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On the normal forms for Pfaffian systems

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Abstract. We discuss local normal forms of Pfaffian systems and obtain a necessary and sufficient condition, in terms of relative polarizations, for the local generators of a Pfaffian system to convert to the contact system on the jet manifold $J^r(\mathbb{R}^h, \mathbb{R}^q)$, or to the Pfaffian system associated to a system of partial differential equations. This result generalizes the Darboux theorem on a Pfaffian equation of constant class.

Key words: Pfaffian system, contact system, relative polarization, local normal form.

1. Introduction

On the jet manifold $J^r(\mathbb{R}^h, \mathbb{R}^q) = (x_{\alpha_1}, z^i, z^i_{\alpha_1}, \ldots, z^i_{\alpha_1\cdots\alpha_r})$ there is a canonical Pfaffian system, called the *contact system*, which is the Pfaffian system generated by the 1-forms

$$\begin{cases} \omega^i = dz^i - \displaystyle\sum_{\alpha_1=1}^{h} z^i_{\alpha_1} dx_{\alpha_1}, \\ \omega^i_{\alpha_1} = dz^i_{\alpha_1} - \displaystyle\sum_{\alpha_2=1}^{h} z^i_{\alpha_1\alpha_2}\, dx_{\alpha_2}, \\ \quad\vdots \qquad\qquad \vdots \\ \omega^i_{\alpha_1\cdots\alpha_{r-1}} = dz^i_{\alpha_1\cdots\alpha_{r-1}} - \displaystyle\sum_{\alpha_r=1}^{h} z^i_{\alpha_1\cdots\alpha_r}\, dx_{\alpha_r}. \end{cases} \tag{1}$$

$$(1 \le i \le q,\ 1 \le \alpha_1, \ldots, \alpha_{r-1} \le h)$$

The contact system restricted to a submanifold of $J^r(\mathbb{R}^h, \mathbb{R}^q)$ is called the *Pfaffian system associated to a system of partial differential equations*.

The study of local normal forms of Pfaffian systems seems to have been initiated by Darboux [1], who showed that a Pfaffian equation $\omega = 0$ of constant class $2h+1$ (that is, $\omega \wedge (d\omega)^h \neq 0$ and $\omega \wedge (d\omega)^{h+1} = 0$) can be

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transformed locally into the contact form

$$\omega = dz - \sum_{i=1}^{h} y_i \, dx_i,$$

on the jet manifold $J^1(\mathbb{R}^h, \mathbb{R})$ (called the *Darboux theorem on a Pfaffian equation of constant class*). Many works have been done concerning local normal forms of Pfaffian systems in various cases. Engel [4], von Weber [13], Bryant [2], Goze-Haraguchi [6], and Kumpera-Rubin [7] gave a necessary and sufficient condition for the local generators of a Pfaffian system to be transformed into the contact system on $J^2(\mathbb{R}, \mathbb{R})$ (called the *Engel normal form*), the contact system on $J^r(\mathbb{R}, \mathbb{R})$ (called the *Cartan-von Weber model*), the contact system on $J^1(\mathbb{R}^h, \mathbb{R}^q)$, $q \geq 3$ (called the *Bryant normal form*), the contact system on $J^1(\mathbb{R}^h, \mathbb{R}^q)$, and the contact system on $J^r(\mathbb{R}, \mathbb{R}^q)$ (called the *extended von Weber model*) respectively. Yamaguchi [14] gave a general criterion for a Pfaffian system to be transformed into the contact system on $J^r(\mathbb{R}^h, \mathbb{R}^q)$ using the theory of exterior differential systems.

The aim of this paper is to give a necessary and sufficient condition for a Pfaffian system to be transformed into the contact system on $J^r(\mathbb{R}^h, \mathbb{R}^q)$ or into the Pfaffian system associated to a system of partial differential equations. To prove this result, we use the properties of relative polarization (see in Lemma 3.1 and Lemma 3.2).

In Section 2, we prepare some basic definitions from the theory of Pfaffian systems, and recall Pfaffian systems on jet manifolds (the contact system and the Pfaffian system associated to a system of partial differential equations).

In Section 3, we define the relative polarization of a Pfaffian system according to Libermann [9], and discuss the Pfaffian systems with relative polarizations. We note that there is not always the relative polarization of a Pfaffian system (see in Proposition 3.3). Given a Pfaffian system with relative polarization, we can choose local coordinates to simplify the local generators of a Pfaffian system. We also evaluate the class and the Engel invariant, which are the invaiants of a Pfaffian system, using the existence of relative polarization.

In Section 4, we prove that under the existence of relative polarization and the condition on the type and the class of a Pfaffian system, the local generators of a Pfaffian system can be transformed into the contact system (1) above, or into the contact system restricted to a submanifold of $J^r(\mathbb{R}^h, \mathbb{R}^q)$ (see in Theorem 4.1).

2. Pfaffian systems

Let M be a C^∞-manifold of dimension n and T^*M its cotangent bundle. Let $C^\infty(M)$ denote the ring of C^∞-functions on M and $\Gamma(T^*M)$ the $C^\infty(M)$-module of global C^∞-sections of T^*M.

A *Pfaffian system* S on M is a $C^\infty(M)$-submodule of $\Gamma(T^*M)$. The system S is said to have *rank* q at $x \in M$ if the values of elements of S (at x) form a q-dimensional subspace of T_x^*M. If the rank of S is constant on M, S is a space of C^∞-sections of a subbundle of T^*M. We usually consider locally, and assume that S is of constant rank and spanned by 1-forms $\omega^1, \dots, \omega^q$.

Let $\Omega^k(M)$ be the space of k-forms on M, $\Omega^*(M) = \oplus_{k=0}^n \Omega^k(M)$ the exterior algebra of differential forms on M, and $\mathscr{S}$ the ideal generated by S.

Definition 1 The *Engel invariant* of a Pfaffian system S is the non-negative integer s_0 defined as

$$s_0 = \min\{s \in \mathbb{Z}\ (\geq 0) \,|\, (d\omega)^{s+1} \equiv 0 \bmod \mathscr{S},\ \omega \in S\}.$$

Let S be a Pfaffian system on M of rank q and $\{\omega^1, \dots, \omega^q\}$ a local basis of S. We define $\mathrm{Char}_x\, S$ by

$$\mathrm{Char}_x\, S = \{X \in T_xM \,|\, \omega^i(X) = 0,\ \iota_X\, d\omega^i \equiv 0 \bmod \mathscr{S},\ 1 \leq i \leq q\},$$

where ι_X is the interior product with respect to the vector X. We denote the annihilator of $\mathrm{Char}_x\, S$ by $C(S)_x$. Note that the dimension of $C(S)_x$ is not necessarily constant even if the rank of S is constant on M. However, we always consider the case where $C(S)_x$ is of constant dimension on M throughout this paper. Then $C(S) := \cup_{x \in M} C(S)_x$ (called the *Cartan system* of S) is a space of C^∞-sections of a subbundle of T^*M.

Definition 2 The *class* of S is the rank of $C(S)$.

The following theorem shows that the class of a Pfaffian system is the minimum number of variables necessary for describing local generators of the system.

Theorem 2.1 (Cartan [3]) *Let S be a Pfaffian system of constant rank. The Cartan system $C(S)$ is the smallest completely integrable Pfaffian system with the property such that if $(x_1, \dots, x_p)$ is a local system of the first integrals of $C(S)$, then there exist local generators of S, which depend only on $(x_1, \dots, x_p)$ and their differentials.*

The *first derived system* of S is defined as

$$S_1 = \{\eta \in S \,|\, d\eta \equiv 0 \bmod \mathscr{S}\}.$$

Definition 3 We say that a Pfaffian system S is *totally regular* if there is a decreasing sequence

$$S = S_0 \supset S_1 \supset \cdots \supset S_{r-1} \supset S_r = S_{r+1} = \cdots$$

of Pfaffian systems of constant rank, where S_k, $1 \leq k \leq r$, is the first derived system of S_{k-1} (called the *kth derived system* of S).

Obviously, there exists the smallest non-negative integer r such that $S_{r+p} = S_{r+p+1}$ for all $p \geq 0$. The integer r satisfying the condition is called the *length* of S.

Definition 4 Let S be a totally regular Pfaffian system of length r

$$C(S) \supset S = S_0 \supset S_1 \supset \cdots \supset S_{r-1} \supset S_r = S_{r+1} = \cdots .$$

The *type* of S is the $(r+2)$-tuple of non-negative integers $(p_0, \dots, p_{r+1})$ defined by

$$\begin{cases} p_0 = \operatorname{rank} S_r, \\ p_k = \operatorname{rank} S_{r-k}/S_{r-k+1}, \ 1 \leq k \leq r, \\ p_{r+1} = \operatorname{rank} C(S)/S. \end{cases}$$

The rank of each derived system S_{r-k}, $1 \leq k \leq r$, and the class of S easily seen to be

$$\operatorname{rank} S_{r-k} = \sum_{i=0}^{k} p_i, \quad \operatorname{class} S = \sum_{i=0}^{r+1} p_i.$$

Let $J^r(\mathbb{R}^h, \mathbb{R}^q)$ be the manifold of r-jets of maps from $\mathbb{R}^h$ to $\mathbb{R}^q$ and

$$\mathbf{x} := (x_{\alpha_1}, z^i, z^i_{\alpha_1}, \dots, z^i_{\alpha_1 \cdots \alpha_r})_{1 \leq i \leq q,\ 1 \leq \alpha_1, \dots, \alpha_r \leq h}$$

the canonical coordinate system of $J^r(\mathbb{R}^h, \mathbb{R}^q)$, where $(x_\alpha) \in \mathbb{R}^h$, $(z^i) \in \mathbb{R}^q$, and $z^i_{\alpha_1\alpha_2}, \dots, z^i_{\alpha_1\dots\alpha_r}$ are symmetric with respect to the subscript $\alpha_1, \dots, \alpha_r$. On $J^r(\mathbb{R}^h, \mathbb{R}^q)$, there is a canonical Pfaffian system, called the contact system, which is given as follows.

Definition 5 The *contact system* on $J^r(\mathbb{R}^h, \mathbb{R}^q)$ is the Pfaffian system generated by the 1-forms $\omega^i, \omega^i_{\alpha_1}, \dots, \omega^i_{\alpha_1\dots\alpha_{r-1}}$, where

$$\begin{cases} \omega^i := dz^i - \displaystyle\sum_{\alpha_1=1}^{h} z^i_{\alpha_1}\, dx_{\alpha_1}, \\ \omega^i_{\alpha_1} := dz^i_{\alpha_1} - \displaystyle\sum_{\alpha_2=1}^{h} z^i_{\alpha_1\alpha_2}\, dx_{\alpha_2}, \\ \quad\vdots \qquad\qquad \vdots \\ \omega^i_{\alpha_1\cdots\alpha_{r-1}} := dz^i_{\alpha_1\cdots\alpha_{r-1}} - \displaystyle\sum_{\alpha_r=1}^{h} z^i_{\alpha_1\cdots\alpha_r}\, dx_{\alpha_r}. \end{cases} \tag{2}$$

$$(1 \le i \le q,\ 1 \le \alpha_1, \dots, \alpha_{r-1} \le h)$$

We denote this contact system on $J^r(\mathbb{R}^h, \mathbb{R}^q)$ by $\Omega^r(\mathbb{R}^h, \mathbb{R}^q)$. We note that the 1-forms $\omega^i_{\alpha_1\dots\alpha_k}$, $2 \le k \le r-1$, are *symmetric* with respect to the subscript $\alpha_1, \dots, \alpha_k$. Namely, $(\alpha_1, \dots, \alpha_k)$ is the same index as $(\beta_1, \dots, \beta_k)$ as a set, then $\omega^i_{\alpha_1\dots\alpha_k}$ and $\omega^i_{\beta_1\dots\beta_k}$ are coincident.

Example 1 $\Omega^r(\mathbb{R}^h, \mathbb{R}^q)$ is a totally regular Pfaffian system of length r. The derived system of $\Omega^r(\mathbb{R}^h, \mathbb{R}^q)$ is computed as follows:

$$\begin{aligned} \Omega^r(\mathbb{R}^h, \mathbb{R}^q)_1 &= \operatorname{span}\{\omega^i, \dots, \omega^i_{\alpha_1\cdots\alpha_{r-2}}\} = \pi^{r*}_{r-1}\Omega^{r-1}(\mathbb{R}^h, \mathbb{R}^q), \\ \Omega^r(\mathbb{R}^h, \mathbb{R}^q)_2 &= \operatorname{span}\{\omega^i, \dots, \omega^i_{\alpha_1\cdots\alpha_{r-3}}\} = \pi^{r*}_{r-2}\Omega^{r-2}(\mathbb{R}^h, \mathbb{R}^q), \\ &\ \ \vdots \\ \Omega^r(\mathbb{R}^h, \mathbb{R}^q)_{r-1} &= \operatorname{span}\{\omega^i\} = \pi^{r*}_{1}\Omega^{1}(\mathbb{R}^h, \mathbb{R}^q), \\ \Omega^r(\mathbb{R}^h, \mathbb{R}^q)_r &= \{0\}, \end{aligned}$$

where $\pi^r_k \colon J^r(\mathbb{R}^h, \mathbb{R}^q) \to J^k(\mathbb{R}^h, \mathbb{R}^q)$ is the natural projection. The rank, the class, and the Engel invariant of each derived system are

$$\operatorname{rank} \Omega^r(\mathbb{R}^h, \mathbb{R}^q)_{r-j} = q\binom{h+j-1}{j-1},$$

$$\operatorname{class}\Omega^r(\mathbb{R}^h,\mathbb{R}^q)_{r-j} = h + q\binom{h+j}{j},$$
$$s_{r-j} = h, \qquad (1 \le j \le r-1)$$

where $\binom{a}{b}$ stands for the binomial coefficients. The type of $\Omega^r(\mathbb{R}^h,\mathbb{R}^q)$ is

$$(p_0, p_1, \ldots, p_r, p_{r+1}) = \bigl(0, q, \ldots, q\tbinom{h+r-2}{h+r-1}, q\tbinom{h+r-1}{h+r}\bigr).$$

We define the Pfaffian system associated to a system of partial differential equations (*PDEs* for short). By *a system of rth order PDEs in* $\mathbb{R}^h$ *and* $\mathbb{R}^q$ *of codimension* σ, we mean a submanifold $\mathscr{L}_{\{F_j\}}$ of $J^r(\mathbb{R}^h,\mathbb{R}^q)$ defined locally by

$$\mathscr{L}_{\{F_j\}} = \{\mathbf{x} \in J^r(\mathbb{R}^h,\mathbb{R}^q) \mid F_j(\mathbf{x}) = 0,\ 1 \le j \le \sigma\},$$

where we require the following regularity condition

$$\operatorname{rank}\frac{\partial(F_1,\ldots,F_\sigma)}{\partial(z^1_{1\cdots1},\ldots,z^i_{\alpha_1\cdots\alpha_r},\ldots,z^q_{h\cdots h})} = \sigma.$$

Let Σ be an open subset of $\mathbb{R}^l$, $l = \dim J^r(\mathbb{R}^h,\mathbb{R}^q) - \sigma$, and $\imath\colon \Sigma \to \mathscr{L}_{\{F_j\}}$ a C^∞-map of maximal rank.

Definition 6 The *Pfaffian system on* Σ *associated to* $\mathscr{L}_{\{F_j\}}$ is the pull back $\imath^*\Omega^r(\mathbb{R}^h,\mathbb{R}^q)$ of $\Omega^r(\mathbb{R}^h,\mathbb{R}^q)$ to Σ. We denote this system by $\Omega(\Sigma^{(r,h,q)}_\sigma) := \imath^*\Omega^r(\mathbb{R}^h,\mathbb{R}^q)$.

Example 2 (Libermann [10]) Consider the Cauchy-Riemann equation

$$\mathscr{L}_{\{F_j\}} = \{\mathbf{x} \in J^1(\mathbb{R}^2,\mathbb{R}^2) \mid z^1_1 - z^2_2 = 0,\ z^1_2 + z^2_1 = 0\}.$$

Take $\Sigma = \mathbb{R}^6 = \{(x_1,x_2,z^1,z^2,z^1_1,z^1_2)\}$. We have the map $\imath\colon \Sigma \to J^1(\mathbb{R}^2,\mathbb{R}^2)$ defined by

$$\imath(x_1,x_2,z^1,z^2,z^1_1,z^1_2) = (x_1,x_2,z^1,z^2,z^1_1,z^1_2,-z^1_2,z^1_1).$$

Therefore

$$\Omega(\Sigma^{(1,2,2)}_2) = \begin{cases} \imath^*\omega^1 = dz^1 - z^1_1\,dx_1 - z^1_2\,dx_2 \\ \imath^*\omega^2 = dz^2 + z^1_2\,dx_1 - z^1_1\,dx_2. \end{cases}$$

Example 3 (Gardner [5]) Consider the Monge-Ampère equation

$$\mathscr{L}_F = \{\mathbf{x} \in J^2(\mathbb{R}^2,\mathbb{R}) \mid (z^1_{11} - f_1)(z^1_{22} - f_3) - (z^1_{12} - f_2)^2 = 0\},$$

where f_i is a function of variables $(x_1, x_2, z^1, z^1_1, z^1_2)$ of $J^2(\mathbb{R}^2, \mathbb{R}) \to J^1(\mathbb{R}^2, \mathbb{R})$. Take $\Sigma = \{(x_1, x_2, z^1, z^1_1, z^1_2, u, v) \in \mathbb{R}^7 \mid (u, v) \neq (0, 0)\}$. We have the map $\imath\colon \Sigma \to J^2(\mathbb{R}^2, \mathbb{R})$ defined by

$$\imath(x_1, x_2, z^1, z^1_1, z^1_2, u, v) = (x_1, x_2, z^1, z^1_1, z^1_2, f_1 + vu^2, f_2 + vu, f_3 + v).$$

Therefore

$$\Omega(\Sigma_1^{(2,2,1)}) = \begin{cases} \imath^*\omega^1 = dz^1 - z^1_1\, dx_1 - z^1_2\, dx_2 \\ \imath^*\omega^1_1 = dz^1_1 - (f_1 + vu^2)\, dx_1 - (f_2 + vu)\, dx_2 \\ \imath^*\omega^1_2 = dz^1_2 - (f_2 + vu)\, dx_1 - (f_3 + v)\, dx_2. \end{cases}$$

3. Pfaffian systems with relative polarizations

In this section we discuss totally regular Pfaffian systems with relative polarizations. A relative polarization, which was introduced by Libermann [9], is a useful notion in our discussion (called *pivot* by Lutz [11] and it is the anologue of *polarization* introduced by Molino [12]). There does not always exist such a relative polarization for every totally regular Pfaffian system. Given a totally regular Pfaffian system with relative polarization, we can choose local coordinates to simplify the local genelators of a Pfaffian system. Also, we evaluate the Engel invariant and the class, which is the minimum number of variables necessary in order to describe the local generators of a Pfaffian system. Here we start with the definition of relative polarizations for totally regular Pfaffian systems.

Definition 7 Let S be a totally regular Pfaffian system on M of length r. We say that S has a *relative polarization* H if there exists a Pfaffian system H of constant rank, satisfying the following conditions:

(i) $S \cap H = 0_M$ (zero section),

(ii) H and $S_i \oplus H$, $0 \leq i \leq r$, are completely integrable, where S_i is the ith derived system of S,

(iii) H is minimal, that is, if a subbundle $\overline{H} \subset H$ satisfies the conditions (i) and (ii), then $\overline{H} = H$.

Example 4 Consider the contact system $\Omega^r(\mathbb{R}^h, \mathbb{R}^q)$ on the jet manifold $J^r(\mathbb{R}^h, \mathbb{R}^q)$. The exterior derivatives of (2) can be written as

$$
\begin{cases}
d\omega^i \equiv \displaystyle\sum_{\alpha_1=1}^{h} dx_{\alpha_1} \wedge \omega^i_{\alpha_1} \mod \mathscr{S}_{r-1}, \\
\quad \vdots \qquad\qquad \vdots \\
d\omega^i_{\alpha_1\cdots\alpha_l} \equiv \displaystyle\sum_{\alpha_{l+1}=1}^{h} dx_{\alpha_{l+1}} \wedge \omega^i_{\alpha_1\cdots\alpha_{l+1}} \mod \mathscr{S}_{r-l-1}, \\
\quad \vdots \qquad\qquad \vdots \\
d\omega^i_{\alpha_1\cdots\alpha_{r-1}} \equiv \displaystyle\sum_{\alpha_r=1}^{h} dx_{\alpha_r} \wedge dz^i_{\alpha_1\cdots\alpha_{r-1}\alpha_r} \mod \mathscr{S}.
\end{cases}
$$

$$(1 \leq i \leq q,\ 1 \leq l \leq r-2,\ 1 \leq \alpha_1, \ldots, \alpha_r \leq h)$$

It can be verified that $\Omega^r(\mathbb{R}^h, \mathbb{R}^q)$ has a relative polarization

$$H = \operatorname{span}\{dx_1, \ldots, dx_h\}.$$

We note that H is the pull back of $T^*\mathbb{R}^h$ by the source projection $J^r(\mathbb{R}^h, \mathbb{R}^q) \to \mathbb{R}^h$.

Example 5 (Libermann [10]) Consider the Pfaffian system S on $\mathbb{R}^7$ of rank 5 defined by 1-forms $\omega^1, \ldots, \omega^5$, satisfying

$$d\omega^1 = \omega^4 \wedge \omega^6 + \omega^3 \wedge \omega^7, \tag{3}$$
$$d\omega^2 = \omega^4 \wedge \omega^7 + \omega^3 \wedge \omega^6, \tag{4}$$
$$d\omega^3 = \omega^5 \wedge \omega^6, \tag{5}$$
$$d\omega^4 = \omega^5 \wedge \omega^7, \tag{6}$$
$$d\omega^5 = \omega^6 \wedge \omega^7, \tag{7}$$

where $\omega^1 \wedge \cdots \wedge \omega^7 \neq 0$. Then S is a totally regular Pfaffian system of length 3 and the derived systems are

$$S_1 = \operatorname{span}\{\omega^1, \ldots, \omega^4\}, \quad S_2 = \operatorname{span}\{\omega^1, \omega^2\}, \quad S_3 = \{0\}.$$

The system S has a relative polarization $H = \operatorname{span}\{\omega^6, \omega^7\}$ of rank 2.

Indeed, differentiating (3)∼(7) we have

$$0 = \omega^4 \wedge d\omega^6 + \omega^3 \wedge d\omega^7, \tag{8}$$
$$0 = \omega^4 \wedge d\omega^7 + \omega^3 \wedge d\omega^6, \tag{9}$$
$$0 = \omega^5 \wedge d\omega^6, \tag{10}$$

$$0 = \omega^5 \wedge d\omega^7, \tag{11}$$

$$0 = d\omega^6 \wedge \omega^7 - \omega^6 \wedge d\omega^7. \tag{12}$$

From (10) and (11) there exist two 1-forms θ^6, $\theta^7 \in \Omega^1(\mathbb{R}^7)$ such that

$$d\omega^6 = \omega^5 \wedge \theta^6, \quad d\omega^7 = \omega^5 \wedge \theta^7. \tag{13}$$

Put the 1-forms θ^i ($i = 6, 7$) as $\theta^i := \sum_{j=1}^7 f^i_j \omega^j$, $f^i_j \in \Omega^0(\mathbb{R}^7)$. Substituting (13) into (8), (9), and (12), we can see that $\theta^6 = f^6_5 \omega^5$, $\theta^7 = f^7_5 \omega^5$. Therefore

$$d\omega^6 = d\omega^7 = 0.$$

Thus the system H is completely integrable. Since $S_i \oplus H$, $0 \le i \le 2$, are also completely integrable by (8)$\sim$(12), it is easy to see that H satisfies the conditions (i)$\sim$(iii).

Let us consider a totally regular Pfaffian system S of length r with relative polarization of rank h. We have the following lemmas on S. The main theorem in Section 4 can be obtained by using these properties. Let M be a C^∞-manifold of dimension n, where n is sufficiently big. In the next lemma, due to Libermann [10], we use

$$x := (x_1, \dots, x_h), \quad y^{(k)} := \big(y^{(k)}_1, \dots, y^{(k)}_{p_k}\big),$$
$$u := (u_1, \dots, u_{n-(h+p_1+\cdots+p_r)})$$

to denote local coordinates of M, where $(p_0, \dots, p_{r+1})$ is the type of S and $k = 1, \dots, r$.

Lemma 3.1 (Libermann) *Let S be a totally regular Pfaffian system on M of length r with $S_r = \{0\}$. If S has a relative polarization H of rank h, then there exist local coordinates $(x, y^{(1)}, \dots, y^{(r)}, u)$ around each point in M such that S is generated by 1-forms*

$$\begin{cases} \omega^{i_1}_{(1)} = dy^{(1)}_{i_1} - \displaystyle\sum_{\alpha=1}^{h} A^{i_1}_\alpha(1)\, dx_\alpha, \\ \vdots \qquad\qquad \vdots \\ \omega^{i_r}_{(r)} = dy^{(r)}_{i_r} - \displaystyle\sum_{\alpha=1}^{h} A^{i_r}_\alpha(r)\, dx_\alpha. \end{cases} \tag{14}$$

$$(1 \le i_1 \le p_1, \dots, 1 \le i_r \le p_r)$$

Each derived system S_{r-k}, $1 \le k \le r-1$, is generated by the 1-forms $\omega^{i_1}_{(1)}, \ldots, \omega^{i_k}_{(k)}$, and each $A^{i_k}_\alpha(k)$ is a function

$$A^{i_k}_\alpha(k) = A^{i_k}_\alpha(k)\big(x, y^{(1)}, \ldots, y^{(k)}, y^{(k+1)}\big).$$

Proof. By the integrability of H and $S_{r-1} \oplus H$, let $(x_1, \ldots, x_h)$ be a family of the first integrals generating H and $\big(y^{(1)}_1, \ldots, y^{(1)}_{p_1+h}\big)$ a family of the first integrals generating $S_{r-1} \oplus H$. Then we can choose, by changing numbering if needed, the local basis $\big\{\omega^1_{(1)}, \ldots, \omega^{p_1}_{(1)}\big\}$ of S_{r-1} such that $\omega^1_{(1)} = \pi_1\big(dy^{(1)}_1\big), \ldots, \omega^{p_1}_{(1)} = \pi_1\big(dy^{(1)}_{p_1}\big)$, where $\pi_1 \colon S_{r-1} \oplus H \to S_{r-1}$ is the natural projection. Therefore we have

$$dy^{(1)}_{i_1} = \omega^{i_1}_{(1)} + \sum_{\alpha=1}^{h} A^{i_1}_\alpha(1)\, dx_\alpha, \qquad 1 \le i_1 \le p_1,$$

where $A^{i_1}_\alpha(1) \in \Omega^0(M)$. Thus the $(r-1)$st derived system S_{r-1} is locally generated by the 1-forms

$$\omega^{i_1}_{(1)} = dy^{(1)}_{i_1} - \sum_{\alpha=1}^{h} A^{i_1}_\alpha(1)\, dx_\alpha, \qquad 1 \le i_1 \le p_1.$$

To consider the $(r-2)$nd derived system S_{r-2}, we take any Pfaffian system T_{r-1} satisfying $S_{r-2} = S_{r-1} \oplus T_{r-1}$. By the integrability of $S_{r-2} \oplus H$, let $\big(y^{(2)}_1, \ldots, y^{(2)}_{p_1+p_2+h}\big)$ be a family of the first integrals generating $S_{r-2} \oplus H$. Then we can choose, by changing numbering if needed, the local basis $\big\{\eta^1_{(2)}, \ldots, \eta^{p_2}_{(2)}\big\}$ of T_{r-1} such that $\eta^1_{(2)} = \pi_2\big(dy^{(2)}_1\big), \ldots, \eta^{p_2}_{(2)} = \pi_2\big(dy^{(2)}_{p_2}\big)$, where $\pi_2 \colon S_{r-2} \oplus H \to T_{r-1}$ is the natural projection. Therefore we have

$$dy^{(2)}_{i_2} = \eta^{i_2}_{(2)} + \sum_{i_1=1}^{p_1} f^{i_2}_{i_1} \omega^{i_1}_{(1)} + \sum_{\alpha=1}^{h} A^{i_2}_\alpha(2)\, dx_\alpha, \qquad 1 \le i_2 \le p_2,$$

where $f^{i_2}_{i_1}$, $A^{i_2}_\alpha(2) \in \Omega^0(M)$. Putting $\omega^{i_2}_{(2)} := \eta^{i_2}_{(2)} + \sum_{i_1=1}^{p_1} f^{i_2}_{i_1} \omega^{i_1}_{(1)}$, we can see that S_{r-2} is locally generated by the 1-forms

$$\begin{cases} \omega^{i_1}_{(1)} = dy^{(1)}_{i_1} - \displaystyle\sum_{\alpha=1}^{h} A^{i_1}_\alpha(1)\, dx_\alpha, \\ \omega^{i_2}_{(2)} = dy^{(2)}_{i_2} - \displaystyle\sum_{\alpha=1}^{h} A^{i_2}_\alpha(2)\, dx_\alpha. \qquad (1 \le i_1 \le p_1,\ 1 \le i_2 \le p_2) \end{cases}$$

After $r-2$ times application of this manipulation, we obtain the local normal form (14) of S. By the definition of the derived system of S_{r-k}, we have

$$d\omega_{(k)}^{i_k}\wedge\omega_{(1)}^{1}\wedge\cdots\wedge\omega_{(1)}^{p_1}\wedge\cdots\wedge\omega_{(k+1)}^{1}\wedge\cdots\wedge\omega_{(k+1)}^{p_{k+1}}=0,\quad 1\le i_k\le p_k.$$

From (14), this implies

$$\sum_{\alpha=1}^{h}(dx_\alpha\wedge dA_\alpha^{i_k}(k))\wedge dy_1^{(1)}\wedge\cdots\wedge dy_{p_1}^{(1)}\wedge\cdots\wedge dy_1^{(k+1)}\wedge\cdots\wedge dy_{p_{k+1}}^{(k+1)}=0.$$

Therefore $A_\alpha^{i_k}(k)$ is a function of the coordinates x, $y^{(1)},\dots,y^{(k+1)}$ as desired. $\square$

Lemma 3.2 *Let S be a totally regular Pfaffian system on M of length r. If S has a relative polarization H of rank h, then the set of exterior derivatives of elements of S_{r-k}, $1\le k\le r$, satisfies the inclusion*

$$dS_{r-k} \bmod \mathscr{S}_{r-k}\subset\mathscr{H}\cap\mathscr{S}_{r-k-1}, \tag{15}$$

where S_{-1} stands for the Cartan system $C(S)$ of S.

Proof. We first consider the case of $1\le k\le r-1$. Let $(p_0,\dots,p_{r+1})$ denote the type of S. Let $\{\omega_{(k)}^1,\dots,\omega_{(k)}^{p_k}\}$ be a local basis of any complementary subspace of S_{r-k+1} in S_{r-k} and $\{\omega_{(k+1)}^1,\dots,\omega_{(k+1)}^{p_{k+1}}\}$ a local basis of any complementary subspace of S_{r-k} in S_{r-k-1}. According to the previous lemma, the exterior derivative of $\omega_{(k)}^{i_k}$, $1\le i_k\le p_k$, can be written in the form

$$\begin{aligned}
d\omega_{(k)}^{i_k} &= \sum_{\alpha=1}^{h} dx_\alpha\wedge dA_\alpha^{i_k}(k)\\
&\equiv \sum_{\alpha=1}^{h}\sum_{j_{k+1}=1}^{p_{k+1}}\frac{\partial A_\alpha^{i_k}(k)}{\partial y_{j_{k+1}}^{(k+1)}}\,dx_\alpha\wedge\omega_{(k+1)}^{j_{k+1}}+\sum_{\alpha,\beta=1}^{h}\Bigg(\frac{\partial A_\alpha^{i_k}(k)}{\partial x_\beta}\\
&\quad+\sum_{j_1=1}^{p_1}\frac{\partial A_\alpha^{i_k}(k)}{\partial y_{j_1}^{(1)}}A_\beta^{j_1}(1)+\cdots+\sum_{j_k=1}^{p_k}\frac{\partial A_\alpha^{i_k}(k)}{\partial y_{j_k}^{(k)}}A_\beta^{j_k}(k)\\
&\quad+\sum_{j_{k+1}=1}^{p_{k+1}}\frac{\partial A_\alpha^{i_k}(k)}{\partial y_{j_{k+1}}^{(k+1)}}A_\beta^{j_{k+1}}(k+1)\Bigg)dx_\alpha\wedge dx_\beta \quad \bmod \mathscr{S}_{r-k}.
\end{aligned}$$

Since $d\omega_{(k)}^{i_k}$ vanishes mod $\mathscr{S}_{r-k-1}$, the terms of $dx_\alpha\wedge dx_\beta$, $1\le\alpha,\,\beta\le h$, do vanish. Hence we have

$$d\omega_{(k)}^{i_k} \equiv \sum_{\alpha=1}^{h} \sum_{i_{k+1}=1}^{p_{k+1}} \frac{\partial A_\alpha^{i_k}(k)}{\partial y_{i_{k+1}}^{(k+1)}} \, dx_\alpha \wedge \omega_{(k+1)}^{i_{k+1}} \quad \bmod \mathscr{S}_{r-k}. \tag{16}$$

Since the 1-forms $dx_1, \dots, dx_h$ and $\omega_{(k+1)}^1, \dots, \omega_{(k+1)}^{p_{k+1}}$ are the local basis of H and of S_{r-k-1}/S_{r-k} respectively, we obtain (15) as claimed. Consider the case of $k = r$. Let $\{du_1, \dots, du_{p_{r+1}-h}\}$ be a local basis of any complementary subspace of $S_0 \oplus H$ in S_{-1}. The exterior derivative of $\omega_{(r)}^{i_r}$, $1 \leq i_r \leq p_r$, can be written in the form

$$\begin{aligned} d\omega_{(r)}^{i_r} \equiv & \sum_{\alpha=1}^{h} \sum_{l=1}^{p_{r+1}-h} \frac{\partial A_\alpha^{i_r}(r)}{\partial u_l} \, dx_\alpha \wedge du_l \\ & + \sum_{\alpha,\beta=1}^{h} \Bigg(\frac{\partial A_\alpha^{i_r}(r)}{\partial x_\beta} + \sum_{j_1=1}^{p_1} \frac{\partial A_\alpha^{i_r}(r)}{\partial y_{j_1}^{(1)}} A_\beta^{i_1}(1) + \cdots \\ & \quad + \sum_{j_r=1}^{p_r} \frac{\partial A_\alpha^{i_r}(r)}{\partial y_{j_r}^{(r)}} A_\beta^{j_r}(r) \Bigg) dx_\alpha \wedge dx_\beta \quad \bmod \mathscr{S}_0. \end{aligned}$$

We obtain the desired inclusion (15) in the same way of the case of $1 \leq i \leq r-1$. $\square$

Proposition 3.1 *Let S be a totally regular Pfaffian system of length r with $S_r = \{0\}$. If S has a relative polarization of rank h, then the Engel invariant s_{r-k} of each derived system S_{r-k}, $1 \leq k \leq r-1$, satisfies the inequality*

$$s_{r-k} \leq h.$$

Proof. Let $\{\omega_{(1)}^{i_1}, \dots, \omega_{(k)}^{i_k} \,|\, 1 \leq i_1 \leq p_1, \dots, 1 \leq i_k \leq p_k\}$ be a local basis of S_{r-k} and $\{\omega_{(k+1)}^1, \dots, \omega_{(k+1)}^{p_{k+1}}\}$ a local basis of any complementary subspace of S_{r-k} in S_{r-k-1}. Then, by the definition of the derived system of S_{r-k}, we have

$$\begin{cases} d\omega_{(1)}^{i_1} \equiv 0 \bmod \mathscr{S}_{r-k}, \\ \quad \vdots \qquad\qquad \vdots \\ d\omega_{(k-1)}^{i_{k-1}} \equiv 0 \bmod \mathscr{S}_{r-k}, \\ d\omega_{(k)}^{i_k} \not\equiv 0 \bmod \mathscr{S}_{r-k}. \qquad (1 \leq i_1 \leq p_1, \dots, 1 \leq i_k \leq p_k) \end{cases}$$

Let $\{dx_1, \dots, dx_h, \omega^{i_1}_{(1)}, \dots, \omega^{i_{k+1}}_{(k+1)}\}$ be a local basis of $S_{r-k-1} \oplus H$. Then by (16) and linear independence of $\{dx_1, \dots, dx_h, \omega^{i_1}_{(1)}, \dots, \omega^{i_{k+1}}_{(k+1)}\}$ we can see that

$$\begin{aligned}(d\omega^{i_k}_{(k)})^h &\equiv \pm h! \sum_{j_1,\dots,j_h=1}^{p_{k+1}} \frac{\partial A^{i_k}_1(k)}{\partial y^{(k+1)}_{j_1}} \cdots \frac{\partial A^{i_k}_h(k)}{\partial y^{(k+1)}_{j_h}} \, \mathbf{dx} \wedge \omega^{j_1}_{(k+1)} \wedge \dots \wedge \omega^{j_h}_{(k+1)} \\ &\not\equiv 0 \mod \mathscr{S}_{r-k},\end{aligned}$$

where $\mathbf{dx}$ stands for a h-form $\mathbf{dx} = dx_1 \wedge \dots \wedge dx_h$. Therefore

$$\begin{aligned}(d\omega^{i_k}_{(k)})^{h+1} &= (d\omega^{i_k}_{(k)})^h \wedge d\omega^{i_k}_{(k)} \\ &= (d\omega^{i_k}_{(k)})^h \wedge \sum_{\alpha=1}^{h} (dx_\alpha \wedge dA^{i_k}_\alpha(k)) \\ &\equiv 0 \mod \mathscr{S}_{r-k}.\end{aligned}$$

This proves Proposition 3.1. □

The following proposition means that the existence of relative polarizations allows us to evaluate the number of variables necessary in order to describe the local generators of each derived system S_{r-k}.

Proposition 3.2 *Let S be a totally regular Pfaffian system on M of length r with $S_r = \{0\}$. If S has a relative polarization H of rank h, then the class of each derived system S_{r-k}, $1 \leq k \leq r-1$, satisfies the inequality*

$$\operatorname{class} S_{r-k} \leq \operatorname{rank} S_{r-k-1} + h.$$

Proof. From Lemma 3.1, we know that there exist local coordinates $(x, y^{(1)}, \dots, y^{(r)}, u)$ in M such that S is generated by the 1-forms

$$\begin{cases} \omega^{i_1}_{(1)} = dy^{(1)}_{i_1} - \displaystyle\sum_{\alpha=1}^{h} A^{i_1}_\alpha(1)\, dx_\alpha, \\ \quad\vdots \qquad\qquad\qquad \vdots \\ \omega^{i_r}_{(r)} = dy^{(r)}_{i_r} - \displaystyle\sum_{\alpha=1}^{h} A^{i_r}_\alpha(r)\, dx_\alpha, \qquad (1 \leq i_1 \leq p_1, \dots, 1 \leq i_r \leq p_r) \end{cases}$$

and the $(r-k)$th derived system S_{r-k} is generated by the 1-forms

$\omega^{i_1}_{(1)}, \dots, \omega^{i_k}_{(k)}$. By the definition of S_{r-k} and Lemma 3.2 the exterior derivatives $d\omega^{i_1}_{(1)}, \dots, d\omega^{i_{k-1}}_{(k-1)}$, and $d\omega^{i_k}_{(k)}$, of S_{r-k} satisfy

$$\begin{cases} d\omega^{i_1}_{(1)} \equiv 0 \bmod \mathscr{S}_{r-k}, \\ \quad \vdots \qquad\qquad \vdots \\ d\omega^{i_{k-1}}_{(k-1)} \equiv 0 \bmod \mathscr{S}_{r-k}, \qquad (1 \le i_1 \le p_1, \dots, 1 \le i_{k-1} \le p_{k-1}) \end{cases}$$

and

$$\begin{aligned} d\omega^{i_k}_{(k)} &\equiv \sum_{\alpha=1}^{h} \sum_{i_{k+1}=1}^{p_{k+1}} \frac{\partial A^{i_k}_{\alpha}(k)}{\partial y^{(k+1)}_{i_{k+1}}} \, dx_\alpha \wedge \omega^{i_{k+1}}_{(k+1)} \\ &\not\equiv 0 \quad \bmod \mathscr{S}_{r-k}, \qquad 1 \le i_k \le p_k. \end{aligned}$$

Hence the condition $\iota_X d\omega^{i_k}_{(k)} \equiv 0 \bmod \mathscr{S}_{r-k}$ implies

$$\sum_{\alpha=1}^{h} \sum_{i_{k+1}=1}^{p_{k+1}} \frac{\partial A^{i_k}_{\alpha}(k)}{\partial y^{(k+1)}_{i_{k+1}}} \Big(dx_\alpha(X) \omega^{i_{k+1}}_{(k+1)} - \omega^{i_{k+1}}_{(k+1)}(X) \, dx_\alpha \Big) = 0.$$

The linear independence of $\{dx_1, \dots, dx_h, \omega^1_{(k+1)}, \dots, \omega^{p_{k+1}}_{(k+1)}\}$ allows us to represent the formula above as

$$\left(\begin{array}{c|c} {}^tA^1 & \\ \vdots & 0 \\ {}^tA^{p_k} & \\ \hline & A^1 \\ 0 & \vdots \\ & A^{p_k} \end{array} \right) \left(\begin{array}{c} dx_1(X) \\ \vdots \\ dx_h(X) \\ \hline \omega^1_{(k+1)}(X) \\ \vdots \\ \omega^{p_{k+1}}_{(k+1)}(X) \end{array} \right) = 0, \tag{17}$$

where A^i, $1 \le i \le p_k$, is a $h \times p_{k+1}$ matrix

$$A^i := \left(\frac{\partial A^i_\alpha(k)}{\partial y^{(k+1)}_j} \right)_{\substack{1 \le \alpha \le h \\ 1 \le j \le p_{k+1}}}$$

and ${}^tA^i$ is the transposed matrix of A^i. The rank of the coefficient matrix is clearly less than or equal to $h + p_{k+1}$. The class of S_{r-k} is equal to the number of independent equations on the vectors $X \in TM$, $\omega^{i_1}_{(1)}(X) = \cdots =$

$\omega^{i_k}_{(k)}(X) = 0$ $(1 \leq i_1 \leq p_1, \ldots, 1 \leq i_k \leq p_k)$, and (17). So we have

$$\operatorname{class} S_{r-k} \leq (p_1 + \cdots + p_k) + (h + p_{k+1}),$$

which is the desired inequality. □

We end this section by giving an example of totally regular Pfaffian system which does not have relative polarization. The following proposition gives a necessary condition for the existence of relative polarizations. We define the *reduced derived system* $\widetilde{S_{r-k}}$ of S_{r-k} by

$$\widetilde{S_{r-k}} = \bigl\{\omega \in S_{r-k-1} \bigm| d\omega \equiv 0 \bmod \mathscr{S}_{r-k-1} + (\mathscr{S}_{r-k-2} \cap \mathscr{S}^2)\bigr\}.$$

Proposition 3.3 (cf. Libermann [9]) *Let S be a totally regular Pfaffian system on M of length r. If the system S has a relative polarization H, then it satisfies the condition*

$$S_{r-k} = \widetilde{S_{r-k}}, \qquad 0 \leq k \leq r-2.$$

Proof. Since $S_{r-k} \subset \widetilde{S_{r-k}}$ holds, it is sufficient to show that $S_{r-k} \supset \widetilde{S_{r-k}}$. We take an element $\widetilde{\omega}$ of $\widetilde{S_{r-k}}$. Then $\widetilde{\omega} \in S_{r-k-1}$ and $d\widetilde{\omega} \bmod \mathscr{S}_{r-k-1} \in \mathscr{S}_{r-k-2} \cap \mathscr{S}^2$. Let $\{\alpha^1, \ldots, \alpha^h\}$ be a local basis of H and $\bigl\{\omega^1_{(k+2)}, \ldots, \omega^{p_{k+2}}_{(k+2)}\bigr\}$ a local basis of any complementary subspace of S_{r-k-1} in S_{r-k-2}. By Lemma 3.2 $d\widetilde{\omega}$ can be written

$$d\widetilde{\omega} \bmod \mathscr{S}_{r-k-1} \equiv \sum_{i=1}^{h} \sum_{j=1}^{p_{k+2}} f^i_j\, \alpha^i \wedge \omega^j_{(k+2)},$$

where $f^i_j \in \Omega^0(M)$. The condition (i) of the relative polarization gives $\alpha^i \wedge \omega^j_{(k+2)} \notin \mathscr{S}^2$, and hence

$$\alpha^i \wedge \omega^j_{(k+2)} \notin \mathscr{S}_{r-k-2} \cap \mathscr{S}^2.$$

Therefore we have

$$d\widetilde{\omega} \equiv 0 \quad \bmod \mathscr{S}_{r-k-1}.$$

This proves Proposition 3.3. □

Example 6 (Kumpera-Ruiz [8]) Consider the Pfaffian system S on $\mathbb{R}^5 = (t, x_1, x_2, x_3, x_4)$ of rank 3, defined by 1-forms

$$\left\{\begin{array}{l} \omega^1 = dx_1 - x_2\,dt, \\ \omega^2 = dx_2 - x_3\,dt, \\ \omega^3 = dt - x_4\,dx_3. \end{array}\right. \tag{18}$$

The system S does not have a relative polarization.

Indeed, put 1-forms ω^4, ω^5 as $\omega^4 := dx_3$ and $\omega^5 := dx_4$. Differentiating (18) we have

$$\left\{\begin{array}{l} d\omega^1 = -\omega^2 \wedge \omega^3 - x_4\,\omega^2 \wedge \omega^4, \\ d\omega^2 = -\omega^4 \wedge \omega^3, \\ d\omega^3 = -\omega^5 \wedge \omega^4. \end{array}\right.$$

The system S is a totally regular Pfaffian system of length 3 and the derived systems are

$$S_1 = \operatorname{span}\{\omega^1, \omega^2\}, \quad S_2 = \operatorname{span}\{\omega^1\}, \quad S_3 = \{0\}.$$

Note that the ideals $\mathscr{S}^2$, $\mathscr{S}_1$, and $\mathscr{S}_2$ are

$$\begin{aligned} \mathscr{S}^2 &= \left\{ \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha^i_j \wedge \omega^i \wedge \omega^j \,\middle|\, \alpha^i_j \in \Omega^*(\mathbb{R}^5) \right\}, \\ \mathscr{S}_1 &= \left\{ \alpha^1 \wedge \omega^1 + \alpha^2 \wedge \omega^2 \,\middle|\, \alpha^i \in \Omega^*(\mathbb{R}^5) \right\}, \\ \mathscr{S}_2 &= \left\{ \alpha \wedge \omega^1 \,\middle|\, \alpha \in \Omega^*(\mathbb{R}^5) \right\}. \end{aligned}$$

We have also $\mathscr{S}_1 \cap \mathscr{S}^2 = \left\{ \sum_{i=1}^{2}\sum_{j=1}^{3} \alpha^i_j \wedge \omega^i \wedge \omega^j \,\middle|\, \alpha^i_j \in \Omega^*(\mathbb{R}^5) \right\}$. Hence

$$\left\{\begin{array}{l} \omega^2 \wedge \omega^3 \in \mathscr{S}_1 \cap \mathscr{S}^2, \\ \omega^2 \wedge \omega^4 \notin \mathscr{S}_2,\ \omega^2 \wedge \omega^4 \notin \mathscr{S}_1 \cap \mathscr{S}^2. \end{array}\right.$$

Therefore

$$d\omega^1 \equiv \left\{\begin{array}{ll} 0 \bmod \mathscr{S}_2 + (\mathscr{S}_1 \cap \mathscr{S}^2), & \text{if } x_4 = 0, \\ x_4\,\omega^4 \wedge \omega^2 \bmod \mathscr{S}_2 + (\mathscr{S}_1 \cap \mathscr{S}^2), & \text{if } x_4 \neq 0, \end{array}\right.$$

and hence $\widetilde{S_3}$ is

$$\widetilde{S_3} = \left\{\begin{array}{ll} \operatorname{span}\{\omega^1\}, & \text{if } x_4 = 0, \\ \{0\}, & \text{if } x_4 \neq 0. \end{array}\right.$$

Thus we have $S_3 \neq \widetilde{S_3}$ and S does not have a relative polarization by Proposition 3.3.

4. The main theorem

Fixing integers h, m, p_1, $r \in \mathbb{N}$ with $m \geq p_1\binom{h+r-1}{r}$, we take a manifold M of dimension $n = h + p_1\binom{h+r}{r} + m$, and consider local coordinates of M, x, z, $z(1)$, $z(k)$, $1 \leq k \leq r$, in the following way:

$$\begin{cases} x := (x_1, \dots, x_h), \\ z := (z^1, \dots, z^{p_1}), \\ z(k) := (\dots, z^i_{\alpha_1 \cdots \alpha_k}, \dots)_{1 \leq i \leq p_1,\, 1 \leq \alpha_1, \dots, \alpha_k \leq h}, \end{cases}$$

where $z^i_{\alpha_1 \cdots \alpha_k}$, $2 \leq k \leq r$, is symmetric with respect to the subscript $\alpha_1, \dots, \alpha_k$. Let $\widetilde{z(r)}$ denote the coordinates $z(r)$ with some σ coordinate functions deleted for $\sigma \geq 0$. We note that the number of elements of the coordinates $z(k)$ and $\widetilde{z(r)}$ are $p_1\binom{h+k-1}{k}$ and $p_1\binom{h+r-1}{r} - \sigma$.

In Theorem 4.1 below, we prove the following. Under the existence of relative polarizations and the condition on the type and the class of a Pfaffian system, there exist local coordinates $(x, z, z(1), \dots, z(r-1), \widetilde{z(r)}, u_1, \dots, u_{m - p_1\binom{h+r-1}{r} + \sigma})$ in M such that the Pfaffian system can be transformed locally into the normal form

$$\begin{cases} \omega^i = dz^i - \displaystyle\sum_{\alpha_1=1}^{h} z^i_{\alpha_1}\, dx_{\alpha_1}, \\ \omega^i_{\alpha_1} = dz^i_{\alpha_1} - \displaystyle\sum_{\alpha_2=1}^{h} z^i_{\alpha_1\alpha_2}\, dx_{\alpha_2}, \\ \quad \vdots \qquad\qquad \vdots \\ \omega^i_{\alpha_1 \cdots \alpha_{r-2}} = dz^i_{\alpha_1 \cdots \alpha_{r-2}} - \displaystyle\sum_{\alpha_{r-1}=1}^{h} z^i_{\alpha_1 \cdots \alpha_{r-1}}\, dx_{\alpha_{r-1}}, \\ \omega^i_{\alpha_1 \cdots \alpha_{r-1}} = dz^i_{\alpha_1 \cdots \alpha_{r-1}} - \displaystyle\sum_{\alpha_r=1}^{h} F^i_{\alpha_1 \cdots \alpha_r}(x, \dots, u_1, \dots)\, dx_{\alpha_r}, \end{cases} \tag{19}$$

$$(1 \leq i \leq p_1,\ 1 \leq \alpha_1, \dots, \alpha_{r-1} \leq h)$$

where $F^i_{\alpha_1 \cdots \alpha_r}$ is a function on M, symmetric with respect to the subscript $\alpha_1, \dots, \alpha_r$, and satisfies the condition

$$\operatorname{rank} \frac{\partial(\ldots, F^i_{\alpha_1\cdots\alpha_r}, \ldots)}{\partial(\widetilde{z(r)})} = p_1 \binom{h+r-1}{r} - \sigma.$$

The Pfaffian system (19) is just the Pfaffian system $\Omega\big(\Sigma_\sigma^{(r,h,p_1)}\big)$ associated to a system of rth order PDEs in $\mathbb{R}^h$ and $\mathbb{R}^{p_1}$ of codimension σ. When $\sigma = 0$, it is just the contact system $\Omega^r(\mathbb{R}^h, \mathbb{R}^{p_1})$ on $J^r(\mathbb{R}^h, \mathbb{R}^{p_1})$.

Theorem 4.1 *For h, m, n, p_1, r, σ as above, let S be a Pfaffian system on M. There exist local coordinates $\big(x, z, z(1), \ldots, z(r-1), \widetilde{z(r)}, u_1, \ldots, u_{m-p_1\binom{h+r-1}{r}+\sigma}\big)$ in M such that S can be transformed locally into the normal form* (19) *if and only if S satisfies the following conditions*:

(i) *S is a totally regular Pfaffian system of length r with $S_r = \{0\}$, that is, there is a decreasing sequence of derived systems*

$$S = S_0 \supset S_1 \supset \cdots \supset S_{r-1} \supset S_r = \{0\},$$

(ii) *S has a relative polarization of rank h,*

(iii) *the type $(p_0, \ldots, p_{r+1})$ of S satisfies*

$$p_i = \begin{cases} 0, & \text{if } i = 0, \\ p_1\binom{h+i-2}{i-1}, & \text{if } 1 \le i \le r, \end{cases}$$

(iv) *for each derived system S_{r-j}, $1 \le j \le r-1$, and S,*

$$\operatorname{class} S_{r-j} = \begin{cases} h + p_1\binom{h+j}{j}, & \text{if } 1 \le j \le r-1, \\ h + p_1\binom{h+r}{r} - \sigma, & \text{if } j = r. \end{cases}$$

Proof. Assume that S admits locally the normal form (19). Then by Example 1 and Example 4, it is easy to see that S satisfies the conditions (i)$\sim$(iv). Now, assuming S is a Pfaffian system satisfying the conditions (i)$\sim$(iv), we prove that S can be transformed locally into the normal form (19). From Lemma 3.1 we know that there exist local coordinates $(x, y^{(1)}, \ldots, y^{(r)}, u)$ in M such that S is generated by the 1-forms

$$\begin{cases} \omega^{i_1}_{(1)} = dy^{(1)}_{i_1} - \displaystyle\sum_{\alpha=1}^{h} A^{i_1}_\alpha(1)\, dx_\alpha, \\ \quad\vdots \qquad\qquad\qquad \vdots \\ \omega^{i_r}_{(r)} = dy^{(r)}_{i_r} - \displaystyle\sum_{\alpha=1}^{h} A^{i_r}_\alpha(r)\, dx_\alpha, \qquad (1 \le i_1 \le p_1, \ldots, 1 \le i_r \le p_r) \end{cases}$$

where $x := (x_1, \dots, x_h)$, $y^{(k)} := \big(y^{(k)}_1, \dots, y^{(k)}_{p_k}\big)$, $1 \le k \le r$, $u := (u_1, \dots, u_m)$, $h + \sum_{k=1}^{r} p_k + m = n$, and $A^{i_k}_\alpha(k)$ is a function $A^{i_k}_\alpha(k) = A^{i_k}_\alpha(x, y^{(1)}, \dots, y^{(k+1)})$.

Let us consider the $(r-1)$st derived system S_{r-1}. According to Lemma 3.2 the exterior derivatives $d\omega^{i_1}_{(1)}$, $1 \le i_1 \le p_1$, of elements of S_{r-1} can be written in the form

$$d\omega^{i_1}_{(1)} \equiv \sum_{\alpha=1}^{h} \sum_{i_2=1}^{p_2} \frac{\partial A^{i_1}_\alpha(1)}{\partial y^{(2)}_{i_2}} \, dx_\alpha \wedge \omega^{i_2}_{(2)} \quad \bmod \mathscr{S}_{r-1}.$$

Thus the condition $\iota_X d\omega^{i_1}_{(1)} \equiv 0 \bmod \mathscr{S}_{r-1}$ is equivalent to

$$\sum_{\alpha=1}^{h} \sum_{i_2=1}^{p_2} \frac{\partial A^{i_1}_\alpha(1)}{\partial y^{(2)}_{i_2}} \Big(dx_\alpha(X) \omega^{i_2}_{(2)} - \omega^{i_2}_{(2)}(X) \, dx_\alpha \Big) = 0.$$

By the linear independence of $\big\{\omega^{i_2}_{(2)}, dx_\alpha\big\}$ we have

$$\begin{cases} \displaystyle\sum_{\alpha=1}^{h} \frac{\partial A^{i_1}_\alpha(1)}{\partial y^{(2)}_{i_2}} \, dx_\alpha(X) = 0, & 1 \le i_1 \le p_1, \ 1 \le i_2 \le p_2, \\ \displaystyle\sum_{i_2=1}^{p_2} \frac{\partial A^{i_1}_\alpha(1)}{\partial y^{(2)}_{i_2}} \, \omega^{i_2}_{(2)}(X) = 0, & 1 \le i_1 \le p_1, \ 1 \le \alpha \le h. \end{cases}$$

So the class of S_{r-1} is equal to the number of independent equations on the vectors $X \in TM$, $\omega^{i_1}_{(1)}(X) = 0$, $1 \le i_1 \le p_1$, and

$$\left(\begin{array}{c|c} {}^tA^1 & \\ \vdots & 0 \\ {}^tA^{p_1} & \\ \hline & A^1 \\ 0 & \vdots \\ & A^{p_1} \end{array} \right) \left(\begin{array}{c} dx_1(X) \\ \vdots \\ dx_h(X) \\ \hline \omega^1_{(2)}(X) \\ \vdots \\ \omega^{p_2}_{(2)}(X) \end{array} \right) = 0$$

where A^{i_1}, $1 \le i_1 \le p_1$, is a $h \times p_2$ matrix

$$A^{i_1} := \left(\frac{\partial A^{i_1}_{\alpha_1}(1)}{\partial y^{(2)}_{i_2}} \right)_{\substack{1 \le \alpha_1 \le h \\ 1 \le i_2 \le p_2}},$$

and ${}^tA^{i_1}$ is the transposed matrix of A^{i_1}. By the assumption that class $S_{r-1} = h + p_1 + p_1h$ and $p_2 = p_1h$, we can see that the rank of the coefficient matrix is equal to $h + p_1h$. The upper left part of the coefficient matrix is a $p_1p_2 \times h$ matrix, so the rank of the upper left part is less than or equal to h. Hence the p_1h-square matrix ${}^t(A^1 \cdots A^{p_1})$ should be nonsingular. Therefore we have

$$\begin{aligned} dA^1_1(1) \wedge \cdots \wedge dA^{p_1}_h(1) &= \pm \det {}^t(A^1 \cdots A^{p_1})\, dy^{(2)}_1 \wedge \cdots \wedge dy^{(2)}_{p_2} \\ &\neq 0. \end{aligned}$$

Let $\mathbf{dx}$, $\mathbf{dy^{(k)}}$, $1 \leq k \leq r$, and $\mathbf{du}$ denote differential forms

$$\begin{aligned} &\mathbf{dx} := dx_1 \wedge \cdots \wedge dx_h, \quad \mathbf{dy^{(k)}} := dy^{(k)}_1 \wedge \cdots \wedge dy^{(k)}_{p_k}, \\ &\mathbf{du} := du_1 \wedge \cdots \wedge du_m \end{aligned}$$

respectively. Since we have

$$\begin{aligned} &\mathbf{dx} \wedge \mathbf{dy^{(1)}} \wedge dA^1_1(1) \wedge \cdots \wedge dA^{p_1}_h(1) \wedge \mathbf{dy^{(3)}} \wedge \cdots \wedge \mathbf{dy^{(r)}} \wedge \mathbf{du} \\ &\quad = \pm \det {}^t(A^1 \cdots A^{p_1})\, \mathbf{dx} \wedge \mathbf{dy^{(1)}} \wedge \mathbf{dy^{(2)}} \wedge \mathbf{dy^{(3)}} \wedge \cdots \wedge \mathbf{dy^{(r)}} \wedge \mathbf{du} \\ &\quad \neq 0, \end{aligned}$$

the functions x, $y^{(1)}$, $A^{i_1}_{\alpha_1}(1)$ $(1 \leq i_1 \leq p_1, 1 \leq \alpha_1 \leq h)$, $y^{(3)}, \ldots, y^{(r)}$, and u form local coordinates in M. Put the 1-forms $\omega^{i_1}_{(1)}$ and the functions $y^{(1)}_{i_1}$, $A^{i_1}_{\alpha_1}(1)$ as $\omega^{i_1} := \omega^{i_1}_{(1)}$, $z^{i_1} := y^{(1)}_{i_1}$, and $z^{i_1}_{\alpha_1} := A^{i_1}_{\alpha_1}(1)$. Then we can choose local coordinates $(x, z, z(1), y^{(3)}, \ldots, y^{(r)}, u)$ in M such that S_{r-1} is locally generated by the 1-forms

$$\omega^{i_1} = dz^{i_1} - \sum_{\alpha_1=1}^{h} z^{i_1}_{\alpha_1}\, dx_{\alpha_1}, \quad 1 \leq i_1 \leq p_1. \tag{20}$$

Let us consider the $(r-2)$nd derived system $S_{r-2} = S_{r-1} \oplus \operatorname{span}\{\omega^{i_2}_{(2)}\}$ in the new coordinates $(x, z, z(1), y^{(3)}, \ldots, y^{(r)}, u)$ of M. Since $A^{i_1}_{\alpha}$ is a function of the variables x, $y^{(1)}$, $y^{(2)}$ on the coordinates $(x, y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(r)}, u)$ and ${}^t(A^1 \cdots A^{p_1})$ is nonsingular, we can see that $y^{(2)}_{i_2} = y^{(2)}_{i_2}(x, z, z(1))$ on the new coordinates $(x, z, z(1), y^{(3)}, \ldots, y^{(r)}, u)$ of M. Hence the local basis $\{\omega^{i_2}_{(2)}\}$, of any complementary subspace of S_{r-1} in S_{r-2} can be written in the form

$$\omega^{i_2}_{(2)} = dy^{(2)}_{i_2} - \sum_{\alpha=1}^{h} A^{i_2}_{\alpha}(2)(x,z,z(1),y^{(3)})\,dx_\alpha, \qquad 1 \le i_2 \le p_2.$$

Note that $dy^{(2)}_{i_2}$ is generated by dx_{α_1}, ω^i, and $dz^i_{\alpha_1}$, hence $\omega^{i_2}_{(2)}$ are expressed as

$$\begin{pmatrix} \omega^1_{(2)} \\ \vdots \\ \omega^{p_2}_{(2)} \end{pmatrix} = X_1 \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^{p_1} \end{pmatrix} + X_2 \begin{pmatrix} dz^1_1 \\ \vdots \\ dz^{p_1}_h \end{pmatrix} - R \begin{pmatrix} dx_1 \\ \vdots \\ dx_h \end{pmatrix},$$

where X_1 and X_2 are a $p_2 \times p_1$ matrix and a $p_2 \times p_1 h$ matrix

$$X_1 := \left(\frac{\partial y^{(2)}_{i_2}}{\partial z^i}\right)_{\substack{1\le i_2\le p_2\\ 1\le i\le p_1}}, \quad X_2 := \left(\frac{\partial y^{(2)}_{i_2}}{\partial z^i_{\alpha_1}}\right)_{\substack{1\le i_2\le p_2\\ 1\le i\le p_1,\,1\le\alpha_1\le h}},$$

respectively, and R is a $p_2 \times h$ matrix. Since X_2 is the inverse matrix of ${}^t(A^1 \cdots A^{p_1})$, we can replace ${}^t\big(\omega^1_{(2)}, \dots, \omega^{p_2}_{(2)}\big)$ by

$$X_2^{-1}\left\{\begin{pmatrix} \omega^1_{(2)} \\ \vdots \\ \omega^{p_2}_{(2)} \end{pmatrix} - X_1 \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^{p_1} \end{pmatrix}\right\}.$$

Then we have

$$\begin{pmatrix} \omega^1_{(2)} \\ \vdots \\ \omega^{p_2}_{(2)} \end{pmatrix} = \begin{pmatrix} dz^1_1 \\ \vdots \\ dz^{p_1}_h \end{pmatrix} - \begin{pmatrix} x^1_1 & \cdots & x^1_h \\ \vdots & & \vdots \\ x^{p_2}_1 & \cdots & x^{p_2}_h \end{pmatrix} \begin{pmatrix} dx_1 \\ \vdots \\ dx_h \end{pmatrix}. \tag{21}$$

where $(x^{i_2}_{\alpha_2}) := X_2^{-1}R$. We put

$$\begin{cases} i_2 := (i-1)h + \alpha_1, \\ \omega^i_{\alpha_1} := \omega^{i_2}_{(2)}, \\ A^i_{\alpha_1\alpha_2} := x^{i_2}_{\alpha_2}. \qquad (1 \le i \le p_1, 1 \le \alpha_1, \alpha_2 \le h), \end{cases}$$

Then the formula (21) can be written in the form

$$\begin{pmatrix} \omega_1^1 \\ \vdots \\ \omega_h^1 \\ \hline \vdots \\ \hline \omega_1^{p_1} \\ \vdots \\ \omega_h^{p_1} \end{pmatrix} = \begin{pmatrix} dz_1^1 \\ \vdots \\ dz_h^1 \\ \hline \vdots \\ \hline dz_1^{p_1} \\ \vdots \\ dz_h^{p_1} \end{pmatrix} - \begin{pmatrix} A_{11}^1 & \cdots & A_{1h}^1 \\ \vdots & & \vdots \\ A_{h1}^1 & \cdots & A_{hh}^1 \\ \hline \vdots & & \vdots \\ \hline A_{11}^{p_1} & \cdots & A_{1h}^{p_1} \\ \vdots & & \vdots \\ A_{h1}^{p_1} & \cdots & A_{hh}^{p_1} \end{pmatrix} \begin{pmatrix} dx_1 \\ \vdots \\ dx_h \end{pmatrix}. \tag{22}$$

By Lemma 3.2 and (22) the exterior derivative $d\omega^i$ can be written

$$\begin{aligned} d\omega^i &\equiv \sum_{\alpha_1=1}^{h} f_{\alpha_1}^i \, dx_{\alpha_1} \wedge \omega_{\alpha_1}^i \\ &\equiv \sum_{\alpha_1=1}^{h} f_{\alpha_1}^i \, dx_{\alpha_1} \wedge \left(dz_{\alpha_1}^i - \sum_{\alpha_2=1}^{h} A_{\alpha_1\alpha_2}^i \, dx_{\alpha_2} \right) \quad \mathrm{mod}\ \mathscr{S}_{r-1}, \end{aligned}$$

where $f_{\alpha_1}^i \in \Omega^0(M)$. Comparing the coefficients between the formulae above and the exterior derivatives of (20), we have

$$A_{\alpha_1\alpha_2}^i - A_{\alpha_2\alpha_1}^i = 0. \qquad (1 \le i \le p_1,\ 1 \le \alpha_1,\ \alpha_2 \le h) \tag{23}$$

Therefore S_{r-2} is locally generated by the 1-forms

$$\begin{cases} \omega^i = dz^i - \displaystyle\sum_{\alpha_1=1}^{h} z_{\alpha_1}^i \, dx_{\alpha_1}, \\ \omega_{\alpha_1}^i = dz_{\alpha_1}^i - \displaystyle\sum_{\alpha_2=1}^{h} A_{\alpha_1\alpha_2}^i \, dx_{\alpha_2} \quad \text{with } A_{\alpha_1\alpha_2}^i = A_{\alpha_2\alpha_1}^i, \end{cases} \tag{24}$$

$$(1 \le i \le p_1,\ 1 \le \alpha_1 \le h)$$

where $A_{\alpha_1\alpha_2}^i$ is a function $A_{\alpha_1\alpha_2}^i = A_{\alpha_1\alpha_2}^i(x, z, z(1), y^{(3)})$.

Let us show that the functions x, z, $z(1)$, $A_{\alpha_1\alpha_2}^i$ $(1 \le i \le p_1,$ $1 \le \alpha_1,\ \alpha_2 \le h)$, $y^{(4)}, \ldots, y^{(r)}$, and u form local coordinates of M. From (24) and the linear independence of $\{dx_\alpha, \omega_{(3)}^{i_3}\}$ the condition $\iota_X d\omega_{\alpha_1}^i \equiv 0$ mod $\mathscr{S}_{r-2}$ is equivalent to

$$\begin{cases} \displaystyle\sum_{\alpha_2=1}^{h} \frac{\partial A^i_{\alpha_1\alpha_2}}{\partial y^{(3)}_{i_3}}\, dx_{\alpha_2}(X) = 0, & 1 \le i_3 \le p_3, \\ \displaystyle\sum_{i_3=1}^{p_3} \frac{\partial A^i_{\alpha_1\alpha_2}}{\partial y^{(3)}_{i_3}}\, \omega^{i_3}_{(3)}(X) = 0, & 1 \le \alpha_2 \le h. \end{cases}$$

Hence the class of S_{r-2} is equal to the number of independent equations on the vectors $X \in TM$, $\omega^i(X) = \omega^i_{\alpha_1}(X) = 0$ $(1 \le i \le p_1,\ 1 \le \alpha_1 \le h)$, and

$$\left(\begin{array}{c|c} {}^tA^1_1 & \\ \vdots & 0 \\ {}^tA^{p_1}_h & \\ \hline & A^1_1 \\ 0 & \vdots \\ & A^{p_1}_h \end{array}\right) \left(\begin{array}{c} dx_1(X) \\ \vdots \\ dx_h(X) \\ \hline \omega^1_{(3)}(X) \\ \vdots \\ \omega^{p_3}_{(3)}(X) \end{array}\right) = 0,$$

where $A^i_{\alpha_1}$ is a $h \times p_3$ matrix

$$A^i_{\alpha_1} := \left(\frac{\partial A^i_{\alpha_1\alpha_2}}{\partial y^{(3)}_{i_3}}\right)_{\substack{1\le\alpha_2\le h \\ 1\le i_3\le p_3}}.$$

By the assumption that class $S_{r-2} = p_1\binom{h+1}{1} + h + p_1\binom{h+1}{2}$ and $p_3 = p_1\binom{h+1}{2}$, we can see that the rank of the coefficient matrix is equal to $h + p_1\binom{h+1}{2}$. The upper left part of the coefficient matrix is a $p_1 h p_3 \times h$ matrix, so the rank of the upper left part is less than or equal to h. Hence we should be

$$\operatorname{rank} {}^t(A^1_1 \cdots A^{p_1}_h) = p_1\binom{h+1}{2}.$$

Let ${}^t(\widetilde{A^1_1 \cdots A^{p_1}_h})$ be any matrix consisting of p_3 independent rows of $p_1 h^2 \times p_3$ matrix ${}^t(A^1_1 \cdots A^{p_1}_h)$. Since $p_3 = p_1\binom{h+1}{2}$, we can see that the $p_1\binom{h+1}{2}$-square matrix ${}^t(\widetilde{A^1_1 \cdots A^{p_1}_h})$ is nonsingular. Hence, for all the functions $A^i_{\alpha_1\alpha_2}$ with $A^i_{\alpha_1\alpha_2} = A^i_{\alpha_2\alpha_1}$ $(1 \le i \le p_1, 1 \le \alpha_1,\ \alpha_2 \le h)$, we have

$$\begin{aligned} dA^1_{11} \wedge \cdots \wedge dA^{p_1}_{hh} &= \pm \det {}^t(\widetilde{A^1_1 \cdots A^{p_1}_h})\, \mathbf{dy^{(3)}} \\ &\ne 0. \end{aligned}$$

Let $\mathbf{dz}$ and $\mathbf{dz(1)}$ denote differential forms

$$\mathbf{dz} := dz^1 \wedge \cdots \wedge dz^{p_1} \ \text{ and } \ \mathbf{dz(1)} := dz^1_1 \wedge \cdots \wedge dz^i_{\alpha_1} \wedge \cdots \wedge dz^{p_1}_h.$$

Since we have

$$\begin{aligned}&\mathbf{dx}\wedge\mathbf{dz}\wedge\mathbf{dz(1)}\wedge dA^1_{11}(1)\wedge\cdots\wedge dA^{p_1}_{hh}(1)\wedge\mathbf{dy^{(4)}}\wedge\cdots\wedge\mathbf{dy^{(r)}}\wedge\mathbf{du}\\&=\pm\det{}^t(\widetilde{A^1_1\cdots A^{p_1}_h})\,\mathbf{dx}\wedge\mathbf{dz}\wedge\mathbf{dz(1)}\wedge\mathbf{dy^{(3)}}\wedge\cdots\wedge\mathbf{dy^{(r)}}\wedge\mathbf{du}\\&\neq 0,\end{aligned}$$

the functions x, z, $z(1)$, $A^i_{\alpha_1\alpha_2}$ $(1\leq i\leq p_1,\ 1\leq\alpha_1,\ \alpha_2\leq h)$, $y^{(4)},\ldots,y^{(r)}$, and u form local coordinates of M. Put the functions $A^i_{\alpha_1\alpha_2}$ as $z^i_{\alpha_1\alpha_2}:=A^i_{\alpha_1\alpha_2}$. Then we can choose local coordinates $(x,z,z(1),z(2),y^{(4)},\ldots,y^{(r)},u)$ in M such that S_{r-2} is locally generated by the 1-forms

$$\begin{cases}\omega^i=dz^i-\displaystyle\sum_{\alpha_1=1}^{h}z^i_{\alpha_1}\,dx_{\alpha_1},\\ \omega^i_{\alpha_1}=dz^i_{\alpha_1}-\displaystyle\sum_{\alpha_2=1}^{h}z^i_{\alpha_1\alpha_2}\,dx_{\alpha_2}\quad\text{with } z^i_{\alpha_1\alpha_2}=z^i_{\alpha_2\alpha_1}.\end{cases}$$
$$(1\leq i\leq p_1,\ 1\leq\alpha_1\leq h)$$

After $r-3$ reiteration of the same manipulation, we can get local coordinates $(x,z,z(1),\ldots,z(r-1),u)$ in M such that the first derived system S_1 is locally generated by the 1-forms

$$\begin{cases}\omega^i=dz^i-\displaystyle\sum_{\alpha_1=1}^{h}z^i_{\alpha_1}\,dx_{\alpha_1},\\ \quad\vdots\qquad\qquad\vdots\\ \omega^i_{\alpha_1\cdots\alpha_{r-2}}=dz^i_{\alpha_1\cdots\alpha_{r-2}}-\displaystyle\sum_{\alpha_{r-1}=1}^{h}z^i_{\alpha_1\cdots\alpha_{r-1}}\,dx_{\alpha_{r-1}},\end{cases}\tag{25}$$
$$(1\leq i\leq p_1,\ 1\leq\alpha_1,\ldots,\alpha_{r-1}\leq h)$$

where the coordinates $z^i_{\alpha_1\alpha_2},\ldots,z^i_{\alpha_1\cdots\alpha_{r-1}}$ are symmetric with respect to the subscript $\alpha_1,\ldots,\alpha_{r-1}$.

Let us consider the system $S=S_1\oplus\operatorname{span}\{\omega^{i_r}_{(r)}\}$ in the new coordinates $(x,z,z(1),\ldots,z(r-1),u)$ of M. Since $A^i_{\alpha_1\cdots\alpha_{r-1}}$ is a function of the variables x, $y^{(1)},\ldots,y^{(r-1)}$, $y^{(r)}$ on the coordinates $(x,y^{(1)},\ldots,y^{(r-1)},y^{(r)},u)$ and the matrix ${}^t(\cdots A^i_{\alpha_1\cdots\alpha_{r-2}}\cdots)$ is nonsingular, we can see that $y^{(r)}_{i_r}=y^{(r)}_{i_r}(x,z,z(1),\ldots,z(r-1))$ on the coordinates $(x,z,z(1),\ldots,z(r-1),u)$

of M. Hence the local basis $\{\omega^{i_r}_{(r)}\}$, of any complementary subspace of S_1 in S can be written in the form

$$\omega^{i_r}_{(r)} = dy^{(r)}_{i_r} - \sum_{\alpha=1}^{h} A^{i_r}_{\alpha}(r)\, dx_\alpha, \qquad 1 \le i_r \le p_r,$$

where $A^{i_r}_{\alpha}(r) = A^{i_r}_{\alpha}(x, z, z(1), \dots, z(r-1), u)$. Note that $dy^{(r)}_{i_r}$ is generated by dx_{α_1}, ω^i, $\omega^i_{\alpha_1\cdots\alpha_k}$, $1 \le k \le r-2$, and $z^i_{\alpha_1\cdots\alpha_{r-1}}$, hence the 1-forms $\omega^{i_r}_{(r)}$ are expressed as

$$\begin{aligned}\begin{pmatrix} \omega^1_{(r)} \\ \vdots \\ \omega^{p_r}_{(r)} \end{pmatrix} &= X_1 \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^{p_1} \end{pmatrix} + \cdots + X_{r-1} \begin{pmatrix} \omega^1_{1\cdots1} \\ \vdots \\ \omega^{p_1}_{h\cdots h} \end{pmatrix} \\ &\quad + X_r \begin{pmatrix} dz^1_{1\cdots1} \\ \vdots \\ dz^{p_1}_{h\cdots h} \end{pmatrix} - R \begin{pmatrix} dx_1 \\ \vdots \\ dx_h \end{pmatrix},\end{aligned}$$

where X_1 and X_k $(2 \le k \le r)$, are a $p_r \times p_1$ matrix and a $p_r \times p_k$ matrix

$$X_1 := \left(\frac{\partial y^{(r)}_{i_r}}{\partial z^i} \right)_{\substack{1 \le i_r \le p_r \\ 1 \le i \le p_1}}, \quad X_k := \left(\frac{\partial y^{(r)}_{i_r}}{\partial z^i_{\alpha_1\cdots\alpha_{k-1}}} \right)_{\substack{1 \le i_r \le p_r \\ 1 \le i \le p_1,\, 1 \le \alpha_1, \dots, \alpha_{k-1} \le h}},$$

and R is a $p_r \times h$ matrix. Since X_r is nonsingular, we can replace ${}^t\big(\omega^1_{(r)} \cdots \omega^{p_r}_{(r)}\big)$ by

$$X_r^{-1} \left\{ \begin{pmatrix} \omega^1_{(r)} \\ \vdots \\ \omega^{p_r}_{(r)} \end{pmatrix} - X_1 \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^{p_1} \end{pmatrix} - \cdots - X_{r-1} \begin{pmatrix} \omega^1_{1\cdots1} \\ \vdots \\ \omega^{p_1}_{h\cdots h} \end{pmatrix} \right\}.$$

Then we have

$$\begin{pmatrix} \omega^1_{(r)} \\ \vdots \\ \omega^{p_r}_{(r)} \end{pmatrix} = \begin{pmatrix} dz^1_{1\cdots1} \\ \vdots \\ dz^{p_1}_{h\cdots h} \end{pmatrix} - \begin{pmatrix} x^1_1 & \cdots & x^1_h \\ \vdots & & \vdots \\ x^{p_r}_1 & \cdots & x^{p_r}_h \end{pmatrix} \begin{pmatrix} dx_1 \\ \vdots \\ dx_h \end{pmatrix}, \tag{26}$$

where $(x^{i_r}_{\alpha_r}) := X_r^{-1} R$. We put

$$\begin{cases} i_r := (i-1)\binom{h+r-2}{r-1} + \sum\limits_{k_1=1}^{\alpha_1-1}\binom{h+r-1-k_1}{r-1} + \cdots \\ \qquad + \sum\limits_{k_{r-3}=1}^{\alpha_{r-3}-1}\binom{h+2-k_{r-3}}{2} + \sum\limits_{k_{r-2}=1}^{\alpha_{r-2}-2}(h-k_{r-2}) + \alpha_{r-1}, \\ \omega^i_{\alpha_1\cdots\alpha_{r-1}} := \omega^{i_r}_{(r)}, \\ A^i_{\alpha_1\cdots\alpha_r} := x^{i_r}_{\alpha_r}. \qquad\qquad (1 \le i \le p_1,\ 1 \le \alpha_1, \dots, \alpha_r \le h) \end{cases}$$

Then the formula (26) can be written in the form

$$\omega^i_{\alpha_1\cdots\alpha_{r-1}} = dz^i_{\alpha_1\cdots\alpha_{r-1}} - \sum_{\alpha_r=1}^{h} A^i_{\alpha_1\cdots\alpha_r}(x, \dots, z(r-1), u)\, dx_{\alpha_r}. \quad (27)$$

Let us show that the functions $A^i_{\alpha_1\cdots\alpha_r}$ are symmetric with respect to the subscript $\alpha_1, \dots, \alpha_r$. By Lemma 3.2 and (27) the exterior derivative $d\omega^i_{\alpha_1\cdots\alpha_{r-2}}$ can be written

$$\begin{aligned} d\omega^i_{\alpha_1\cdots\alpha_{r-2}} &\equiv \sum_{\alpha_{r-1}=1}^{h} f^i_{\alpha_1\cdots\alpha_{r-1}}\, dx_{\alpha_{r-1}} \wedge \omega^i_{\alpha_1\cdots\alpha_{r-1}} \\ &\equiv \sum_{\alpha_{r-1}=1}^{h} f^i_{\alpha_1\cdots\alpha_{r-1}}\, dx_{\alpha_{r-1}} \\ &\quad \wedge \left(dz^i_{\alpha_1\cdots\alpha_{r-1}} - \sum_{\alpha_r=1}^{h} A^i_{\alpha_1\cdots\alpha_r}\, dx_{\alpha_r} \right) \quad \bmod \mathscr{S}, \end{aligned}$$

where $f^i_{\alpha_1\cdots\alpha_{r-1}} \in \Omega^0(M)$. Comparing the coefficients between the formulae above and the exterior derivatives of (25), we can see that

$$A^i_{\alpha_1\cdots\alpha_{r-1}\alpha_r} - A^i_{\alpha_1\cdots\alpha_r\alpha_{r-1}} = 0. \quad (1 \le i \le p_1,\ 1 \le \alpha_1, \dots, \alpha_r \le h)$$

Note that the 1-forms $\omega^i_{\alpha_1\cdots\alpha_{r-1}}$ are symmetric with respect to the subscript $\alpha_1, \dots, \alpha_{r-1}$, hence the functions $A^i_{\alpha_1\cdots\alpha_r}$ are also symmetric. Therefore the system S is locally generated by the 1-forms (25) and

$$\omega^i_{\alpha_1\cdots\alpha_{r-1}} = dz^i_{\alpha_1\cdots\alpha_{r-1}} - \sum_{\alpha_r=1}^{h} A^i_{\alpha_1\cdots\alpha_r}\, dx_{\alpha_r},$$
$$(1 \le i \le p_1,\ 1 \le \alpha_1, \dots, \alpha_r \le h)$$

where $A^i_{\alpha_1\cdots\alpha_r}$ is a function on M and symmetric with respect to the subscript $\alpha_1, \ldots, \alpha_r$.

By the computation of $\iota_X d\omega^i_{\alpha_1\cdots\alpha_{r-1}} \equiv 0 \bmod \mathscr{S}$, it is easy to see that the class of S is equal to the number of independent equations on the vectors $X \in TM$, $\omega^i(X) = \cdots = \omega^i_{\alpha_1\cdots\alpha_{r-1}}(X) = 0$ $(1 \leq i \leq p_1,\ 1 \leq \alpha_1, \ldots, \alpha_{r-1} \leq h)$, and

$$\left(\begin{array}{c|c} {}^tA^1_{1\cdots1} & \\ \vdots & 0 \\ {}^tA^{p_1}_{h\cdots h} & \\ \hline & A^1_{1\cdots1} \\ * & \vdots \\ & A^{p_1}_{h\cdots h} \end{array}\right) \left(\begin{array}{c} dx_1(X) \\ \vdots \\ dx_h(X) \\ \hline du_1(X) \\ \vdots \\ du_m(X) \end{array}\right) = 0,$$

where $A^i_{\alpha_1\cdots\alpha_{r-1}}$ is an $h \times m$ matrix

$$A^i_{\alpha_1\cdots\alpha_{r-1}} := \left(\frac{\partial A^i_{\alpha_1\cdots\alpha_{r-1}\alpha_r}}{\partial u_k}\right)_{\substack{1\leq\alpha_r\leq h \\ 1\leq k\leq m}}.$$

By the assumption that $\operatorname{class} S = p_1\binom{h+r-1}{r-1} + h + p_1\binom{h+r-1}{r} - \sigma$ and $\operatorname{rank} S = p_1\binom{h+r-1}{r-1}$, we can see that the rank of the coefficient matrix is equal to $h + p_1\binom{h+r-1}{r} - \sigma$. The upper left part of the coefficient matrix is a $p_1\binom{h+r-1}{r}m \times h$ matrix, so we have

$$\operatorname{rank} {}^t(A^1_{1\cdots1} \cdots A^i_{\alpha_1\cdots\alpha_r} \cdots A^{p_1}_{h\cdots h}) = p_1\binom{h+r-1}{r} - \sigma. \tag{28}$$

Let $\mathbf{A}$ be any matrix consisting of the $p_1\binom{h+r-1}{r} - \sigma$ independent rows of the $p_1\binom{h+r-1}{r}h \times m$ matrix ${}^t(A^1_{1\cdots1} \cdots A^i_{\alpha_1\cdots\alpha_r} \cdots A^{p_1}_{h\cdots h})$. Let $dA^1_{1\cdots1} \widetilde{\wedge \cdots \wedge} dA^{p_1}_{h\cdots h}$ be $p_1\binom{h+r-1}{r} - \sigma$ independent differential forms excluding σ 1-forms from $dA^1_{1\cdots1} \wedge \cdots \wedge dA^i_{\alpha_1\cdots\alpha_r} \wedge \cdots \wedge dA^{p_1}_{h\cdots h}$. Then we can see that the $\left\{p_1\binom{h+r-1}{r} - \sigma\right\}$-square matrix $\mathbf{A}$ is nonsingular. Hence, for all the functions $A^i_{\alpha_1\cdots\alpha_r}$ $(1 \leq i \leq p_1,\ 1 \leq \alpha_1, \ldots, \alpha_r \leq h)$, we have

$$\begin{aligned} dA^1_{1\cdots1} \widetilde{\wedge \cdots \wedge}\, dA^{p_1}_{h\cdots h} &= \pm \det \mathbf{A}\, du_{m-p_1\binom{h+r-1}{r}+\sigma+1} \wedge \cdots \wedge du_m \\ &\neq 0. \end{aligned}$$

Let $\widetilde{\mathbf{du}}$ and $\mathbf{du}$ denote differential forms

$$\begin{cases} \widetilde{\mathbf{du}} := du_{m-p_1\binom{h+r-1}{r}+\sigma+1} \wedge \cdots \wedge du_m, \\ \mathbf{du} := du_1 \wedge \cdots \wedge du_{m-p_1\binom{h+r-1}{r}+\sigma}. \end{cases}$$

Let $\widetilde{A(r)}$ be $p_1\binom{h+r-1}{r}-\sigma$ independent functions excluding σ functions from $(\ldots, A^i_{\alpha_1\cdots\alpha_r}, \ldots)$. Since we have

$$\begin{aligned} &\mathbf{dx} \wedge \mathbf{dz} \wedge \mathbf{dz(1)} \wedge \cdots \wedge \mathbf{dz(r-1)} \wedge dA^1_{1\cdots 1} \widetilde{\wedge \cdots \wedge} dA^{p_1}_{h\cdots h} \wedge \mathbf{du} \\ &\quad = \pm \det \mathbf{A}\, \mathbf{dx} \wedge \mathbf{dz} \wedge \mathbf{dz(1)} \wedge \cdots \mathbf{dz(r-1)} \wedge \widetilde{\mathbf{du}} \wedge \mathbf{du} \\ &\quad \neq 0, \end{aligned}$$

the functions $x,\ z, \ldots, z(r-1), \widetilde{A(r)}, u_1, \ldots, u_{m-p_1\binom{h+r-1}{r}+\sigma}$ form local coordinates in M. Therefore we can get local coordinates $\big(x, z, z(1), \ldots, z(r-1), \widetilde{A(r)}, u_1, \ldots, u_{m-p_1\binom{h+r-1}{r}+\sigma}\big)$ in M such that S is locally generated by the 1-forms (25) and

$$\omega^i_{\alpha_1\cdots\alpha_{r-1}} = dz^i_{\alpha_1\cdots\alpha_{r-1}} - \sum_{\alpha_r=1}^{h} A^i_{\alpha_1\cdots\alpha_r}\big(x, \ldots, u_{m-p_1\binom{h+r-1}{r}+\sigma}\big)\, dx_{\alpha_r},$$
$$(1 \leq i \leq p_1,\ 1 \leq \alpha_1, \ldots, \alpha_{r-1} \leq h)$$

By (28), the function $A^i_{\alpha_1\cdots\alpha_r}$ satisfies

$$\operatorname{rank} \frac{\partial(\ldots, A^i_{\alpha_1\cdots\alpha_r}, \ldots)}{\partial(\widetilde{A(r)})} = p_1\binom{h+r-1}{r} - \sigma.$$

This completes the proof of Theorem 4.1. □

Corollary 4.2 *Let S be a Pfaffian system on M. Then there exist local coordinates $\big(x, z, z(1), \ldots, z(k-1), \widetilde{z(k)}, y^{(k+1)}_1, \ldots, y^{(k+1)}_{p_1\binom{h+k-2}{k-1}+\sigma}, y^{(k+2)}, \ldots, y^{(r)}, u\big)$ in M such that the $(r-k)$th derived system S_{r-k} can be transformed locally into the Pfaffian sysytem $\Omega\big(\Sigma^{(k,h,p_1)}_\sigma\big)$ associated to a system of kth order PDEs in $\mathbb{R}^h$ and $\mathbb{R}^{p_1}$ of codimension σ, if and only if S satisfies the following conditions*:

(i) *S is a totally regular Pfaffian system of length r with $S_r = \{0\}$,*

(ii) *S has a relative polarization of rank h,*

(iii) *the type* $(p_0, \ldots, p_{r+1})$ *of* S *satisfies*

$$p_i = \begin{cases} 0, & \text{if } i = 0, \\ p_1\binom{h+i-2}{i-1}, & \text{if } 1 \le i \le k+1, \end{cases}$$

(iv) *for each derived system* S_{r-j}, $1 \le j \le k$,

$$\operatorname{class} S_{r-j} = \begin{cases} h + p_1\binom{h+j}{j}, & \text{if } 1 \le j \le k-1, \\ h + p_1\binom{h+k}{k} - \sigma, & \text{if } j = k. \end{cases}$$

Proof. Assume that the $(r-k)$th derived system S_{r-k} of S admits $\Omega\big(\Sigma_\sigma^{(k,h,p_1)}\big)$. Then by Example 1 and Example 4, it is easy to see that S satisfies the conditions (i)∼(iv). Now, assuming S is a Pfaffian system satisfying the conditions (i)∼(iv), we prove that S_{r-k} can be transformed locally into the normal form $\Omega\big(\Sigma_\sigma^{(k,h,p_1)}\big)$. From Lemma 3.1 we know that there exist local coordinates $(x, y^{(1)}, \ldots, y^{(r)}, u)$ in M such that S is generated by the 1-forms (14). Applying Theorem 4.1 to the $(r-k+1)$st derived system S_{r-k+1}, there exist local coordinates $(x, z, z(1), \ldots, z(k-1), y^{(k+1)}, \ldots, y^{(r)}, u)$ in M such that S_{r-k+1} is generated by the 1-forms

$$\begin{cases} \omega^i = dz^i - \displaystyle\sum_{\alpha_1=1}^{h} z^i_{\alpha_1}\, dx_{\alpha_1}, \\ \quad\vdots \qquad\qquad \vdots \\ \omega^i_{\alpha_1\cdots\alpha_{k-2}} = dz^i_{\alpha_1\cdots\alpha_{k-2}} - \displaystyle\sum_{\alpha_{k-1}=1}^{h} z^i_{\alpha_1\cdots\alpha_{k-1}}\, dx_{\alpha_{k-1}}, \end{cases}$$
$$(1 \le i \le p_1,\ 1 \le \alpha_1, \ldots, \alpha_{k-2} \le h)$$

where the coordinates $z^i_{\alpha_1\alpha_2}, \ldots, z^i_{\alpha_1\cdots\alpha_{k-1}}$ are symmetric with respect to the subscript $\alpha_1, \ldots, \alpha_{k-1}$. Let us consider the system $S_{r-k} = S_{r-k+1} \oplus \operatorname{span}\big\{\omega^{i_k}_{(k)}\big\}$ in the new coordinates $(x, z, z(1), \ldots, z(k-1), y^{(k+1)}, \ldots, y^{(r)}, u)$ of M. The proof is completely analogous to that of the construction of $S = S_1 \oplus \operatorname{span}\big\{\omega^{i_r}_{(r)}\big\}$ on Theorem 4.1 (pp. 838–842). □

Example 7 We consider Example 5 in Section 3 again. Applying Corollary 4.2 to $S_2 = \operatorname{span}\{\omega^1, \omega^2\}$, we have coordinates $\big(x_1, x_2, z^1, z^2, \widetilde{z^3}, \widetilde{z^4}, y_1^{(3)}\big)$ in $\mathbb{R}^7$ such that S_2 can be transformed into the Pfaffian system $\Omega\big(\Sigma_2^{(1,2,2)}\big)$, which is a Pfaffian system associated to a system of first order PDEs in $\mathbb{R}^2$

and $\mathbb{R}^2$ of codimension 2:

$$S_2 = \begin{cases} \omega^1 = dz^1 - F_1^1(x_1, x_2, z^1, z^2, \widetilde{z^3}, \widetilde{z^4})\, dx_1 \\ \qquad - F_2^1(x_1, x_2, z^1, z^2, \widetilde{z^3}, \widetilde{z^4})\, dx_2, \\ \omega^2 = dz^2 - F_1^2(x_1, x_2, z^1, z^2, \widetilde{z^3}, \widetilde{z^4})\, dx_1 \\ \qquad - F_2^2(x_1, x_2, z^1, z^2, \widetilde{z^3}, \widetilde{z^4})\, dx_2, \end{cases}$$

with

$$\operatorname{rank} \frac{\partial(F_1^1, F_2^1, F_1^2, F_2^2)}{\partial(\widetilde{z^3}, \widetilde{z^4})} = 2. \tag{29}$$

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