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Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces

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Abstract. We prove that b is in $BMO(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ of the multiplication operator by b and the fractional integral operator I_α is bounded from the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$, where $1 < p < \infty$, $0 < \alpha < n$, $0 < \lambda < n - \alpha p$, $1/q = 1/p - \alpha/n$ and $\lambda/p = \mu/q$. Also we will show that b is in $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ or from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, where α and β satisfy some conditions.

Key words: commutator, fractional integral operator, the classical Morrey space, higher order commutator.

1. Introduction

Let I_α, $0 < \alpha < n$, be the fractional integral operator of order α, defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

For a locally integrable function b, the commutator is defined by

$$[b, I_\alpha]f(x) := b(x) I_\alpha f(x) - I_\alpha(bf)(x).$$

The commutator $[b, I_\alpha]$ was introduced by Chanillo [2].

Adams [1] showed that the fractional integral operator is bounded from the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. Chiarenza and Frasca [3] gave an another proof of the previous result.

Recently, Di Fazio and Ragusa [6] showed that if b is in $BMO(\mathbb{R}^n)$, then the commutator $[b, I_\alpha]$ is bounded from the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, and conversely, under some restricted condition on α, if the commutator $[b, I_\alpha]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, then $b \in BMO(\mathbb{R}^n)$.

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Moreover Paluszyński [14] showed that if $p < n/(\alpha+\beta)$, then b is in the (homogeneous) Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, $1/p - 1/r = (\alpha+\beta)/n$.

The aim of this paper is to prove that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ for some appropriate indices p, q, λ, μ and α. Therefore our result will mean to remove some restriction from the result of Di Fazio and Ragusa [6].

Also we show that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$ or from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ for some appropriate indices p, q, λ, μ, α and β.

We will give an answer to a problem posed by Yasuo Komori and Takahiro Mizuhara [10, Problem 1, p.352].

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2. Definitions and notation

Throughout this paper all notation is standard or will be defined as needed. All cubes are assumed to have their sides parallel to the coordinate axes. $Q = Q(x_0, t)$ denotes the cube centered at x_0 with side length t. Given a Lebesgue measurable set E, χ_E will denote the characteristic function of E and $|E|$ is the Lebesgue measure of E. The letter C will be used for various constants, and may change from one occurrence to another.

Definition 2.1 (classical Morrey space) Let $1 \le p < \infty$, $0 \le \lambda$. We define the classical Morrey space by

$$L^{p,\lambda}(\mathbb{R}^n) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)\colon \|f\|_{L^{p,\lambda}} < \infty\},$$

where

$$\|f\|_{L^{p,\lambda}} := \sup_{\substack{x_0\in\mathbb{R}^n \\ t>0}} \left(\frac{1}{t^\lambda}\int_{Q(x_0,t)} |f(x)|^p\,dx\right)^{1/p}.$$

For the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$, the next results are well-known:

If $1 \le p < \infty$, then we have $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ when $\lambda = n$, and if $n < \lambda$, then we have $L^{p,\lambda}(\mathbb{R}^n) = \{0\}$. Therefore we consider the case only $0 < \lambda < n$.

Definition 2.2 (John-Nirenberg space) $BMO(\mathbb{R}^n)$ is the John-Nirenberg space. That is, $BMO(\mathbb{R}^n)$ is a Banach space, modulo constants, with the norm $\|\cdot\|_*$ defined by

$$\|b\|_* := \sup_Q \frac{1}{|Q|}\int_Q |b(x) - b_Q|\,dx,$$

where

$$b_Q := \frac{1}{|Q|}\int_Q b(y)\,dy$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Definition 2.3 (Lipschitz space) We define the (homogeneous) Lipschitz space of order β, $0 < \beta < 1$, by

$$\dot{\Lambda}_\beta(\mathbb{R}^n) = \{f\colon |f(x) - f(y)| \le C|x-y|^\beta\}$$

and the smallest constant $C > 0$ is the Lipschitz norm $\|\cdot\|_{\dot{\Lambda}_\beta}$.

We recall the definitions of some maximal functions.

Definition 2.4 Given a locally integrable function f and α, $0 \le \alpha < n$, define the fractional maximal function by

$$M_\alpha f(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\alpha/n}}\int_Q |f(y)|\,dy$$

when $0 < \alpha < n$. If $\alpha = 0$ then $M_0 f = Mf$ denotes the usual Hardy-Littlewood maximal function. Also define the sharp maximal function by

$$M^\sharp f(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y) - f_Q|\,dy.$$

In both definitions, the supremum is taken over all Q containing x.

Remark As well known, the sharp maximal function was introduced by Fefferman and Stein [7]. The fractional maximal function was used by Muckenhoupt and Wheeden [13].

The blocks and the space generated by blocks were introduced by Long [12]. See also Komori and Mizuhara [10].

Definition 2.5 Let $1 \le q < r \le \infty$. A function $g(x)$ on $\mathbb{R}^n$ is called a (q,r)-block, if there exists a cube $Q(x_0,t)$ such that

$$\operatorname{supp}(g) \subset Q(x_0,t), \quad \|g\|_{L^r} \le t^{n(1/r-1/q)}.$$

Definition 2.6 Let $1 \le q < r \le \infty$. We define the space generated by blocks by

$$h_{q,r}(\mathbb{R}^n) := \left\{ f = \sum_{j=1}^{\infty} m_j g_j : \ g_j \text{ are } (q,r)\text{-blocks}, \right.$$
$$\left. \|f\|_{h_{q,r}} = \inf \sum_{j=1}^{\infty} |m_j| < \infty \right\},$$

where the infimum extends over all representations $f = \sum_{j=1}^{\infty} m_j g_j$.

3. Theorems

The $L^{p,\lambda}$ theory about the fractional integral operator I_α is as follows:

Theorem A (Adams [1]) *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$ and $1/q = 1/p - \alpha/(n-\lambda)$. Then there exists a constant $C > 0$ independent of f such that*

$$\|I_\alpha f\|_{L^{q,\lambda}} \le C \|f\|_{L^{p,\lambda}}$$

for every $f \in L^{p,\lambda}(\mathbb{R}^n)$.

This proof depends on the basic idea due to Hedberg [8]. We have the following theorem from Theorem A using Hölder's inequality, which was obtained by S. Spanne but published by Peetre [16].

Theorem B *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$. Set $1/q = 1/p - \alpha/n$ and $\mu = n\lambda/(n - \alpha p)$ (i.e. $\lambda/p = \mu/q$). Then there exists a constant $C > 0$ independent of f such that*

$$\|I_\alpha f\|_{L^{q,\mu}} \le C \|f\|_{L^{p,\lambda}}$$

for every $f \in L^{p,\lambda}(\mathbb{R}^n)$.

Remark We note that the fractional maximal operator M_α is bounded form $L^{p,\lambda}$ to $L^{q,\lambda}$ or from $L^{p,\lambda}$ to $L^{q,\mu}$ since the pointwise inequality $M_\alpha f(x) \le I_\alpha(|f|)(x)$. More generally, $M_{\alpha,r} f(x) \le I_{\alpha,r}(|f|)(x)$, where

$M_{\alpha,r}f(x) = M_{\alpha r}(|f|^r)(x)^{1/r}$ and $I_{\alpha,r}(|f|)(x) = I_{\alpha r}(|f|^r)(x)^{1/r}$.

Note that Theorem B was originally showed for the Morrey-Campanato spaces on a bounded domain with a more general index λ.

Theorem C (Di Fazio and Ragusa [6]) *Let* $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, $1/q = 1/p - \alpha/(n-\lambda)$.

If b *is in* $BMO(\mathbb{R}^n)$, *then the commutator* $[b, I_\alpha]$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$.

Conversely if $n - \alpha$ *is an even integer and* $[b, I_\alpha]$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$ *for indices* p, q, λ *as above, then* $b \in BMO(\mathbb{R}^n)$.

In the case of different indices, we have the following results. In the following, we assume that $f \in C_c^\infty(\mathbb{R}^n)$, the space of infinitely differentiable functions with compact support.

Theorem 3.1 (Main Theorem) *Let* $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, $1/q = 1/p - \alpha/n$ *and* $\mu = n\lambda/(n - \alpha p)$ (*i.e.* $\lambda/p = \mu/q$).

Then the following conditions are equivalent:

(a) $b \in BMO(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\mu}(\mathbb{R}^n)$.

Furthermore we get the following results when $\alpha < n(1/p - 1/q)$.

Theorem 3.2 *Let* $1 < p < q < \infty$, $0 < \alpha$, $0 < \beta < 1$, $0 < \alpha + \beta = n(1/p - 1/q) < n$, $0 < \lambda < n - (\alpha + \beta)p$ *and* $\mu/q = \lambda/p$.

Then the following conditions are equivalent:

(a) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\mu}(\mathbb{R}^n)$.

Theorem 3.3 *Let* $1 < p < q < \infty$, $0 < \alpha$, $0 < \beta < 1$ *and* $0 < \alpha + \beta = (1/p - 1/q)(n - \lambda) < n$.

Then the following conditions are equivalent:

(a) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$.

Theorem D (Komori and Mizuhara [10]) *Let* $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$ *and* $1/q = 1/p - \alpha/(n - \lambda)$.

Then the following conditions are equivalent:

(a) $b \in BMO(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$.

Remark Our proof is an another proof of Theorem D due to Komori and Mizuhara [10]. Our method is direct, but there have used the factorization theorem for $H^1(\mathbb{R}^n)$.

4. Technical lemmas

We need some lemmas in order to prove our theorems.

Lemma 4.1 *The following are true*:

(1) *For each p, $1 < p < \infty$, there exists a constant C_p such that*

$$\sup_Q \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^p \, dx \right)^{1/p} \le C_p \|b\|_*.$$

(2) *Given α, $0 < \alpha < n$, there exists a constant C such that for any cube Q and a nonnegative function f*

$$\int_Q I_\alpha f(x) \, dx \le C|Q|^{\alpha/n} \int_{\mathbb{R}^n} f(x) \, dx. \tag{4.1}$$

The first follows from the John-Nirenberg lemma. For a detailed proof of (1), for example, see [5, Chapter 6]. For a proof of (2), see [4, Lemma 5.2.(1)].

As well known, the idea of relating commutators with the sharp maximal operator is due to Strömberg (cf. [9]).

Lemma 4.2 *Let $0 < \alpha < n$, $1 < r < \infty$ and $b \in BMO(\mathbb{R}^n)$. Then there exists a constant $C > 0$ independent of b and f such that*

$$M^\sharp([b, I_\alpha](f))(x) \le C\|b\|_* \{I_\alpha(|f|)(x) + I_{\alpha,r}(|f|)(x)\} \tag{4.2}$$

for almost all x and every $f \in C_c^\infty(\mathbb{R}^n)$.

This lemma is similar to the result due to Cruz-Uribe and Fiorenza [4].

Proof. We first note that $I_\alpha(|f|)$ is in the Muckenhoupt class A_1 (see Sawyer [17]); there exists a constant C such that $M(I_\alpha(|f|))(x) \le CI_\alpha(|f|)(x)$ for almost every x. Therefore it satisfies the reverse Hölder inequality for some index $s > 1$. Fix $x \in \mathbb{R}^n$ and fix a cube Q containing x. Then it will suffice to prove for some complex constant c_Q that there exists C such that

$$\frac{1}{|Q|} \int_Q \big|[b, I_\alpha]f(y) - c_Q\big| \, dy \le C\|b\|_* \{I_\alpha(|f|)(x) + I_{\alpha,r}(|f|)(x)\}. \tag{4.3}$$

Decompose f as f_1+f_2, where $f_1=f\chi_{Q^*}$ and Q^* is the cube with the same center as Q whose sides are $3\sqrt{n}$ times as long. Let $c_Q=I_\alpha((b-b_{Q^*})f_2)(x_0)$. Since $[b,I_\alpha]f=[(b-b_{Q^*}),I_\alpha]f$, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \big|[b,I_\alpha]f(y)-c_Q\big|\,dy\\
&\quad\le \frac{1}{|Q|}\int_Q |b(y)-b_{Q^*}|\,|I_\alpha f(y)|\,dy+\frac{1}{|Q|}\int_Q |I_\alpha((b-b_{Q^*})f_1)(y)|\,dy\\
&\qquad\quad+\frac{1}{|Q|}\int_Q \big|I_\alpha((b-b_{Q^*})f_2)(y)-I_\alpha((b-b_{Q^*})f_2)(x_0)\big|\,dy\\
&\quad=\mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

We estimate each integral in turn. For I, using Hölder's inequality with exponent s satisfying the reverse Hölder inequality, Lemma 4.1 (1) and $I_\alpha(|f|)\in A_1$ we have

$$\begin{aligned}
\mathrm{I}&\le\left(\frac{1}{|Q|}\int_Q |b(y)-b_{Q^*}|^{s'}\,dy\right)^{1/s'}\left(\frac{1}{|Q|}\int_Q |I_\alpha f(y)|^s\,dy\right)^{1/s}\\
&\le C\|b\|_*\left(\frac{1}{|Q|}\int_Q I_\alpha(|f|)(y)\,dy\right)\\
&\le C\|b\|_* M(I_\alpha(|f|))(x)\\
&\le C\|b\|_* I_\alpha(|f|)(x).
\end{aligned}$$

To estimate II, we apply Höder's inequality with exponemt r and (4.1). Then we have

$$\begin{aligned}
\mathrm{II}&\le\frac{1}{|Q|}\int_Q I_\alpha(|(b-b_{Q^*})f_1|)(y)\,dy\\
&\le C|Q|^{\alpha/n}\frac{1}{|Q|}\int_{\mathbb{R}^n}|b(y)-b_{Q^*}|\,|f_1(y)|\,dy\\
&=C\frac{|Q^*|^{\alpha/n}}{|Q^*|}\int_{Q^*}|b(y)-b_{Q^*}|\,|f(y)|\,dy\\
&\le C|Q^*|^{\alpha/n}\left(\frac{1}{|Q^*|}\int_{Q^*}|b(y)-b_{Q^*}|^{r'}\,dy\right)^{1/r'}\left(\frac{1}{|Q^*|}\int_{Q^*}|f(y)|^r\,dy\right)^{1/r}\\
&\le C\|b\|_* M_{\alpha,r}f(x)\\
&\le C\|b\|_* I_{\alpha,r}(|f|)(x).
\end{aligned}$$

The last inequality follows from the remark below Theorem B.

Finally, we estimate the third integral. By the mean value theorem, if $|x| > 2|y|$ then there exists γ, $0 \le \gamma \le 1$, such that

$$\left| \frac{1}{|x|^{n-\alpha}} - \frac{1}{|x+y|^{n-\alpha}} \right| \le C \frac{|y|}{|x+\gamma y|^{n-\alpha+1}} \le C \frac{|y|}{|x|^{n-\alpha+1}}.$$

If $y \in Q$ and $z \in \mathbb{R}^n \setminus 2^k Q^*$, then $|x_0 - z| > 2^{k+1}|y - x_0|$ by geometric observation. Hence we can control III pointwise by

$$\begin{aligned}
&\left| I_\alpha((b - b_{Q^*})f_2)(y) - I_\alpha((b - b_{Q^*})f_2)(x_0) \right| \\
&\quad \le \int_{\mathbb{R}^n \setminus Q^*} \left| \frac{1}{|y-z|^{n-\alpha}} - \frac{1}{|x_0 - z|^{n-\alpha}} \right| |b(z) - b_{Q^*}|\, |f(z)|\, dz \\
&\quad \le C \sum_{k=0}^{\infty} \int_{2^{k+1}Q^* \setminus 2^k Q^*} |b(z) - b_{Q^*}|\, |f(z)|\, \frac{|y - x_0|}{|x_0 - z|^{n-\alpha+1}}\, dz \\
&\quad \le C \sum_{k=0}^{\infty} 2^{-k} \frac{1}{|2^{k+1}Q^*|^{1-\alpha/n}} \int_{2^{k+1}Q^*} |b(z) - b_{Q^*}|\, |f(z)|\, dz \\
&\quad \le C \sum_{k=0}^{\infty} 2^{-k} |2^{k+1}Q^*|^{\alpha/n} \left(\frac{1}{|2^{k+1}Q^*|} \int_{2^{k+1}Q^*} |b(z) - b_{Q^*}|^{r'} dz \right)^{1/r'} \\
&\qquad\qquad \times \left(\frac{1}{|2^{k+1}Q^*|} \int_{2^{k+1}Q^*} |f(z)|^r\, dz \right)^{1/r} \\
&\quad \le C \|b\|_* M_{\alpha,r} f(x) \\
&\quad \le C \|b\|_* I_{\alpha,r}(|f|)(x),
\end{aligned}$$

where we have uesed Hölder's inequality. The last inequality follows from the remark below Theorem B. Combining these estimates, we get the desired pointwise inequality. □

Lemma 4.3 (Di Fazio and Ragusa [6]) *Let $1 < p < \infty$, $0 < \lambda < n$. Then there exists a constant $C > 0$ independent of f such that*

$$\|Mf\|_{L^{p,\lambda}} \le C \|M^\sharp f\|_{L^{p,\lambda}}$$

for every $f \in L^{p,\lambda}(\mathbb{R}^n)$.

Lemma 4.4 (Komori and Mizuhara [10]) *Let $1 \le p < \infty$, $0 < \lambda < n$ and $1 \le q < r \le \infty$. Then we have*

$$\|\chi_{Q(x_0,t)}\|_{L^{p,\lambda}} \leq C_n t^{(n-\lambda)/p}, \quad \|\chi_{Q(x_0,t)}\|_{h_{q,r}} \leq C_n t^{n/q}$$

where $C_n > 0$ depends only on n.

Lemma 4.5 (Komori and Mizuhara [10], Long [12]) *Let $1 \leq q < p' < \infty$, $q = np/(np - n + \lambda)$ and $1/p + 1/p' = 1$. Then the Banach space dual of $h_{q,p'}(\mathbb{R}^n)$ is isomorphic to $L^{p,\lambda}(\mathbb{R}^n)$.*

See Komori and Mizuhara [10] for Lemmas 4.4 and 4.5. The following lemma can be found in [14, Lemma 1.5.].

Lemma 4.6 (cf. Paluszyński [14]) *For $0 < \beta < 1$ and $1 < q \leq \infty$, we have*

$$\begin{aligned}\|f\|_{\dot{\Lambda}_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q|\, dx \\ &\approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^q\, dx \right)^{1/q},\end{aligned}$$

for $q = \infty$ the formula should be interpreted appropriately, where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

5. Proof of theorems

Proof of Theorem 3.1. (a) $\Rightarrow$ (b): Let $1 < r < p$. From Lemmas 4.2 and 4.3 we get

$$\begin{aligned}\|[b, I_\alpha]f\|_{L^{q,\mu}} &\leq \|M([b, I_\alpha](f))\|_{L^{q,\mu}} \\ &\leq C\|M^\sharp([b, I_\alpha](f))\|_{L^{q,\mu}} \\ &\leq C\|b\|_*\{\|I_\alpha(|f|)\|_{L^{q,\mu}} + \|I_{\alpha,r}(|f|)\|_{L^{q,\mu}}\} \\ &= C\|b\|_*\left\{\|I_\alpha(|f|)\|_{L^{q,\mu}} + \|I_{\alpha r}(|f|^r)\|^{1/r}_{L^{q/r,\mu}}\right\} \\ &\leq C\|b\|_*\|f\|_{L^{p,\lambda}}.\end{aligned}$$

The last inequality follows from Theorem B. This completes the proof of (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a): We use the same argument as Janson [9]. Choose $0 \neq z_0 \in \mathbb{R}^n$ such that $0 \notin Q(z_0, 2)$. Then for $x \in Q(z_0, 2)$, $|x|^{n-\alpha} \in C^\infty(Q(z_0, 2))$. Hence, considering a cut function on the cube $Q(z_0, 2 + \delta)$ for sufficiently small $\delta > 0$, $|x|^{n-\alpha}$ can be written as the absolutely convergent Fourier series;

$$|x|^{n-\alpha} = \sum_{m\in\mathbb{Z}^n} a_m e^{i\langle v_m, x\rangle}$$

with $\sum_m |a_m| < \infty$, where the exact form of the vectors v_m is unrelated.

For any $x_0 \in \mathbb{R}^n$ and $t > 0$, let $Q = Q(x_0, t)$ and $Q^{z_0} = Q(x_0 + z_0 t, t)$. Let $s(x) = \overline{\operatorname{sgn}(\int_{Q^{z_0}}(b(x)-b(y))\,dy)}$. If $x\in Q$ and $y \in Q^{z_0}$, then $(y-x)/t \in Q(z_0, 2)$. Hence we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |b(x) - b_{Q^{z_0}}|\,dx \\
&= \frac{1}{|Q|}\frac{1}{|Q^{z_0}|}\int_Q \left|\int_{Q^{z_0}} (b(x)-b(y))\,dy\right| dx \\
&= \frac{1}{t^{2n}}\int_Q s(x)\left(\int_{Q^{z_0}} (b(x)-b(y))|x-y|^{\alpha-n}|x-y|^{n-\alpha}\,dy\right) dx \\
&= \frac{t^{n-\alpha}}{t^{2n}}\int_Q s(x)\left(\int_{Q^{z_0}} (b(x)-b(y))|x-y|^{\alpha-n}\left|\frac{x-y}{t}\right|^{n-\alpha} dy\right) dx \\
&= t^{-n-\alpha}\sum_{m\in\mathbb{Z}^n} a_m \int_Q s(x) \\
&\qquad \times \left(\int_{Q^{z_0}} (b(x)-b(y))|x-y|^{\alpha-n} e^{i\langle v_m, y/t\rangle}\,dy\right) e^{-i\langle v_m, x/t\rangle}\,dx \\
&\le t^{-n-\alpha}\left|\sum_{m\in\mathbb{Z}^n} a_m \int_{\mathbb{R}^n} s(x)[b, I_\alpha](\chi_{Q^{z_0}} e^{i\langle v_m,\,\cdot/t\rangle})(x)\chi_Q(x) e^{-i\langle v_m, x/t\rangle} dx\right| \\
&\le t^{-n-\alpha}\sum_{m\in\mathbb{Z}^n} |a_m| \left\|[b, I_\alpha](\chi_{Q^{z_0}} e^{i\langle v_m,\,\cdot/t\rangle})\right\|_{L^{q,\mu}} \cdot \|\chi_Q\|_{h_{nq/(nq-n+\mu),q'}} \\
&\le t^{-n-\alpha}\sum_{m\in\mathbb{Z}^n} |a_m|\, \|[b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\mu}} \cdot \|\chi_{Q^{z_0}}\|_{L^{p,\lambda}} \cdot \|\chi_Q\|_{h_{nq/(nq-n+\mu),q'}} \\
&\le t^{-n-\alpha}\sum_{m\in\mathbb{Z}^n} |a_m|\, \|[b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\mu}} \cdot C_n t^{(n-\lambda)/p} \cdot C'_n t^{(nq-n+\mu)/q} \\
&= C\|[b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\mu}}.
\end{aligned}$$

The second inequality follows from Lemma 4.5, the third inequality follows from Lemma 4.4. Therefore we get

$$\frac{1}{|Q|}\int_Q |b(x) - b_Q|\,dx \le \frac{2}{|Q|}\int_Q |b(x) - b_{Q^{z_0}}|\,dx$$

$$\leq 2C\|[b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\mu}}.$$

This implies that $b \in BMO(\mathbb{R}^n)$ and $\|b\|_* \leq C\|[b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\mu}}$, and the proof of the theorem is completed. $\square$

Proof of Theorem 3.2. (a) $\Rightarrow$ (b): Let $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. Then we get

$$\begin{aligned}
|[b, I_\alpha]f(x)| &= \left|\int_{\mathbb{R}^n} \frac{(b(x)-b(y))f(y)}{|x-y|^{n-\alpha}}\,dy\right| \\
&\leq \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|\cdot|f(y)|}{|x-y|^{n-\alpha}}\,dy \\
&\leq C\|b\|_{\dot{\Lambda}_\beta}\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-(\alpha+\beta)}}\,dy = C\|b\|_{\dot{\Lambda}_\beta} I_{\alpha+\beta}(|f|)(x)
\end{aligned}$$

for almost all $x \in \mathbb{R}^n$. Therefore we have, from Theorem B

$$\|[b, I_\alpha]f\|_{L^{q,\mu}} \leq C'\|b\|_{\dot{\Lambda}_\beta}\|I_{\alpha+\beta}(|f|)\|_{L^{q,\mu}} \leq C\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^{p,\lambda}}.$$

(b) $\Rightarrow$ (a): We can prove using an argument similar to the proof of Theorem 3.1. For completeness we give a proof.

Let Q and Q^{z_0} be same cubes as the proof of (b) $\Rightarrow$ (a) in Theorem 3.1. Then we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |b(x) - b_{Q^{z_0}}|\,dx \\
&\quad= \frac{1}{t^{2n}}\int_Q s(x)\left(\int_{Q^{z_0}} (b(x)-b(y))|x-y|^{\alpha-n}|x-y|^{n-\alpha}\,dy\right)dx \\
&\quad= \frac{t^{n-\alpha}}{t^{2n}}\int_Q s(x)\left(\int_{Q^{z_0}} (b(x)-b(y))|x-y|^{\alpha-n}\left|\frac{x-y}{t}\right|^{n-\alpha}dy\right)dx \\
&\quad\leq t^{-n-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\,\|[b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\mu}}\cdot C_n t^{(n-\lambda)/p}\cdot C'_n t^{(nq-n+\mu)/q} \\
&\quad= C\|[b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\mu}}t^\beta.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x) - b_Q|\,dx &\leq \frac{2}{|Q|^{1+\beta/n}}\int_Q |b(x) - b_{Q^{z_0}}|\,dx \\
&\leq 2C\|[b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\mu}}.
\end{aligned}$$

From Lemma 4.6, we have $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $\|b\|_{\dot{\Lambda}_\beta} \leq C\|[b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\mu}}$. This complete the proof. $\square$

Theorem 3.3 is shown in the same argument as the proof of Theorem 3.2. We omit this proof.

6. Boundedness of higher order commutator on classical Morrey spaces

We will consider a higher order commutator operator defined by

$$[b, I_\alpha]^k f(x) := \int_{\mathbb{R}^n} \frac{\Delta_h^k b(x) f(h)}{|h|^{n-\alpha}}\, dh,$$

where

$$\Delta_h^1 b(x) = \Delta_h b(x) = b(x+h) - b(x),$$
$$\Delta_h^{k+1} b(x) = \Delta_h^k b(x) - \Delta_h^k b(y), \qquad k \geq 1.$$

Let $0 < \beta < k \leq n$, k an integer and n be the dimension of the whole space. We now try to define the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ again. For $\beta > 0$, we say $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if

$$\|b\|_{\dot{\Lambda}_\beta} = \sup_{\substack{x,h \in \mathbb{R}^n \\ x \neq h}} \frac{|\Delta_h^k b(x)|}{|h|^\beta} < \infty, \qquad k \geq 1.$$

Theorem 6.1 *Suppose the same condition as* Theorem 3.2. *The following conditions are equivalent*:
(a) *$b = b_1 + P$, where $b_1 \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and P is a polynomial of degree less than k.*
(b) *$[b, I_\alpha]^k$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$.*

If $k = [\beta] + 1$, then (a) of theorem says that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

The proof of (a) $\Rightarrow$ (b) will be omitted since we can prove the same argument as Theorem 3.2. The part of (b) $\Rightarrow$ (a) is based on the following results for the Besov spaces.

Lemma 6.2 (Paluszyński and Taibleson [15]) *Let $0 < \beta < k$, with k an integer. Suppose $f \in \mathcal{S}' \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The following conditions are equivalent*:
(a) *$f = f_1 + P$, where $f_1 \in \dot{B}^{\beta,\infty}_\infty(\mathbb{R}^n)$ $(= \dot{\Lambda}_\beta(\mathbb{R}^n))$ and P is a polynomial of degree less than k.*
(b) *There exists $z_0 \in \mathbb{R}^n$ such that*

$$\sup_{t>0} t^{-\beta} \sup_{x_0\in\mathbb{R}^n} \frac{1}{|Q|} \frac{1}{|Q^{z_0}|} \left(\int_Q \left| \int_{Q^{z_0}} (\Delta^k_{(y-x)/k} f(x))\, dy \right| dx \right) \le C < \infty,$$

where $Q = Q(x_0, t)$, *and* $Q^{z_0} = Q(x_0 + z_0 t, t)$.

If these conditions hold then $\|f\|_{\dot{B}^{\beta,\infty}_{\infty}}$ *is comparable with the best possible* C *in* (b).

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