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The Hausdorff measure of a Sierpinski-like fractal

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Abstract. Let S be a Sierpinski-like fractal with the compression ratio $1/3$, N be the set of all the basic triangles to generate S. In this paper, by the mass distribution principle, the exact value of the Hausdorff measure of S, $H(S) = 1$, is obtained, and the fact that the Hausdorff measure of S can be determined by the net measure $H_N(S)$ is shown, and the best coverings of S that are nontrivial are also obtained.

Key words: self-similar set, Sierpinski-like fractal, Hausdorff measure, mass distribution principle.

It is well known that it is one of the most important subjects to calculate or estimate the Hausdorff measures of fractal sets in fractal geometry. But, generally speaking, it is very difficult to calculate or estimate the Hausdorff measures of fractal sets, even for simple sets. For a self-similar set satisfying the open set condition, we know that its Hausdorff dimension equals to its similarity dimension. However, there are not many results on the computation and estimation of the Hausdorff measure for such fractal sets except for a few fractal sets on a line, like the Cantor set ([1, 2]). For the Sierpinski gasket S with the compression ratio $1/2$ and the Hausdorff dimension $\log_2 3$, Marion showed in [3] that $H^s(S) \leq 3^s/6 \approx 0.9508$, and conjectured that the upper bound is its exact Hausdorff measure. Zhou pointed out in [4] that the conjecture is not true by showing that $H^s(S) \leq 25/22(6/7)^s \approx 0.8900$. In the reference [5], the upper bound is improved to $H^s(S) \leq (1927233/1509380)(61/80)^s \approx 0.8308$.

In this paper, we study a Sierpinski-like fractal S with the compression ratio $1/3$ and the Hausdorff dimension $s = 1$. By the mass distribution principle, the exact value of the Hausdorff measure of S, $H(S) = 1$, is obtained, and the fact that the Hausdorff measure of S can be determined by the net measure $H_N(S)$ is shown, and the best coverings of S that are nontrivial are also obtained.

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1. The generation of a Sierpinski-like fractal

The terminology and the basic definitions in this paper can be found in references [1-2]. The generation of the Sierpinski-like fractal can be described as follows.

Take S_0 to be an equilateral triangle with side length 1 in the Euclidean plane R^2 and delete all but the three corner equilateral triangles with side length 3^{-1} to obtain S_1. Continue in the way, replacing at the kth stage each equilateral triangle of S_{k-1} by three corner equilateral triangles of side length 3^{-k} to get S_k. So we obtain S_k $(k = 1, 2, \ldots)$: $S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_k \supset \cdots$ (See Fig. 1).

Let $S = \bigcap_{k=0}^{\infty} S_k$. Since each S_k $(k = 1, 2, \ldots)$ is compact and non-empty, S is compact and non-empty. We call S a Sierpinski-like fractal. S can be considered as a self-similar set generated by three similitudes with the scale factor 1/3. Since S satisfies the open set condition, the Hausdorff dimension of S equals to its similarity dimension $s = -\lg 3/\lg(1/3) = 1$.

For each $k \geq 0$, S_k consists of 3^k equilateral triangles with side length 3^{-k}. Any one of such equilateral triangles is called the kth-stage basic triangle, and is denoted by $\triangle_k$. It is obvious that the diameter of $\triangle_k$ equals to $\mid \triangle_k \mid = 3^{-k}$. We denote by N_k the class of the kth-stage basic triangles, and $N = \bigcup_{k=0}^{\infty} N_k$ the class of all the basic triangles. Obviously, N is a net for S, and any two basic triangles of N are either disjoint or else one is contained in the other.

Since the Hausdorff dimension of S is $s = 1$, in the sequel of this paper, $H(S)$ will always denote the Hausdorff 1-dimensional measure of S, and $H_N(S)$ the 1-dimensional net measure of S determined by the net N.

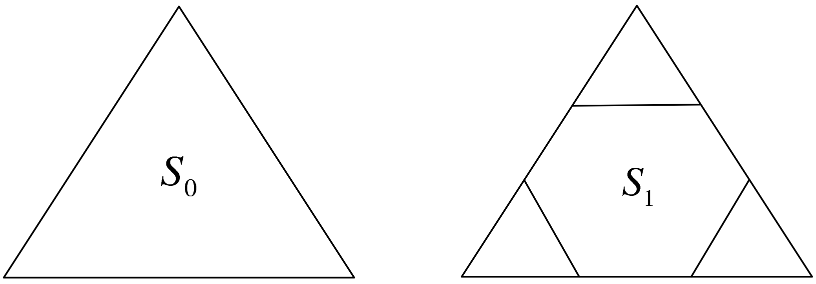

Fig. 1.

2. Some lemmas

Lemma 1 *Let* $\triangle ABC$ *be an equilateral triangle with side length* d*, and* U *be an open set contained in* $\triangle ABC$ *whose closure* $\overline{U}$ *intersects with the three sides of* $\triangle ABC$*. Then the diameter of* U *satisfies* $|U| \geq d/2$*.*

Proof. See Fig. 2. Suppose that the closure $\overline{U}$ of U intersects with the three sides of $\triangle ABC$ at D, E, F respectively. If $\overline{U}$ intersects with any of the three sides of $\triangle ABC$ at more than one point, we take any of such intersect points. Since U is open, $|\overline{U}| = |U|$. It is obvious that $|\overline{U}| \geq |\triangle DEF|$, and the diameter $|\triangle DEF|$ of $\triangle DEF$ equals to the maximum among the three side lengths of $\triangle DEF$. We can see that among the three sides of $\triangle DEF$ there is at least one of them which is not less than $d/2$. If not, $|DE|$, $|EF|$ and $|DF|$ are all less than $d/2$. By the reference [6], for the area S_{DEF} of $\triangle DEF$ we have

$$S_{DEF} \leq \frac{\sqrt{3}}{4}(|DE| \cdot |EF| \cdot |DF|)^{2/3} < \frac{\sqrt{3}}{4}\left(\frac{d^3}{8}\right)^{2/3} = \frac{\sqrt{3}}{16}d^2. \tag{1}$$

On the other hand, let $|AD| = a$, $|BE| = b$, $|CF| = c$, We have

$$\begin{aligned} S_{DEF} &= S_{ABC} - S_{ADF} - S_{BED} - S_{CFE} \\ &= \frac{\sqrt{3}}{4}d^2 - \frac{\sqrt{3}}{4}[a(d-c) + b(d-a) + c(d-b)]. \end{aligned}$$

Set $f(a, b, c) = a(d-c) + b(d-a) + c(d-b)$. It is easy to see that $f(a, b, c)$ will achieve its maximum $3/4d^2$ when $a = b = c = d/2$. Therefore,

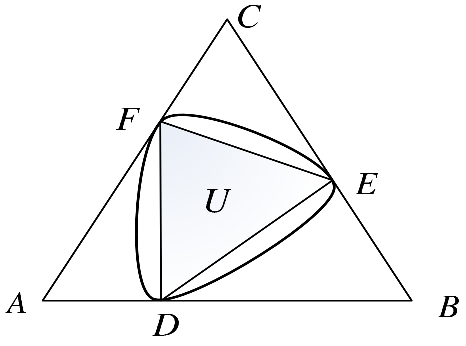

Fig. 2.

$$S_{DEF} \geq \frac{\sqrt{3}}{4}d^2 - \frac{\sqrt{3}}{4}\frac{3}{4}d^2 = \frac{\sqrt{3}}{16}d^2. \tag{2}$$

(2) is contrary to (1). So $|\triangle DEF| \geq d/2$, and hence $|U| \geq d/2$. This completes the proof of the Lemma. □

Lemma 2 $H(S) \leq 1$.

Since S can be covered by 3^k kth-stage basic triangles of length 3^{-k}, by the definition of the Hausdorff measure, Lemma 2 holds obviously.

We define a distribution function μ on R^2 such that

$$\mu(S_0) = 1, \quad \mu(\triangle_k) = |\triangle_k|^s = 3^{-k} \ (k \geq 0), \quad \mu(R^2 - S) = 0. \tag{3}$$

It is easy to see that the restriction of μ on S is a mass distribution. By Proposition 1.7 in [1], the definition of μ may be extended to all subsets of R^2 so that μ becomes a measure with support S. From the definition of μ, the following lemma holds obviously.

Lemma 3 *Let* $\alpha = \{U_i\}$ *be an arbitrary* (*countable*) *covering of* S *which consists of the basic triangles, and in which each of* α *cannot contain completely the other. Then*

$$\mu\Big(\bigcup_{U_i \in \alpha} U_i\Big) = 1. \tag{4}$$

That is

$$\sum_{U_i \in \alpha} |U_i|^s = |S|^s = 1. \tag{5}$$

Corollary 1 $H_N(S) = |S|^s = 1$.

Proof. By the definition of the net measure, it is enough to prove the above equality for all coverings that consist of the basic triangles in which no one is contained in the other. By Lemma 3, it follows that Corollary 1 holds. □

3. The Hausdorff measure of the Sierpinski-like fractal

Theorem 1 *Let* U *be any open subset of* R^2, *then*

$$\mu(U) \leq |U|^s. \tag{6}$$

Proof. By the definition of μ, it is obvious that $\mu(U) \leq 1$. So the Theorem holds when $|U| \geq 1$. Next, let $|U| < 1$, then there exists an integer $k \geq 0$ such that $3^{-(k+1)} \leq |U| < 3^{-k}$. It is easy to see that U can only intersect with one kth-stage basic triangle, denoted by $\triangle$, and cannot completely contain $\triangle$.

Since $\triangle$ contains three $(k+1)$th-stage basic triangle, next we will prove the Theorem by three cases. For the convenience of the following discussion, we denote by $\triangle^l_m$, $\triangle^r_m$, $\triangle^\mu_m$ $(m > k)$ the mth-stage basic triangle on the bottom left, the bottom right and the top in $\triangle$, respectively.

1. U intersects with one $(k+1)$th-stage basic triangle in $\triangle$. Then

$$\mu(U) \leq |\triangle_{k+1}|^s = 3^{-(k+1)}.$$

Since$|U| \geq 3^{-(k+1)}$, (6) holds.

2. U intersects with two $(k+1)$th-stage basic triangles in $\triangle$. Let us say that U intersects with $\triangle^l_{k+1}$, $\triangle^r_{k+1}$ (See Fig. 3). Since U is an open set, $|\overline{U}| = |U| < 3^{-k}$, and hence $\overline{U}$ and U cannot contain both the bottom left vertex A and the bottom right vertex B of $\triangle$.

2.1. If A and B are not contained in $\overline{U}$, then there exist positive integers n, m such that

$$U \cap \triangle^l_{k+n} \neq \emptyset, U \cap \triangle^l_{k+n+1} = \emptyset; U \cap \triangle^r_{k+m} \neq \emptyset, U \cap \triangle^r_{k+m+1} = \emptyset.$$

2.1.1. Assume that $n > 1$ and $m \geq 1$. It is easy to see that

$$|U| \geq |\triangle| - |\triangle^l_{k+n}| - |\triangle^r_{k+m}| = (1 - 3^{-n} - 3^{-m})3^{-k}.$$

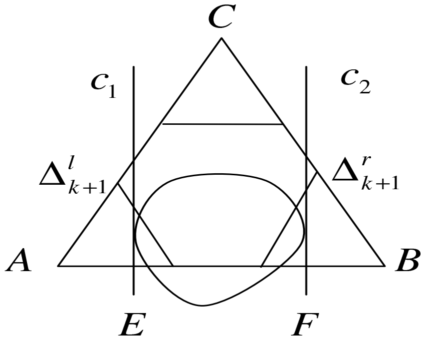

Fig. 3.

Since $\triangle^l_{k+1}$ contains 3^n $(k+n+1)$th-stage basic triangles, and $\triangle^r_{k+1}$ contains 3^m $(k+m+1)$th-stage basic triangles, it follows that

$$\begin{aligned}\mu(U) &\le (3^n-1)|\triangle_{k+n+1}|^s + (3^m-1)|\triangle_{k+m+1}|^s\\ &= \Big(\frac{2}{3}-\frac{1}{3^{n+1}}-\frac{1}{3^{m+1}}\Big)3^{-k}.\end{aligned}$$

Since $n > 1$ and $m \ge 1$, $1-3^{-n}-3^{-m} > 2/3-1/3^{n+1}-1/3^{m+1}$. So $\mu(U) \le |U|^s$, that is, (6) holds.

2.1.2. Assume $n = m = 1$. Now draw two vertical lines c_1 and c_2 to the bottom side of $\triangle$ such that they intersect with the boundary of U and U is located between them. Let the distance between c_1 and the bottom right vertex E of $\triangle^l_{k+1}$ be a, and the distance between c_2 and the bottom left vertex F of $\triangle^r_{k+1}$ be b (See Fig. 3), then $|U| = (|\triangle|-|\triangle^l_{k+1}|-|\triangle^r_{k+1}|)+ a+b = (3^{-k}-2\cdot 3^{-(k+1)})+a+b = 3^{-(k+1)}+a+b$.

If

$$\begin{aligned}0 < a &\le \frac{(3^{-(k+1)}-3^{-(k+2)})}{2} = 3^{-(k+2)},\\ 0 < b &\le \frac{(3^{-(k+1)}-3^{-(k+2)})}{2} = 3^{-(k+2)},\end{aligned}$$

then $|U| > 3^{-(k+1)}$, and U only intersects with one $\triangle_{k+2}$ in $\triangle^l_{k+1}$ and $\triangle^r_{k+1}$, respectively. It follows that $\mu(U) \le 2|\triangle_{k+2}|^s = 2/93^{-k}$. Since $3^{-(k+1)} > (2/9)3^{-k}$, we know that (6) holds.

If $0 < a \le 3^{-(k+2)}$, $b > 3^{-(k+2)}$, then $|U| > 3^{-(k+1)}+3^{-(k+2)} = 4\cdot 3^{-(k+2)}$, and U intersects with one $\triangle_{k+2}$ in $\triangle^l_{k+1}$ and with two $\triangle_{k+2}$ in $\triangle^r_{k+1}$. It follows that $\mu(U) \le 3|\triangle_{k+2}|^s = 3^{-(k+1)}$. Since $4\cdot 3^{-(k+2)} > 3^{-(k+1)}$, we know that (6) holds.

If $a > 3^{-(k+2)}$, $b > 3^{-(k+2)}$, then $|U| > 3^{-(k+1)}+2\cdot 3^{-(k+2)} = 5\cdot 3^{-(k+2)}$, and U intersects with two $\triangle_{k+2}$ in $\triangle^l_{k+1}$ and $\triangle^r_{k+1}$, respectively. It follows that $\mu(U) \le 4|\triangle_{k+2}|^s = 4\cdot 3^{-(k+2)}$. Since $5\cdot 3^{-(k+2)} > 4\cdot 3^{-(k+2)}$, we know that (6) holds.

2.2. If either A or B is contained in $\overline{U}$, let us say $A \notin \overline{U}$, $B \in \overline{U}$, then there exists a positive integer n such that $U \bigcap \triangle^l_{k+n} \ne \emptyset$, $U \bigcap \triangle^l_{k+n+1} = \emptyset$. It is easy to see that $|U| \ge |\triangle| - |\triangle^l_{k+n}| = (1-3^{-n})3^{-k}$, and

$$\mu(U) \le (3^n-1)|\triangle_{k+n+1}|^s + |\triangle^r_{k+1}|^s = \Big(\frac{2}{3}-\frac{1}{3^{n+1}}\Big)3^{-k}.$$

Since $1-3^{-n} > 2/3 - 1/3^{n+1}$, we know that (6) holds.

3. U intersects with three $(k+1)$th-stage basic triangles in $\triangle$ (See Fig. 4). Since $|\overline{U}| = |U| < 3^{-k}$, $\overline{U}$ and U cannot contain any two points of the bottom left vertex A and the bottom right vertex B and the top vertex C of $\triangle$.

3.1. If A, B and C are not contained in $\overline{U}$, then there exist positive integers n, m, l such that

$$\begin{aligned} &U \cap \triangle^l_{k+n} \neq \emptyset, \quad U \cap \triangle^l_{k+n+1} = \emptyset;\\ &U \cap \triangle^r_{k+m} \neq \emptyset, \quad U \cap \triangle^r_{k+m+1} = \emptyset;\\ &U \cap \triangle^u_{k+l} \neq \emptyset, \quad U \cap \triangle^u_{k+l+1} = \emptyset. \end{aligned}$$

Now draw three straight lines paralleling the sides of $\triangle$ such that they intersect with the boundary of U, and U is contained in the equilateral triangle V bounded by the above three straight lines (See Fig. 4). Denote the side length of V by$|V|$. By Lemma 1, we know that $|U| \geq |V|/2$.

Note that each side of V consists of three parts. It is easy to see that

$$\begin{aligned} |V| \geq &(3^{-k} - 3^{-(k+n)} - 3^{-(k+m)}) + 3^{-(k+m)}\\ &\qquad + (3^{-k} - 3^{-(k+m)} - 3^{-(k+l)}) \qquad (7)\\ = &2\cdot 3^{-k} - (3^{-(k+n)} + 3^{-(k+m)} + 3^{-(k+l)}). \end{aligned}$$

Next, we will discuss by four cases.

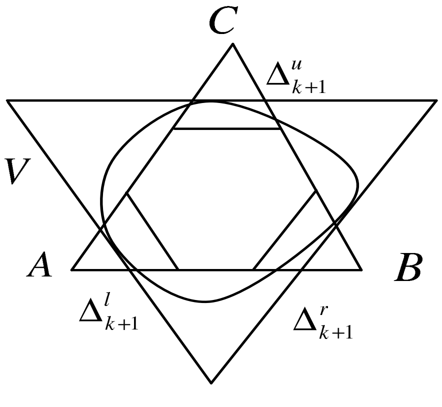

Fig. 4.

3.1.1. Assume that U only intersects with one $\triangle_{k+n+1}$ in $\triangle^l_{k+n}$, one $\triangle_{k+m+1}$ in $\triangle^r_{k+m}$, and one $\triangle_{k+l+1}$ in $\triangle^u_{k+l}$ respectively, then

$$\begin{aligned}\mu(U) &\le (3^n-2)|\triangle_{k+n+1}|^s + (3^m-2)|\triangle_{k+m+1}|^s + (3^l-2)|\triangle_{k+l+1}|^s \\ &= \Big(1-\frac{2}{3^{n+1}}-\frac{2}{3^{m+1}}-\frac{2}{3^{l+1}}\Big)3^{-k},\end{aligned}$$

and $|U| \ge |V|/2 \ge \big(1-(1/2)(3^{-n}+3^{-m}+3^{-l})\big)3^{-k}$. We know that (6) holds because of $1-1/2(3^{-n}+3^{-m}+3^{-l}) > 1-2/3^{n+1}-2/3^{m+1}-2/3^{l+1}$.

3.1.2. Assume that U intersects with two $\triangle_{k+n+1}$ in $\triangle^l_{k+n}$ and with only one $\triangle_{k+m+1}$ in $\triangle^r_{k+m}$ and one $\triangle_{k+l+1}$ in $\triangle^u_{k+l}$ respectively.

If there exists an integer $t \ge 2$ such that U cannot intersect with those two $(k+n+t)$th-stage basic triangles, one of which is on the bottom left of the $(k+n+1)$th-stage basic triangle that is on the top of $\triangle^l_{k+n}$, and the other is on the bottom left of the $(k+n+1)$th-stage basic triangle that is on the bottom right of $\triangle^l_{k+n}$. In fact, if t satisfies the above condition, so does t' for any $t' > t$. We choose t to be the minimum of these t'. It is easy to see that

$$\begin{aligned}|V| &\ge 2\cdot 3^{-k} - (3^{-(k+n)}+3^{-(k+m)}+3^{-(k+l)}) \\ &\qquad\qquad +(|\triangle_{k+n+1}|-|\triangle_{k+n+t-1}|), \\ |U| &\ge \frac{|V|}{2} \ge 3^{-k} - \frac{1}{2}(3^{-(k+n)}+3^{-(k+m)}+3^{-(k+l)}) \\ &\qquad\qquad +\frac{1}{2}(|\triangle_{k+n+1}|-|\triangle_{k+n+t-1}|) \\ &= \Big(1-\frac{1}{2}(3^{-n}+3^{-m}+3^{-l})+\frac{1}{2}(3^{-(n+1)}-3^{-(n+t-1)})\Big)3^{-k},\end{aligned}$$

and

$$\begin{aligned}\mu(U) &\le (3^{-n}-1)|\triangle_{k+n+1}|^s - 2|\triangle_{k+n+t}|^s \\ &\qquad\qquad +(3^m-2)|\triangle_{k+m+1}|^s + (3^l-2)|\triangle_{k+l+1}|^s \\ &= \Big(1-\frac{1}{3^{n+1}}-\frac{2}{3^{m+1}}-\frac{2}{3^{l+1}}-\frac{2}{3^{n+t}}\Big)3^{-k}.\end{aligned}$$

Since

$$\begin{aligned}&1-\frac{1}{2}(3^{-n}+3^{-m}+3^{-l})+\frac{1}{2}(3^{-(n+1)}-3^{-(n+t-1)}) \\ &\qquad\qquad > 1-\frac{1}{3^{n+1}}-\frac{2}{3^{m+1}}-\frac{2}{3^{l+1}}-\frac{2}{3^{n+t}},\end{aligned}$$

We know that (6) holds.

If the above t does not exist, it is easy to see that

$$|V| \geq 2\cdot 3^{-k} - (3^{-(k+n)} + 3^{-(k+m)} + 3^{-(k+l)}) + |\triangle_{k+n+1}|,$$
$$|U| \geq \frac{|V|}{2} \geq \Big(1 - \frac{1}{2}(3^{-n} + 3^{-m} + 3^{-l}) + \frac{1}{2}3^{-(n+1)}\Big)3^{-k},$$

and

$$\begin{aligned}\mu(U) &\leq (3^n - 1)|\triangle_{k+n+1}|^s + (3^m - 2)|\triangle_{k+m+1}|^s + (3^l - 2)|\triangle_{k+l+1}|^s\\ &= \Big(1 - \frac{1}{3^{n+1}} - \frac{2}{3^{m+1}} - \frac{2}{3^{l+1}}\Big)3^{-k}.\end{aligned}$$

Since

$$1 - \frac{1}{2}(3^{-n} + 3^{-m} + 3^{-l}) + \frac{1}{2}3^{-(n+1)} > 1 - \frac{1}{3^{n+1}} - \frac{2}{3^{m+1}} - \frac{2}{3^{l+1}},$$

We know that (6) holds.

3.1.3. Assume that U intersects with two $\triangle_{k+n+1}$ in $\triangle^l_{k+n}$ and two $\triangle_{k+m+1}$ in $\triangle^r_{k+m}$ respectively, and with only one $\triangle_{k+l+1}$ in $\triangle^u_{k+l}$.

3.1.4. Assume that U intersects with two $\triangle_{k+n+1}$ in $\triangle^l_{k+n}$, two $\triangle_{k+m+1}$ in $\triangle^r_{k+m}$ and two $\triangle_{k+l+1}$ in $\triangle^u_{k+l}$ respectively.

By the symmetry of $\triangle^l_{k+1}$, $\triangle^r_{k+1}$, $\triangle^u_{k+1}$, the proofs of 3.1.3 and 3.1.4 are entirely similar to the discussion of 3.1.2.

3.2. If only one of A, B and C is contained in $\overline{U}$, let us say $C \in \overline{U}$ then there exist positive integers n, m such that

$$U \cap \triangle^l_{k+n} \neq \emptyset, U \cap \triangle^l_{k+n+1} = \emptyset, U \cap \triangle^r_{k+m} \neq \emptyset, U \cap \triangle^r_{k+m+1} = \emptyset.$$

In this case, we have

$$|V| \geq 2\cdot 3^{-k} - (3^{-(k+n)} + 3^{-(k+m)}),$$
$$\text{and}\quad |U| \geq \frac{|V|}{2} \geq \Big(1 - \frac{1}{2}(3^{-n} + 3^{-m})\Big)3^{-k}.$$

Note that $\mu(U \cap \triangle^u_{k+1}) \leq |\triangle^u_{k+1}|^s$. The rest of the proof is the same as 3.1.

This completes the proof of Theorem 1. □

Theorem 2 $H(S) = H_N(S) = 1$.

Proof. By Lemma 2 and Corollary 1, we have $H(S) \leq H_N(S) = 1$. To prove the opposite inequality, we need consider all δ-coverings of S by the definition of the Hausdorff measure. Since the class of all sets is completely equivalent to the class of all the open sets ([1-2]), it is enough to prove $H(S) \geq H_N(S) = 1$ for all open δ-coverings of S. Now, let U be any open subset of R^2, then (6) holds for U. According to the mass distribution principle (See [1]), we have

$$H(S) \geq \mu(S) = 1. \tag{8}$$

So we obtain $H(S) = H_N(S) = 1$. $\square$

In reference [7], the authors pose eight open problems on the exact value of the Hausdorff measure. The first open problem is as follows.

Problem 1 Under what conditions is there a covering of E, say $\alpha = \{U_i\}$, such that $H^s(E) = \sum_{U_i \in \alpha} |U_i|^s$?

Such a covering of E is called a best covering.

From the references [1-2], It is easy to see that the Cantor set C possesses a best covering $\{C\}$, and that the class of the kth-stage basic intervals to generate C is all the best covering of C.

For the Sierpinski-like fractal S, we can obtain the following result by Lemma 3 and Theorem 2.

Corollary 2 *Let $\alpha = \{U_i\}$ be any covering of S consisting of basic triangles in which each of α cannot completely contain the other, then $\alpha = \{U_i\}$ is the best covering of S.*

Note that $\alpha = \{U_i\}$ in Corollary 2 may be infinite, so it is non-trivial.

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