cdots + k\varepsilon v_5,$$

where $u_0^2 + u_1^2 = 1$. Next,let φ be a homogeneous isometric imbedding from $S^1 \times \boldsymbol{R}^5$ to $\boldsymbol{R}^7$. Then there exists an orthonormal basis $\begin{pmatrix} e_1 & e_2 & e_3 & \dots & e_7 \end{pmatrix}$ of $\boldsymbol{R}^7$ such that

$$\varphi(x_0, x_1, y_1, \dots, y_5) = e_1 x_0 + e_2 x_1 + e_3 y_1 + \cdots + e_7 y_5,$$

where $x_0^2 + x_1^2 = 1$. By Proposition 3.1, there exists a $g \in G_2$ such that $g(i) = e_1$, $g(j) = e_2$. From this, we have

$$span_{\boldsymbol{R}}\{g(k), \dots, g(k\varepsilon)\} = span_{\boldsymbol{R}}\{e_3, \dots, e_7\}.$$

Therefore there exists an $A \in SO(5)$ such that

$$(g(k), \dots, g(k\varepsilon)) = (e_3, \dots, e_7)A.$$

We set the diffeomorphism $\psi : S^1 \times \boldsymbol{R}^5 \to S^1 \times \boldsymbol{R}^5$ by

$$\psi(u_0, u_1, v_1, \dots, v_5) = (u_0, u_1, (v_1, \dots, v_5)^t A).$$

Then we have

$$g(\varphi_0(u_0, u_1, v_1, \dots, v_5)) = \varphi(\psi(u_0, u_1, v_1, \dots, v_5)).$$

Therefore the induced almost complex structure of φ_0 coincides with that of φ. □

4.1.3 $\boldsymbol{R^1} \times \boldsymbol{S^5}$

Proposition 4.4 *Let* $\psi_5 : \boldsymbol{R}^1 \times S^5 \hookrightarrow \operatorname{Im}\mathfrak{C}$ *be an imbedding given by*

$$\psi_5(x, z_0, z_1, z_2) = \varepsilon x + E \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} + \bar{E} \begin{pmatrix} \bar{z_0} \\ \bar{z_1} \\ \bar{z_2} \end{pmatrix}.$$

where $x \in \boldsymbol{R}^1$, z_0, z_1, $z_2 \in \boldsymbol{C}$, $|z_0|^2 + |z_1|^2 + |z_2|^2 = 1$, *and* $E = (E_1, E_2, E_3)$. *Then, we have*

$$tr\,{}^t\bar{\mathfrak{B}}\mathfrak{B} = \frac{9}{16}, \quad tr\,{}^t\bar{\mathfrak{A}}\mathfrak{A} = \frac{1}{16}.$$

The automorphism group of the induced almost Hermitian structure coincide with $\boldsymbol{R}^1 \times SU(3)$ $(\subset \boldsymbol{R}^1 \times SO(6))$ *and it acts transitively on* $\boldsymbol{R}^1 \times S^5$*. The induced almost Hermitian structure is unique up to the action of* G_2*.*

Proof. Let $\rho_{SU(3)}$ be the representation of $SU(3)$ to $End_{\boldsymbol{R}}(\boldsymbol{C} \otimes_{\boldsymbol{R}} \operatorname{Im} \mathfrak{C})$ defined by

$$\rho_{SU(3)}(U)(v) = \begin{pmatrix} \varepsilon & E & \bar{E} \end{pmatrix} \begin{pmatrix} 1 & 0_{1\times 3} & 0_{1\times 3} \\ 0_{3\times 1} & U & 0_{3\times 3} \\ 0_{3\times 1} & 0_{3\times 3} & \bar{U} \end{pmatrix} \begin{pmatrix} v_0 \\ \vdots \\ v_6 \end{pmatrix},$$

for any $v = v_0\varepsilon + \sum_{i=1}^{3} v_i E_i + \sum_{i=1}^{3} v_{i+3}\overline{E_i} \in \boldsymbol{C} \otimes_{\boldsymbol{R}} \operatorname{Im} \mathfrak{C}$. We represent the imbedding ψ_5 by using $\rho_{SU(3)}$. For any $U \in SU(3)$ we set $\tilde{\psi_5} : \boldsymbol{R} \times SU(3) \hookrightarrow End(\boldsymbol{C} \otimes_{\boldsymbol{R}} \operatorname{Im} \mathfrak{C})$ as

$$\tilde{\psi_5}(x, U) = (0; \varepsilon, E, \bar{E}) \left(\begin{array}{c|c|c|c} 1 & 0 & 0_{1\times 3} & 0_{1\times 3} \\ \hline x & 1 & 0_{1\times 3} & 0_{1\times 3} \\ \hline 0_{3\times 1} & 0_{3\times 1} & U & 0_{3\times 3} \\ \hline 0_{3\times 1} & 0_{3\times 1} & 0_{3\times 3} & \bar{U} \end{array} \right).$$

where $x \in \boldsymbol{R}$. Then we see that

$$\psi_5(x, z_0, z_1, z_2) = \tilde{\psi_5}(x, U)(p_0).$$

where $p_0 = {}^t\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$. We note that $S^5 = \{\rho_{SU(3)}(U)(i) \mid U \in SU(3) \subset M_{3\times 3}(\mathbf{C})\}$. Therefore we have

$$T_{\rho_{SU(3)}(U)(i)} S^5 = span_{\boldsymbol{R}} \{ \rho_{SU(3)}(U)(j),\ \rho_{SU(3)}(U)(k),\ \rho_{SU(3)}(U)(i\varepsilon),$$
$$\rho_{SU(3)}(U)(j\varepsilon),\ \rho_{SU(3)}(U)(k\varepsilon) \}$$

The unit normal vector field ξ is given by $\rho_{SU(3)}(U)(i)$, and we set $e_4 = \xi = \rho_{SU(3)}(U)(i)$. We put the orthonormal frame field of $T_{e_4}(\boldsymbol{R}^1 \times S^5)$ by

$$e_1 = \rho_{SU(3)}(U)(i\varepsilon), \quad e_2 = \rho_{SU(3)}(U)(j), \quad e_3 = -\rho_{SU(3)}(U)(k\varepsilon),$$
$$e_5 = \rho_{SU(3)}(U)(\varepsilon), \quad e_6 = -\rho_{SU(3)}(U)(k), \quad e_7 = \rho_{SU(3)}(U)(j\varepsilon).$$

Then $(e_1, \dots, e_7)$ is a G_2-frame field. In Section 3.2, we set

$$f_1 = \frac{1}{2}\big(e_1 - \sqrt{-1}e_5\big), \quad f_2 = \frac{1}{2}\big(e_2 - \sqrt{-1}e_6\big), \quad f_3 = -\frac{1}{2}\big(e_3 - \sqrt{-1}e_7\big).$$

To calculate the second fundamental form, we note that

$$d\psi_5 = \varepsilon dx + d\xi, \quad d\xi = e_1 \otimes \mu^1 + e_2 \otimes \mu^2 + e_7 \otimes \mu^3 - e_6 \otimes \mu^4 - e_3 \otimes \mu^5$$

where $\mu^1, \dots, \mu^5$ are $\boldsymbol{R}$-valued 1-forms of S^5. The dual 1-forms ω^i $(i \in \{1,2,3\})$ are given by

$$\omega^1 = \mu^1 - \sqrt{-1}dx, \quad \omega^2 = \mu^2 - \sqrt{-1}\mu^4, \quad \omega^3 = \mu^5 + \sqrt{-1}\mu^3,$$

Also the 1-forms θ^i $(i \in \{1,2,3\})$ which satisfy $d\xi = \sum_{i=1}^3 f_i(-2\sqrt{-1}\theta^i) + \overline{f_i}(2\sqrt{-1}\bar{\theta}^i)$ are obtained by

$$\sqrt{-1}\theta^1 = -\frac{1}{2}\mu^1, \quad \sqrt{-1}\theta^2 = -\frac{1}{2}\big(\mu^2 - \sqrt{-1}\mu^4\big),$$
$$\sqrt{-1}\theta^3 = -\frac{1}{2}\big(\mu^5 + \sqrt{-1}\mu^3\big).$$

Hence we get

$$\sqrt{-1}\theta = -\frac{1}{4}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{array}\right)\begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}.$$

Lastly we obtain the second fundamental form by

$$\mathfrak{B} = -\frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathfrak{A} = -\frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the above arguments and the definition of ψ_5, the automorphism group of $\boldsymbol{R}^1 \times S^5$ is $\boldsymbol{R}^1 \rtimes SU(3)$. By the similar arguments of the proof of Proposition 4.3, we obtain the uniqueness of the almost complex structre of $\boldsymbol{R}^1 \times S^5$. □

4.1.4 S^6

Proposition 4.5 *Let $\psi_6 : S^6 \hookrightarrow \mathrm{Im}\,\mathfrak{C}$ be the mapping from S^6 to $\mathrm{Im}\,\mathfrak{C}$, defined by*

$$\psi_6(\theta, q_1, q_2) = \cos\theta(q_1 i \bar{q_1}) + \sin\theta(q_2 i \bar{q_1})\varepsilon,$$

where $(\theta, q_1, q_2) \in S^1 \times S^3 \times S^3$. Then we have

$$tr\,{}^t\bar{\mathfrak{B}}\mathfrak{B} = \frac{3}{4}, \quad tr\,{}^t\bar{\mathfrak{A}}\mathfrak{A} = 0.$$

The automorphism group of the induced almost complex structure coincides with G_2, and it acts transitively on S^6. The induced almost complex structure is unique up to the action of G_2.

Proof. Since the immersion ψ_6 is totally umbilic, we get $d\psi_6 = d\xi$. Then we have

$$\sqrt{-1}\theta = \left(-\frac{1}{2}I_3 \,\middle|\, 0_{3\times 3} \right) \begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}.$$

Hence we obtain

$$\mathfrak{B} = -\frac{1}{2}I_3, \quad \mathfrak{A} = 0_{3\times 3}.$$

It is well known that the automorphism group of the induced almost complex structure coincides with G_2 ([4]). □

4.2. Non-homogeneous induced almost complex structure on $\boldsymbol{R}^2 \times S^4$

4.2.1 $\boldsymbol{R}^2 \times S^4$

Theorem 4.1 *Let $\psi_4 : \boldsymbol{R}^2 \times S^4 \hookrightarrow \mathrm{Im}\,\mathfrak{C}$ be the mapping from $\boldsymbol{R}^2 \times S^4$ to $\mathrm{Im}\,\mathfrak{C}$, defined by*

$$\psi_4(x_1, x_2, y_0, y_1 q) = y_0 i + x_1 j + x_2 k + y_1 q\varepsilon,$$

where $(x_1, x_2) \in \boldsymbol{R}^2$, $y_0^2 + y_1^2 = 1$, and $q \in S^3 \subset \boldsymbol{H}$, where S^3 is a 3-dimensional unit sphere in $\boldsymbol{H}$. Then we have

$$tr\,{}^t\bar{\mathfrak{B}}\mathfrak{B} = \frac{1}{8}(3 + y_0^2), \quad tr\,{}^t\bar{\mathfrak{A}}\mathfrak{A} = \frac{1}{8}y_1^2.$$

The automorphism group of the induced almost complex structure is $\boldsymbol{R}^2 \rtimes U(2)$ $(\subset (\boldsymbol{R}^2 \rtimes SO(2)) \times SO(5))$. *Therefore, it does not act transitively on* $\boldsymbol{R}^2 \times S^4$.

Proof. We construct the G_2-frame filed on $\boldsymbol{R}^2 \times S^4$. Let $e_4 = \xi = y_0 i + y_1 q\varepsilon$ be a unit normal vector field on $\boldsymbol{R}^2 \times S^4$. Next we put $e_1 = j$, then, we have $e_5 = e_1 e_4 = -y_0 k + y_1 (qj)\varepsilon$. Moreover, we put $e_2 = (qi)\varepsilon$. Then we obtain $\{e_3, e_6, e_7\}$ as $e_3 = e_1 e_2 = (qk)\varepsilon$, $e_6 = e_2 e_4 = -y_1 i + y_0 q\varepsilon$, $e_7 = e_2 e_5 = -y_1 k - y_0 (qj)\varepsilon$. From which, $(e_1, \ldots, e_7)$ is a G_2-valued function on $\boldsymbol{R}^2 \times S^4$. Next, we set the complex-valued G_2-frame field on $\boldsymbol{R}^2 \times S^4$ as

$$\begin{cases} f_1 = \dfrac{1}{2}\big(j - \sqrt{-1}(-y_0 k + y_1 (qj)\varepsilon)\big), \\ f_2 = \dfrac{1}{2}\big((qi)\varepsilon - \sqrt{-1}(-y_1 i + y_0 q\varepsilon)\big), \\ f_3 = -\dfrac{1}{2}\big((qk)\varepsilon + \sqrt{-1}(y_1 k + y_0 (qj)\varepsilon)\big). \end{cases}$$

Then we have, $J f_i = \sqrt{-1} f_i$. To calculate the forms ω^i for any $i \in \{1, 2, 3\}$. Since

$$d\psi_4 = i dy_0 + j dx_1 + k dx_2 + (q\varepsilon) dy_1 + y_1 (dq)\varepsilon,$$

we see that

$$\begin{aligned} \omega^1 &= dx_1 + \sqrt{-1}\big(- y_0 dx_2 + y_1^2 \langle \bar{q} dq, j\rangle\big), \\ \omega^2 &= y_1 \langle \bar{q} dq, i\rangle + \sqrt{-1}(-y_1 dy_0 + y_0 dy_1), \\ \omega^3 &= -y_1 \langle \bar{q} dq, k\rangle + \sqrt{-1}(y_1 dx_2 + y_0 y_1 \langle \bar{q} dq, j\rangle). \end{aligned}$$

In the same way, we get

$$d\xi = i dy_0 + (q\varepsilon) dy_1 + y_1 (dq)\varepsilon.$$

Therefore

$$\sqrt{-1}\theta^1 = -\frac{\sqrt{-1}}{2} y_1^2 \langle \bar{q} dq, j\rangle,$$

$$\sqrt{-1}\theta^2 = -\frac{1}{2}y_1\langle \bar{q}dq, i\rangle - \frac{\sqrt{-1}}{2}(-y_1dy_0 + y_0dy_1),$$

$$\sqrt{-1}\theta^3 = \frac{1}{2}y_1\langle \bar{q}dq, k\rangle - \frac{\sqrt{-1}}{2}y_0y_1\langle \bar{q}dq, j\rangle.$$

Hence, we have

$$\sqrt{-1}\theta = -\frac{1}{4}\left(\begin{array}{ccc|ccc} y_1^2 & 0 & y_0y_1 & -y_1^2 & 0 & -y_0y_1 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ y_0y_1 & 0 & 1+y_0^2 & -y_0y_1 & 0 & y_1^2 \end{array}\right)\begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}.$$

We obtain lastly

$$\mathfrak{B} = -\frac{1}{4}\begin{pmatrix} y_1^2 & 0 & y_0y_1 \\ 0 & 2 & 0 \\ y_0y_1 & 0 & 1+y_0^2 \end{pmatrix}, \quad \mathfrak{A} = -\frac{1}{4}\begin{pmatrix} -y_1^2 & 0 & -y_0y_1 \\ 0 & 0 & 0 \\ -y_0y_1 & 0 & y_1^2 \end{pmatrix}. \qquad \Box$$

From above arguments and the results, the induced almost complex structure is not homogeneous. Next, we shall prove

Proposition 4.6 *The induced almost complex structure on $\boldsymbol{R}^2 \times S^4$ is unique up to the action of G_2.*

Proof. Let φ_0 be the fixed immersion from $\boldsymbol{R}^2 \times S^4$ to $\operatorname{Im}\mathfrak{C}$ by

$$\varphi_0(u_1, u_2, v_0, \dots, v_4) = iu_1 + ju_2 + kv_0 + \cdots + k\varepsilon v_4,$$

where $(u_1, u_2) \in \boldsymbol{R}^2$ and $\sum_{i=0}^4 v_i^2 = 1$. Next we take an isometric immersion φ from $\boldsymbol{R}^2 \times S^4$ to $\boldsymbol{R}^7$. Then there exists an orthonormal basis $(e_1\ e_2\ e_3\ \dots\ e_7)$ of $\boldsymbol{R}^7$ such that

$$\varphi(x_1, x_2, y_0, \dots, y_4) = e_1x_1 + e_2x_2 + e_3y_0 + \cdots + e_7y_4,$$

where $(x_1, x_2) \in \boldsymbol{R}^2$ and $\sum_{i=0}^4 y_i^2 = 1$. By Proposition 3.1, there exists a $g \in G_2$ satisfying

$$g(i) = e_1, \quad g(j) = e_2.$$

Also, we have

$$span_{\boldsymbol{R}}\{g(k),\ldots,g(k\varepsilon)\} = span_{\boldsymbol{R}}\{e_3,\ldots,e_7\}.$$

Therefore, there exists an $A \in SO(5)$ such that

$$(g(k),\ldots,g(k\varepsilon)) = (e_3,\ldots,e_7)A.$$

We define the diffeommorphism ψ of $\boldsymbol{R}^2 \times S^4$ as follows

$$\psi(u_1,u_2,v_0,\ldots,v_4) = (u_1,u_2,(v_0,\ldots,v_4)^t A).$$

Then we have

$$g(\varphi_0(u_1,u_2,v_0,\ldots,v_4)) = \varphi(\psi(u_1,u_2,v_0,\ldots,v_4)).$$

Therefore the induce almost complex structure of φ_0 coincides with that of φ. □

4.3. 1-parameter family of homogeneous almost complex structures on $S^2 \times \mathbf{R}^4$

4.3.1 $S^2 \times \mathbf{R}^4$

In this section, we give the explicit representation of G_2-frame fields on $S^2 \times \mathbf{R}^4 \subset \mathrm{Im}\mathfrak{C}$, and the G_2-invariants. Let $q \in S^3(\subset \mathbf{H})$ be the unit quaternion. We define the map $\pi : S^3 \to S^2$ such that $\pi(q) = qi\bar{q}$, which is called the Hopf map.

Proposition 4.7 *Let $\varphi_{2,\alpha}$ be the 1-parameter family of imbeddings from $S^2 \times \mathbf{R}^4$ to $Im\mathfrak{C}$, as follows*

$$\begin{aligned}\varphi_{2,\alpha}(qi\bar{q},\tilde{y}) &= \cos(\alpha)qi\bar{q} + \sin(\alpha)(qi\bar{q})\varepsilon + y_0\varepsilon + y_1(-\sin(\alpha)i + \cos(\alpha)i\varepsilon)\\ &\quad + y_2(-\sin(\alpha)j + \cos(\alpha)j\varepsilon) + y_3(-\sin(\alpha)k + \cos(\alpha)k\varepsilon).\end{aligned} \tag{4.1}$$

where $qi\bar{q} \in S^2$ and $\tilde{y} = (y_0,y_1,y_2,y_3) \in \mathbf{R}^4$, for some fixed $\alpha \in [0,\pi/3]$. Then, we have

$$tr({}^t\overline{\mathfrak{B}}\mathfrak{B}) = \frac{1}{8}(1+\cos^2(3\alpha)), \quad tr({}^t\overline{\mathfrak{A}}\mathfrak{A}) = \frac{1}{8}(1-\cos^2(3\alpha)).$$

The automorphism group of the induced almost Hermitian structure coincides with $SU(2) \ltimes \boldsymbol{R}^4(\subset SO(3) \times (SO(4) \ltimes \boldsymbol{R}^4))$ and it acts transitively on

$S^2 \times \boldsymbol{R}^4$ *for any* $\alpha \in [0, \pi/3]$.

From which, we have

Theorem 4.2 *For* $\alpha \in \mathbf{R}$ $(0 \leq \alpha \leq \pi/3)$, *let* $(S^2 \times \mathbf{R}^4, \varphi_{2,\alpha})$ *be defined as in Proposition* 4.7. *The family of the imbeddings* $\varphi_{2,\alpha}$ *induce the 1-parameter family of the almost complex structures* J_α *on* $S^2 \times \mathbf{R}^4$, *which are not* G_2-*congruent to each other. Moreover the induced almost Hermitian structure* $(J_\alpha, \langle\ ,\ \rangle)$ *is* $(1,2)$*-symplectic iff* $\alpha = 0$ *or* $\pi/3$.

We here note that $\varphi_{2,\alpha}$ and $\varphi_{2,\alpha+\pi/3}$ are G_2-congruent. The almost Hermitian manifold $(M, J, \langle\ ,\ \rangle)$ is said to be *(1,2)-symplectic* if $(d\omega)^{(1,2)} = 0$, where $\omega = \langle J,\ \rangle$ is the canonical 2-form (or Kähler form) on M. In our situation, $(d\omega)^{(1,2)} = 0$, is equivalent to $\mathfrak{A} = 0$.

Proof. First we note that the imbeddings are equivariant in the following sense. Let $\rho_{III} : Sp(1) \to G_2$ be the representation of the Lie subgroup $Sp(1)$ of G_2, which is defined by

$$\rho_{III}(q)(a + b\varepsilon) = qa\overline{q} + (qb\overline{q})\varepsilon, \tag{4.2}$$

where $a, b \in \mathbf{H}$ (see [7]). In fact, we see that ρ_{III} satisfies

$$\rho_{III}(q)(a + b\varepsilon)\rho_{III}(q)(c + d\varepsilon) = \rho_{III}(q)\big(ac - \bar{d}b + (da + b\bar{c})\varepsilon\big),$$

for any $a, b, c, d \in \mathbf{H}$. From (4.1) and (4.2), it follows immediately that the imbedding $\varphi_{2,\alpha}$ is rewritten as

$$\begin{aligned}\varphi_{2,\alpha}(qi\overline{q}, \tilde{y}) = \rho_{III}(q)(\cos(\alpha)i + \sin(\alpha)i\varepsilon) + y_0\varepsilon + y_1(-\sin(\alpha)i + \cos(\alpha)i\varepsilon)\\ + y_2(-\sin(\alpha)j + \cos(\alpha)j\varepsilon) + y_3(-\sin(\alpha)k + \cos(\alpha)k\varepsilon).\end{aligned} \tag{4.3}$$

Therefore, we see that the imbeddings are equivariant and the induced almost Hermitian structures are homogeneous for all $\alpha \in [0, \pi/3]$. In fact, we define the G_2-frame field by

$$\xi = \{\rho_{III}(q)(\cos(\alpha)i + \sin(\alpha)i\varepsilon)\},$$

$$f_1 = \frac{1}{2}\big\{\rho_{III}(q)(-\sin(\alpha)i + \cos(\alpha)i\varepsilon - \sqrt{-1}(\varepsilon))\big\},$$

$$f_2 = \frac{1}{2}\{\rho_{III}(q)(j - \sqrt{-1}(-\cos(\alpha)k + \sin(\alpha)k\varepsilon))\},$$

$$f_3 = -\frac{1}{2}\{\rho_{III}(q)(-\sin(\alpha)k - \cos(\alpha)k\varepsilon - \sqrt{-1}j\varepsilon)\}.$$

Then we see that $(f_1,\ f_2,\ f_3)$ is a $SU(3)-$frame field on $\varphi_{2,\alpha}(S^2 \times \mathbf{R}^4)$.

To calculate the G_2 invariants, we define the local 1-forms μ_1, μ_2 on S^2 by

$$\mu_1 = \langle d(qi\overline{q}), qj\overline{q}\rangle, \quad \mu_2 = \langle d(qi\overline{q}), qk\overline{q}\rangle.$$

Then, we obtain

$$\begin{aligned}\omega^1 &= dy_1 - \sqrt{-1}dy_0,\\ \omega^2 &= \cos(\alpha)\mu_1 - \sin(\alpha)dy_2 + \sqrt{-1}(-\cos(2\alpha)\mu_2 + \sin(2\alpha)dy_3),\\ \omega^3 &= \sin(2\alpha)\mu_2 + \cos(2\alpha)dy_3 - \sqrt{-1}(\sin(\alpha)\mu_1 + \cos(\alpha)dy_2),\end{aligned}$$

at $q = 1$. Since

$$d\xi = \cos(\alpha)(j \otimes \mu_1 + k \otimes \mu_2) + \sin(\alpha)(j\varepsilon \otimes \mu_1 + k\varepsilon \otimes \mu_2),$$

at $q = 1$. Hence we have

$$\begin{aligned}\mathfrak{B} &= -\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos^2(\alpha) + \cos^2(2\alpha) & \frac{\sqrt{-1}}{2}(\sin(2\alpha) - \sin(4\alpha)) \\ 0 & -\frac{\sqrt{-1}}{2}(\sin(2\alpha) - \sin(4\alpha)) & \sin^2(\alpha) + \sin^2(2\alpha) \end{pmatrix},\\ \mathfrak{A} &= -\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos^2(\alpha) - \cos^2(2\alpha) & -\frac{\sqrt{-1}}{2}(\sin(2\alpha) + \sin(4\alpha)) \\ 0 & -\frac{\sqrt{-1}}{2}(\sin(2\alpha) + \sin(4\alpha)) & -\sin^2(\alpha) + \sin^2(2\alpha) \end{pmatrix}.\end{aligned} \tag{4.4}$$

Therefore from (4.4), we get the G_2 invariants on $S^2 \times \mathbf{R}^4$ given by

$$tr({}^t\overline{\mathfrak{B}}\mathfrak{B}) = \frac{1}{8}(1 + \cos^2(3\alpha)), \quad tr({}^t\overline{\mathfrak{A}}\mathfrak{A}) = \frac{1}{8}(1 - \cos^2(3\alpha)). \qquad \square$$

Proposition 4.8 *Let φ be any isometric imbedding from $S^2 \times \mathbf{R}^4$ to $Im\mathfrak{C}$. Then there exist a $g \in G_2$ and $\alpha \in [0, \pi/3]$ such that $g \circ \varphi = \varphi_{2,\alpha}$. Hence the* *moduli space (up to the action of G_2) of isometric imbedddings from $S^2\times\mathbf{R}^4$ to Im$\mathfrak{C}$ coincides with $\{\varphi_{2,\alpha}|\alpha\in[0,\pi/3]\}$.*

Proof. If S^2 is included in an associative 3-plane, then the imbedding from $S^2\times\mathbf{R}^4$ to Im$\mathfrak{C}$ is G_2-congruent to $\varphi_0(S^2\times\mathbf{R}^4)$. By Proposition 3.2, we may assume that S^2 is included in the 3-dimensional vector space

$$\mathrm{span}_{\mathbf{R}}\{g(i),\ g(j),\ g(\cos\theta k+\sin\theta\varepsilon)\},$$

for some $\theta\in[0,\pi/2]$. By changing the basis of the 3-dimensional subspace in Im$\mathfrak{C}$ suitably, we may assume that

$$S^2\subset\mathrm{span}_{\mathbf{R}}\{\cos\alpha i+\sin\alpha i\varepsilon,\ \ \cos\alpha j+\sin\alpha j\varepsilon,\ \ \cos\alpha k+\sin\alpha k\varepsilon\},$$

for some $\alpha\in[0,\pi/3]$. Hence we get the desired result. □

4.4. Deformation of almost complex structures on $S^3\times R^3$

4.4.1 $S^3\times R^3$

The purpose of this section is to prove the following

Theorem 4.3 *Let $\varphi_{3,\alpha}:S^3\times\boldsymbol{R}^3\to\mathrm{Im}\,\mathfrak{C}$ be a 1-parameter family of imbeddings defined by*

$$\begin{aligned}&\varphi_{3,\alpha}(q_0,q_1,q_2,q_3,x_1,x_2,x_3)\\&\quad=x_1(\cos\alpha i+\sin\alpha\varepsilon)+x_2j+x_3k+q_0(-\sin\alpha i+\cos\alpha\varepsilon)+\mathfrak{q}\varepsilon,\end{aligned}$$

where $\mathfrak{q}=q_1i+q_2j+q_3k$, $\sum_{i=0}^{3}{q_i}^2=1$, $(x_1,x_2,x_3)\in\boldsymbol{R}^3$ and $\alpha(0\le\alpha\le\pi/2)$ is a parameter of the deformation. Then we have

$$\begin{aligned}tr({}^t\bar{\mathfrak{B}}\mathfrak{B})&=\frac{1}{16}\big(2(1-{q_1}^2)\sin^2\alpha+3\big),\\tr({}^t\bar{\mathfrak{A}}\mathfrak{A})&=\frac{1}{16}\big(-2(1-{q_1}^2)\sin^2\alpha+3\big).\end{aligned}$$

From which, we can easily see that

Corollary 4.1 *There exists a 1-parameter family of induced almost complex structures J_α on $S^3\times\boldsymbol{R}^3$, for any α $(0\le\alpha\le\pi/2)$, which are not G_2 equivalent. Moreover, the induced almost complex structures J_α $(0<\alpha\le\pi/2)$ are not homogeneous.*

Proof. First we construct the G_2-frame field on $\varphi_{3,\alpha}(S^3 \times \boldsymbol{R}^3)$. We put $\mu = q_0 \cos\alpha + \mathfrak{q}$. Moreover, we set $e_4 = \xi = -q_0 \sin\alpha i + (q_0 \cos\alpha + \mathfrak{q})\varepsilon = -q_0 \sin\alpha i + \mu\varepsilon$, and we take $e_1 = j$, and put $e_5 = e_1 e_4 = q_0 \sin\alpha k + (\mu j)\varepsilon$. Next, we set $e_2 = \frac{1}{A}(\mu i)\varepsilon$, where $A = \sqrt{1 - {q_0}^2 \sin^2\alpha}$. Then e_2 is orthogonal to the associated 3-plane $span_{\boldsymbol{R}}\{e_1, e_4, e_5\}$. Also we put $\{e_3, e_6, e_7\}$ as

$$e_3 = e_1 e_2 = \frac{1}{A}(\mu k)\varepsilon, \quad e_6 = e_2 e_4 = -\frac{1}{A}\big(A^2 i + q_0 \sin\alpha \mu\varepsilon\big),$$
$$e_7 = e_3 e_4 = -\frac{1}{A}\big(A^2 k - q_0 \sin\alpha(\mu j)\varepsilon\big),$$

then we obtain the G_2-frame field $\{e_1, e_2, \ldots, e_7\}$. We now set

$$\begin{cases} f_1 = \dfrac{1}{2}\big(j - \sqrt{-1}(q_0 \sin\alpha k + (\mu j)\varepsilon)\big), \\ f_2 = \dfrac{1}{2A}\big\{(\mu i)\varepsilon + \sqrt{-1}(A^2 i + q_0 \sin\alpha \mu\varepsilon\big\}, \\ f_3 = -\dfrac{1}{2A}\big\{(\mu k)\varepsilon + \sqrt{-1}(A^2 k - q_0 \sin\alpha(\mu j)\varepsilon)\big\}. \end{cases}$$

We calculate the second fundamental forms of $\varphi_{3,\alpha}$. Since we have

$$\begin{aligned} d\varphi_{3,\alpha} &= (\cos\alpha i + \sin\alpha\varepsilon)dx_0 + j dx_1 + k dx_2 \\ &\qquad + (-\sin\alpha i + \cos\alpha\varepsilon)dq_0 + (d\mathfrak{q})\varepsilon, \end{aligned}$$

we get

$$\begin{aligned} \omega^1 &= dx_1 - \sqrt{-1}\big(\sin\alpha(q_2 dx_0 - q_0 dx_2) + \cos\alpha(q_2 dq_0 - q_0 dq_2) - \langle \bar{\mathfrak{q}} d\mathfrak{q}, j\rangle\big), \\ \omega^2 &= -\frac{1}{A}\big\{\big(q_1 \sin\alpha dx_0 + \cos\alpha(q_1 dq_0 - q_0 dq_1) - \langle \bar{\mathfrak{q}} d\mathfrak{q}, i\rangle\big) \\ &\qquad + \sqrt{-1}\big(\cos\alpha dx_0 - \sin\alpha(|\mathfrak{q}|^2 dq_0 - q_0\langle \bar{\mathfrak{q}} d\mathfrak{q}, 1\rangle)\big)\big\}, \\ \omega^3 &= \frac{1}{A}\big\{\big(q_3 \sin\alpha dx_0 + \cos\alpha(q_3 dq_0 - q_0 dq_3) - \langle \bar{\mathfrak{q}} d\mathfrak{q}, k\rangle\big) \\ &\qquad + \sqrt{-1}\big(q_0 q_2 \sin^2\alpha dx_0 + A^2 dx_2 \\ &\qquad\qquad + q_0 \sin\alpha(\cos\alpha(q_2 dq_0 - q_0 dq_2) - \langle \bar{\mathfrak{q}} d\mathfrak{q}, j\rangle\big)\big\}. \end{aligned}$$

On the other hand, we take exterior derivative of the unit normal vector field ξ, then we get

$$d\xi = (-\sin\alpha i + \cos\alpha\varepsilon)dq_0 + (d\mathfrak{q})\varepsilon.$$

Therefore, we have

$$\begin{aligned}
\sqrt{-1}\theta^1 &= \frac{\sqrt{-1}}{2}\big(\cos\alpha(q_2 dq_0 - q_0 dq_2) - \langle\bar{\mathfrak{q}}d\mathfrak{q}, j\rangle\big),\\
\sqrt{-1}\theta^2 &= \frac{1}{2A}\big\{\big(\cos\alpha(q_1 dq_0 - q_0 dq_1) - \langle\bar{\mathfrak{q}}d\mathfrak{q}, i\rangle\big)\\
&\qquad - \sqrt{-1}\big(\sin\alpha(|\mathfrak{q}|^2 dq_0 - q_0\langle\bar{\mathfrak{q}}d\mathfrak{q}, 1\rangle)\big)\big\},\\
\sqrt{-1}\theta^3 &= -\frac{1}{2A}\big\{\big(\cos\alpha(q_3 dq_0 - q_0 dq_3) - \langle\bar{\mathfrak{q}}d\mathfrak{q}, k\rangle\big)\\
&\qquad + \sqrt{-1}\big(q_0\sin\alpha(\cos\alpha(q_2 dq_0 - q_0 dq_2) - \langle\bar{\mathfrak{q}}d\mathfrak{q}, j\rangle\big)\big\}.
\end{aligned}$$

Hence we have

$$\sqrt{-1}\theta^1 = -\frac{1}{2}\big\{\omega^1 - dx_1 + \sqrt{-1}\sin\alpha(q_2 dx_0 - q_0 dx_2)\big\}, \tag{4.5}$$

$$\sqrt{-1}\theta^2 = -\frac{1}{2}\left\{\omega^2 + \frac{1}{A}(q_1\sin\alpha + \sqrt{-1}\cos\alpha)dx_0\right\}, \tag{4.6}$$

$$\sqrt{-1}\theta^3 = -\frac{1}{2}\left\{\omega^3 - \frac{1}{A}\big(\sin\alpha(q_3 + \sqrt{-1}q_0 q_2\sin\alpha)dx_0 + \sqrt{-1}A^2 dx_2\big)\right\}. \tag{4.7}$$

Now, we want to know the (local complexified) vector fields $\{v_1, v_2, v_3\}$ on $S^3 \times \boldsymbol{R}^3$, which satisfy $\varphi_{3,\alpha_*}(v_i) = f_i$ $(i = 1, 2, 3)$. We set

$$\begin{aligned}
&E_1 = \left(\frac{\partial}{\partial x_0}\right)_p, \quad E_2 = \left(\frac{\partial}{\partial x_1}\right)_p, \quad E_3 = \left(\frac{\partial}{\partial x_2}\right)_p,\\
&E_4 = (q_0 + \mathfrak{q})i, \quad E_5 = (q_0 + \mathfrak{q})j, \quad E_6 = (q_0 + \mathfrak{q})k.
\end{aligned}$$

The tangent space $T_p(S^3 \times \boldsymbol{R}^3)$ at $p \in S^3 \times \boldsymbol{R}^3$ is given by

$$T_p(S^3 \times \boldsymbol{R}^3) = \mathrm{span}_{\boldsymbol{R}}\big\{E_1,\ E_2,\ E_3,\ E_4,\ E_5,\ E_6\big\}.$$

The elements of the image $\varphi_{3,\alpha_*}(T_p(S^3 \times \boldsymbol{R}^3))$ are given by

$$\varphi_{3,\alpha_*}\left(\frac{\partial}{\partial x_0}\right) = \cos\alpha i + \sin\alpha\varepsilon, \quad \varphi_{3,\alpha_*}\left(\frac{\partial}{\partial x_1}\right) = j, \quad \varphi_{3,\alpha_*}\left(\frac{\partial}{\partial x_2}\right) = k,$$

$$\begin{aligned}\varphi_{3,\alpha_*}((q_0+\mathfrak{q})i) &= \frac{d}{d\theta}\big(\varphi_{3,\alpha}(\cos\theta(q_0+\mathfrak{q}) + \sin\theta(q_0+\mathfrak{q})i)\big)\big|_{\theta=0} \\ &= q_1 \sin\alpha i + (q_1(1-\cos\alpha) + (q_0+\mathfrak{q})i)\varepsilon.\end{aligned}$$

In the same way

$$\begin{aligned}\varphi_{3,\alpha_*}((q_0+\mathfrak{q})j) &= q_2 \sin\alpha i + (q_2(1-\cos\alpha) + (q_0+\mathfrak{q})j)\varepsilon, \\ \varphi_{3,\alpha_*}((q_0+\mathfrak{q})k) &= q_3 \sin\alpha i + (q_3(1-\cos\alpha) + (q_0+\mathfrak{q})k)\varepsilon.\end{aligned}$$

Since $\langle \varphi_{3,\alpha_*}(E_i), \varphi_{3,\alpha_*}(E_j)\rangle = \delta_{ij}$, we have

$$\begin{aligned}\left\langle \varphi_{3,\alpha_*}(v_1), \varphi_{3,\alpha_*}\left(\frac{\partial}{\partial x_0}\right)\right\rangle &= -\frac{\sqrt{-1}}{2}\langle \mu j, 1\rangle \sin\alpha = \frac{\sqrt{-1}}{2} q_2 \sin\alpha, \\ \left\langle \varphi_{3,\alpha_*}(v_1), \varphi_{3,\alpha_*}\left(\frac{\partial}{\partial x_1}\right)\right\rangle &= \frac{1}{2}, \\ \left\langle \varphi_{3,\alpha_*}(v_1), \varphi_{3,\alpha_*}\left(\frac{\partial}{\partial x_2}\right)\right\rangle &= -\frac{\sqrt{-1}}{2} q_0 \sin\alpha.\end{aligned}$$

Therefore we obtain

$$f_1 = \varphi_{3,\alpha_*}(v_1) = \varphi_{3,\alpha_*}\left(\frac{\sqrt{-1}}{2} q_2 \sin\alpha \frac{\partial}{\partial x_0} + \frac{1}{2}\frac{\partial}{\partial x_1} - \frac{\sqrt{-1}}{2} q_0 \sin\alpha \frac{\partial}{\partial x_2} + \tilde{v_1}\right),$$

where $\tilde{v_1}$ is a some (complexfied) vector filed on S^3. Hence

$$v_1 = \frac{\sqrt{-1}}{2} q_2 \sin\alpha \frac{\partial}{\partial x_0} + \frac{1}{2}\frac{\partial}{\partial x_1} - \frac{\sqrt{-1}}{2} q_0 \sin\alpha \frac{\partial}{\partial x_2} + \tilde{v_1}. \tag{4.8}$$

In the same way, we get

$$v_2 = -\frac{1}{2A}\big(q_1 \sin\alpha - \sqrt{-1}\cos\alpha\big)\frac{\partial}{\partial x_0} + \tilde{v_2}, \tag{4.9}$$

$$v_3 = \frac{1}{2A}\big(q_3 \sin\alpha - \sqrt{-1}q_0 q_2 \sin^2\alpha\big)\frac{\partial}{\partial x_0} - \frac{\sqrt{-1}}{2}a\frac{\partial}{\partial x_2} + \tilde{v_3}, \tag{4.10}$$

where $\tilde{v_2}$, $\tilde{v_3}$ are some (complexified) vector fields on S^3. Since $\omega^i(v_j) = \delta^i_j$, and, from (3.1), (4.8), (4.9), (4.10), we obtain

$$\begin{aligned}
\mathfrak{B}^1_1 &= \sqrt{-1}\theta^1(v_1) = -\frac{1}{2}\{\omega^1 - dx_1 + \sqrt{-1}\sin\alpha(q_2 dx_0 - q_0 dx_2)\} \\
&\qquad \times \left(\frac{\sqrt{-1}}{2}q_2 \sin\alpha\frac{\partial}{\partial x_0} + \frac{1}{2}\frac{\partial}{\partial x_1} - \frac{\sqrt{-1}}{2}q_0 \sin\alpha\frac{\partial}{\partial x_2} + \tilde{v_1}\right) \\
&= \frac{1}{4}\big(1 - ({q_0}^2 + {q_2}^2)\sin^2\alpha\big), \\
\mathfrak{B}^2_2 &= \sqrt{-1}\theta^2(v_2) = -\frac{1}{2}\left\{1 - \frac{1}{2A^2}(({q_1}^2 - 1)\sin^2\alpha + 1)\right\}, \\
\mathfrak{B}^3_3 &= \sqrt{-1}\theta^3(v_3) = -\frac{1}{2}\left\{1 - \frac{A^2}{2} - \frac{\sin^2\alpha}{2A^2}({q_3}^2 + {q_0}^2{q_2}^2\sin^2\alpha)\right\}, \\
\mathfrak{B}^2_1 &= \sqrt{-1}\theta^1(v_2) = \frac{q_2\sin\alpha}{4A}\big(\cos\alpha + \sqrt{-1}q_1\sin\alpha\big), \\
\mathfrak{B}^3_1 &= \sqrt{-1}\theta^1(v_3) = -\frac{\sin\alpha}{4A}\{q_0(({q_0}^2 + {q_2}^2)\sin^2\alpha - 1) + \sqrt{-1}q_2 q_3\sin^2\alpha\}, \\
\mathfrak{B}^3_2 &= \sqrt{-1}\theta^2(v_3) = -\frac{\sin\alpha}{4A^2}\{\sin\alpha(q_1 q_3 + q_0 q_2\cos\alpha) \\
&\qquad + \sqrt{-1}(q_3\cos\alpha - q_0 q_1 q_2\sin^2\alpha))\}.
\end{aligned}$$

If we put $X = \sin^2\alpha$, then we have

$$\begin{aligned}
16A^4|\mathfrak{B}^1_1| &= (1 - {q_0}^2 X)\{1 - ({q_0}^2 + {q_2}^2)X\}^2 \\
&= X^4\{{q_0}^4({q_0}^2 + {q_2}^2)\} + X^3\{-2{q_0}^2({q_0}^2 + {q_2}^2)(2{q_0}^2 + {q_2}^2)\} \\
&\quad + X^2\{6{q_0}^2({q_0}^2 + {q_2}^2) + {q_2}^4\} + X\{-2(2{q_0}^2 + {q_2}^2)\} + 1, \\
16A^4|\mathfrak{B}^2_2| &= \{-(2{q_0}^2 + ({q_1}^2 - 1)) + 1\}^2 \\
&= X^2\{2{q_0}^2 + ({q_1}^2 - 1)\} + X\{-2(2{q_0}^2 + ({q_1}^2 - 1))\} + 1,
\end{aligned}$$

$$\begin{aligned}
16A^4|\mathfrak{B}_3^3| &= \big\{-{q_0}^2({q_0}^2+{q_2}^2)X^2-{q_3}^2X+1\big\}^2\\
&= X^4\big\{{q_0}^4({q_0}^2+{q_2}^2)^2\big\}+X^3\big\{2{q_0}^2{q_3}^2({q_0}^2+{q_2}^2)\big\}\\
&\quad +X^2\big\{-2{q_0}^2({q_0}^2+{q_2}^2)+{q_3}^4\big\}+X\big\{-2{q_3}^2\big\}+1,\\
32A^4|\mathfrak{B}_1^2| &= 2{q_2}^2X(1-{q_0}^2X)\{({q_1}^2-1)X+1\}\\
&= X^3\big\{-2{q_0}^2{q_2}^2({q_1}^2-1)\big\}+\big\{2{q_2}^2(-{q_0}^2+({q_1}^2-1))\big\}\\
&\quad +X\big\{2{q_2}^2\big\},\\
32A^4|\mathfrak{B}_1^3| &= 2X(1-{q_0}^2X)\big\{{q_0}^2\big(({q_0}^2+{q_2}^2)X-1\big)^2+{q_2}^2{q_3}^2X\big\}\\
&= X^4\big\{-2{q_0}^4({q_0}^2+{q_2}^2)^2\big\}\\
&\quad +X^3\big\{2{q_0}^2\big(({q_0}^2+{q_2}^2)(3{q_0}^2+{q_2}^2)^2-{q_2}^2{q_3}^2\big)\big\}\\
&\quad +X^2\big\{2\big(-{q_0}^2({q_0}^2+2({q_0}^2+{q_2}^2))+{q_2}^2{q_3}^2\big)\big\}+X\big\{2{q_0}^2\big\},\\
32A^4|\mathfrak{B}_2^3| &= 2X\big\{X(q_1q_3+q_0q_2\cos\alpha)^2+(q_3\cos\alpha-q_0q_1q_2X)^2\big\}\\
&= X^3\big\{2{q_0}^2{q_2}^2({q_1}^2-1)\big\}+X^2\big\{2({q_3}^2({q_1}^2-1)+{q_0}^2{q_2}^2)\big\}\\
&\quad +X\big\{2{q_3}^2\big\}.
\end{aligned}$$

Hence

$$\begin{aligned}
tr({}^t\bar{\mathfrak{B}}\mathfrak{B}) &= \frac{1}{16A^4}\big\{X^3\big(2{q_0}^4(1-{q_1}^2)\big)+X^2\big({q_0}^2(3{q_0}^2+4({q_1}^2-1))\big)\\
&\qquad +X\big(-2(3{q_0}^2+{q_1}^2-1)\big)+3\big\}\\
&= \frac{1}{16}\big(2(1-{q_1}^2)X+3\big).
\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}
tr({}^t\bar{\mathfrak{A}}\mathfrak{A}) &= \frac{1}{16A^4}\big\{X^3\big(-2{q_0}^4(1-{q_1}^2)\big)+X^2\big(4{q_0}^2(1-{q_1}^2)+3{q_0}^4\big)\\
&\qquad +X\big(-2(3{q_0}^2+(1-{q_1}^2))\big)+3\big\}\\
&= \frac{1}{16}\big(-2(1-{q_1}^2)X+3\big).
\end{aligned}$$

□

Proposition 4.9 *Let φ be any homogeneous isometric imbedding from $S^3\times\mathbf{R}^3$ to $Im\mathfrak{C}$. Then there exist a $g\in G_2$ and $\alpha\in[0,\pi]$ such that*

$$g \circ \varphi = \varphi_\alpha.$$

Proof. We fix the immersion φ_0 from $S^3 \times \boldsymbol{R}^3$ to $\mathrm{Im}\,\mathfrak{C}$, as

$$\varphi_0(u_0, \ldots, u_3, v_1, v_2, v_3) = u_0 i + \cdots + u_3 \varepsilon + v_1 i\varepsilon + v_2 j\varepsilon + v_3 k\varepsilon,$$

where $\sum_{i=0}^3 u_i^2 = 1$ and $(v_1, v_2, v_3) \in \boldsymbol{R}^3$.

Let $\varphi : S^3 \times \boldsymbol{R}^3 \to \boldsymbol{R}^7$ be an arbitrary immersion. Then there exists an orthonormal frame $\{e_1, e_2, e_3, \ldots, e_7\}$ of $\boldsymbol{R}^7$ satisfying

$$\varphi_0(x_0, \ldots, x_3, y_1, y_2, y_3) = x_0 e_1 + \cdots + x_3 e_4 + y_1 e_5 + y_2 e_6 + y_3 e_7,$$

where $\sum_{i=0}^3 x_i^2 = 1$ and $(y_1, y_2, y_3) \in \boldsymbol{R}^3$. If we set $V = span_{\boldsymbol{R}}\{e_5, e_6, e_7\}$, then by Proposition 3.2, we have

$$V = span_{\boldsymbol{R}}\{g(i),\ g(j),\ g(\cos\theta k + \sin\theta\varepsilon)\}.$$

In the same argument of the proof of Proposition 4.8, we get the desired result. □

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