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# Forward limit sets of singularities for the Lozi family

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**Abstract.** The Lozi family is a two-parameter family of piecewise affine uniformly hyperbolic maps on $\mathbb{R}^2$ with strange attractors. We find an open set $\mathcal{O}$ in the parameter space such that, for almost every parameter in $\mathcal{O}$, the forward limit set of a point in the $y$-axis which is a singularity in a trapping region coincides with the strange attractor. This is an extension of the corresponding result about turning orbits in the dynamical core of tent maps on $\mathbb{R}$ by Brucks and Misiurewicz.

*Key words*: Lozi attractors, singularity set for piecewise hyperbolic maps, $\omega$-limit set.

## 1. Introduction

The Lozi map is a homeomorphism on $\mathbb{R}^2$ given by

$$f_{a,b}(x, y) = (1 - a|x| + y, bx)$$

for $(x, y) \in \mathbb{R}^2$ where $a$ and $b$ are real parameters. This family was introduced by Lozi [12] as an piecewise affine analogue of the Hénon family, which is now one of the central subjects of study in dynamical system theory [6, 7, 15, 16, 17, 18, 19]. Misiurewicz showed that the map $f_{a,b}$ admits a unique strange attractor $\Lambda_{a,b}$ if $(a, b)$ belongs to the open set $\mathcal{M}$ defined by the inequalities:

$$\begin{cases} 0 < b < 1,\ a > b+1,\ 2a+b < 4, \\ a\sqrt{2} > b+2,\ b < (a^2-1)/(2a+1). \end{cases} \tag{1}$$

This strange attractor (the Lozi attractor) has "almost" hyperbolic structure, that is, there is a uniform hyperbolic structure out of the $y$-axis where the Lozi maps are not differentiable, see [13]. However, by the influence of the singularities in the $y$-axis, the dynamics of the Lozi maps are quite delicate [8, 9, 10, 11].

To state our main result, we describe trapping region and singularity set of the Lozi maps, as follows. For any $(a, b) \in \mathcal{M}$, $f_{a,b}$ has a saddle fixed

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point $p_{a,b} = (1/(1+a-b), b/(1+a-b))$ which is contained in the first quadrant. The unstable set $W^u(p_{a,b})$ of $p_{a,b}$ contains the line segment that connects $p_{a,b}$ and the point $z_{a,b} = ((2+a+\sqrt{a^2+4b})/(2+2a-2b), 0)$ on the $x$-axis. The triangle $T_{a,b}$ with vertices $z_{a,b}$, $f_{a,b}(z_{a,b})$ and $f^2_{a,b}(z_{a,b})$ is a *trapping region* of $f_{a,b}$, that is, $f_{a,b}(T_{a,b}) \subset T_{a,b}$. By [13], the Lozi attractor is given by

$$\Lambda_{a,b} = \bigcap_{i \geq 0} f^i_{a,b}(T_{a,b}),$$

which coinsides with the closure of $W^u(p_{a,b})$. We denote by $\mathcal{Y}_{a,b}$ the segment of the $y$-axis in $T_{a,b}$. In this paper, we consider the forward orbits of the singularities in $\mathcal{Y}_{a,b}$, and show that their $\omega$-limit sets coincide with $\Lambda_{a,b}$ for almost every $(a, b)$ in some parameter region. The main result is

**Main Theorem** *There exists an open set $\mathcal{O} \subset \mathcal{M}$ whose closure contains $(2, 0)$ such that, for Lebesugue almost every $((a, b), z) \in \{((a, b), z) \mid (a, b) \in \mathcal{O}, z \in \mathcal{Y}_{a,b}\}$, the $\omega$-limit set $\omega(z, f_{a,b})$ coincides with the Lozi attractor $\Lambda_{a,b}$.*

When $b = 0$, the Lozi maps are equivalent to the tent maps $t_a(x) = 1 - a|x|$. It has a turning point $x = 0$ whose forward orbit can not escape from its dynamical core $\Lambda(t_a) = [t_a^2(0), t_a(0)]$ for $1 < a < 2$. Brucks and others [3] found a $G_\delta$-dense subset of $a \in [\sqrt{2}, 2]$ such that the forward orbit of $x = 0$ is dense in $\Lambda(t_a)$. Brucks and Misiurewicz showed in [4] that almost every $a \in [\sqrt{2}, 2]$ satisfies $\omega(0, t_a) = \Lambda(t_a)$. The main theorem of this paper is an extension of this result to the 2-dimensional context. In the 1-dimensional case, Brucks and Buczolich [1] showed that the complement of such parameters is $\sigma$-porous, and Bruin [5] showed that, for almost every parameter value, the turning orbit is typical for an absolutely continuous invariant probability measure. (See also [2].) However, the corresponding results for the Lozi family are not known.

## 2. Proof of Main Theorem

Let $f_{a,b}$ be the Lozi family, and $\mathcal{M}$ the parameter set defined by (1). Let $\mathcal{O} \subset \mathcal{M}$ be a small open set which is specified concretely in the following sections. At this stage, it is enough to keep in mind that it is small and $(2, 0) \in \mathrm{cl}(\mathcal{O})$, where $\mathrm{cl}(\cdot)$ is the closure of the corresponding set. Let us fix a point $(a_0, b_0) \in \mathcal{O}$ arbitrarily in the argument belows. Let $I \subset \mathbb{R}$ be a neighborhood of $a_0$ such that $I \times \{b_0\} \subset \mathcal{O}$. For any $a \in I$, we abbreviate $f_{a,b_0}$ to $f_a$. For each $a \in I$, $f_a$ has a saddle fixed point

$$p_a = \left( \frac{1}{1+a-b_0}, \frac{b_0}{1+a-b_0} \right).$$

The stable and unstable set of $p_a$ are denoted by $W^s(p_a)$ and $W^u(p_a)$, respectively. As illustrated in the Fig. 1, $W^s(p_a)$ contains the line segment $\mathcal{S}_a$ connecting $p_a$ and the point

$$w_a = \left( 0, \frac{2b_0 - a - \sqrt{a^2+4b_0}}{2(1+a-b_0)} \right)$$

on the $y$-axis. Also, $W^u(p_a)$ contains the line segment $\mathcal{U}_a$ that connects $p_a$ and the point

$$z_a = \left( \frac{2 + a + \sqrt{a^2+4b_0}}{2(1+a-b_0)}, 0 \right)$$

on the $x$-axis. Since the expanding eigenvalue of $(Df_a)_{p_a}$ is negative, we get

$$W^u(p_a) = \bigcup_{i \geq 0} f_a^i(\mathcal{U}_a).$$

Since $(a, b_0) \in \mathcal{O}$, we can check that

$$(w_a)_y < \left(f_a^2(z_a)\right)_y, \quad \left(f_a^{-1}(w_a)\right)_y > (f_a(z_a))_y$$

where $(\,.\,)_y$ is the $y$-coordinate of the corresponding point. Therefore, $\mathcal{S}_a$ and $f_a^2(\mathcal{U}_a)$ intersect transversely, and $f_a^{-1}(\mathcal{S}_a)$ and $f_a(\mathcal{U}_a)$ intersect transversely for every $a \in I$, as in Fig. 1. The triangle $T_a$ with vertexes $z_a$, $f_a(z_a)$ and $f_a^2(z_a)$ is a trapping region. The Lozi attractor is given by

$$\Lambda_a = \bigcap_{i \geq 0} f_a^i(T_a).$$

Let us fix a point $z \in \mathcal{Y}_a$ where $\mathcal{Y}_a = \{y\text{-axis}\} \cap T_a$. For $a \in I$ and $i \geq 0$, we put

$$\varphi_i(a) := f_a^i(z).$$

Set $\tilde{\mathcal{U}} = \bigcup_{a \in I} \mathcal{U}_a$, and consider its cover $\mathcal{H}$ which consists of all open balls whose radii and central coordinates are both rational, and whose intersec-

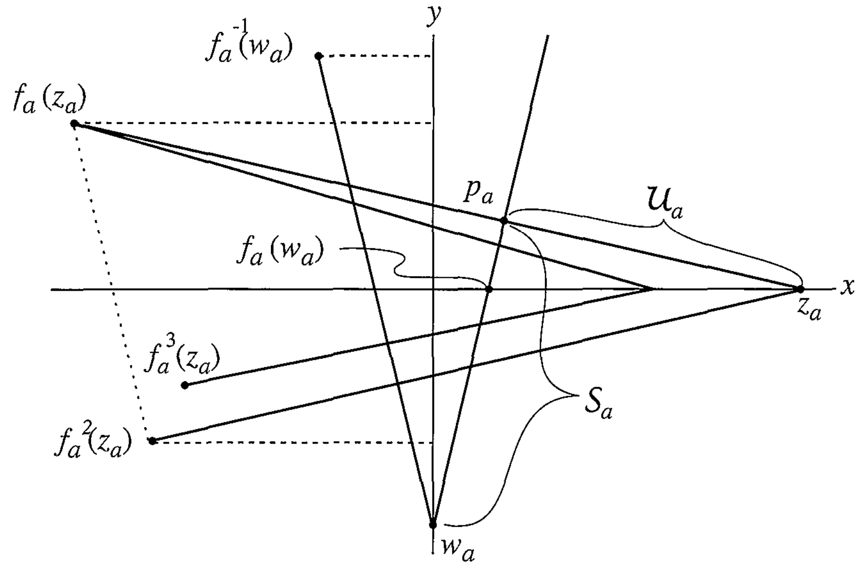


Fig. 1.

tion with $\tilde{\mathcal{U}}$ is non-empty. See Fig. 2. For $H \in \mathcal{H}$, we define

$$I_H = \{a \in I \mid \mathcal{U}_a \cap H \neq \varnothing\},$$

and

$$A_H = \{a \in I_H \mid \varphi_i(a) \notin H \quad \text{for} \quad \forall i \geq 0\}.$$

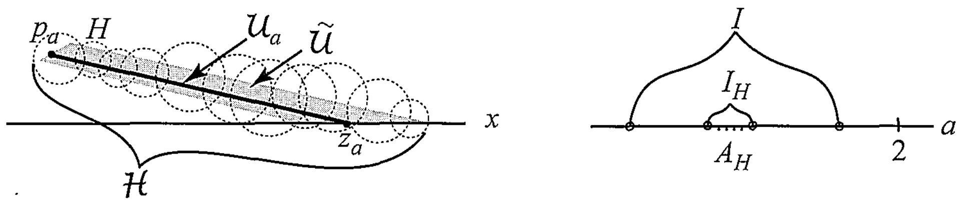


Fig. 2.

The next lemma is essential in the proof of the main theorem. We denote by $\mu$ the 1-dimensional Lebesgue measure on $I$.

**Lemma 1** *For any $H \in \mathcal{H}$, $\mu(A_H) = 0$.*

The proof of Lemma 1 is given based on the following claims: *for almost every $a \in I_H$ and every neighborhood $U$ of the point $a$, there exist integer $\nu > 0$ and closed interval $J \subset U$ including the point $a$ such that*

(A) *$\varphi_\nu(J)$ intersects with $f_a^{-1}(\mathcal{S}_a)$, one of the endpoints of $\varphi_\nu(J)$ belongs to the $y$-axis, and* $1/2 < \mathrm{Length}(\varphi_\nu(J)) < 4$, *as in* Fig. 3, (*see* Theorem 6), *where* Length$(J)$ *is the length of* $J$;

(B) *for any $a_1, a_2 \in J$,*

$$\frac{|d\varphi_\nu(a_1)/da|}{|d\varphi_\nu(a_2)/da|} < 2,$$

(*see* Proposition 4),

whose proofs will be presented in the following sections.

*Proof of Lemma* 1. Suppose that there exists an open set $H \in \mathcal{H}$ such that $\mu(A_H) > 0$. Take a Lebesgue density point $\alpha$ of $A_H$. By the inclination lemma [14] and the piecewise hyperbolic structure of Lozi maps, for any line segment $l \subset H$ intersecting with the unstable segments $\mathcal{U}_\alpha$ transversally, there exists an integer $k > 0$ such that $f_\alpha^{-k}(l)$ becomes a V-shaped segment which is piecewise $C^1$-close to $f_\alpha^{-1}(\mathcal{S}_\alpha)$, as shown in Fig. 3.

Since $f_\alpha^{-k}(l)$ is compact, and $H$ is an open set, there is a $c > 0$ such that

$$N_{2c}(f_\alpha^{-k}(l)) \subset f_\alpha^{-k}(H),$$

where $N_{2c}(f_\alpha^{-k}(l))$ is a $2c$-neighborhood of $f_\alpha^{-k}(l)$. If a neighborhood $U$ of $\alpha$ is sufficiently small, then for any $a \in U$

$$N_c(f_\alpha^{-k}(l)) \subset f_a^{-k}(H). \tag{2}$$

By claim (A) above, there exists a segment $L \subset J \subset U$ such that

$$\varphi_\nu(L) \subset \varphi_\nu(J) \cap \bigcap_{a \in J} f_a^{-k}(H) \neq \varnothing.$$

From (2),

$$\mathrm{Length}(\varphi_\nu(L)) > c > 0, \tag{3}$$

where Length$(\,\cdot\,)$ is the length of a given arc.

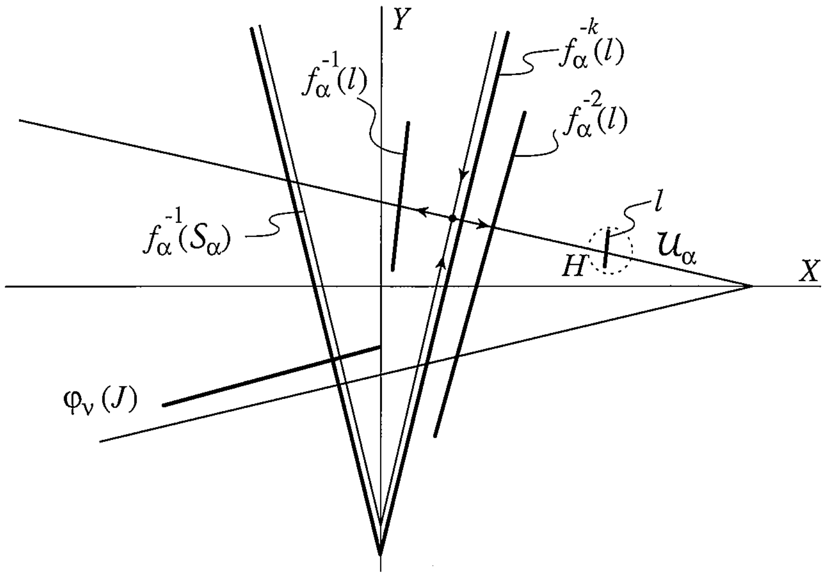


Fig. 3.

By claim (B) above, we have

$$\frac{\text{Length}(\varphi_\nu(L))/\mu(L)}{\text{Length}(\varphi_\nu(J))/\mu(J)} = \frac{|\tau_\nu(a_1)|}{|\tau_\nu(a_2)|} < 2,$$

where $\tau_\nu(a_i) = d\varphi_\nu(a_i)/da$. For every $a \in L$, we have $\varphi_\nu(a) \in \varphi_\nu(L)$, and $\varphi_{\nu+k}(a) \in H$. Therefore, such a parameter value $a$ is not contained in $A_H$. Thus,

$$\frac{\mu(L)}{\mu(J)} < \frac{\mu(J \setminus A_H)}{\mu(J)} = 1 - \frac{\mu(J \cap A_H)}{\mu(J)},$$

and hence

$$\frac{\text{Length}(\varphi_\nu(L))}{2\,\text{Length}(\varphi_\nu(J))} < 1 - \frac{\mu(J \cap A_H)}{\mu(J)}.$$

Since the diameter of the trapping region is smaller than 4, we have $\text{Length}(\varphi_\nu(J)) < 4$. Therefore, using (3), we obtain

$$\frac{\mu(J \cap A_H)}{\mu(J)} < 1 - \frac{\text{Length}(\varphi_\nu(L))}{8} < 1 - \frac{c}{8}.$$

However, since $\alpha$ is a Lebesugue density point of $A_H$, we have

$$\frac{\mu(J \cap A_H)}{\mu(J)} > 1 - \frac{c}{8}$$

for every interval $J \subset U$, if $U$ is sufficiently small. This is a contradiction.

□

*Proof of Main Theorem.* For any $H \in \mathcal{H}$, from the above Lemma 1, we have $\mu(A_H) = 0$. Since $\mathcal{H}$ is countable,

$$\mu\Big(\bigcup_{H\in\mathcal{H}} A_H\Big) \le \sum_{H\in\mathcal{H}} \mu(A_H) = 0.$$

That is, for almost every $a \in I_H$, there exists $i \ge 0$ such that $\varphi_i(a) = f_a^i(z) \in H$ where $z \in \mathcal{Y}_a$. Since this holds for each element of $\mathcal{H}$, we get

$$\mathcal{U}_a \subset \omega(z, f_a).$$

Thus, since $W^u(p_a) = \bigcup_{i\ge 0} f_a^i(\mathcal{U}_a)$, we obtain

$$\mathrm{cl}\,(W^u(p_a)) \subset \omega(z, f_a).$$

Remember that, at the beginning of this section, $(a_0, b_0)$ is an arbitrary point in $\mathcal{O}$. For almost every point $(a, b_0)$ of the horizontal parameter segment in $\mathcal{O}$, the above claim is true. Hence, the main theorem is proved.

□

## 3. Estimations of parameter dependence

In this section, we first define the open set $\mathcal{O} \subset \mathcal{M}$ of parameters in the main theorem. After that we set an open interval $I$ and a constant $b_0$ such that $I \times \{b_0\} \subset \mathcal{O}$. The goal of this section is to show the Proposition 4 which is used in the proof of Lemma 1.

To begin with, we assume that $\mathcal{O}$ satisfies $(2, 0) \in \mathrm{cl}(\mathcal{O})$ and it is sufficiently small such that, for any $(a, b) \in \mathcal{O}$,

$$f_{a,b}^i(\mathcal{Y}_{a,b}) \cap \mathcal{C} = \varnothing, \quad 1 \le \forall i \le 10; \tag{4}$$

$$\sup\{|x| : (x, y) \in T_{a,b}\} < 1.05; \tag{5}$$

$$1.9 < \lambda_{a,b} < 2,$$

where $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 : |x| < 1/2\}$ and $\lambda_{a,b} = (a + \sqrt{a^2 - 4b})/2$. Let us define $\tilde{\lambda}_{a,b} = (a - \sqrt{a^2 - 4b})/2$ which satisfies $0 < \tilde{\lambda}_{a,b} < 1 < \lambda_{a,b}$. If a point $\mathbf{x} = (x, y)$ is not contained in the $y$-axis, by [13], each cone

$$C^u = \{(x, y) \in T_{\mathbf{x}}\mathbb{R}^2 \,:\, |y| \le \tilde{\lambda}_{a,b}|x|\},$$
$$C^s = \{(x, y) \in T_{\mathbf{x}}\mathbb{R}^2 \,:\, |y| \ge \lambda_{a,b}|x|\},$$

is invariant by $(Df_{a,b})_{\mathbf{x}}$ and $(Df_{a,b})_{\mathbf{x}}^{-1}$, respectively, and it holds

$$|(Df_{a,b})_{\mathbf{x}}\mathbf{u}| \ge \lambda_{a,b}|\mathbf{u}|, \quad |(Df_{a,b})_{\mathbf{x}}^{-1}\mathbf{s}| \le \tilde{\lambda}_{a,b}^{-1}|\mathbf{s}|$$

for any $\mathbf{u} \in C^u$ and $\mathbf{s} \in C^s$.

We abbreviate $f_{a,b_0} = f_a$ as $b_0 > 0$ is fixed small. For a fixed $z \in \mathcal{Y}_{a,b_0}$ and each $i \ge 0$, we put

$$\varphi_i(a) = f_a^i(z)$$

and

$$\tau_i = \tau_i(a) = \frac{d\varphi_i(a)}{da}.$$

If the Jacobian $(Df_a)_j$, $0 \le j \le i$, make sense, we have

$$\begin{cases} \tau_1 & = & (0, 0) \\ \tau_{j+1} & = & (Df_a)_j\tau_j + \eta_{j+1} \quad \text{for } 0 \le j \le i \end{cases} \tag{6}$$

where $\eta_{j+1} = (-|x_j|, 0)$ and $x_j$ is the $x$-coordinate of $f_a^j(z)$. We say that $\tau_i(a)$ is *well-defiend* if $\tau_j(a)$ is given by (6) for all $1 \le j \le i$.

To estimate $\tau_i$, let us introduce a pair of *reference vectors* $(\mathbf{u}_i, \mathbf{s}_i)$ as follows. Let $i_0 \ge 2$ be an integer such that $(x_i, y_i) \notin y$-axis for each $2 \le i \le i_0$, that is, $\tau_i$ is well-defined for all $2 \le i \le i_0$. As shown in the Fig. 4, we first define

$$\mathbf{u}_2 = (-1.05, 0), \quad \mathbf{s}_2 = \frac{|\mathbf{u}_2|}{|\mathbf{e}_2|}\mathbf{e}_2,$$

where

$$\mathbf{e}_2 = \left(Df_a^{i_0-2}\right)_2^{-1}\begin{pmatrix}0\\1\end{pmatrix} \in C^s.$$

For each $2 \le i \le i_0$, we define

$$\mathbf{u}_{i+1} = (Df_a)_i\mathbf{u}_i, \quad \mathbf{s}_{i+1} = \frac{|\mathbf{u}_{i+1}|}{|(Df_a)_i\mathbf{s}_i|}(Df_a)_i\mathbf{s}_i.$$

So, these reference vectors satisfy

- $\mathbf{u}_i \in C^u$, $\mathbf{s}_i \in C^s$ and $|\mathbf{u}_i| = |\mathbf{s}_i|$;
- $|\mathbf{u}_{i+1}| \geq \lambda_{a,b}|\mathbf{u}_i|$ and $|\mathbf{s}_{i+1}| \geq \lambda_{a,b}|\mathbf{s}_i|$.

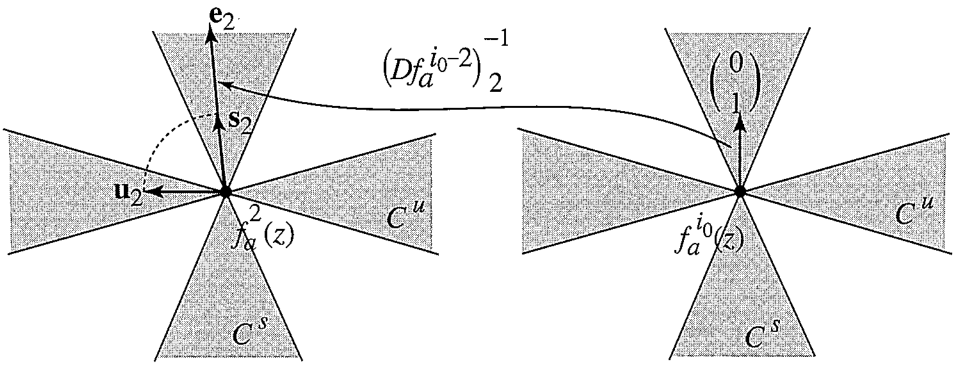


Fig. 4.

Since $\eta_i = (-|x_{i-1}|, 0)$ and (5), we have

$$|\mathbf{u}_i| \geq 1.05 > |\eta_i|,$$

for each $3 \leq i \leq i_0$. Then, there exist $\xi_i$, $\tilde{\xi}_i \in \mathbb{R}$ such that

$$\eta_i = \xi_i \mathbf{u}_i + \tilde{\xi}_i \mathbf{s}_i.$$

Since the slope of the central line of the cones $C^u$ and $C^s$ tend to 0 and 2 as $(a, b) \to (2, 0)$ respectively, if the open set $\mathcal{O}$ is sufficiently small, then we have

$$1.1|\eta_i| > |\xi_i \mathbf{u}_i| > |\tilde{\xi}_i \mathbf{s}_i|$$

for each $3 \leq i \leq i_0$. We get

$$1.1|\mathbf{u}_2| \geq 1.1|\eta_i| > |\xi_i \mathbf{u}_i| = |\xi_i||(Df_a^{i-2})_2 \mathbf{u}_2| > \lambda_{a,b}^{i-2}|\xi_i||\mathbf{u}_2|.$$

Therefore,

$$|\tilde{\xi}_i| < |\xi_i| < \frac{1.1}{\lambda_{a,b}^{i-2}}. \tag{7}$$

We can confirm that $\xi_3 > |\tilde{\xi}_3|$ and $\xi_4 > |\tilde{\xi}_4|$. Using the above decompositions by reference vectors, provided $\tau_i$ is well-defined, we obtain

$$\begin{aligned}\tau_i =& (\xi_2 + \xi_3 + \xi_4 + \cdots + \xi_i)\mathbf{u}_i \\ &+ \left( \frac{|(Df_a)_{i-1}\mathbf{s}_{i-1}|}{|\mathbf{u}_i|} \cdots\cdots\cdots\cdots \frac{|(Df_a)_3\mathbf{s}_3|}{|\mathbf{u}_4|}\tilde{\xi}_3 \right. \\ &\quad + \frac{|(Df_a)_{i-1}\mathbf{s}_{i-1}|}{|\mathbf{u}_i|} \cdots\cdots\cdots \frac{|(Df_a)_3\mathbf{s}_3|}{|\mathbf{u}_4|}\tilde{\xi}_4 \\ &\quad + \cdots\cdots\cdots\cdots \\ &\quad + \frac{|(Df_a)_{i-1}\mathbf{s}_{i-1}|}{|\mathbf{u}_i|}\tilde{\xi}_{i-1} \\ &\quad \left. + \tilde{\xi}_i \right)\mathbf{s}_i.\end{aligned}$$

We moreover assume that $\mathcal{O}$ is so small that, for any $(a, b) \in \mathcal{O}$,

$$f_{a,b}(\mathcal{Y}_{a,b}) \subset (0.92, \infty) \times \{0\}.$$

Hence,

$$|\xi_2| = \frac{|\eta_2|}{|\mathbf{u}_2|} = \frac{|x_1|}{|\mathbf{u}_2|} > \frac{0.92}{1.05} > 0.87. \tag{8}$$

By $\tilde{\lambda}_{a,b} \searrow 0$ and $\lambda_{a,b} \nearrow 2$ as $(a, b) \to (2, 0)$, if the open set $\mathcal{O}$ is sufficiently small, then the following condition can be held:

$$\frac{\xi_2 + (\xi_3 - |\tilde{\xi}_3|) + (\xi_4 - |\tilde{\xi}_4|) - \Gamma(a, b)}{\xi_2 + \xi_3 + |\tilde{\xi}_3| + \xi_4 + |\tilde{\xi}_4| + \Gamma(a, b)} \cdot \lambda_{a,b} > 1.18 \tag{9}$$

where

$$\Gamma(a, b) = \frac{1.1(1 + 2\tilde{\lambda}_{a,b})}{\lambda_{a,b}^2(\lambda_{a,b} - 1.1)}.$$

**Lemma 2** *If we take sufficiently small $\mathcal{O}$ with $(2, 0) \in \mathrm{cl}(\mathcal{O}) \subset \mathcal{M}$, there is an integer $i_0 > 2$ such that*

- *if $\tau_i(a)$ is well-defined for $a \in I$ and $i \geq 2$, then*

$$0.1\lambda_a^{i-2} < |\tau_i(a)| < 4(1 + \sqrt{2})^{i-1}; \tag{10}$$

- *if $\tau_i(a)$ is well-defined for $a \in I$ and $i \geq i_0$, then*

$$0.1\lambda_a^{i-2} < \sqrt{2}|\Pi_x(\tau_i(a))| \tag{11}$$

*where $\Pi_x$ is a canonical projection from $\mathbb{R}^2$ to the $x$-axis.*

*Proof.* From the above expression of $\tau_i$ by reference vectors, we get

$$|\tau_i| \geq \left\{ \xi_2 + \left(\xi_3 - |\tilde{\xi}_3|\right) + \left(\xi_4 - |\tilde{\xi}_4|\right) - \sum_{n=5}^{i}\left(|\xi_n| + |\tilde{\xi}_n|\right) \right\} |\mathbf{u}_i|.$$

By (7), we have

$$\begin{aligned}\sum_{n=5}^{i}\left(|\xi_n| + |\tilde{\xi}_n|\right) &< \frac{2 \cdot (1.1/\lambda_a^3)}{1 - (1.1/\lambda_a)} \\ &< \frac{2.2}{\lambda_a^2(\lambda_a - 1.1)} < \frac{2.2}{1.9^2(1.9 - 1.1)} < 0.77,\end{aligned}$$

Hence, by (8),

$$\begin{aligned}|\tau_i| &\geq \left\{0.87 + \left(\xi_3 - |\tilde{\xi}_3|\right) + \left(\xi_4 - |\tilde{\xi}_4|\right) - 0.77\right\} |\mathbf{u}_i| \\ &\geq 0.1\lambda_a^{i-2}|\mathbf{u}_2| \\ &> 0.1\lambda_a^{i-2}.\end{aligned}$$

The second inequality of (10) is obtained as follows. Since

$$\|(Df_a)_i\| = \frac{a + \sqrt{a^2 + 4b}}{2} < 1 + \sqrt{2},$$

we get, for every $i \geq 2$,

$$|\tau_i| \leq \|(Df_a)_{i-1}\| |\tau_{i-1}| + |\eta_i| < \left(1 + \sqrt{2}\right)|\tau_{i-1}| + 4.$$

Then,

$$|\tau_i| \leq \left(1 + \sqrt{2}\right)^{i-2}|\tau_2| + 4\left\{(1 + \sqrt{2})^{i-3} + \cdots + 1\right\}.$$

Since $|\tau_2| = |\eta_2| < 4$,

$$|\tau_i| \leq 4\left\{(1 + \sqrt{2})^{i-2} + (1 + \sqrt{2})^{i-3} + \cdots + 1\right\} < 4(1 + \sqrt{2})^{i-1}.$$

Hence, (10) is obtained.

Denote that $\tilde{C}^{u+} = \left\{(x, y) \in T_{\mathbf{x}}\mathbb{R}^2 : |y| \leq x\right\}$, $\tilde{C}^{u-} = \left\{(x, y) \in T_{\mathbf{x}}\mathbb{R}^2 : |y| \leq -x\right\}$ and $\tilde{C}^u = \tilde{C}^{u+} \cup \tilde{C}^{u-}$. Note that $(Df_a)_{\mathbf{x}}\mathbf{v} \in \mathrm{Int}(\tilde{C}^u)$ for any nonzero $\mathbf{v} \in \tilde{C}^u$ if $(a, b)$ is close to $(2, 0)$. Since $\eta_{i+1} = (-|x_i|, 0)$ and $|x_i| < 2$ for any $i > 0$, there exists a constant $u_0 > 0$ such that for any $\mathbf{u} \in \tilde{C}^u$ with $|\mathbf{u}| \geq u_0$,

$$(Df_a)_{\mathbf{x}}\mathbf{u} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \in \mathrm{Int}(\tilde{C}^u).$$

By (10), if $(a, b)$ is close to $(2, 0)$, then there exists an integer $i_0 > 2$ such that

- $\tau_2, \cdots, \tau_{i_0-1} \in \tilde{C}^{u-}$ and $\tau_{i_0} \in \tilde{C}^{u+}$;
- $|\tau_i| > u_0$ for every $i \geq i_0$, as $\tau_i$ is well-defined.

This implies that the norm of $\Pi_x(\tau_i(a))$ is greater than $|\tau_i(a)|/\sqrt{2}$ for any $i \geq 0$. Therefore, the proof is now complete. □

If $\tau_i(a)$ is well-defined, then one can get

$$\tau_i'(a) = \frac{d\tau_i(a)}{da} = \frac{d^2\varphi_i(a)}{da^2} = \left(\frac{d^2x_i}{da^2}, \frac{d^2y_i}{da^2}\right).$$

By direct calculations, we have

$$\begin{aligned}\frac{d^2x_{i+1}}{da^2} &= -\mathrm{sgn}(x_i)\cdot a \cdot \frac{d^2x_i}{da^2} + \frac{d^2y_i}{da^2} - 2\cdot \mathrm{sgn}(x_i)\cdot \frac{dx_i}{da} \\ \frac{d^2y_{i+1}}{da^2} &= b\cdot \frac{d^2x_i}{da^2},\end{aligned}$$

that is,

$$\tau_{i+1}'(a) = (Df_a)_i\tau_i'(a) + 2\begin{pmatrix}-\mathrm{sgn}(x_i)(dx_i/da)\\ 0\end{pmatrix}.$$

**Lemma 3** *If $\tau_i(a)$ is well-defined for $a \in I$, then*

$$|\tau_j'(a)| < 8j\big(1+\sqrt{2}\big)^j$$

*for $j = 1, \ldots, i$.*

*Proof.* We prove it by induction. Since $|\tau_1'(a)| = 0$, the claim holds for $j = 1$. Suppose that it holds for $1 \leq j < i$. Using $|(Df_a)_j| < 1+\sqrt{2}$ and Lemma 2, we have

$$\begin{aligned}|\tau_{j+1}'(a)| &\leq \|(Df_a)_j\||\tau_j'(a)| + 2\left|\frac{dx_j}{da}\right| \\ &< \big(1+\sqrt{2}\big)\cdot 8n\big(1+\sqrt{2}\big)^j + 2|\tau_j| \\ &< 8(j+1)\big(1+\sqrt{2}\big)^{j+1}\end{aligned}$$

Then the claim holds for $j+1$, Therefore, the lemma is true for each $j = 1, \ldots, i$. □

**Proposition 4** *For any $\gamma > 0$ there exists $i_1 \geq 1$ such that if $\tau_i$ is well-defined on a closed interval $J \subset I$ for $i \geq i_1$, then*

$$\frac{|\tau_i(a_1)|}{|\tau_i(a_2)|} \leq 1 + \gamma$$

*for any $a_1$, $a_2 \in J$.*

*Proof.* When $a_1 = a_2$, the lemma is trivial. Let $a_1$, $a_2 \in J$ with $a_1 \neq a_2$. Using Lemma 3, we have

$$\frac{|\tau_i(a_1)| - |\tau_i(a_2)|}{|a_1 - a_2|} \leq \sup_{a \in J} |\tau_i'(a)| < 8i(1 + \sqrt{2})^i.$$

If $\tau_i$ is well-defined on $J$ for $i \geq i_0$, by Lemma 2 (11),

$$\frac{4}{|a_1 - a_2|} \geq \frac{|\Pi_x(\varphi_i(a_1))| - |\Pi_x(\varphi_i(a_2))|}{|a_1 - a_2|}$$
$$\geq \inf_{a \in J} |\Pi_x(\tau_i(a))| > \frac{0.1\lambda_{\tilde{a}}^{i-2}}{\sqrt{2}},$$

for some $\tilde{a} \in J$. Then, we have

$$|\tau_i(a_1)| - |\tau_i(a_2)| < 8i(1 + \sqrt{2})^i |a_1 - a_2| < \frac{32\sqrt{2}i(1 + \sqrt{2})^i}{0.1\lambda_{\tilde{a}}^{i-2}}.$$

Note that $\lambda_a^2 > (7/5)(1 + \sqrt{2})$ for any $a \in J$. Then, using Lemma 2 (10), we get

$$\frac{|\tau_i(a_1)|}{|\tau_i(a_2)|} - 1 < \frac{32\sqrt{2}i(1 + \sqrt{2})^i}{0.1\lambda_{\tilde{a}}^{i-2}} \frac{1}{0.1\lambda_{a_2}^{i-2}} < 6400(1 + \sqrt{2})^2 i \left(\frac{5}{7}\right)^{i-2}.$$

So, for any $\gamma > 0$, we take an integer $i_1 \geq i_0$ such that, for any $i \geq i_1$,

$$6400(1 + \sqrt{2})^2 i \left(\frac{5}{7}\right)^{i-2} \leq \gamma.$$

□

**Lemma 5** *There exists an integer $i_2 > 0$ and $\zeta > 1.15$ such that, for any $a \in I$,*

$$\frac{|\tau_{i+1}(a)|}{|\tau_i(a)|} > \zeta$$

*if $\tau_{i+1}$ is well-defined for given $i \geq i_2$.*

*Proof.* By (6),

$$|\tau_{i+1}| \geq |(Df_a)_i\tau_i| - |\eta_{i+1}| > |(Df_a)_i\tau_i| - 4.$$

Then, using Lemma 2 , we get

$$\frac{|\tau_{i+1}|}{|\tau_i|} > \frac{|(Df_a)_i\tau_i|}{|\tau_i|} - \frac{4}{|\tau_i|} > \frac{|(Df_a)_i\tau_i|}{|\tau_i|} - \frac{4}{0.1\lambda_a^{i-2}}.$$

If

$$\frac{|(Df_a)_i\tau_i|}{|\tau_i|} > 1.18, \tag{12}$$

then we can get an integer $i_2 > 0$ such that, for $i \geq i_2$,

$$\frac{|(Df_a)_i\tau_i|}{|\tau_i|} - \frac{40}{\lambda_a^{i-2}} > 1.18 - \frac{40}{\lambda_a^{i-2}} > 1.15.$$

Let us show that (12) is true as follow. By the linear decomposition of $\tau_i$, we get

$$\begin{aligned}|\tau_i| &\leq \left\{\xi_2 + \xi_3 + |\tilde{\xi}_3| + \xi_4 + |\tilde{\xi}_4| + \sum_{n=5}^{i}\left(|\xi_n| + |\tilde{\xi}_n|\right)\right\}|\mathbf{u}_i| \\ &\leq \left\{\xi_2 + \xi_3 + |\tilde{\xi}_3| + \xi_4 + |\tilde{\xi}_4| + \frac{1.1(1+2\tilde{\lambda}_a)}{\lambda_a^2(\lambda_a - 1.1)}\right\}|\mathbf{u}_i|,\end{aligned}$$

and

$$\begin{aligned}|(Df_a)_i\tau_i| &\geq \left\{\xi_2 + (\xi_3 - |\tilde{\xi}_3|) + (\xi_4 - |\tilde{\xi}_4|) - \sum_{n=5}^{i}\left(|\xi_n| + |\tilde{\xi}_n|\right)\right\}|\mathbf{u}_{i+1}| \\ &\geq \left\{\xi_2 + (\xi_3 - |\tilde{\xi}_3|) + (\xi_4 - |\tilde{\xi}_4|) - \frac{1.1(1+2\tilde{\lambda}_a)}{\lambda_a^2(\lambda_a - 1.1)}\right\}\lambda_a|\mathbf{u}_i|.\end{aligned}$$

Then, by (9), we get

$$\frac{|(Df_a)_i\tau_i|}{|\tau_i|} > \frac{\xi_2 + (\xi_3 - |\tilde{\xi}_3|) + (\xi_4 - |\tilde{\xi}_4|) - \dfrac{1.1(1+2\tilde{\lambda}_a)}{\lambda_a^2(\lambda_a - 1.1)}}{\xi_2 + \xi_3 + |\tilde{\xi}_3| + \xi_4 + |\tilde{\xi}_4| + \dfrac{1.1(1+2\tilde{\lambda}_a)}{\lambda_a^2(\lambda_a - 1.1)}} \cdot \lambda_a > 1.18.$$

This completes the proof. □

## 4. Usefulness and maturity of parameter arcs

The concepts of usefulness and maturity for parameter intervals of tent maps are introduced in [4]. Let us extend these concepts to the Lozi family.

For $k \geq 1$, the parameter interval $I$ is called *$k$-useful* if

- $\tau_k$ is well-defined on $I$,
- there exists $a_0 \in \partial I$ such that $\varphi_k(a_0) \in \{y\text{-axis}\}$,

where $\partial I$ is the set of endpoints of $I$. If there exist several $k$'s for which $I$ is $k$-useful, we call the largest one *order* of $I$ which is denoted by $\mathrm{Ord}(I)$, whose finitude is ensured by Lemma 2.

Next, we extend the concept of maturity to a subinterval of a useful $I$ of order $N$. Let $\tilde{I} \subset I$ be an open interval. We say that $\tilde{I}$ is *$k$-mature* if

- there exists some $k \geq N$ such that $\tilde{I}$ is $k$-useful,
- there exist $\tilde{a} \in \tilde{I}$ and $(0, \tilde{y}) \in \{y\text{-axis}\}$ such that
$$\varphi_k(\tilde{a}) = f_{\tilde{a}}^k(0, y) = f_{\tilde{a}}^m(0, \tilde{y})$$
for some $m \in \{1, 2, \ldots, 9\}$.

A point of $I$ which does not belong to any mature subset of $I$ is called *bad*, and a set of all bad points of $I$ is denoted by $\mathcal{B}$.

We define partitions of $I$ inductively. Let $k > 0$ be an integer such that $\varphi_k(I) \cap \{y\text{-axis}\} = \varnothing$ where $\varphi_k(I) = \{\varphi_k(a) : a \in I\}$. Using Lemma 2, we have the smallest integer $h > 0$ such that $\varphi_{k+h}(I)$ intersects transversely at one point of the $y$-axis. So, by this intersection, $\varphi_{k+h}(I)$ is divided into two adjacent segments which are images of two adjacent $(k+h)$-useful open intervals of $I$, respectively, denoted by $J_1$ and $J_2$. We now get the *first partition* $\mathcal{P}_1 = \{J_1, J_2\}$ of $I$. If $J_i \in \mathcal{P}_1$ is mature, we set $\rho(J_i) = \{J_i\}$; otherwise, by similar steps, we can divide $J_i$ into two $(k+h')$-useful, $h' > h$, arcs $J_{i1}$ and $J_{i2}$, and set $\rho(J_i) = \{J_{i1}, J_{i2}\}$. Then we get the *second partition* $\mathcal{P}_2 = \bigcup_{J_i \in \mathcal{P}_1} \rho(J_i)$. Similarly, for every $n \geq 3$, we obtain the *partition* $\mathcal{P}_n = \bigcup_{J \in \mathcal{P}_{n-1}} \rho(J)$ of $I$.

We claim the following:

**Theorem 6** *For almost every $a \in I$, there is a $k$-mature $\tilde{I} \subset I$ with $a \in \tilde{I}$ such that*
$$|(\varphi_k(\tilde{a}))_x| \geq \frac{1}{2}$$
*for some $\tilde{a} \in \partial\tilde{I}$, where $(\,.\,)_x$ is the $x$-coordinate of the corresponding point. That is, one of the endpoints of $\varphi_k(\mathrm{cl}(\tilde{I}))$ keeps away from the $y$-axis at*

*least by* $1/2$.

We will deduce this theorem from Proposition 8 and Proposition 9 stated later. To prove these propositions, we prepare a lemma. Let us define the function $\psi_n$ on $\mathcal{B}$ by

$$\psi_n(a) = \int_{\mathcal{B}\cap J} |\tau_k|\, d\mu$$

where $J$ is an element of $\mathcal{P}_n$ that contains $a$ and $k$ is the order of $J$. We set

$$N = \max\{i_1,\, i_2\},$$

where $i_1$ and $i_2$ are given in Proposition 4 and Lemma 5 , respectively. Remember that the constants $\gamma > 0$ and $\zeta > 1.15$ are also presented in Proposition 4 and Lemma 5, respectively.

**Lemma 7** *For* $n \geq N$, *let* $J \in \mathcal{P}_n$ *be a* $k$*-useful interval. If* $\mu(\mathcal{B}\cap J) > 0$, *then*

$$\int_{\mathcal{B}\cap J} \psi_{n+1} d\mu \geq \sigma \int_{\mathcal{B}\cap J} \psi_n d\mu$$

*where*

$$\sigma = \min\left\{\zeta,\, \frac{\zeta^{10}}{2(2+\gamma)}\right\}.$$

*Proof.* From $\mu(\mathcal{B}\cap J) > 0$, $J$ is not mature. Then, there is the smallest integer $m \geq 0$ such that $\xi_{k+m}(J)$ intersects $Y$. Thus, we get the first partition $\rho(J) = \{J_1,\, J_2\}$. Obviously, $\mathrm{Ord}(J_i) \geq k+m$. Without loss of generality, we may assume that

$$\int_{\mathcal{B}\cap J_1} |\tau_{k+m}|\, d\mu \geq \int_{\mathcal{B}\cap J_2} |\tau_{k+m}|\, d\mu. \tag{13}$$

By Lemma 5, we have

$$\psi_{n+1}(a) = \int_{\mathcal{B}\cap J_i} |\tau_{\mathrm{Ord}}(J_i)|\, d\mu \geq \int_{\mathcal{B}\cap J_i} |\tau_{k+m}|\, d\mu \tag{14}$$

for any $a \in J$ and $i = 1, 2$. Also from Lemma 5, we have

$$|\tau_{k+m}| > \zeta^m |\tau_k| > \zeta |\tau_k|. \tag{15}$$

Since $\mu(\mathcal{B}\cap J) > 0$, it is impossible that both $J_1$ and $J_2$ are mature. First, we suppose that $\mathcal{B}\cap J_2 = \varnothing$, that is, $\mathcal{B}\cap J = \mathcal{B}\cap J_1$. Then using (14) and (15) we get

$$\psi_{n+1}(a) \geq \int_{\mathcal{B}\cap J_1} |\tau_{k+m}| d\mu > \zeta \int_{\mathcal{B}\cap J} |\tau_k| d\mu = \zeta \psi_n(a)$$

for each $a \in \mathcal{B} \cap J_1$. Hence, we get the claim of this lemma.

Next, we suppose that $\mathcal{B} \cap J_1 \neq \varnothing$ and $\mathcal{B} \cap J_2 \neq \varnothing$. Then $J_1$ and $J_2$ are both immature but $(k+m)$-useful. There exist $a_0 \in \partial J$ and $\tilde{y}$ such that $\varphi_k(a_0) = (0, \tilde{y}) \in \{y\text{-axis}\}$. Then,

$$\begin{aligned}\varphi_{m+k}(a_0) = f_{a_0}^{m+k}(0, y) &= f_{a_0}^m \circ f_{a_0}^k(0, y) \\ &= f_{a_0}^m(\varphi_k(a_0)) = f_{a_0}^m(0, \tilde{y}).\end{aligned}$$

If $m \in \{1, 2, \ldots, 9\}$, $J_i$ is mature. This is a contradiction. Thus we have $m \geq 10$. From (15), for any $m \geq 10$, we get

$$|\tau_{k+m}| > \zeta^{10} |\tau_k|. \tag{16}$$

By the mean value theorem, there exists $a^{(i)} \in \mathcal{B} \cap J_i$ such that

$$\int_{\mathcal{B}\cap J_i} |\tau_{k+m}(a)| d\mu = \left|\tau_{k+m}(a^{(i)})\right| \mu(\mathcal{B} \cap J_i).$$

By Proposition 4 , for $k + m \geq N$, we have

$$\frac{\int_{\mathcal{B}\cap J_1} |\tau_{k+m}(a)| d\mu}{\int_{\mathcal{B}\cap J_2} |\tau_{k+m}(a)| d\mu} \cdot \frac{\mu(\mathcal{B} \cap J_2)}{\mu(\mathcal{B} \cap J_1)} = \frac{\left|\tau_{k+m}(a^{(1)})\right|}{\left|\tau_{k+m}(a^{(2)})\right|} < 1 + \gamma.$$

Then, using (13) we get

$$\frac{\mu(\mathcal{B} \cap J_2)}{\mu(\mathcal{B} \cap J_1)} < 1 + \gamma.$$

Since $\mu(\mathcal{B} \cap J) = \mu(\mathcal{B} \cap J_1) + \mu(\mathcal{B} \cap J_2)$, we have

$$(2 + \gamma)\mu(\mathcal{B} \cap J_1) > \mu(\mathcal{B} \cap J). \tag{17}$$

Hence, using (14), (16) and (17), we get

$$\begin{aligned}\int_{\mathcal{B}\cap J_1} \psi_{n+1} d\mu &\geq \mu(\mathcal{B} \cap J_1) \int_{\mathcal{B}\cap J_1} |\tau_{k+m}| d\mu \\ &\geq \frac{\mu(\mathcal{B} \cap J)}{2 + \gamma} \cdot \frac{1}{2} \int_{\mathcal{B}\cap J} |\tau_{k+m}| d\mu \\ &\geq \frac{\mu(\mathcal{B} \cap J)}{2(2 + \gamma)} \cdot \zeta^{10} \int_{\mathcal{B}\cap J} |\tau_k| d\mu\end{aligned}$$

$$=\frac{\zeta^{10}}{2(2+\gamma)}\int_{\mathcal{B}\cap J}\psi_n d\mu.$$

□

Since $\zeta > 1.15$, see Lemma 5, we have

$$\zeta^{10} > 4.$$

In Proposition 4, $\gamma > 0$ can be arbitrarily small. Then, we have

$$\zeta^{10} > 2(2+\gamma) \tag{18}$$

**Proposition 8** *For almost every $a \in I$, there exist $k \geq N$ and open interval $\tilde{I} \subset I$ with $a \in \tilde{I}$ such that $\tilde{I}$ is $k$-mature.*

*Proof.* We just show that $\mu(\mathcal{B}) = 0$. Let $\mathcal{P}_n$ be a partition of $I$. Now suppose $\mu(\mathcal{B}) > 0$. Then, there exists some $k$-useful $J \in \mathcal{P}_n$ such that $\mu(\mathcal{B}\cap J) > 0$. Since $\tau_k$ is the tangent vector of $\varphi_k$, we have

$$\psi_n(a) = \int_{\mathcal{B}\cap J}|\tau_k|d\mu \leq \int_J |\tau_k|d\mu = \text{Length}(\varphi_k(J)),$$

where $\text{Length}(\varphi_k(J))$ is bounded by some constant $K$ independent of $n$ because of the trapping region. Then we have for all $n \geq N$

$$\int_{\mathcal{B}}\psi_n(a)d\mu < K\cdot\mu(\mathcal{B}). \tag{19}$$

By Lemma 7 and (18), there is $\sigma > 1$ such that, for all $n \geq N$,

$$\int_{\mathcal{B}}\psi_{n+1}d\mu \geq \sigma\int_{\mathcal{B}}\psi_n d\mu.$$

This means that $\int_{\mathcal{B}}\psi_n d\mu$ increases exponentially for $n \geq N$, which contradicts (19). Then we have $\mu(\mathcal{B}) = 0$. □

**Proposition 9** *Let $\tilde{I} \subset I$ be a $k$-mature interval which is obtained for almost every $a \in I$ in* Proposition 8. *Then, there exists $\tilde{a} \in \partial\tilde{I}$ such that*

$$|(\varphi_k(\tilde{a}))_x| \geq \frac{1}{2}.$$

*Proof.* Since $\tilde{I}$ is $k$-mature, there exist $\tilde{a} \in \partial\tilde{I}$ and $(0, \tilde{y}) \in \mathcal{Y}_{\tilde{a},b}$ such that

$$\varphi_k(\tilde{a}) = f_{\tilde{a}}^m(0, \tilde{y})$$

where $m \in \{1, 2, \ldots, 9\}$. By (4), we have the fact that

$$f_{\tilde{a}}^{m}(0, \tilde{y}) \notin \left\{(x, y) \in \mathbb{R}^2 \ : \ |x| < \frac{1}{2}\right\}.$$

Then,

$$\varphi_k(\tilde{a}) \notin \left\{(x, y) \in \mathbb{R}^2 \ : \ |x| < \frac{1}{2}\right\}.$$

□

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