Hokkaido Mathematical Journal Vol. 35 (2006) p. 457–469

Approximations of hypersingular integral equations by the quadrature method

E.G. Ladopoulos and V.A. Zisis

(Received November 10, 2004; Revised December 22, 2004)

Abstract. A numerical method is proposed and investigated for the hypersingular integral equations defined in Banach spaces. The hypersingular integral equations belong to a wider class of singular integral equations having much more stronger singularities.

The proposed approximation method is an extension beyond the quadrature method. Moreover an error estimates theory is introduced for the hypersingular integral equations by proving the proper theorem. Finally, the inequalities valid between the exact solutions of the hypersingular integral equations and the corresponding approximate solutions, are proposed and proved.

Key words: hypersingular integral equations, singularity, quadrature method, error estimate, Banach spaces.

1. Introduction

The hypersingular integral equations consist to a wider class of singular integral equations. In particular the kernel of such integral equations has a stronger singularity as compared to the finite-part singular integral equations. Hence, there is a big interest for the numerical evaluation of the hypersingular integral equations, as closed form solutions are not possible to be determined.

J. Hadamard [1], [2] was the first scientist who introduced the concept of finite - part integrals, and L. Schwartz [3] studied very basic properties of them. Several years later, H.R. Kutt [4] proposed some algorithms for the numerical solution of the finite-part singular integrals and studied the difference between a finite - part integral and a "generalized principal value integral".

On the other hand, M.A. Golberg [5] investigated the convergence of several numerical methods for the solution of finite-part integrals. He proposed a method, which was an extension beyond the Galerkin and collocation methods [6]. Also, A.C. Kaya and F. Erdogan [7], [8] introduced and

2000 Mathematics Subject Classification : 65L10, 65R20.

investigated several problems of elasticity and fracture mechanics, which are reduced to the solution of finite-part singular integral equations.

Moreover, by E.G. Ladopoulos [9]–[15] were proposed several numerical methods for the solution of the finite-part singular integral equations of the first and the second kind. He also applied this type of singular integral equations to the solution of very important problems of elasticity, fracture mechanics and aerodynamics. Beyond the above, E.G. Ladopoulos, V.A. Zisis and D. Kravvaritis [16], [17] used functional analysis for the solution of finite-part singular integral equations. Hence, they studied such type of singular integral equations defined in general Hilbert spaces and L_p spaces and applied them to several crack problems.

In the present research are introduced and investigated the hypersingular integral equations, which have stronger singularity in comparison to the finite-part singular integral equations. Thus, the hypersingular integral equations belong to a wider class of integral equations with kernels of very strong singularities.

A numerical method is proposed for the solution of the hypersingular integral equations, defined in Banach spaces. The proposed approximation method is an extension beyond the quadrature method.

Furthermore, an error estimates theory is proposed for the hypersingular integral equations, by proving the corresponding theorem. Thus the inequalities which are valid between the exact solutions of the hypersingular integral equations and the corresponding approximate solutions, are investigated and proved.

The hypersingular integral equations are used for the solution of several important problems of engineering mechanics, and especially in the theories of elasticity, fracture mechanics, fluid mechanics and aerodynamics.

2. Numerical evaluation methods for hypersingular integral equations

Definition 2.1 An equation of the following form is called hypersingular integral equation:

$$\int_a^b \frac{u(x)dx}{|x-t|^{\lambda}} = f(t), \quad 1 < \lambda < 3 \tag{2.1}$$

where $u(x)$ is the unknown function and $f(t)$ is a known function such as $f(t) \in C^{\infty}[\alpha, b]$.

Theorem 2.1 *Let the hypersingular integral equation* (2.1) *and suppose that following conditions are satisfied*:

$$u(x) = \begin{cases} k_1(x-a)^{(\lambda-1)/2} + k_2(x-a)^{(\lambda+1)/2} + \omega_1(x), \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for } x = a \\ k_1'(b-x)^{(\lambda-1)/2} + k_2'(b-x)^{(\lambda+1)/2} + \omega_2(x), \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for } x = b \end{cases} \tag{2.2}$$

where the functions $\omega_1''(x), \omega_2''(x)$ *are Hölder-continuous with exponent* $\varepsilon > 0$.

Then the hypersingular integral equation (2.1) *is approximated by the quadrature formula*:

$$R(t_j) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{u(t_i)}{|x - t_j|^{\lambda}} dx \tag{2.3}$$

and its error function $\Delta(t_j)$ *satisfies the estimate*:

$$|\Delta(t_j)| \le \left\{ \begin{array}{ll} D\left(\dfrac{h^{3-\lambda}}{\delta_j^{(5-\lambda)/2}} + \dfrac{h}{\delta_j^{(\lambda+1)/2}} \right), & \textit{for } \lambda \neq 2 \\ Dh \dfrac{|\ln h|}{\delta_j^{3/2}}, & \textit{for } \lambda = 2 \end{array} \right\} \tag{2.4}$$

where $h = (b-a)/n$, D *a constant and* δ_j *the distance of the point* t_j *from the boundary of the segment* $[\alpha, b]$.

Proof. The hypersingular integral in the left hand side of (2.1) is understood in its principal value sense: [5], [15]

$$\begin{aligned} I(t) &= \int_a^b \frac{u(x)}{|x-t|^{\lambda}} dx \\ &= \lim_{\varepsilon \to 0} \left[\int_a^{t-\varepsilon} \frac{u(x) - u(t)}{|x-t|^{\lambda}} dx + \int_{t+\varepsilon}^b \frac{u(x) - u(t)}{|x-t|^{\lambda}} dx \right. \\ &\quad \left. + u(t) \int_a^{t-\varepsilon} \frac{dx}{|x-t|^{\lambda}} + u(t) \int_{t+\varepsilon}^b \frac{dx}{|x-t|^{\lambda}} + 2u(t) \frac{\varepsilon^{-\lambda+1}}{1-\lambda} \right] \end{aligned} \tag{2.5}$$

For the numerical evaluation of the integral $I(t)$, then the following points are used:

$$x_i = a + ih,\ i = 0,\ 1,\ \ldots,\ n \quad \text{and}$$
$$t_j = a + \Big(j + \frac{1}{2}\Big)h,\ j = 0,\ 1,\ \ldots,\ n-1$$

where $h = (b-a)/n$ and the quadrature formula (2.3) is applied.

Furthermore consider the point t_j be at the distance δ_j from the boundary of the segment $[a,\ b]$, where $\delta_j \geq 5h$.

If $\Delta(t_j)$ is the error function, then it is valid:

$$\begin{aligned}|\Delta(t_j)| &= |I(t_j) - R(t_j)| - \left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \frac{u(x)-u(t_i)}{|x-t_j|^{\lambda}}dx\right| \\ &\leq \left|\sum_{s_i\in[a,b]}\int_{s_i}\frac{u(x)-u(t_i)}{|x-t_j|^{\lambda}}dx\right| + \left|\sum_{s_i\in N_h}\int_{s_i}\frac{u(x)-u(t_i)}{|x-t_j|^{\lambda}}dx\right| \\ &= \Delta_1 + \Delta_2 \end{aligned} \tag{2.6}$$

where the set N_h consists of segments $s_i = [x_i,\ x_{i+1}]$.

Beyond the above, consider the following equality to be valid:

$$\begin{aligned}&\sum_{s_i\in N_h}\int_{s_i}\frac{u'(t_j)(x-t_i)}{|x-t_j|^{\lambda}}dx \\ &= \sum_{s_i\in N_h}\int_{s_i}\frac{u'(t_j)(x-t_j)}{|x-t_j|^{\lambda}}dx + \sum_{s_i\in N_h}\int_{s_i}\frac{u'(t_j)(t_j-t_i)}{|x-t_j|^{\lambda}}dx \\ &= \Gamma_1 + \Gamma_2 \end{aligned} \tag{2.7}$$

Since the set N_h is symmetric with respect to t_j, then follows: $\Gamma_1 = \Gamma_2 = 0$.

The following formulae hold:

$$u(x) - u(t_i) - u'(t_j)(x-t_i) = u''(z_{ij})(z_i - t_j)(x-t_i) = a_{ij} \tag{2.8}$$

where $x \in [x_i,\ x_{i+1}]$, $t_i \in [x_i,\ x_{i+1}]$ and $z_{ij} \in [z_i,\ t_j]$.

On the other hand, since $u(x)$ satisfies condition (2.2), then the following inequality is valid:

$$|u''(z_{ij})| \leq D\delta_j^{(\lambda-5)/2} \tag{2.9}$$

from which follows:

$$\begin{aligned}\Delta_2 = &\left| \sum_{s_i \in N_h} \int_{s_i} \frac{u'(t_j)(x-t_i)}{|x-t_j|^\lambda} dx + \sum_{s_i \in N_h,\, i \neq j} \int_{s_i} \frac{a_{ij}(x)}{|x-t_j|^\lambda} dx \right. \\ &\left. + \int_{s_i}^{x_{j+1}} \frac{a_{jj}(x)}{|x-t_j|^\lambda} dx \right| \\ \leq D &\left[\frac{h}{\delta_j^{(5-\lambda)/2}} \int_{t_j+h/2}^{t_j+\delta_j/2-h} \frac{dx}{|x-t_j|^{\lambda-1}} \right. \\ &\left. + \frac{1}{\delta_j^{(5-\lambda)/2}} \int_{t_j-h/2}^{t_j+h/2} \frac{dx}{|x-t_j|^{\lambda-2}} \right] \end{aligned} \tag{2.10}$$

and thus:

$$\Delta_2 \leq \begin{cases} D\left(\dfrac{h^{3-\lambda}}{\delta_j^{(5-\lambda)/2}} + \dfrac{h}{\delta_j^{(\lambda+1)/2}} \right), & \text{for } \lambda \neq 2 \\ Dh \dfrac{|\ell n h|}{d_j^{3/2}}, & \text{for } \lambda = 2 \end{cases} \tag{2.11}$$

By the same way by applying (2.2), then follows inequality:

$$\begin{aligned}\Delta_1 \leq \max_s &\left\{ \left| \sum_{s_i \in [a,\,b]} \int_{s_i} \frac{g_s(x) - g_s(t_i)}{|x-t_j|^\lambda} dx \right| \right. \\ &\left. + \left| \sum_{s_i \in [a,\,b]} \int_{s_i} \frac{d_s(x) - d_s(t_i)}{|x-t_j|^\lambda} dx \right| \right\} \\ = {\Delta_1}' + {\Delta_1}'', &\quad s = 1,\, 2 \end{aligned} \tag{2.12}$$

where:

$$\begin{aligned} g_1(x) &= k_1 (x-a)^{(\lambda-1)/2} \\ g_2(x) &= k_1' (b-x)^{(\lambda-1)/2} \\ d_1(x) &= k_2 (x-a)^{(\lambda+1)/2} + \omega_1(x) \\ d_2(x) &= k_2' (b-x)^{(\lambda+1)/2} + \omega_2(x) \end{aligned} \tag{2.13}$$

Moreover, following inequality holds for the functions $g_s(x)$:

$$|g_s(x) - g_s(t_i)| \leq Dh, \quad \text{for } x \in s_i \tag{2.14}$$

and thus:

$$\Delta_1'' \leq \frac{D_1 h}{\delta_j^{(\lambda+1)/2}} \tag{2.15}$$

By denoting further by Ω_1 the set of segments $s_i \in [a, b]$ which are on the left from point t_j and by Ω_2 the set of such segments which are on the right from t_j, then follows:

$$\begin{aligned} \Delta_1{}' &\leq \left| \sum_{s_i \in \Omega_1} \int_{s_i} \frac{g_s(x) - g_s(t_i)}{|x - t_j|^{\lambda}} dx \right| + \left| \sum_{s_i \in \Omega_2} \int_{s_i} \frac{g_s(x) - g_s(t_i)}{|x - t_j|^{\lambda}} dx \right| \\ &= Z_1 + Z_2 \end{aligned} \tag{2.16}$$

On the other hand, by applying the generalized mean-value theorem we have:

$$\begin{aligned} Z_1 &= h^{(\lambda-1)/2} \left| \sum_{s_i \in \Omega_1} \left[(i + \xi_i)^{(\lambda-1)/2} - \left(i + \frac{1}{2} \right)^{(\lambda-1)/2} \right] \int_{s_i} \frac{dx}{|x - t_j|^{\lambda}} \right| \\ &\leq \frac{D h^{(\lambda-1)/2}}{m} \sum_{i=0}^{m} \frac{1}{(i+1)^{(3-\lambda)/2}} \int_{\alpha}^{t_j - \delta_j/2 + h} \frac{dx}{|x - t_j|^{\lambda}} \end{aligned} \tag{2.17}$$

where m is the number of segments that belong to Ω_1, $mh > \delta_j/2 - h$ and $0 < \xi_i < 1$.

Thus inequality (2.17) reduces to:

$$Z_1 \leq \frac{D_1 h}{\delta_j^{(\lambda+1)/2}} \tag{2.18}$$

By the same way can be proved a similar inequality for Z_2. Hence, from inequalities (2.15) and (2.18) follows:

$$\Delta_1 \leq \frac{D_1 h}{\delta_j^{(\lambda-1)/2}} \tag{2.19}$$

and the estimate (2.4) is proved. □

3. Error estimates for hypersingular integral equations

Theorem 3.1 *Consider the hypersingular integral equation* (2.1) *where* $f(t) \in C^{\infty}[a, b]$, *with an approximate solution* $u_h(t_i)$ *given from the system:*

$$\sum_{i=0}^{n-1} u_h(t_i) \int_{x_i}^{x_i+1} \frac{dx}{|x-t_j|^\lambda} = f(t_j), \quad j = 0, 1, \ldots, n-1 \tag{3.1}$$

Then the values $u(t_k)$ *of an exact solution to* (2.1) *and the values* $u_h(t_k)$ *of the approximate solution obtained from* (3.1) *satisfy the following inequalities*:

$$\begin{aligned} &|u(t_k) - u_h(t_k)| \le Dh^{(\lambda-1)/2}, \quad 1 < \lambda < 2 \\ &|u(t_k) - u_h(t_k)| \le Dh|\ln h|^2, \quad \lambda = 2 \\ &|u(t_k) - u_h(t_k)| \le Dh^{(3-\lambda)/2}, \quad 2 < \lambda < 3 \end{aligned} \tag{3.2}$$

for $k = 0, 1, \ldots, n-1$, *where* $h = (b-a)/n$ *and* D *a constant.*

Proof. Let the system of equations:

$$\begin{aligned} &\sum_{i=0}^{n-1} u_h(t_i) c_{ij} = f(t_j) \\ &\sum_{i=0}^{n-1} u(t_i) c_{ij} = f(t_j) + \Delta f, \quad j = 0, 1, \ldots, n-1 \end{aligned} \tag{3.3}$$

where Δf is the error of the quadrature formula.

From (3.3) one obtains:

$$\sum_{i=0}^{n-1} [u(t_i) - u_h(t_i)] d_{ij} = \Delta f(t_j) \tag{3.4}$$

and thus:

$$\Delta u(t_k) = u(t_k) - u_h(t_k) = \sum_{\ell=0}^{n-1} x_{k\ell} \Delta f(t_\ell) \tag{3.5}$$

The general Fourier operator will be further used:

$$\frac{1}{2\pi} \int_0^{2\pi} H(\varphi) x(\varphi) M_n(y-\varphi) d\varphi = F(y) \tag{3.6}$$

where:

$$H(\varphi) = \sum_{\ell=-\infty}^{\infty} d_{\ell 0} e^{i\ell\varphi}$$

$$x(\varphi) = \sum_{\ell=0}^{n-1} u_h(t_\ell) e^{i\ell\varphi}$$
$$F(y) = \sum_{\ell=0}^{n-1} f(t_\ell) e^{i\ell\varphi} \tag{3.7}$$
$$M_n(y) = \sum_{\ell}^{n-1} e^{i\ell y}$$
$$d_{ij} = \int_{x_i}^{x_i+1} \frac{dx}{|x - t_j|^\lambda}.$$

Furthermore eqn (3.6) reduces to the following equation:

$$\frac{1}{2\pi} \int_0^{2\pi} H_n(\varphi) x(\varphi) M_n(y - \varphi) d\varphi = F(y) \tag{3.8}$$

in which:

$$H_n(\varphi) = \sum_{\ell=-n}^{n} d_{\ell 0} e^{i\ell\varphi} \tag{3.9}$$

Also in order to be obtained estimates for the inverse matrix, an approximation formula is necessary for the Fourier coefficients $x_{k\ell}^{(n)}$ of the function $x_k^{(\varphi)} = \sum_{\ell=0}^{n-1} x_{k\ell}(n) e^{i\ell\varphi}$, where x_k denotes a solution of the following equation:

$$\frac{1}{2\pi} \int_0^{2\pi} H_n(\varphi) x_k(\varphi) M_n(y - \varphi) d\varphi = e^{iky}, \qquad k = 0, 1, \ldots, n-1 \tag{3.10}$$

Let us consider the equation:

$$\frac{1}{2\pi} \int_0^{2\pi} H_n(\varphi) g_k(\varphi) M_n(y - \varphi) d\varphi = e^{iky} \tag{3.11}$$

where:

$$H_n(\varphi) = \sum_{\ell=-n}^{n} a_\ell e^{i\ell\varphi}$$
$$a_\ell = -c_{\ell_0} h^{\lambda-1} \tag{3.12}$$

and the solutions $x_k(\varphi)$ and $g_k(\varphi)$ are related by the following formula:

$$x_k(\varphi) = -g_k(\varphi)h^{\lambda-1} \tag{3.13}$$

Moreover in order to study the properties of the function $H_n(\varphi)$ consider the set of segments $[\pi/(16n),\ \pi/(8n)]$, $n = 1, 2, \ldots$. which forms a covering of the half - open interval $(0,\ \pi/8]$. By choosing an arbitrary $x \in (0,\ \pi/8]$, then there exists a minimal number N so that $x \in [\pi/(16N),\ \pi/(8N)]$.

For $\pi/(16N) \le \varphi \le \pi/(8N)$ and $N \le \ell \le 2N$ the inequality $\sin^2(\ell\varphi/2) > \delta > 0$ is obtained, from which follows:

$$H(N,\ \varphi) = -2\sum_{\ell=N}^{2N} a_\ell \sin^2\Big(\frac{\ell\varphi}{2}\Big) \ge 2\delta \sum_{\ell=N}^{2N} |a_e| \ge D\varphi^{\lambda-1} \tag{3.14}$$

By choosing $D > 0$ not depending on N, then from (3.14) follows that inequality:

$$H(\varphi) > D(\varphi)^{\lambda-1} \tag{3.15}$$

holds for $0 \le \varphi \le \pi/8$.

Similarly following inequality:

$$H(\varphi) > D(2\pi - \varphi)^{\lambda-1} \tag{3.16}$$

holds for $15\pi/8 \le \varphi \le 2\pi$.

Beyond the above consider $H_n(\varphi) = a_0 + \sum_{\ell=1}^{n} a_\ell \cos(\ell\varphi)$. Because of (3.16), inequality $H_n(\varphi) - H_n(0) > D\varphi^{\lambda-1}$ holds for $\pi/(16n) \le \varphi \le \pi/8$.

From the inequality

$$H_n(0) \ge D_1 \sum_{\ell=n+1}^{\infty} \ell^{-\lambda} \ge D_2(n+1)^{1-\lambda}$$

follows that there exists a constant $B_0 > 0$ not depending on n and for which the inequality $H_n(0) \ge B_0\varphi^{\lambda-1}$ holds for $0 \le \varphi \le \pi/(16n)$.

Thus by setting $B_0{}^* = \min(B_0/2,\ D)$ one has:

$$H_n(\varphi) \ge \frac{1}{2}H_n(0) + B_0{}^*\varphi^{\lambda-1} \tag{3.17}$$

for $0 \le \varphi \le \pi/8$.

Consider further the linear space E spanned by the functions $e^{i\ell_\varphi}$, $\ell =$

0, 1, ..., $n-1$ and having the following norms:

$$\begin{aligned} \|x\|_1 &= \left(\int_0^{2\pi} H_n(\varphi)|x(\varphi)|^2 d\varphi\right)^{1/2} \\ \|x\|_2 &= \max_{\|y(\varphi)\|=1} \left|\int_0^{2\pi} x(\varphi)\overline{y}(\varphi)d\varphi\right| \end{aligned} \tag{3.18}$$

By $E(\|\cdot\|_1)$ and $E(\|\cdot\|_2)$ we denote the linear spaces E having the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively.

Furthermore the operator $\Psi(x) = (2\pi)^{-1}\int_0^{2\pi} H_n(\varphi)x(\varphi)B_n(\Psi-\varphi)d\varphi$ maps isometrically the space $E(\|\cdot\|_1)$ into $E(\|\cdot\|_2)$.

Thus following inequality holds:

$$\|e^{ik\varphi}\|_2 \le D\left(\int_0^{2\pi} \frac{d\varphi}{H_n(\varphi)}\right)^{1/2} \tag{3.19}$$

From (3.17) and (3.19) follows:

$$\|g_k(\varphi)\|_1 = \|e^{ik\varphi}\|_2 \le \begin{cases} D, & \text{for} \quad 1<\lambda<2 \\ D\sqrt{|\ell n h|}, & \text{for} \quad \lambda=2 \\ D/h^{(\lambda-2)/2} & \text{for} \quad 2<\lambda<3 \end{cases} \tag{3.20}$$

In order to calculate the Fourier cofficients $x(\varphi)$ of the functions $x(\varphi)$, $\|x(\varphi)\|_1 = 1$, one has:

$$\begin{aligned} |x_\ell| &= \frac{1}{2\pi}\left|\int_0^{2\pi} x(\varphi)e^{-i\ell\varphi}d\varphi\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{b(\varphi)}{\sqrt{H_n}(\varphi)}d\varphi \le D\left(\int_0^{2\pi}\frac{d\varphi}{H_n(\varphi)}\right)^{1/2} \end{aligned} \tag{3.21}$$

and thus $|x_\ell|$ satisfy estimates which are similar to (3.20).

Hence, the Fourier coefficients $x_{k\ell}$ of the functions $x_\chi(\varphi)$ which are solutions of the problem (3.10), $x_k(\varphi) = g_k(\varphi)h^{\lambda-1}$ satisfy following inequalities:

$$\begin{aligned} &|x_{k\ell}| \le Dh^{\lambda-1}, \quad 1<\lambda<2 \\ &|x_{k\ell}| \le Dh|\ell n h|, \quad \lambda=2 \\ &|x_{k\ell}| \le Dh, \quad 2<\lambda<3 \end{aligned} \tag{3.22}$$

As $x_{k\ell}$ belong to the kth row of the inverse matrix, then equations

(3.22) denote estimates for the elements of the inverse matrix for (3.1).

Finally from eqs (2.4) and (3.22) we obtain:

$$|\Delta u(t_k)| \leq \sum_{j=0}^{n-1} Dh^{\lambda-1}\left[\frac{h^{3-\lambda}}{\delta_j^{(5-\lambda)/2}} + \frac{h}{\delta_j^{(\lambda+1)/2}}\right]$$

$$\leq D_1\left[\sum_{j=0}^{n/2} \frac{h^2}{h^{(5-\lambda)/2}(j+1)^{(5-\lambda)/2}} + \sum_{j=0}^{n/2} \frac{h^\lambda}{h^{(1+\lambda)/2}(j+1)^{(\lambda+1)/2}}\right]$$

$$\leq D_2 h^{(\lambda-1)/2} \tag{3.23}$$

Also, the proof for $\lambda = 2$ and $2 < \lambda < 3$ is done by the same way and thus the Theorem is proved. □

4. Conclusions

An approximation method has been proposed for the numerical evaluation of the hypersingular integral equations defined in Banach spaces. The hypersingular integral equations are a very special class of singular integral equations having kernels with very strong singularities, as compared to the known finite-part singular integral equations.

The numerical method which was used is an extension beyond the quadrature method. It was therefore proved that the quadrature method is a suitable approximation method for the numerical solution of the hypersingular integral equations. Same method has been successfully used in the past for the numerical evaluation of the non-linear singular integral equations [18]–[28].

Also, an error estimates theory was proposed for the hypersingular integral equations, by proving the suitable theorems. Thus it was shown that same inequalities are valid between the exact solutions of the hypersingular integral equations and the corresponding numerical solutions.

Finally, the hypersingular integral equations are very important for the solution of basic problems of engineering mechanics and mathematical physics, like for example problems of elasticity, fracture mechanics, fluid mechanics and aerodynamics. Hence, there is a big interest for further research on the numerical evaluation of the hypersingular integral equations.

References

[1] Hadamard J., *Lectures on Cauchy's Problem in Linear Partial Differential Equations*. Yale University Press, Yale, 1932.

[2] Hadamard J., *Le Probléme de Cauchy et les Équations aux Derivées Partielles Linéaires Hyperboliques*. Hermann, Paris, 1932.

[3] Schwartz L., *Théorie des Distributions*. Hermann, Paris, 1966.

[4] Kutt H.R., *The numerical evaluation of principal value integrals, by finite-part integration*. Num. Math. **24** (1975), 205–210.

[5] Golberg M.A., *The convergence of several algorithms for solving integral equations with finite-part integrals*. J. Int. Eq. **5** (1983), 329–340.

[6] Golberg M.A., *The numerical solution of Cauchy singular integral equations with constant coefficients*. J. Int. Eq. **9** (1985), 127–151.

[7] Kaya A.C. and Erdogan F., *On the solution of integral equations with strongly singular kernels*. Q. Appl. Math. **45** (1987), 105–122.

[8] Kaya A.C. and Erdogan F., *On the solution of integral equations with a generalized Cauchy kernel*. Q. Appl. Math. **45** (1987), 455–469.

[9] Ladopoulos E.G., *On the numerical solution of the finite-part singular integral equations of the first and the second kind used in fracture mechanics*. Comp. Meth. Appl. Mech. Engng **65** (1987), 253–266.

[10] Ladopoulos E.G., *On the numerical evaluation of the general type of finite-part singular integrals and integral equations used in fracture mechanics*. J. Engng Fract. Mech. **31** (1988), 315–337.

[11] Ladopoulos E.G., *The general type of finite-part singular integrals and integral equations with logarithmic singularities used in fracture mechanics*. Acta Mech. **75** (1988), 275–285.

[12] Ladopoulos E.G., *Finite-part singular integro-differential equations arising in two-dimensional aerodynamics*. Arch. Mech. **41** (1989), 925–936.

[13] Ladopoulos E.G., *Systems of finite-part singular integral equations in L_p applied to crack problems*. J. Engng Fract. Mech. **48** (1994), 257–266.

[14] Ladopoulos E.G., *New aspects for the generalization of the Sokhotski-Plemelj formulae for the solution of finite-part singular integrals used in fracture mechanics*. Int. J. Fract. **54** (1992), 317–328.

[15] Ladopoulos E.G., *Singular Integral Equations, Linear and Non-linear Theory and its Applications in Science and Engineering*. Springer-Verlag, Berlin, New York, 2000.

[16] Ladopoulos E.G., Zisis V.A. and Kravvaritis D., *Singular integral equations in Hilbert space applied to crack problems*. Theor. Appl. Fract. Mech. **9** (1988), 271–281.

[17] Ladopoulos E.G., Kravvaritis D. and Zisis V.A., *Finite-part singular integral representation analysis in L_p of two-dimensional elasticity problems*. J. Engng Fract. Mech. **43** (1992), 445–454.

[18] Ladopoulos E.G., *Non-linear integro-differential equations used in orthotropic shallow spherical shell analysis*. Mech. Res. Commun. **18** (1991), 111–119.

[19] Ladopoulos E.G., *Non-linear integro-differential equations in sandwich plates stress analysis*. Mech. Res. Commun. **21** (1994), 95–102.

[20] Ladopoulos E.G., *Non-linear singular integral computational analysis for unsteady flow problems*. Renew. Energy **6** (1995), 901–906.

[21] Ladopoulos E.G., *Non-linear singular integral representation for unsteady inviscid flowfields of 2-D airfoils*. Mech. Res. Commun. **22** (1995), 25–34.

[22] Ladopoulos E.G., *Non-linear singular integral representation analysis for inviscid flowfields of unsteady airfoils*. Int. J. Non-Lin. Mech. **32** (1997), 377–384.

[23] Ladopoulos E.G., *Collocation approximation methods for non-linear singular integro-differential equations in Banach spaces*. J. Comp. Appl. Math. **79** (1997), 289–297.

[24] Ladopoulos E.G., *Non-linear multidimensional singular integral equations in 2-dimensional fluid mechanics analysis*. Int. J. Non-Lin. Mech. **35** (2000), 701–708.

[25] Ladopoulos E.G., *Non-linear singular integral equations in elastodynamics, by using Hilbert transformations*. Nonlin. Anal., Real World Appl. **6** (2005), 531–536.

[26] Ladopoulos E.G. and Zisis V.A., *Non-linear singular integral approximations in Banach spaces*. Nonlin. Anal., Th. Meth. Appl. **26** (1996), 1293–1299.

[27] Ladopoulos E.G. and Zisis V.A., *Existence and uniqueness for non-linear singular integral equations used in fluid mechanics*. Appl. Math. **42** (1997), 345–367.

[28] Ladopoulos E.G. and Zisis V.A., *Non-linear finite-part singular integral equations arising in two-dimensional fluid mechanics*. Nonlin. Anal., Th. Meth. Appl. **42** (2000), 277–290.

E.G. Ladopoulos
Interpaper Research Organization
56, Anagnostopoulou Str.
Athens, GR - 106 72, Greece

V.A. Zisis
Department of Applied Mathematics
the National Technical University of Athens
Athens GR - 157 73, Greece