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The global existence theorem for quasi-linear wave equations with multiple speeds

Kunio HIDANO

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Abstract. The Cauchy problem is studied for systems of quasi-linear wave equations with multiple speeds. We pursue the extension of the excellent method of Klainerman and Sideris to its limit, and a unified proof is given to previous results of Agemi-Yokoyama, Hoshiga-Kubo, Kovalyov, and Yokoyama.

Key words: global existence, quasi-linear wave equations, non-resonance.

1. Introduction

The well-known commuting vector fields method of John and Klainerman has brought remarkable progress in the study of large-time existence of small amplitude solutions to quasi-linear wave equations. Its feature lies in the fact that only the energy integral argument together with the Klainerman inequality [20], [22] suffices to prove almost global existence for perturbed classical wave equations if quasi-linear perturbation terms are quadratic in three space dimensions or cubic in two space dimensions. The Klainerman inequality contains the boost operators which are members of the generators of Lorentz rotations in the Minkowski space. In the analysis of elastic equations as well as systems of multiple-speed wave equations, a lack of the Lorentz invariance used to be compensated for by direct estimates of the fundamental solution to obtain some L^∞-decay estimates in the late 1980's [13], [24]. In the middle of 1990's Klainerman and Sideris threw a new light on the difficulty of lack of the Lorentz invariance. They have successfully overcome it, and proved almost global existence for quadratic quasi-linear wave equations in three space dimensions without relying on direct estimates of the fundamental solution [23]. In their analysis Klainerman and Sideris made use of only the invariance of the D'Alembertian operator under Euclid rotations, space-time scaling in addition to space-time translations. Though the spatial divergence form is assumed in the

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nonlinearity, the equation of nonlinear elasticity just fits in and Klainerman and Sideris have succeeded in obtaining the same almost global existence theorem as in John [13], without troublesome estimates of the fundamental solution. The global existence has been subsequently established by Sideris [28], [30] in line with the enterprise of Klainerman and Sideris [23], and by Agemi [1] in line with the thought of John [13], under what they called the null condition, which reminds us of the null condition of Christodoulou [5] and Klainerman [21] for relativistic wave equations.

From the point of view of non-resonance phenomena of the wave propagation with different speeds, the analysis of systems of nonlinear wave equations with multiple speeds has also attracted much attention. And this is actually the main problem which the present paper treats. Because of the nature of multiple speeds, a lack of the Lorentz invariance occurs, and this kind of difficulty was first overcome by Kovalyov who employed direct estimates of the fundamental solution to get some L^∞-time decay estimates. In [24] Kovalyov proved global existence of small amplitude solutions to a system of multiple-speed wave equations with some quadratic semi-linear terms in three space dimensions or cubic semi-linear terms in two space dimensions. In Proposition 2.1 of [2] Agemi and Yokoyama have introduced some non-resonance conditions to systems of multiple-speed, cubic quasi-linear wave equations in two space dimensions with the help of what they called "John-Shatah's observation" (see [14]). A decisive result has been obtained by Hoshiga and Kubo [10], who have considered full systems of cubic, quasi-linear equations with semi-linear terms and proved global existence of small solutions in two space dimensions under the non-resonance conditions found by Agemi and Yokoyama. As for full systems of multiple-speed, quadratic quasi-linear equations in three space dimensions, Yokoyama has shown a prominent global existence theorem under non-resonance conditions [33]. It may be safe to say that all the proofs of Agemi and Yokoyama [2], Hoshiga and Kubo [10], and Yokoyama [33] lie in the same line as in Kovalyov [24] where the energy integral argument was carried out with the help of some L^∞ decay estimates which follow from direct estimates of the fundamental solution.

Interestingly enough from the point of view of technical innovations, Sideris and Tu have recently applied the techniques of Klainerman and Sideris [23], which have been refined significantly by Sideris [29], [30], to the problem of wave propagations with different speeds [31]. Sideris and

Tu have found an essentially simplified proof of the outstanding theorem of Yokoyama [33], when the semi-linear terms are completely omitted. In this paper we revisit the paper of Sideris and Tu to aim at getting completely the same result as in Yokoyama [33].[1] Our another purpose is to explain how to prove the same result as in Hoshiga and Kubo [10] by expanding the enterprise of Klainerman and Sideris. To get things straight, we pursue the extension of the excellent method of Klainerman and Sideris to its limit and give a unified proof of the previous results [2], [10], [24] and [33], without relying on direct estimates of the fundamental solution. In the sequels [8]-[9] the Klainerman-Sideris method will be employed to show a simple proof of the global or almost global existence theorem of Klainerman [20], and the method will be combined with space-time L^2-estimates of Keel, Smith and Sogge [17] to give an essentially new proof of global solvability of quadratic, semi-linear equations in three space dimensions.

Though we mostly follow the thought of Klainerman and Sideris [23], Sideris and Tu [31] and rely on their tools, further efforts are demanded to treat semi-linear terms such as $H(u, u)$ in (3.3), (3.4). This is because Sobolev-type inequalities in [31] and [30] are not sufficient to get desirable time decay estimates of the semi-linear terms, especially in two space dimensions. We overcome this difficulty by the introduction of Sobolev-type inequalities which are new in the enterprise of Klainerman and Sideris, and follow from radius-angular mixed-norm inequalities containing fractional-order Sobolev norms (see the proof of (4.20)).

Last but not least the author remarks that he does not mean to make little of the idea of direct estimates of the fundamental solution. Indeed, he has learned that an L^1-L^∞ estimate for inhomogeneous wave equations plays a role in the attempt to combine the vector fields method with local energy decay estimates for the analysis of quasi-linear equations in exterior domains [18]. There may exist problems for which direct estimates of the fundamental solution are still valuable.

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[1] Sogge has recently combined both the techniques in [31] and [18] to accomplish the same purpose [32].

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This paper is organized as follows. After explaining the notations in the next section, we state the main theorem in Section 3. Useful Sobolev-type inequalities as well as crucial estimates of the null forms are collected in Section 4. Weighted L^2-norms are shown to be bounded by generalized energies in Section 5. In Section 6 we carry out the energy estimates to complete the proof of the main theorem. In the final section some comments are added in accordance with referee's instructions to mention the application of the Klainerman-Sideris method to equations with the nonlinear terms including the unknown functions themselves, say $F(u, \partial u, \partial^2 u)$, because there exist many papers covering this type of nonlinearity.

2. Notation

Mostly following Sideris and Tu [31], we explain the notation used in this paper. Let n denote the space dimensions. Repeated indices are summed if lowed and uppered. Greek indices range from 0 to n, and roman indices from 1 to m. We shall consider systems of m quasi-linear equations. Points in $\mathbb{R}^{n+1}_+$ are denoted by $(x^0, x^1, \dots, x^n) = (t, x)$. In addition to the usual partial differential operators $\partial_\alpha = \partial/\partial x^\alpha$ $(\alpha = 0, 1, \dots, n)$ with the abbreviation $\partial = (\partial_0, \partial_1, \dots, \partial_n) = (\partial_0, \nabla)$, we shall use the generators of Euclid rotations $\Omega = x^1\partial_2 - x^2\partial_1$ for $n = 2$, $\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla$ for $n = 3$, and the generator of space-time scaling $S = x^\alpha \partial_\alpha$. The set of these $\nu = (n^2 + n + 4)/2$ vector fields are denoted by $\Gamma = \{\Gamma_0, \Gamma_1, \dots, \Gamma_{\nu-1}\} = \{\partial, \Omega, S\}$. We employ the multi-index notation in Sideris and Tu [31] to mean, for $a = (a_1, \dots, a_\kappa)$ a sequence of indices $a_i \in \{0, \dots, \nu - 1\}$ of length $|a| = \kappa$,

$$\Gamma^a = \Gamma_{a_\kappa} \cdots \Gamma_{a_1}.$$

It is convenient to set $\Gamma^a = 1$ if $|a| = 0$. Suppose that b and c are disjoint subsequences of a, allowing that $|b| = 0$ or $|c| = 0$. We say $b + c = a$ if $|b| + |c| = |a|$, $b + c < a$ if $|b| + |c| < |a|$.

The D'Alembertian, which acts on vector-valued functions $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}^m$, is denoted by

$$\Box = \operatorname{Diag}(\Box_1, \dots, \Box_m), \quad \Box_k = \frac{\partial^2}{\partial t^2} - c_k^2 \Delta.$$

Associated with this operator, the energy is defined as

$$E_1\big(u(t)\big) = \frac{1}{2} \sum_{k=1}^{m} \int_{\mathbb{R}^n} \big(|\partial_t u^k(t, x)|^2 + c_k^2 |\nabla u^k(t, x)|^2\big) dx,$$

and the generalized energy is defined as

$$E_\kappa\big(u(t)\big) = \sum_{|a| \le \kappa - 1} E_1(\Gamma^a u(t)), \quad \kappa = 2, 3, \dots .$$

Allowing a higher-order energy to grow polynomially in time but bounding a lower-order one uniformly in time, we build up a series of estimates of the generalized energies. The auxiliary norm

$$M_\kappa\big(u(t)\big) = \sum_{k=1}^{m} \sum_{|a|=2} \sum_{|b| \le \kappa - 2} \|\langle c_k t - |x| \rangle \partial^a \Gamma^b u^k(t)\|_{L^2(\mathbb{R}^n)},$$
$$\kappa = 2, 3, \dots$$

plays an intermediate role. Here and later on as well we use the notation $\langle A \rangle = \sqrt{1 + |A|^2}$ for a scalar or a vector A.

3. Results

We consider the Cauchy problem for a full system of quasi-linear wave equations with semi-linear terms

$$\Box u = F(\partial u, \partial^2 u) \quad \text{in } \mathbb{R}^{1+n}_+ \tag{3.1}$$

($n = 2, 3$) subject to the initial data

$$u(0) = \varphi, \quad \partial_t u(0) = \psi. \tag{3.2}$$

We assume the k-th component of the vector function F to be of the form:

If $n=2$, then $F^k(\partial u,\, \partial^2 u)=G^k(u,\, u,\, u)+H^k(u,\, u,\, u)$, where

$$\begin{aligned} G^k(u,\, v,\, w) &= G_{ij}^{k,\,\alpha\beta\gamma\delta}\partial_\alpha u^i \partial_\beta v^j \partial_\gamma \partial_\delta w^k, \\ H^k(u,\, v,\, w) &= H_{ijl}^{k,\,\alpha\beta\gamma}\partial_\alpha u^i \partial_\beta v^j \partial_\gamma w^l, \end{aligned} \tag{3.3}$$

and, if $n=3$, then $F^k(\partial u,\, \partial^2 u)=G^k(u,\, u)+H^k(u,\, u)$, where

$$G^k(u,\, v)=G_i^{k,\,\alpha\beta\gamma}\partial_\alpha u^i \partial_\beta \partial_\gamma v^k, \quad H^k(u,\, v)=H_{ij}^{k,\,\alpha\beta}\partial_\alpha u^i \partial_\beta v^j \tag{3.4}$$

for real constants $G_{ij}^{k,\,\alpha\beta\gamma\delta}$, $H_{ijl}^{k,\,\alpha\beta\gamma}$, $G_i^{k,\,\alpha\beta\gamma}$ and $H_{ij}^{k,\,\alpha\beta}$. In the definition of G^k of (3.3), a term is called non-resonant if $(i,\, j)\neq(k,\, k)$ in its coefficient. If $(i,\, j,\, l)\neq(k,\, k,\, k)$ in the definition of $H^k(u,\, v,\, w)$, such terms are also called non-resonant. Similarly in (3.4), we say the corresponding terms are non-resonant if $i\neq k$ in the coefficient of $G_i^{k,\,\alpha\beta\gamma}$ or $(i,\, j)\neq(k,\, k)$ in $H_{ij}^{k,\,\alpha\beta}$. The remaining terms are called resonant.

As is often supposed in the theory of hyperbolic equations, the highest-order terms are assumed to appear diagonally in the nonlinear terms. Since our proof is based on the energy integral method, we of course suppose

$$G_{ij}^{k,\,\alpha\beta\gamma\delta}=G_{ij}^{k,\,\alpha\beta\delta\gamma} \quad \text{if } n=2, \quad G_i^{k,\,\alpha\beta\gamma}=G_i^{k,\,\alpha\gamma\beta} \quad \text{if } n=3. \tag{3.5}$$

We are now in a position to recall the null condition in the setting of multiple speeds which is proposed by Agemi and Yokoyama [2] for $n=2$, Sideris and Tu [31], Yokoyama [33] for $n=3$: For every $k=1,\, \ldots,\, m$

$$G_{kk}^{k,\,\alpha\beta\gamma\delta}X_\alpha X_\beta X_\gamma X_\delta=0, \quad H_{kkk}^{k,\,\alpha\beta\gamma}X_\alpha X_\beta X_\gamma=0 \tag{3.6}$$

for all $X\in\mathcal{N}_k$ if $n=2$,

$$G_k^{k,\,\alpha\beta\gamma}X_\alpha X_\beta X_\gamma=0, \quad H_{kk}^{k,\,\alpha\beta}X_\alpha X_\beta=0 \tag{3.7}$$

for all $X\in\mathcal{N}_k$ if $n=3$, where $\mathcal{N}_k$ is the hypersurface defined as

$$\mathcal{N}_k=\{X\in\mathbb{R}^{n+1}\,:\, X_0^2-c_k^2(X_1^2+\cdots+X_n^2)=0\}. \tag{3.8}$$

For the initial data φ, ψ, we assume φ^k, $\psi^k\in C_0^\infty(\mathbb{R}^n)$, $k=1,\, \ldots,\, m$. The assumption of infinite times differentiability is not essential for the proof of global existence. In our theorem, however, the assumption of compactness of the support cannot be replaced by any other suitable decay condition at spatial infinity for $n=2$.

The main theorem of this paper is stated as follows.

Theorem *Assume $n = 2, 3$. Suppose that the null condition* (3.6)-(3.7) *as well as the symmetry condition* (3.5) *are satisfied. Let $\kappa \geq 9$. There exist positive constants ε, ε_1 and A with the following property: Suppose that smooth, compactly supported initial data satisfy*

$$4E_{\kappa-2}(u(0)) \exp\bigl[A\bigl\{2E_\kappa\bigl(u(0)\bigr)\bigr\}^\theta\bigr] \leq 3\varepsilon^2 \tag{3.9}$$

($\theta = 1$ if $n = 2$, $\theta = 1/2$ if $n = 3$). *For $n = 2$, assume in addition*

$$M_{\kappa-2}(u(0)) \leq \varepsilon_1. \tag{3.10}$$

Then the problem (3.1)-(3.2) *has a unique global in time solution satisfying*

$$E_\kappa\bigl(u(t)\bigr) \leq CE_\kappa\bigl(u(0)\bigr)\langle t\rangle^{C\varepsilon^{2\theta}} \quad \text{and} \quad E_{\kappa-2}\bigl(u(t)\bigr) < 4\varepsilon^2 \tag{3.11}$$

for all $t > 0$.

4. Preliminaries

In this section we collect several lemmas concerning commutation relations, some estimates of the null forms, and the Sobolev-type inequalities.

We begin with the commutation relations. Let $[\,\cdot\,,\,\cdot\,]$ be the commutator. In addition to the well-known facts

$$[\partial_\alpha, \Box] = 0, \quad [\Omega_{ij}, \Box] = 0, \quad [S, \Box] = -2\Box, \tag{4.1}$$

we need the commutation relations of the vector fields Γ with respect to the nonlinear terms. Recall the nonlinear terms $G = (G^1, \ldots, G^m)$ and $H = (H^1, \ldots, H^m)$ defined in (3.3)-(3.4). Part (i) of the following lemma implies that the null structure is preserved upon differentiation, and Part (ii) together with (4.1) inductively shows that, for any a, the nonlinearities of the equations (4.6), (4.7) also possess the null structure.

Lemma 4.1 (i) *For any Γ^a the following equalities hold:*

$$\Gamma^a G(u, v, w) = \sum_{b+c+d+e=a} G_e(\Gamma^b u, \Gamma^c v, \Gamma^d w), \tag{4.2}$$

$$\Gamma^a H(u, v, w) = \sum_{b+c+d+e=a} H_e(\Gamma^b u, \Gamma^c v, \Gamma^d w) \tag{4.3}$$

if $n = 2$, and

$$\Gamma^a G(u, v) = \sum_{b+c+d=a} G_d(\Gamma^b u, \Gamma^c v), \tag{4.4}$$

$$\Gamma^a H(u,\, v) = \sum_{b+c+d=a} H_d(\Gamma^b u,\, \Gamma^c v) \tag{4.5}$$

if $n=3$. *Here each* G_e *(resp.* H_e*) is a cubic nonlinear term of the form which* G *(resp.* H*) has in* (3.3)*, and each* G_d *(resp.* H_d*) is a quadratic nonlinear term of the form which* G *(resp.* H*) has in* (3.4)*. In particular,* $G_e = G$*,* $H_e = H$ *if* $b+c+d=a$ *in* (4.2)-(4.3)*,* $G_d = G$*,* $H_d = H$ *if* $b+c=a$ *in* (4.4)-(4.5)*. Moreover, if the original nonlinearities* G *and* H *have the null structure* (3.6)-(3.7)*, then so does each new nonlinearity* G_e*,* H_e*,* G_d*, and* H_d*.*

(ii) *Let* u *be a smooth solution of* (3.1)-(3.3)*. Then, for any* Γ^a*, the equalities*

$$\begin{aligned}\Box\Gamma^a u = & \sum_{b+c+d+e=a} G_e(\Gamma^b u,\, \Gamma^c v,\, \Gamma^d w) \\ & + \sum_{b+c+d+e=a} H_e(\Gamma^b u,\, \Gamma^c v,\, \Gamma^d w) - [\Gamma^a,\, \Box]u \quad \text{if} \quad n=2 \end{aligned} \tag{4.6}$$

$$\begin{aligned}\Box\Gamma^a u = & \sum_{b+c+d=a} G_d(\Gamma^b u,\, \Gamma^c v) \\ & + \sum_{b+c+d=a} H_d(\Gamma^b u,\, \Gamma^c v) - [\Gamma^a,\, \Box]u \quad \text{if} \quad n=3, \end{aligned} \tag{4.7}$$

hold.

Proof. See Lemma 4.1 of Sideris and Tu [31]. □

The next lemma, which crucially comes into play in the estimates of lower-order energies, is the statement of gain of additional decay in nonlinearities with the null structure (3.6)-(3.7).

Lemma 4.2 *For any smooth scalar functions* u*,* v*,* w *and* z*, the following inequalities hold for* $r \geq c_k t/2$:

$$\begin{aligned} & |G_{kk}^{k,\,\alpha\beta\gamma\delta}\partial_\alpha u \partial_\beta v \partial_\gamma \partial_\delta w| \\ & \leq C\langle t\rangle^{-1}\big[|\Gamma u||\partial v||\partial^2 w| + |\partial u||\Gamma v||\partial^2 w| \\ & \qquad + |\partial u||\partial v||\partial\Gamma w| + \langle c_k t - r\rangle|\partial u||\partial v||\partial^2 w|\big], \end{aligned} \tag{4.8}$$

$$\begin{aligned} & |G_{kk}^{k,\,\alpha\beta\gamma\delta}\partial_\alpha u \partial_\beta v \partial_\gamma w \partial_\delta z| \\ & \leq C\langle t\rangle^{-1}\big[|\Gamma u||\partial v||\partial w||\partial z| + |\partial u||\Gamma v||\partial w||\partial z| + |\partial u||\partial v||\Gamma w||\partial z| \\ & \qquad + |\partial u||\partial v||\partial w||\Gamma z| + \langle c_k t - r\rangle|\partial u||\partial v||\partial w||\partial z|\big], \end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&|G_{kk}^{k,\,\alpha\beta\gamma\delta}\partial_\alpha\partial_\gamma u\partial_\beta v\partial_\delta w|\\
&\le C\langle t\rangle^{-1}\big[|\partial\Gamma u||\partial v||\partial w| + |\partial^2 u||\Gamma v||\partial w|\\
&\qquad + |\partial^2 u||\partial v||\Gamma w| + |\langle c_k t - r\rangle|\partial^2 u||\partial v||\partial w|\big], \qquad (4.10)\\
&|G_{kk}^{k,\,\alpha\beta\gamma\delta}\partial_\alpha u\partial_\gamma\partial_\beta v\partial_\delta w|\\
&\le C\langle t\rangle^{-1}\big[|\Gamma u||\partial^2 v||\partial w| + |\partial u||\partial\Gamma v||\partial w|\\
&\qquad + |\partial u||\partial^2 v||\Gamma w| + \langle c_k t - r\rangle|\partial u||\partial^2 v||\partial w|\big], \qquad (4.11)\\
&|H_{kkk}^{k,\,\alpha\beta\gamma}\partial_\alpha u\partial_\beta v\partial_\gamma w|\\
&\le C\langle t\rangle^{-1}\big[|\Gamma u||\partial v||\partial w| + |\partial u||\Gamma v||\partial w|\\
&\qquad + |\partial u||\partial v||\Gamma w| + \langle c_k t - r\rangle|\partial u||\partial v||\partial w|\big] \qquad (4.12)
\end{aligned}
$$

if $n = 2$*, and*

$$
\begin{aligned}
&|G_k^{k,\,\alpha\beta\gamma}\partial_\alpha u\partial_\beta\partial_\gamma v|\\
&\le C\langle t\rangle^{-1}\big[|\Gamma u||\partial^2 v| + |\partial u||\partial\Gamma v| + \langle c_k t - r\rangle|\partial u||\partial^2 v|\big], \qquad (4.13)\\
&|G_k^{k,\,\alpha\beta\gamma}\partial_\alpha u\partial_\beta v\partial_\gamma w|\\
&\le C\langle t\rangle^{-1}\big[|\Gamma u||\partial v||\partial w| + |\partial u||\Gamma v||\partial w| + |\partial u||\partial v||\Gamma w|\\
&\qquad + \langle c_k t - r\rangle|\partial u||\partial v||\partial w|\big], \qquad (4.14)\\
&|G_k^{k,\,\alpha\beta\gamma}\partial_\gamma\partial_\alpha u\partial_\beta v|\\
&\le C\langle t\rangle^{-1}\big[|\partial\Gamma u||\partial v| + |\partial^2 u||\Gamma v| + \langle c_k t - r\rangle|\partial^2 u||\partial v|\big], \qquad (4.15)\\
&|H_{kk}^{k,\,\alpha\beta}\partial_\alpha u\partial_\beta v|\\
&\le C\langle t\rangle^{-1}\big[|\Gamma u||\partial v| + |\partial u||\Gamma v| + \langle c_k t - r\rangle|\partial u||\partial v|\big] \qquad (4.16)
\end{aligned}
$$

if $n = 3$.

Proof. The proof of (4.13)-(4.14) is given by Sideris and Tu [31]. The proof of the others is similar. □

The following lemma is concerned with Sobolev-type inequalities.

Lemma 4.3 (i) *Let* $n = 2$*. The following inequalities hold for any smooth vector-valued functions* $u : \mathbb{R}_+^{2+1} \to \mathbb{R}^m$*, provided that the norms on the right-hand side are finite*:

$$\langle r\rangle^{1/2}|\partial u(t,\,x)| \le CE_3(u(t)), \tag{4.17}$$

$$\langle r\rangle^{1/2}\langle c_j t - r\rangle^{1/2}|\partial u^j(t,x)| \le CE_2^{1/2}(u(t)) + CM_3(u(t)), \tag{4.18}$$

$$\langle r\rangle^{1/2}\langle c_j t - r\rangle|\partial^2 u^j(t,x)| \le CM_4(u(t)), \tag{4.19}$$

$$\langle r\rangle^{\varepsilon}\langle c_jt-r\rangle|\partial u^j(t,x)|$$
$$\leq C\sum_{j=1}^{m}\sum_{|a|\leq 1}\|\langle c_jt-r\rangle\partial\Omega^a u^j(t)\|_{L^2}^{\varepsilon}\|\langle c_jt-r\rangle\partial\Omega^a u^j(t)\|_{\dot H^1_2}^{1-\varepsilon}$$
$$+CM_3(u(t)), \tag{4.20}$$

where $0<\varepsilon<1/2$.

(ii) *Let* $n=3$. *The following inequalities hold for any smooth vector-valued functions* $u:\mathbb{R}^{3+1}_+\to\mathbb{R}^m$, *provided that the norms on the right-hand side are finite*:

$$\langle r\rangle^{1/2}|u(t,x)|\leq CE_2^{1/2}(u(t)), \tag{4.21}$$
$$\langle r\rangle^{1/2}\langle c_jt-r\rangle|\partial u^j(t,x)|\leq CE_2^{1/2}(u(t))+CM_3(u(t)), \tag{4.22}$$
$$\langle r\rangle|\partial u(t,x)|\leq CE_3^{1/2}(u(t)), \tag{4.23}$$
$$\langle r\rangle\langle c_jt-r\rangle^{1/2}|\partial u^j(t,x)|\leq CE_3^{1/2}(u(t))+CM_3(u(t)), \tag{4.24}$$
$$\langle r\rangle\langle c_jt-r\rangle|\partial^2 u^j(t,x)|\leq CM_4(u(t)). \tag{4.25}$$

Proof. Part (i) is proved in Lemma 1 of Sideris [29] except (4.20), and Part (ii) is shown in Lemma 6.1 of Sideris and Tu [31] except (4.22) (see also Proposition 3.3 of Sideris [30]). Hence (4.20) and (4.22) remain to be shown. The proof of (4.20) starts with the following radius-angular mixed-norm inequality which has been proved in Hidano [6] (see also Lemma 3.4 in [7]):

$$r^{n/2-s}\left(\int_{S^{n-1}}|v(r\omega)|^2d\omega\right)^{1/2}\leq C\|v\|_{\dot H^s_2},\quad \frac{1}{2}<s<\frac{n}{2}. \tag{4.26}$$

Set $\varepsilon=1-s$ for $1/2<s<1$. Let us simply denote u^j, c_j by u, c, respectively. For $r\geq 1$ it follows from (4.26) and the Sobolev embedding on S^1 that

$$r^{\varepsilon}|\langle ct-r\rangle\partial u(t,x)|$$
$$\leq C\sum_{|a|=1}\sum_{|b|\leq 1}r^{\varepsilon}\left(\int_{S^1}|\langle ct-r\rangle\partial^a\Omega^b u(t,r\omega)|^2d\omega\right)^{1/2}$$
$$\leq C\sum_{|a|=1}\sum_{|b|\leq 1}\|\langle ct-r\rangle\partial^a\Omega^b u(t)\|_{\dot H^{1-\varepsilon}_2}$$
$$\leq C\sum_{|a|=1}\sum_{|b|\leq 1}\|\langle ct-r\rangle\partial^a\Omega^b u(t)\|_{L^2}^{\varepsilon}\|\langle ct-r\rangle\partial^a\Omega^b u(t)\|_{\dot H^1_2}^{1-\varepsilon}. \tag{4.27}$$

On the other hand, for $r \le 1$, we proceed as follows. Let Φ be a smooth, compactly supported function in $\mathbb{R}^2$ such that $\Phi(x) = 1$ for $|x| \le 1$, $= 0$ for $|x| \ge 2$. Without loss of generality we may assume $ct \ge 3$. It then follows from the Sobolev embedding and (4.27) that

$$\begin{aligned}
&|\langle ct-r\rangle \partial u(t,x)| \le (1+ct)|\Phi(x)\partial u(t,x)| \\
&\le C(1+ct)\|\Phi\partial u(t)\|_{H^2} \le C(1+ct)\sum_{i=1}^{2}\|\nabla^i(\Phi\partial u(t))\|_{L^2} \\
&\le C(1+ct)\sum_{i=1}^{2}\|\nabla^i\partial u(t)\|_{L^2(|x|<2)} + C(1+ct)\sup_{1<|x|<2}|\partial u(t,x)| \\
&\le C\sum_{i=1}^{2}\|\langle ct-r\rangle\nabla^i\partial u(t)\|_{L^2} + C\sup_{|x|>1}|\langle ct-r\rangle\partial u(t,x)| \\
&\le CM_3\big(u(t)\big) \\
&\quad + C\sum_{|a|=1}\sum_{|b|\le 1}\|\langle ct-r\rangle\partial^a\Omega^b u(t)\|_{L^2}^{\varepsilon}\|\langle ct-r\rangle\partial^a\Omega^b u(t)\|_{\dot H^1_2}^{1-\varepsilon}. \quad (4.28)
\end{aligned}$$

Therefore, the inequality (4.20) is a consequence of (4.27)-(4.28).

As for (4.22), we start with the following inequality

$$r^{1/2}\left(\int_{S^2}|v(r\omega)|^4 d\omega\right)^{1/4} \le C\|v\|_{\dot H^1_2} \tag{4.29}$$

(see (3.16) of Sideris [30]). For $r \ge 1$, (4.22) is an immediate consequence of the Sobolev embedding on S^2 and (4.29). For $r \le 1$ we have only to modify the argument in (4.28) properly. □

Lemma 4.4 *Let $n = 2$. Suppose that, for every $t > 0$, smooth and scalar functions $u = u(t,x)$ are compactly supported in $\{x \in \mathbb{R}^2 : |x| \le ct + R\}$ with suitable constants $c > 0$ and $R > 0$. Then the inequality*

$$\left\|\frac{1}{\langle ct-r\rangle}u(t)\right\|_{L^2(\mathbb{R}^2)} \le C_R\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \tag{4.30}$$

holds for a constant C_R with $C_R \to \infty$ as $R \to \infty$.

Proof. See Lemma 1.2 of Lindblad [26], where this lemma is proved for $n = 3$. As is pointed out in, e.g., Lemma 3.3 of Katayama [16], this lemma is actually valid for $n = 2$ as well. □

5. Weighted L^2-estimates

It is necessary to bound the weighted L^2-norm $M_\kappa\big(u(t)\big)$ by $E_\kappa^{1/2}\big(u(t)\big)$ for the completion of the energy integral argument. We carry out this by starting with the next crucial inequality due to Klainerman and Sideris [23], estimating the nonlinear terms carefully, and doing a bootstrap argument.

Lemma 5.1 (Klainerman–Sideris inequality) *Let $\kappa \geq 2$. The inequality*

$$M_\kappa\big(u(t)\big) \leq C\bigg(E_\kappa^{1/2}\big(u(t)\big) + \sum_{|a|\leq \kappa-2} \|(t+r)\Box\Gamma^a u(t)\|_{L^2}\bigg) \qquad (5.1)$$

holds for any smooth function u with the finite norms on the right-hand side.

Proof. See Lemma 3.1 of Klainerman and Sideris [23] and Lemma 7.1 of Sideris and Tu [31]. Note that their proof is obviously valid for $n=2$ as well as $n=3$. $\Box$

Lemma 5.2 *Let u be a smooth solution of* (3.1)-(3.4). *Set $\kappa' = [(\kappa-1)/2]+3$. Then for all $|a|\leq\kappa-2$*

$$\begin{aligned}
&\|(t+r)\Box\Gamma^a u(t)\|_{L^2}\\
&\leq C\big(E_{\kappa'}^{1/2}\big(u(t)\big) + M_{\kappa'}\big(u(t)\big)\big)^2 E_\kappa^{1/2}\big(u(t)\big) + CE_{\kappa'}\big(u(t)\big)M_\kappa\big(u(t)\big)\\
&\quad + CE_{\kappa'}^{1/2}\big(u(t)\big)E_\kappa^{1/2}\big(u(t)\big)M_{\kappa'}\big(u(t)\big) \quad \textit{if} \quad n=2 \qquad (5.2)\\
&\|(t+r)\Box\Gamma^a u(t)\|_{L^2}\\
&\leq CE_{\kappa'}^{1/2}\big(u(t)\big)E_\kappa^{1/2}\big(u(t)\big) + CM_{\kappa'}\big(u(t)\big)E_\kappa^{1/2}\big(u(t)\big)\\
&\quad + CE_{\kappa'}^{1/2}(u(t))M_\kappa(u(t)) \quad \textit{if} \quad n=3. \qquad (5.3)
\end{aligned}$$

Proof. We may focus on the estimate of the L^2-norm of $t\Box\Gamma^a u(t)$ because that of $r\Box\Gamma^a u(t)$ is treated in a similar (in fact, easier) way. Set $p=[(\kappa-1)/2]$. We start with $n=2$. It immediately follows from (4.6) that

$$t\|\Box\Gamma^a u(t)\|_{L^2} \leq C \sum_{\substack{|b|+|c|+|d|\\ \leq\kappa-2}} t\big[\|\partial\Gamma^b u^i(t)\partial\Gamma^c u^j(t)\partial^2\Gamma^d u^l(t)\|_{L^2} + \|\partial\Gamma^b u^i(t)\partial\Gamma^c u^j(t)\partial\Gamma^d u^l(t)\|_{L^2}\big]. \quad (5.4)$$

Supposing $|b|+|c|\leq p$ without loss of generality, we bound the second norm as

$$\begin{aligned}\cdots\le&\langle t\rangle^{-1}\|\langle r\rangle^{1/2}\langle c_i t-r\rangle^{1/2}\partial\Gamma^b u^i(t)\|_{L^\infty}\\&\times\|\langle r\rangle^{1/2}\langle c_j t-r\rangle^{1/2}\partial\Gamma^c u^j(t)\|_{L^\infty}\|\partial\Gamma^d u^l(t)\|_{L^2}\\\le&\langle t\rangle^{-1}C\big(E_{\kappa'}^{1/2}(u(t))+M_{\kappa'}(u(t))\big)^2E_\kappa^{1/2}(u(t))\end{aligned}\tag{5.5}$$

by (4.18). For the first norms on the right-hand side of (5.4), we sort them out into two groups: $|b|+|c|\le p$ or $|d|\le p-1$. The first group is estimated as

$$\begin{aligned}\cdots\le&\langle t\rangle^{-1}\|\langle r\rangle^{1/2}\partial\Gamma^b u^i(t)\|_{L^\infty}\|\langle r\rangle^{1/2}\partial\Gamma^c u^j(t)\|_{L^\infty}\\&\times\|\langle c_l t-r\rangle\partial^2\Gamma^d u^l(t)\|_{L^2}\\\le&C\langle t\rangle^{-1}E_{\kappa'}(u(t))M_\kappa(u(t))\end{aligned}\tag{5.6}$$

by (4.17). Otherwise, assuming $|b|\le p$ as well as $|d|\le p-1$ without loss of generality, we get

$$\begin{aligned}\cdots\le&\langle t\rangle^{-1}\|\langle r\rangle^{1/2}\partial\Gamma^b u^i(t)\|_{L^\infty}\|\partial\Gamma^c u^j(t)\|_{L^2}\\&\times\|\langle r\rangle^{1/2}\langle c_l t-r\rangle\partial^2\Gamma^d u^l(t)\|_{L^\infty}\\\le&C\langle t\rangle^{-1}E_{\kappa'}^{1/2}(u(t))M_{\kappa'}(u(t))E_{\kappa-1}^{1/2}(u(t))\end{aligned}\tag{5.7}$$

by (4.17), (4.19), which completes the proof of (5.2).

As for $n=3$ we first observe by (4.7) that the L^2-norm of $t\Box\Gamma^a u$ is estimated as in (5.4). Since the quadratic, quasi-linear terms have already been treated by Sideris and Tu [31], we have only to bound the L^2-norm of quadratic, semi-linear terms such as

$$t\|\partial\Gamma^b u^i(t)\partial\Gamma^c u^j(t)\|_{L^2},\quad |b|+|c|\le\kappa-2.\tag{5.8}$$

Assuming $|b|\le p$ without loss of generality, we estimate (5.8) as

$$\begin{aligned}\cdots\le&\big(\|\langle c_i t-r\rangle\partial\Gamma^b u^i(t)\|_{L^\infty}+\|\langle r\rangle\partial\Gamma^b u^i(t)\|_{L^\infty}\big)\|\partial\Gamma^c u^j(t)\|_{L^2}\\\le&C\big(E_{\kappa'}^{1/2}(u(t))+M_{\kappa'}(u(t))\big)E_{\kappa-1}^{1/2}(u(t))\\&+CE_{\kappa'}^{1/2}(u(t))E_{\kappa-1}^{1/2}(u(t))\end{aligned}\tag{5.9}$$

by (4.22) and (4.23). □

Lemma 5.3 *Let $\kappa\ge 9$, $\mu=\kappa-2$. There exist small, positive constants ε_0, ε_1 with the following property: Suppose that, for a local smooth solution u of (3.1)-(3.4), the supremum of $E_\mu^{1/2}(u(t))$ on an interval $[0,T)$ is sufficiently small so that*

$$\sup_{0\le t<T} E_\mu^{1/2}\big(u(t)\big) \le \varepsilon_0 \tag{5.10}$$

may hold. For $n=2$, *assume in addition*

$$M_\mu\big(u(0)\big) \le \varepsilon_1. \tag{5.11}$$

Then

$$M_\mu\big(u(t)\big) \le CE_\mu^{1/2}\big(u(t)\big), \quad 0\le t<T \tag{5.12}$$

and

$$M_\kappa\big(u(t)\big) \le CE_\kappa^{1/2}\big(u(t)\big), \quad 0\le t<T \tag{5.13}$$

hold with a constant C *independent of* T.

Remark This lemma is actually valid for $\kappa \ge 8$. We have assumed $\kappa\ge 9$ for the latter use.

Proof. Set $\mu' = [(\mu-1)/2]+3$. We start with $n=2$. It follows from Lemma 5.1 and Lemma 5.2 that for $0\le t<T$

$$\begin{aligned} M_\mu\big(u(t)\big) \le & CE_\mu^{1/2}(u(t)) + C\big(E_{\mu'}^{1/2}(u(t)) \\ & + M_{\mu'}(u(t))\big)^2 E_\mu^{1/2}(u(t)) + CE_{\mu'}(u(t))M_\mu(u(t)) \\ & + CE_{\mu'}^{1/2}(u(t))E_\mu^{1/2}(u(t))M_{\mu'}(u(t)) \\ \le & CE_\mu^{1/2}(u(t)) + C\big(\varepsilon_0 + M_{\mu'}(u(t))\big)^2 E_\mu^{1/2}(u(t)) \\ & + C\varepsilon_0^2 M_\mu(u(t)) + C\varepsilon_0^2 M_{\mu'}(u(t)) \\ \le & CE_\mu^{1/2}(u(t)) + C\varepsilon_0^2 M_\mu(u(t)) + CM_\mu^2(u(t)), \end{aligned} \tag{5.14}$$

which immediately yields

$$M_\mu(u(t)) \le CE_\mu^{1/2}(u(t)) + CM_\mu^2(u(t)) \le C\varepsilon_0 + CM_\mu^2(u(t)) \tag{5.15}$$

because ε_0 is sufficiently small. Since $M_\mu\big(u(0)\big)$ is small enough and $M_\mu\big(u(t)\big)$ is continuous on $[0,\, T)$, we see that

$$M_\mu\big(u(t)\big) \le C(\varepsilon_0), \quad 0\le t<T \tag{5.16}$$

for a constant $C(\varepsilon_0)$ with $C(\varepsilon_0)\to 0$ as $\varepsilon_0\to 0$. Inserting (5.16) into (5.15), we finally get the estimate (5.12). As for (5.13), we first note that the inequality $\kappa' := [(\kappa-1)/2]+3 \le \mu$ holds. Proceeding as in (5.14) and using

(5.12), we easily see that

$$M_\kappa\big(u(t)\big) \le CE_\kappa^{1/2}\big(u(t)\big) + C\varepsilon_0^2 M_\kappa\big(u(t)\big) + CE_\kappa^{1/2}\big(u(t)\big), \qquad (5.17)$$

which yields (5.13).

As far as the case $n = 3$ is concerned, we have only to repeat essentially the same argument as in Lemma 7.3 of Sideris and Tu [31]. $\square$

6. Energy estimates

Following the strategy in Sideris [30], Sideris and Tu [31], we accomplish the energy integral argument by deriving a pair of coupled differential inequalities for a higher-order energy $E_\kappa\big(u(t)\big)$, $\kappa \ge 9$ and a lower-order energy $E_\mu\big(u(t)\big)$, $\mu = \kappa - 2$. Since the equation is quasi-linear, we must actually consider modified energies which are equivalent to the original ones for small solutions.

For initial data (φ, ψ) with $(\varphi^k, \psi^k) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ $(k = 1, \dots, m)$, let us assume $E_\mu^{1/2}\big(u(0)\big) < \varepsilon$ for a sufficiently small $\varepsilon > 0$ such that $2\varepsilon \le \varepsilon_0$ (see (5.10) for ε_0). By the standard local existence theorem we know that a unique smooth solution exists locally in time. Suppose that T_0 is the supremum of all $T > 0$ for which $E_\mu^{1/2}\big(u(t)\big) < 2\varepsilon$ for all $0 \le t < T$. It is shown that $E_\mu^{1/2}\big(u(t)\big) < 2\varepsilon$ on the closed interval $0 \le t \le T_0$, therefore we can continue the local solution to all time.

Suppose $0 \le t < T_0$ in what follows. Denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the scalar product in $\mathbb{R}^m$, we have for each $l = 1, \dots, \kappa$ $(\kappa \ge 9)$

$$\begin{aligned}
E_l'\big(u(t)\big) &= \sum_{|a| \le l-1} \int_{\mathbb{R}^2} \langle \Box \Gamma^a u(t),\, \partial_t \Gamma^a u(t)\rangle dx \\
&= \sum_{\substack{1 \le k \le m \\ |a| = l-1}} \int_{\mathbb{R}^2} G_{ij}^{k,\,\alpha\beta\gamma\delta} \partial_\alpha u^i \partial_\beta u^j \partial_\gamma \partial_\delta \Gamma^a u^k \partial_t \Gamma^a u^k dx \\
&\quad + \sum_{\substack{b+c+d+e=a \\ |a| \le l-1,\, d \ne a}} \int_{\mathbb{R}^2} \langle G_e(\Gamma^b u,\, \Gamma^c u,\, \Gamma^d u),\, \partial_t \Gamma^a u\rangle dx \\
&\quad + \sum_{b+c+d+e=a} \int_{\mathbb{R}^2} \langle H_e(\Gamma^b u,\, \Gamma^c u,\, \Gamma^d u),\, \partial_t \Gamma^a u\rangle dx \\
&\quad - \int_{\mathbb{R}^2} \langle [\Gamma^a,\, \Box] u,\, \partial_t \Gamma^a u\rangle dx, \quad \text{if} \quad n = 2,
\end{aligned} \qquad (6.1)$$

$$
\begin{aligned}
E_l'(u(t)) = &\sum_{\substack{1\le k\le m\\ |a|=l-1}} \int_{\mathbb{R}^3} G_i^{k,\,\alpha\beta\gamma}\partial_\alpha u^i \partial_\beta\partial_\gamma \Gamma^a u^k \partial_t \Gamma^a u^k dx\\
&+\sum_{\substack{b+c+d=a\\ |a|\le l-1,\ c\ne a}} \int_{\mathbb{R}^3} \langle G_d(\Gamma^b u,\ \Gamma^c u),\ \partial_t\Gamma^a u\rangle dx\\
&+\sum_{b+c+d=a} \int_{\mathbb{R}^3} \langle H_d(\Gamma^b u,\ \Gamma^c u),\ \partial_t\Gamma^a u\rangle dx\\
&-\int_{\mathbb{R}^3} \langle [\Gamma^a,\ \Box]u,\ \partial_t\Gamma^a u\rangle dx, \quad \text{if} \quad n=3. \qquad (6.2)
\end{aligned}
$$

The loss of derivatives which has occurred in the first term on each right-hand side is prevented by the symmetry condition (3.5) as follows:

$$
\begin{aligned}
&\int_{\mathbb{R}^2} G_{ij}^{k,\,\alpha\beta\gamma\delta}\partial_\alpha u^i \partial_\beta u^j \partial_\gamma\partial_\delta \Gamma^a u^k \partial_t \Gamma^a u^k \, dx\\
=&\int_{\mathbb{R}^2} G_{ij}^{k,\,\alpha\beta\gamma\delta}\partial_\gamma\big(\partial_\alpha u^i \partial_\beta u^j \partial_\delta \Gamma^a u^k \partial_t \Gamma^a u^k\big) dx\\
&-\int_{\mathbb{R}^2} G_{ij}^{k,\,\alpha\beta\gamma\delta}\Big[\partial_\gamma\big(\partial_\alpha u^i \partial_\beta u^j\big)\partial_\delta \Gamma^a u^k \partial_t \Gamma^a u^k\\
&\qquad\qquad + \partial_\alpha u^i \partial_\beta u^j \partial_\delta \Gamma^a u^k \partial_t\partial_\gamma \Gamma^a u^k\Big] dx\\
=&\partial_t \int_{\mathbb{R}^2} G_{ij}^{k,\,\alpha\beta 0\delta}\partial_\alpha u^i \partial_\beta u^j \partial_\delta \Gamma^a u^k \partial_t \Gamma^a u^k \, dx\\
&-\int_{\mathbb{R}^2} G_{ij}^{k,\,\alpha\beta\gamma\delta}\partial_\gamma\big(\partial_\alpha u^i \partial_\beta u^j\big)\partial_\delta \Gamma^a u^k \partial_t \Gamma^a u^k \, dx\\
&-\int_{\mathbb{R}^2} \frac{1}{2} G_{ij}^{k,\,\alpha\beta\gamma\delta}\partial_\alpha u^i \partial_\beta u^j \partial_t\big(\partial_\delta \Gamma^a u^k \partial_\gamma \Gamma^a u^k\big) dx\\
=&\partial_t \int_{\mathbb{R}^2} \frac{1}{2} G_{ij}^{k,\,\alpha\beta\lambda\delta}\eta_\lambda^\gamma \partial_\alpha u^i \partial_\beta u^j \partial_\delta \Gamma^a u^k \partial_\gamma \Gamma^a u^k \, dx\\
&-\int_{\mathbb{R}^2} G_{ij}^{k,\,\alpha\beta\gamma\delta}\partial_\gamma\big(\partial_\alpha u^i \partial_\beta u^j\big)\partial_\delta \Gamma^a u^k \partial_t \Gamma^a u^k dx\\
&+\int_{\mathbb{R}^2} \frac{1}{2} G_{ij}^{k,\,\alpha\beta\gamma\delta}\partial_t\big(\partial_\alpha u^i \partial_\beta u^j\big)\partial_\delta \Gamma^a u^k \partial_\gamma \Gamma^a u^k \, dx. \qquad (6.3)
\end{aligned}
$$

($\eta_\lambda^\gamma = \mathrm{diag}(1,\ -1,\ -1)$) for $n=2$, and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} G_i^{k,\,\alpha\beta\gamma}\partial_\alpha u^i\partial_\beta\partial_\gamma\Gamma^a u^k\partial_t\Gamma^a u^k\,dx\\
=&\partial_t\int_{\mathbb{R}^3}\frac{1}{2}G_i^{k,\,\alpha\beta\lambda}\eta_\lambda^\gamma\partial_\alpha u^i\partial_\beta\Gamma^a u^k\partial_\gamma\Gamma^a u^k\,dx\\
&-\int_{\mathbb{R}^3} G_i^{k,\,\alpha\beta\gamma}\partial_\gamma\partial_\alpha u^i\partial_\beta\Gamma^a u^k\partial_t\Gamma^a u^k\,dx\\
&+\int_{\mathbb{R}^3}\frac{1}{2}G_i^{k,\,\alpha\beta\gamma}\partial_t\partial_\alpha u^i\partial_\beta\Gamma^a u^k\partial_\gamma\Gamma^a u^k\,dx
\end{aligned}\tag{6.4}
$$

($\eta_\lambda^\gamma = \mathrm{diag}(1,\,-1,\,-1,\,-1)$) for $n=3$ as was shown in Sideris and Tu [31] on page 484. Therefore, introducing the modified energy

$$
\begin{aligned}
\tilde{E}_l\big(u(t)\big):=&E_l\big(u(t)\big)\\
&-\sum_{\substack{|a|=l-1\\1\le k\le m}}\int_{\mathbb{R}^2}\frac{1}{2}G_{ij}^{k,\,\alpha\beta\lambda\delta}\eta_\lambda^\gamma\partial_\alpha u^i\partial_\beta u^j\partial_\delta\Gamma^a u^k\partial_\gamma\Gamma^a u^k\,dx
\end{aligned}\tag{6.5}
$$

for $n=2$, and

$$
\begin{aligned}
\tilde{E}_l\big(u(t)\big):=&E_l\big(u(t)\big)\\
&-\sum_{\substack{|a|=l-1\\1\le k\le m}}\int_{\mathbb{R}^3}\frac{1}{2}G_i^{k,\,\alpha\beta\lambda}\eta_\lambda^\gamma\partial_\alpha u^i\partial_\beta\Gamma^a u^k\partial_\gamma\Gamma^a u^k\,dx
\end{aligned}\tag{6.6}
$$

for $n=3$, we finally have

$$
\begin{aligned}
&\tilde{E}_l'\big(u(t)\big)\\
=&\sum_{\substack{b+c+d+e=a\\|a|\le l-1,\,d\ne a}}\int_{\mathbb{R}^2}\langle G_e(\Gamma^b u,\,\Gamma^c u,\,\Gamma^d u),\,\partial_t\Gamma^a u\rangle\,dx\\
&-\sum_{\substack{|a|=l-1\\1\le k\le m}}\int_{\mathbb{R}^2}G_{ij}^{k,\,\alpha\beta\gamma\delta}\partial_\gamma\big(\partial_\alpha u^i\partial_\beta u^j\big)\partial_\delta\Gamma^a u^k\partial_t\Gamma^a u^k\,dx\\
&+\sum_{\substack{|a|=l-1\\1\le k\le m}}\frac{1}{2}\int_{\mathbb{R}^2}G_{ij}^{k,\,\alpha\beta\gamma\delta}\partial_t\big(\partial_\alpha u^i\partial_\beta u^j\big)\partial_\delta\Gamma^a u^k\partial_\gamma\Gamma^a u^k\,dx\\
&+\sum_{b+c+d+e=a}\int_{\mathbb{R}^2}\langle H_e(\Gamma^b u,\,\Gamma^c u,\,\Gamma^d u),\,\partial_t\Gamma^a u\rangle\,dx\\
&-\int_{\mathbb{R}^2}\langle[\Gamma^a,\,\Box]u,\,\partial_t\Gamma^a u\rangle\,dx\quad\text{if}\quad n=2,
\end{aligned}\tag{6.7}
$$

$$
\begin{aligned}
&\tilde{E}_l'\big(u(t)\big)\\
&= \sum_{\substack{b+c+d=a\\ |a|\le l-1,\, c\neq a}} \int_{\mathbb{R}^3} \langle G_d(\Gamma^b u,\, \Gamma^c u),\, \partial_t \Gamma^a u\rangle\, dx\\
&\quad - \sum_{\substack{|a|=l-1\\ 1\le k\le m}} \int_{\mathbb{R}^3} G_i^{k,\,\alpha\beta\gamma} \partial_\gamma \partial_\alpha u^i \partial_\beta \Gamma^a u^k \partial_t \Gamma^a u^k\, dx\\
&\quad + \sum_{\substack{|a|=l-1\\ 1\le k\le m}} \int_{\mathbb{R}^3} \frac{1}{2} G_i^{k,\,\alpha\beta\gamma} \partial_t \partial_\alpha u^i \partial_\beta \Gamma^a u^k \partial_\gamma \Gamma^a u^k\, dx\\
&\quad + \sum_{b+c+d=a} \int_{\mathbb{R}^3} \langle H_d(\Gamma^b u,\, \Gamma^c u),\, \partial_t \Gamma^a u\rangle\, dx\\
&\quad - \int_{\mathbb{R}^3} \langle [\Gamma^a,\, \Box] u,\, \partial_t \Gamma^a u\rangle\, dx \quad \text{if} \quad n=3. \qquad (6.8)
\end{aligned}
$$

We also note that, under the smallness of $E_\mu^{1/2}\big(u(t)\big)$ $(0 \le t < T_0)$ with $\mu = \kappa - 2$, the inequality

$$
\frac{1}{2} E_l\big(u(t)\big) \le \tilde{E}_l\big(u(t)\big) \le 2E_l\big(u(t)\big), \quad l = 1, \ldots, \kappa \qquad (6.9)
$$

holds by the Sobolev embedding.

We plan our energy integral method, allowing the higher-order energy $E_\kappa\big(u(t)\big)$ $(\kappa \ge 9)$ to grow polynomially in time but bounding the lower-order energy $E_\mu\big(u(t)\big)$ $(\mu = \kappa - 2)$ uniformly in time. Let us start with the estimate of the higher-order energy. Setting $l = \kappa$ in (6.7), we have for $n = 2$

$$
\begin{aligned}
&\tilde{E}_\kappa'\big(u(t)\big)\\
&\le \sum_{i,j,l} \sum_{|a|\le\kappa-1} \sum_{\substack{|b|+|c|+|d|\le|a|\\ d\neq a}} \|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial^2 \Gamma^d u^l\|_{L^2} \|\partial\Gamma^a u\|_{L^2}\\
&\quad + \sum_{i,j,l} \sum_{|a|\le\kappa-1} \sum_{|b|+|c|+|d|\le|a|} \|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial\Gamma^d u^l\|_{L^2} \|\partial\Gamma^a u\|_{L^2}. \qquad (6.10)
\end{aligned}
$$

Set $q = [\kappa/2]$. Note that $q + 3 \le \mu$ because of $\kappa \ge 9$. Supposing $|b| + |c| \le q$ without loss of generality, we bound the second term as

$$
\begin{aligned}
&\|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial\Gamma^d u^l\|_{L^2} \\
&\le C\langle t\rangle^{-1}\|\langle r\rangle^{1/2}\langle c_i t - r\rangle^{1/2}\partial\Gamma^b u^i\|_{L^\infty} \\
&\qquad\qquad \times \|\langle r\rangle^{1/2}\langle c_j t - r\rangle^{1/2}\partial\Gamma^c u^j\|_{L^\infty}\|\partial\Gamma^d u^l\|_{L^2} \\
&\le C\langle t\rangle^{-1}\big(E_\mu^{1/2}(u(t)) + M_\mu(u(t))\big)^2 E_\kappa^{1/2}(u(t)) \\
&\le C\langle t\rangle^{-1}E_\mu(u(t))E_\kappa^{1/2}(u(t)).
\end{aligned} \tag{6.11}
$$

Here we have employed (5.12) at the third inequality. As for the first terms on the right-hand side of (6.10), we sort them out into two groups: $|b| + |c| \le q$ or $|d| \le q-1$. The first group is estimated as in (5.6) and (6.11):

$$
\begin{aligned}
&\|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial^2\Gamma^d u^l\|_{L^2} \\
&\le C\langle t\rangle^{-1}\big(E_\mu^{1/2}(u(t)) + M_\mu(u(t))\big)^2 M_\kappa(u(t)) \\
&\le C\langle t\rangle^{-1}E_\mu(u(t))E_\kappa^{1/2}(u(t)).
\end{aligned} \tag{6.12}
$$

Otherwise, assuming $|b| \le q$ in addition to $|d| \le q-1$ without loss of generality, we get as in (5.7)

$$
\begin{aligned}
&\|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial^2\Gamma^d u^l\|_{L^2} \\
&\le C\langle t\rangle^{-1}E_\mu^{1/2}(u(t))M_\mu(u(t))E_{\kappa-1}^{1/2}(u(t)) \\
&\le C\langle t\rangle^{-1}E_\mu(u(t))E_\kappa^{1/2}(u(t)).
\end{aligned} \tag{6.13}
$$

Taking account of the equivalence between E_l and $\tilde{E}_l$, we get from (6.10)-(6.13)

$$
\tilde{E}_\kappa'(u(t)) \le C\langle t\rangle^{-1}\tilde{E}_\mu(u(t))\tilde{E}_\kappa(u(t)). \tag{6.14}
$$

Let us turn our attention to $n = 3$. It follows from (6.8) that

$$
\begin{aligned}
&\tilde{E}_\kappa'(u(t)) \\
&\le C\sum_{i,j}\sum_{|a|\le\kappa-1}\sum_{\substack{|b|+|c|\le|a| \\ c\ne a}}\|\partial\Gamma^b u^i \partial^2\Gamma^c u^j\|_{L^2}\|\partial\Gamma^a u\|_{L^2} \\
&\quad + C\sum_{i,j}\sum_{|a|\le\kappa-1}\sum_{|b|+|c|\le|a|}\|\partial\Gamma^b u^i \partial\Gamma^c u^j\|_{L^2}\|\partial\Gamma^a u\|_{L^2}.
\end{aligned} \tag{6.15}
$$

For the first term which is the contribution from the quasi-linear parts, we may follow Sideris and Tu [31] to get

$$\|\partial\Gamma^b u^i \partial^2\Gamma^c u^j\|_{L^2} \le C\langle t\rangle^{-1} E_\mu^{1/2}\big(u(t)\big) E_\kappa^{1/2}\big(u(t)\big) \tag{6.16}$$

(see [31] on page 485). On the other hand, the second term is treated in quite the same way as in (5.8)-(5.9):

$$\begin{aligned}
&\|\partial\Gamma^b u^i \partial\Gamma^c u^j\|_{L^2}\\
&\le C\langle t\rangle^{-1}\big[\big(E_\mu^{1/2}\big(u(t)\big) + M_\mu\big(u(t)\big)\big) E_\kappa^{1/2}\big(u(t)\big)\\
&\qquad\qquad + E_\mu^{1/2}\big(u(t)\big) E_\kappa^{1/2}\big(u(t)\big)\big]\\
&\le C\langle t\rangle^{-1} E_\mu^{1/2}\big(u(t)\big) E_\kappa^{1/2}\big(u(t)\big).
\end{aligned} \tag{6.17}$$

Taking account of (6.9) again, we finally have

$$\tilde{E}_\kappa'\big(u(t)\big) \le C\langle t\rangle^{-1} \tilde{E}_\mu^{1/2}\big(u(t)\big) \tilde{E}_\kappa\big(u(t)\big). \tag{6.18}$$

Note The almost global existence theorem of John and Klainerman which was shown under the hypothesis (H_1) of [15] follows from (6.18).

Lower-order Energy The crucial part in the proof of global existence is to bound the lower-order energy $E_\mu(u(t))$ ($\mu = \kappa - 2$) uniformly in time. For the purpose we exploit an improved decay rate of solutions inside the cone as well as the difference of propagation speeds to sharpen the decay estimates presented above, when $|a| \le \mu$.

Set $c_0 := \min\{c_i/2 : i = 1, \ldots, m\}$ and $\mu = \kappa - 2$ ($\kappa \ge 9$). Setting $l = \mu$ in (6.7) and (6.8), we estimate the resulting terms on the right-hand side. Divide the integral region $\mathbb{R}^2$ or $\mathbb{R}^3$ into two parts: inside the cone $\{(t, x) : |x| \le c_0 t\}$ and away from the origin $\{(t, x) : |x| \ge c_0 t\}$.

Inside the cone Here we exploit an improved decay rate of solutions. Let us start with $n = 2$. The contribution from the quasi-linear terms is bounded by

$$\sum_{i,j,l}\sum_{|a|\le\mu-1}\sum_{\substack{|b|+|c|+|d|\le|a|\\ d\ne a}} \|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial^2\Gamma^d u^l\|_{L^2(r<c_0t)} \|\partial\Gamma^a u\|_{L^2}. \tag{6.19}$$

We may suppose $|b| \le [\mu/2]$ without loss of generality. It then follows from (4.18) and (5.12) that

$$
\begin{aligned}
&\|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial^2\Gamma^d u^l\|_{L^2}\\
&\le \langle t\rangle^{-3/2}\|\langle c_i t-r\rangle^{1/2}\partial\Gamma^b u^i\|_{L^\infty(r<c_0t)}\|\partial\Gamma^c u^j\|_{L^\infty}\\
&\qquad\qquad \times \|\langle c_l t-r\rangle\partial^2\Gamma^d u^l\|_{L^2(r<c_0t)}\\
&\le C\langle t\rangle^{-3/2}\big(E^{1/2}_{|b|+2}\big(u(t)\big)+M_{|b|+3}\big(u(t)\big)\big)E^{1/2}_{|c|+3}\big(u(t)\big)M_\mu\big(u(t)\big)\\
&\le C\langle t\rangle^{-3/2}E^{1/2}_\kappa\big(u(t)\big)E_\mu\big(u(t)\big),
\end{aligned}
\tag{6.20}
$$

where we have used $|b|+3\le[\mu/2]+3\le\mu$, $|c|+3\le\kappa$. Concerning the contribution from the semi-linear parts, we see from (6.7) that it is bounded by

$$
\sum_{i,j,l}\sum_{|a|\le\mu-1}\sum_{|b|+|c|+|d|\le|a|}\|\partial\Gamma^b u^i\partial\Gamma^c u^j\partial\Gamma^d u^l\|_{L^2(r<c_0t)}\|\partial\Gamma^a u\|_{L^2}. \tag{6.21}
$$

Since u is supported in $\{|x|\le c^*t+R\}$ for a constant $R>0$ and $c^*:=\max\{c_i : i=1,\,\dots,\,m\}$, it follows from (4.20) that for $|x|<c_0t$

$$
\begin{aligned}
&\langle t\rangle^{1-\eta}|\partial\Gamma^b u^i(t,\,x)|\\
&\le C\big(E^{1/2}_{|b|+2}\big(u(t)\big)+M_{|b|+3}\big(u(t)\big)\big),\quad 0<\eta<\frac{1}{2}.
\end{aligned}
\tag{6.22}
$$

Assuming $|b|+|c|\le[\mu/2]$ without loss of generality, we proceed

$$
\begin{aligned}
&\|\partial\Gamma^b u^i\partial\Gamma^c u^j\partial\Gamma^d u^l\|_{L^2(r<c_0t)}\\
&\le C\langle t\rangle^{-(3/2)+\eta}\|\langle t\rangle^{1-\eta}\partial\Gamma^b u^i\|_{L^\infty(r<c_0t)}\\
&\qquad\qquad\qquad \times\|\langle c_j t-r\rangle^{1/2}\partial\Gamma^c u^j\|_{L^\infty(r<c_0t)}\|\partial\Gamma^d u^l\|_{L^2}\\
&\le C\langle t\rangle^{-(3/2)+\eta}\big(E^{1/2}_{|b|+2}\big(u(t)\big)+M_{|b|+3}\big(u(t)\big)\big)\\
&\quad\times\big(E^{1/2}_{|c|+2}\big(u(t)\big)+M_{|c|+3}\big(u(t)\big)\big)E^{1/2}_\mu\big(u(t)\big)\\
&\le C\langle t\rangle^{-(3/2)+\eta}E^{3/2}_\mu\big(u(t)\big).
\end{aligned}
\tag{6.23}
$$

Hence the estimate inside the cone has been finished.

Turning our attention to $n=3$, we see from (6.8) with $l=\mu$ that the contribution from the quasi-linear terms is bounded by

$$
\sum_{i,j}\sum_{|a|\le\mu-1}\sum_{\substack{|b|+|c|\le|a|\\ c\ne a}}\|\partial\Gamma^b u^i\partial^2\Gamma^c u^j\|_{L^2(r<c_0t)}\|\partial\Gamma^a u\|_{L^2}, \tag{6.24}
$$

which is estimated as

$$
\dots\le C\langle t\rangle^{-3/2}E^{1/2}_\kappa\big(u(t)\big)E_\mu\big(u(t)\big) \tag{6.25}
$$

(see [31] on page 486).

On the other hand, the contribution from the semi-linear terms is bounded by

$$\sum_{i,j}\sum_{|a|\le\mu-1}\sum_{|b|+|c|\le|a|}\|\partial\Gamma^b u^i\partial\Gamma^c u^j\|_{L^2(r<c_0t)}\|\partial\Gamma^a u\|_{L^2}, \tag{6.26}$$

which we deal with as follows. Suppose $|b|\le[\mu/2]$ without loss of generality. Employing (4.24) and the Hardy inequality, we get

$$\begin{aligned}
&\|\partial\Gamma^b u^i\partial\Gamma^c u^j\|_{L^2(r<c_0t)}\\
&\le\langle t\rangle^{-3/2}\|r\langle c_it-r\rangle^{1/2}\partial\Gamma^b u^i\|_{L^\infty(r<c_0t)}\left\|\frac{1}{r}\langle c_jt-r\rangle\partial\Gamma^c u^j\right\|_{L^2(r<c_0t)}\\
&\le C\langle t\rangle^{-3/2}\big(E^{1/2}_{|b|+3}(u(t))+M_{|b|+3}(u(t))\big)\\
&\qquad\times\big(E^{1/2}_{|c|+1}(u(t))+M_{|c|+2}(u(t))\big)\\
&\le C\langle t\rangle^{-3/2}E^{1/2}_{\mu}(u(t))E^{1/2}_{\kappa}(u(t)),
\end{aligned}\tag{6.27}$$

as desired.

Away from the origin Here the difference of propagation speeds comes into play. Moreover, we employ the null condition (3.6) for the estimates of resonance terms.

Non-resonance Let us start with non-resonance terms. Our task for $n=2$ is to estimate the contribution from quasi-linear terms

$$\sum_{(i,j,l)\ne(k,k,k)}\sum_{|a|\le\mu-1}\sum_{\substack{|b|+|c|+|d|\le|a|\\ d\ne a}}\|\partial\Gamma^b u^i\partial\Gamma^c u^j\partial^2\Gamma^d u^l\partial\Gamma^a u^k\|_{L^1(r>c_0t)} \tag{6.28}$$

as well as the contribution from semi-linear terms

$$\sum_{(i,j,l)\ne(k,k,k)}\sum_{|a|\le\mu-1}\sum_{|b|+|c|+|d|\le|a|}\|\partial\Gamma^b u^i\partial\Gamma^c u^j\partial\Gamma^d u^l\partial_t\Gamma^a u^k\|_{L^1(r>c_0t)}. \tag{6.29}$$

In estimating the L^1-norm in (6.28) we separate two cases: $i=j=l$ or otherwise. In the former case, noting $i\ne k$, we have

$$
\begin{aligned}
&\|\partial\Gamma^b u^i \partial\Gamma^c u^i \partial^2\Gamma^d u^i \partial\Gamma^a u^k\|_{L^1(r>c_0t)}\\
&\le \langle t\rangle^{-3/2}\|\langle r\rangle^{1/2}\partial\Gamma^b u^i\|_{L^\infty}\|\partial\Gamma^c u^i\|_{L^2}\\
&\qquad\qquad \times \|\langle c_i t - r\rangle\partial^2\Gamma^d u^i\|_{L^2}\|\langle r\rangle^{1/2}\langle c_k t - r\rangle^{1/2}\partial\Gamma^a u^k\|_{L^\infty}\\
&\le C\langle t\rangle^{-3/2}E^{1/2}_{|b|+3}\big(u(t)\big)E^{1/2}_{|c|+1}\big(u(t)\big)\\
&\qquad\qquad \times M_{|d|+2}\big(u(t)\big)\big(E^{1/2}_{|a|+2} + M_{|a|+3}\big(u(t)\big)\big)\\
&\le C\langle t\rangle^{-3/2}E^{1/2}_{\mu+2}\big(u(t)\big)E^{1/2}_{\mu}\big(u(t)\big)M_\mu\big(u(t)\big)E^{1/2}_{\mu+2}\big(u(t)\big)\\
&\le C\langle t\rangle^{-3/2}E_\kappa\big(u(t)\big)E_\mu\big(u(t)\big).
\end{aligned}
\tag{6.30}
$$

Otherwise, it is easy to get

$$
\begin{aligned}
&\|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial^2\Gamma^d u^l \partial\Gamma^a u^k\|_{L^1(r>c_0t)}\\
&\le \langle t\rangle^{-3/2}\|\langle r\rangle^{1/2}\langle c_i t - r\rangle^{1/2}\partial\Gamma^b u^i\|_{L^\infty}\\
&\quad \times \|\langle r\rangle^{1/2}\langle c_j t - r\rangle^{1/2}\partial\Gamma^c u^j\|_{L^\infty}\|\langle c_l t - r\rangle\partial^2\Gamma^d u^l\|_{L^2}\|\partial\Gamma^a u^k\|_{L^2}\\
&\le C\langle t\rangle^{-3/2}E_\kappa\big(u(t)\big)E_\mu\big(u(t)\big).
\end{aligned}
\tag{6.31}
$$

As for (6.29), we may suppose $i \ne k$ without loss of generality. We separate two cases: $|b| \le [\mu/2]-1$ or $|c|+|d| \le [\mu/2]$. If $|b| \le [\mu/2]-1$, we may suppose $|c| \le [\mu/2]$ as well, and obtain

$$
\begin{aligned}
&\|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial\Gamma^d u^l \partial_t\Gamma^a u^k\|_{L^1(r>c_0t)}\\
&\le \langle t\rangle^{-3/2}\|\langle r\rangle^{1/2}\langle c_i t - r\rangle^{1/2}\partial\Gamma^b u^i\|_{L^\infty(r>c_0t)}\\
&\qquad\qquad \times \|\partial\Gamma^c u^j\partial\Gamma^d u^l\|_{L^1}\|\langle r\rangle^{1/2}\langle c_k t - r\rangle^{1/2}\partial_t\Gamma^a u^k\|_{L^\infty(r>c_0t)}\\
&\le C\langle t\rangle^{-3/2}\big(E^{1/2}_{|b|+2}\big(u(t)\big) + M_{|b|+3}\big(u(t)\big)\big)E^{1/2}_\mu\big(u(t)\big)E^{1/2}_\kappa\big(u(t)\big)\\
&\qquad\qquad \times \big(E^{1/2}_{|a|+2}\big(u(t)\big) + M_{|a|+3}\big(u(t)\big)\big)\\
&\le C\langle t\rangle^{-3/2}E_\kappa\big(u(t)\big)E_\mu\big(u(t)\big).
\end{aligned}
\tag{6.32}
$$

Otherwise, we have $|c|+|d| \le [\mu/2]$ and it suffices to modify the argument in (6.32) properly. Therefore the estimates of non-resonance terms have been completed for $n = 2$.

For $n = 3$ the estimates are carried out as follows. We first collect the contributions from quasi-linear terms

$$
\sum_{(i,j)\ne(k,k)}\ \sum_{|a|\le\mu-1}\ \sum_{\substack{|b|+|c|\le|a|\\ c\ne a}} \|\partial\Gamma^b u^i \partial^2\Gamma^c u^j \partial\Gamma^a u^k\|_{L^1(r>c_0t)}
\tag{6.33}
$$

and from semi-linear terms

$$\sum_{(i,j)\neq(k,k)} \sum_{|a|\le\mu-1} \sum_{|b|+|c|\le|a|} \|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial_t \Gamma^a u^k\|_{L^1(r>c_0t)}. \tag{6.34}$$

We again separate two cases in estimating the L^1-norm in (6.33): $i=j$ or otherwise. In the former case, because of $i\neq k$, we can proceed as

$$\begin{aligned}
&\|\partial\Gamma^b u^i \partial^2\Gamma^c u^i \partial\Gamma^a u^k\|_{L^1(r>c_0t)}\\
&\le \langle t\rangle^{-3/2}\|\partial\Gamma^b u^i\|_{L^2}\|\langle c_it-r\rangle\partial^2\Gamma^c u^i\|_{L^2}\|\langle r\rangle\langle c_kt-r\rangle^{1/2}\partial\Gamma^a u^k\|_{L^\infty}\\
&\le C\langle t\rangle^{-3/2}E_\mu^{1/2}(u(t))M_\mu(u(t))\big(E_{|a|+3}^{1/2}(u(t))+M_{|a|+3}(u(t))\big)\\
&\le C\langle t\rangle^{-3/2}E_\mu(u(t))E_\kappa^{1/2}(u(t)).
\end{aligned} \tag{6.35}$$

Otherwise,

$$\begin{aligned}
&\|\partial\Gamma^b u^i \partial^2\Gamma^c u^j\|_{L^2(r>c_0t)}\|\partial\Gamma^a u^k\|_{L^2}\\
&\le \langle t\rangle^{-3/2}\|\langle r\rangle\langle c_it-r\rangle^{1/2}\partial\Gamma^b u^i\|_{L^\infty}\|\langle c_jt-r\rangle\partial^2\Gamma^c u^j\|_{L^2}E_\mu^{1/2}(u(t))\\
&\le C\langle t\rangle^{-3/2}\big(E_{|b|+3}^{1/2}(u(t))+M_{|b|+3}(u(t))\big)M_{|c|+2}(u(t))E_\mu^{1/2}(u(t))\\
&\le C\langle t\rangle^{-3/2}E_\kappa^{1/2}(u(t))E_\mu(u(t)).
\end{aligned} \tag{6.36}$$

As for the estimate of (6.34), we may suppose $i\neq k$. If the propagation speeds satisfy $c_i<c_k$, then

$$\begin{aligned}
&\|\partial\Gamma^b u^i \partial\Gamma^c u^j \partial_t\Gamma^a u^k\|_{L^1(r>c_0t)}\\
&\le \langle t\rangle^{-3/2}\|\partial\Gamma^b u^i\partial\Gamma^c u^j\|_{L^1}\|\langle r\rangle\langle c_kt-r\rangle^{1/2}\partial\Gamma^a u^k\|_{L^\infty(c_0t<r<(c_i+c_k)t/2)}\\
&\quad+\langle t\rangle^{-3/2}\|\partial\Gamma^c u^j\partial\Gamma^a u^k\|_{L^1}\|\langle r\rangle\langle c_it-r\rangle^{1/2}\partial\Gamma^b u^i\|_{L^\infty(r>(c_i+c_k)t/2)}\\
&\le C\langle t\rangle^{-3/2}E_\kappa^{1/2}(u(t))E_\mu(u(t))
\end{aligned} \tag{6.37}$$

by (4.24). Otherwise, we have $c_i>c_k$ and it is enough to modify (6.37) properly. Hence we have finished the estimates of non-resonance terms away from the origin.

Resonance The resonance terms remain to be estimated away from the origin. It is just the place where the null condition comes into play. Without the null condition, the solution may become singular in finite time (see, e.g., John [12]). In view of Lemma 4.2 and (6.7)-(6.8), the estimate is reduced to bounding

$$\begin{aligned}
&\sum_{1\le k\le m}\sum_{|a|\le\mu-1}\sum_{\substack{|b|+|c|+|d|\le|a|\\ d\neq a}}\langle t\rangle^{-1}\bigl(\|\Gamma^{b+1}u^k\partial\Gamma^c u^k\partial^2\Gamma^d u^k\|_{L^2(r>c_0t)}\\
&+\|\partial\Gamma^b u^k\partial\Gamma^c u^k\partial\Gamma^{d+1}u^k\|_{L^2(r>c_0t)}\\
&+\|\langle c_kt-r\rangle\partial\Gamma^b u^k\partial\Gamma^c u^k\partial^2\Gamma^d u^k\|_{L^2(r>c_0t)}\bigr)\|\partial\Gamma^a u\|_{L^2}\\
&+\sum_{1\le k\le m}\sum_{|a|\le\mu-1}\sum_{|b|+|c|+|d|\le|a|}\langle t\rangle^{-1}\bigl(\|\Gamma^{b+1}u^k\partial\Gamma^c u^k\partial\Gamma^d u^k\|_{L^2(r>c_0t)}\\
&+\|\langle c_kt-r\rangle\partial\Gamma^b u^k\partial\Gamma^c u^k\partial\Gamma^d u^k\|_{L^2(r>c_0t)}\bigr)\|\partial\Gamma^a u\|_{L^2}
\end{aligned}\tag{6.38}$$

for $n=2$, and

$$\begin{aligned}
&\sum_{1\le k\le m}\sum_{|a|\le\mu-1}\sum_{\substack{|b|+|c|\le|a|\\ c\neq a}}\langle t\rangle^{-1}\bigl(\|\Gamma^{b+1}u^k\partial^2\Gamma^c u^k\|_{L^2(r>c_0t)}\\
&\qquad+\|\partial\Gamma^b u^k\partial\Gamma^{c+1}u^k\|_{L^2(r>c_0t)}\\
&\qquad+\|\langle c_kt-r\rangle\partial\Gamma^b u^k\partial^2\Gamma^c u^k\|_{L^2(r>c_0t)}\bigr)\|\partial\Gamma^a u\|_{L^2}\\
&+\sum_{1\le k\le m}\sum_{|a|\le\mu-1}\sum_{|b|+|c|\le|a|}\langle t\rangle^{-1}\bigl(\|\Gamma^{b+1}u^k\partial\Gamma^c u^k\|_{L^2(r>c_0t)}\\
&\qquad+\|\langle c_kt-r\rangle\partial\Gamma^b u^k\partial\Gamma^c u^k\|_{L^2(r>c_0t)}\bigr)\|\partial\Gamma^a u\|_{L^2}
\end{aligned}\tag{6.39}$$

for $n=3$. Here, by $b+1$, we mean any sequence of length $|b|+1$.

Using (4.19) and Lemma 4.4, we estimate the first norm on the right-hand side of (6.38)

$$\begin{aligned}
&\langle t\rangle^{-1/2}\left\|\frac{1}{\langle c_kt-r\rangle}\Gamma^{b+1}u^k\partial\Gamma^c u^k\langle r\rangle^{1/2}\langle c_kt-r\rangle\partial^2\Gamma^d u^k\right\|_{L^2(r>c_0t)}\\
&\le\langle t\rangle^{-1/2}\left\|\frac{1}{\langle c_kt-r\rangle}\Gamma^{b+1}u^k\right\|_{L^2}\|\partial\Gamma^c u^k\|_{L^\infty}\\
&\qquad\times\|\langle r\rangle^{1/2}\langle c_kt-r\rangle\partial^2\Gamma^d u^k\|_{L^\infty}\\
&\le C\langle t\rangle^{-1/2}E_\kappa^{1/2}\bigl(u(t)\bigr)E_\mu\bigl(u(t)\bigr).
\end{aligned}\tag{6.40}$$

Assuming $|b|\le|c|$ without loss of generality, we estimate the second and third norms on the right-hand side of (6.38) as

$$
\begin{aligned}
&\langle t\rangle^{-1/2}\|\langle r\rangle^{1/2}\partial\Gamma^b u^k\|_{L^\infty(r>c_0t)}\|\partial\Gamma^c u^k\|_{L^\infty}\|\partial\Gamma^{d+1}u^k\|_{L^2}\\
&+\langle t\rangle^{-1/2}\|\langle r\rangle^{1/2}\partial\Gamma^b u^k\|_{L^\infty(r>c_0t)}\|\partial\Gamma^c u^k\|_{L^\infty}\|\langle c_kt-r\rangle\partial^2\Gamma^d u^k\|_{L^2}\\
&\le C\langle t\rangle^{-1/2}E_\kappa^{1/2}\big(u(t)\big)E_\mu\big(u(t)\big). \qquad (6.41)
\end{aligned}
$$

The remaining terms in (6.38) are estimated as

$$
\begin{aligned}
&\langle t\rangle^{-1}\bigg\|\frac{1}{\langle c_kt-r\rangle}\Gamma^{b+1}u^k\langle r\rangle^{1/2}\langle c_kt-r\rangle^{1/2}\\
&\qquad\times\partial\Gamma^c u^k\langle r\rangle^{1/2}\langle c_kt-r\rangle^{1/2}\partial\Gamma^d u^k\bigg\|_{L^2(r>c_0t)}\\
&+\langle t\rangle^{-1}\|\langle r\rangle^{1/2}\langle c_kt-r\rangle^{1/2}\\
&\qquad\times\partial\Gamma^b u^k\langle r\rangle^{1/2}\langle c_kt-r\rangle^{1/2}\partial\Gamma^c u^k\partial\Gamma^d u^k\|_{L^2(r>c_0t)}\\
&\le C\langle t\rangle^{-1}E_\kappa^{1/2}\big(u(t)\big)E_\mu\big(u(t)\big), \qquad (6.42)
\end{aligned}
$$

thanks to (4.18) and Lemma 4.4.

Collecting (6.40)-(6.42), we have shown that (6.38) is estimated as

$$
\cdots\le C\langle t\rangle^{-3/2}E_\kappa^{1/2}\big(u(t)\big)E_\mu^{3/2}\big(u(t)\big). \qquad (6.43)
$$

Only (6.39) remains to be bounded. The first term of (6.39), which is the contribution from the quasi-linear terms, has already been estimated by Sideris and Tu [31] as

$$
\cdots\le C\langle t\rangle^{-3/2}E_\kappa^{1/2}E_\mu\big(u(t)\big). \qquad (6.44)
$$

The second term of (6.39), which is the contribution from the semi-linear terms, is estimated as

$$
\begin{aligned}
&\langle t\rangle^{-1/2}\|\langle r\rangle^{1/2}\Gamma^{b+1}u^k\partial\Gamma^c u^k\|_{L^2(r>c_0t)}\\
&+\langle t\rangle^{-1/2}\|\langle r\rangle^{1/2}\langle c_kt-r\rangle\partial\Gamma^b u^k\partial\Gamma^c u^k\|_{L^2(r>c_0t)}\\
&\le C\langle t\rangle^{-1/2}E_\kappa^{1/2}\big(u(t)\big)E_\mu^{1/2}\big(u(t)\big), \qquad (6.45)
\end{aligned}
$$

thanks to (4.21) and (4.22). We have therefore proved that (6.39) is bounded as

$$
\cdots\le C\langle t\rangle^{-3/2}E_\kappa^{1/2}\big(u(t)\big)E_\mu\big(u(t)\big) \qquad (6.46)
$$

as desired.

Collecting the estimates of $\tilde{E}_\mu'\big(u(t)\big)$ and taking (6.9) into account, we have obtained

$$\begin{aligned}
&\tilde{E}'_\mu\big(u(t)\big)\\
&\le C\langle t\rangle^{-3/2}\tilde{E}^{1/2}_\kappa\big(u(t)\big)\tilde{E}^{3/2}_\mu\big(u(t)\big)\\
&\quad + C\langle t\rangle^{-(3/2)+\eta}\tilde{E}^{2}_\mu\big(u(t)\big)\\
&\quad + C\langle t\rangle^{-3/2}\tilde{E}_\kappa\tilde{E}_\mu\big(u(t)\big)\\
&\le C\langle t\rangle^{-(3/2)+\eta}\tilde{E}_\kappa\big(u(t)\big)\tilde{E}_\mu\big(u(t)\big)
\end{aligned}\tag{6.47}$$

for $n=2$, and

$$\tilde{E}'_\mu\big(u(t)\big)\le C\langle t\rangle^{-3/2}\tilde{E}^{1/2}_\kappa\big(u(t)\big)\tilde{E}_\mu\big(u(t)\big)\tag{6.48}$$

for $n=3$.

Now we are ready to complete the proof of our main theorem. Assuming $E^{1/2}_\mu\big(u(t)\big)<2\varepsilon$ $(0\le t<T_0)$ for a sufficiently small ε such that $2\varepsilon\le\varepsilon_0$ for ε_0 in (5.10), we get from (6.14), (6.18)

$$\begin{aligned}
&\tilde{E}_\kappa\big(u(t)\big)\le\tilde{E}_\kappa\big(u(0)\big)\langle t\rangle^{C\varepsilon^2}\quad\text{if } n=2,\\
&\tilde{E}_\kappa\big(u(t)\big)\le\tilde{E}_\kappa\big(u(0)\big)\langle t\rangle^{C\varepsilon}\quad\text{if } n=3.
\end{aligned}$$

Inserting this into (6.47)-(6.48), we have

$$\frac{1}{2}E_\mu\big(u(t)\big)\le\tilde{E}_\mu\big(u(t)\big)\le\tilde{E}_\mu\big(u(0)\big)\exp\big[A\tilde{E}^\theta_\kappa\big(u(0)\big)\big]\tag{6.49}$$

$(\theta=1$ for $n=2$, $=1/2$ for $n=3)$. Recalling the assumption on the size of data (3.9), we have finally obtained

$$E^{1/2}_\mu(u(t))\le\sqrt{3}\varepsilon,\ \ 0\le t<T_0.\tag{6.50}$$

The last inequality proves that the norm $E^{1/2}_\mu\big(u(t)\big)$ is strictly smaller than 2ε on the closed interval $[0,\,T_0]$. The proof of the main theorem has been completed. □

7. A note on the Klainerman-Sideris method

In obedience to the referee's report we give some comments on the Klainerman-Sideris method. It indeed seems interesting to consider further applications of the superb method to the problem of global existence for systems of equations which include the unknown u itself in the nonlinear term. An example of physical importance has been studied by Ozawa, Tsutaya and Tsutsumi [27], where it is shown by the method of the $X_{s,b}$-space due to Bourgain [3]-[4], Kenig, Ponce and Vega [19] that the Klein-Gordon-Zakharov system, which has different speeds and includes unknown functions themselves in the quadratic nonlinear term, has a unique small global solution in three space dimensions. On the one hand, the Klainerman-Sideris inequality includes the radial vector field $S = x^\alpha \partial_\alpha$, on the other hand the use of radial vector fields is not compatible with the analysis of Klein-Gordon equations because of the presence of the mass term. Unfortunately, the excellent method of Klainerman and Sideris does not seem efficient in the analysis of such an intriguing system of physical importance, which all the more underscores the necessity for making efforts to make new progress in the method of commuting vector fields.

Finally the author would like to take this opportunity to mention that, in the absence of mass terms, the problem of global solvability of systems of fully nonlinear wave equations with multiple speeds has been studied in the elaborate works due to Hoshiga and Kubo [11], Kubota and Yokoyama [25] in two and three space dimensions, respectively. In their analysis the use of radial vector fields has made a comeback due to the absence of mass terms. They have also made better use of some refined L^∞-estimates of solutions themselves to inhomogeneous equations which directly follow from estimates of the fundamental solution.

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Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji
Tokyo 192-0397, Japan

Current address:
Department of Mathematics
Faculty of Education, Mie University
1515 Kamihama, Tsu
Mie 514-8507, Japan