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On the connected components of a global semianalytic subset of an analytic surface

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Abstract. A global semianalytic subset of a real analytic manifold is a finite union of finite intersections of the solutions of equations and inequalities of real analytic functions on the manifold. Is a union of connected components of a global semianalytic set again global semianalytic? We consider a two-dimensional global semianalytic set such that the normalization of the Zariski closure of it is affine. We show that a union of connected components of it is again global semianalytic. We also give some partial results on connected components of global semianalytic subset of a three-dimensional analytic manifold.

Key words: Global Semianalytic Set.

1. Introduction

A global semianalytic sets are recently studied by real algebraic and analytic geometers, for instance, [2], [3], [4], [5], [8], [10], [11] and [19]. A subset X of a real analytic manifold M is called *global semianalytic* if there exist finitely many real analytic functions f_i, g_{ij} on M with

$$X = \bigcup_{i=1}^{m} \{x \in M;\ f_i(x) = 0,\ g_{i1}(x) > 0, \dots,\ g_{in}(x) > 0\}.$$

Consider the case where M is a connected paracompact real analytic manifold and let N be a coherent analytic subset of M. The notation $\mathcal{O}_M$ denotes the sheaf of real analytic functions on M and $\mathcal{I}_N \subset \mathcal{O}_M$ denotes the sheaf of real analytic functions vanishing on N. A subset X of N of the form

$$X = \bigcup_{i=1}^{m} \{x \in N;\ f_i(x) = 0,\ g_{i1}(x) > 0, \dots,\ g_{in}(x) > 0\}$$

is also called *global semianalytic*. Here f_i and g_{ij} are global sections of the

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sheaf $\mathcal{O}_M/\mathcal{I}_N$. A global semianalytic subset of N is also a global semianalytic subset of M by Cartan's theorem A and B ([9]). We consider global semianalytic sets in the present paper.

It is well-known for real algebraic and analytic geometers that a connected component and the closure of a semialgebraic set are again semialgebraic ([6]) and that the similar statement holds true for semianalytic sets ([16]). As in the semialgebraic case and semianalytic case, it is natural to study the problem whether the closure and a union of connected components of a global semianalytic set is again global semianalytic. Indeed, it is already known that any union of connected components and the closure of a boundary bounded global semianalytic subset of M are again global semianalytic ([1], [19]). In the non-compact case, Andradas and Castilla showed that any union of connected components of a global semianalytic subset of M is again global semianalytic when $\dim(M) \leq 2$ and so is the closure when $\dim(M) \leq 3$ ([2], [11]). In the present paper, we consider the problem whether any union of connected components of a global semianalytic set is again global semianalytic.

We first consider a connected paracompact real analytic manifold M and a coherent real analytic subset A of M. Let $A^{\mathbb{C}}$ be the complexification of A. Consider the (complex) normalization $\pi\colon \widehat{A^{\mathbb{C}}} \to A^{\mathbb{C}}$ defined in [14, Section 8.3]. Then $A' := \pi^{-1}(A)$ is a coherent and normal real analytic variety whose complexification is $\widehat{A^{\mathbb{C}}}$ ([15, Theorem IV.3.14]). We call A' (real) *normalization* of A. A real analytic space A' is called *affine* if there exists a closed embedding of A' into $\mathbb{R}^n$ for some $n \in \mathbb{N}$. We consider the real analytic space A of dimension 2 whose normalization is affine. The following theorem is our first main theorem.

Theorem 1.1 *Let M be a connected paracompact real analytic manifold. Let $A \subset M$ be a coherent analytic subset of dimension ≤ 2. Assume that the normalization of A is affine. Consider a global semianalytic subset X of A and a union W of connected components of X. Then W is again a global semianalytic subset of M.*

We next consider a 3-dimensional connected paracompact real analytic manifold. The *Zariski closure* of a subset A of a real analytic manifold M means the smallest coherent analytic subset of M containing A in the present paper. The second main theorem is as follows.

Theorem 1.2 *Let M be a connected paracompact real analytic manifold of dimension 3 and X be a global semianalytic subset of M. Consider a union W of connected components of X. Assume that $\dim(\overline{W} \cap \overline{X \setminus W}) \leq 1$ and the Zariski closure of $\overline{W} \cap \overline{X \setminus W}$ has at most finitely many analytically irreducible components of dimension 1. Here $\overline{\cdot}$ denotes the closure in M. Then W is again global semianalytic.*

We mainly consider a coherent real analytic subset in the present paper. Hence we simply call a coherent analytic subset *analytic*. The notation $\overline{X}$ denotes the closure of a set X. We consider the sheaf of real analytic functions on a real analytic manifold M in the present paper. The symbol $\mathcal{O}_M$ denotes it and $\mathcal{O}_{M,x}$ is the stalk of $\mathcal{O}_M$ at $x \in M$. When X is a coherent real analytic subset of M, $\mathcal{I}_X$ denotes the ideal sheaf of all real analytic functions on M vanishing on X. The symbol $\mathcal{I}_{X,x}$ also denotes the stalk at $x \in M$. The global sections $H^0(M, \mathcal{O}_M/\mathcal{I}_X)$ is denoted by $C^\omega(X)$. The notation $\overline{U}^Z$ represents the Zariski closure of U.

When M is a connected paracompact real analytic manifold, M is a closed real analytic submanifold of a Euclidean space by [13]. Hence Cartan's theorem A and B hold true in M by [9]. We use this fact in the proof of the present paper in many times.

2. Low Dimensional Case

Disjoint closed subanalytic subsets of a connected paracompact real analytic manifold are separated by a single real analytic function. More precisely,

Lemma 2.1 *Let M be a connected paracompact real analytic manifold. Let X and Y be disjoint closed subanalytic subsets of M. Then there exists a real analytic function f with $f > 0$ on X and $f < 0$ on Y.*

Proof. We may assume that M is a closed real analytic submanifold of a Euclidean space $\mathbb{R}^q$ by [13]. There exists a continuous function $d\colon M \to \mathbb{R}$ with $d(x) = 1$ for all $x \in X$ and $d(y) = -1$ for all $y \in Y$. Indeed, define d_1 and d_2 as the distance functions from X and Y, respectively, then $d := (d_2 - d_1)/(d_2 + d_1)$ satisfies the requirement. Let $\widetilde{d}$ be a continuous extension of d to $\mathbb{R}^q$. Then there exists a real analytic function f on $\mathbb{R}^q$ with $|f(x) - \widetilde{d}(x)| < 1/2$ for all $x \in \mathbb{R}^q$ by [21]. Then f satisfies the requirement. $\square$

We next introduce the results of [2] and [3].

Lemma 2.2 *Let M be a connected paracompact real analytic manifold. A semianalytic subset of an analytic set of dimension 1 is a global semianalytic subset of M.*

Proof. See [2, Lemma 3.1]. □

Remark 2.3 Let M be a connected paracompact real analytic manifold. An open semianalytic set and an open global semianalytic set are strictly open [3].

The following two lemmas are generalizations of the lemmas of [2].

Lemma 2.4 *Let M be a real analytic manifold and $p \in M$. Let F_p and F'_p be closed semianalytic germs at p such that $F_p \cap \overline{F'_p} = \{p\}$. Then there exists an open germ G_p with $F'_p \subset G_p \cup \{p\}$ and $F_p \cap \overline{G_p} = \{p\}$.*

Proof. Since the statement of this lemma is local, we may assume that $M = \mathbb{R}^q$. There exist germs f_{ij} of real analytic functions with

$$F'_p = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} \{f_{ij} \geq 0\}$$

by [1, Corollary VIII.3.2].

We fix $i = 1, \ldots, m$ and set $F'_{ip} := \bigcap_{j=1}^{n} \{f_{ij}(x) \geq 0\}$. Let d be the germ of the real analytic function defined by $d(x_1, \ldots, x_q) = \sum_{l=1}^{q} (x_l - p_l)^2$, where $p = (p_1, \ldots, p_q)$. Consider the subanalytic function $\phi_i \colon [0, r) \to \mathbb{R}$ defined by

$$\phi_i(t) = \max\bigl\{\min\{f_{i1}(x), \ldots, f_{in}(x)\};\ x \in F_p,\ d(x) = t\bigr\},$$

where r is a small positive number. By the definition, $\phi_i(0) = 0$ and $\phi_i(t) < 0$ for all $0 < t < r$. Since a 1-dimensional subanalytic set is semianalytic, ϕ_i is semianalytic. Hence, considering the Puiseux series expansion of the graph of ϕ_i, it is obvious that there exists a positive number C_i and a natural number n_i with $\phi_i(t) + C_i t^{n_i} < 0$ for all $0 < t < r$. Set $g_{ij}(x) := f_{ij}(x) + C_i d(x)^{n_i}$. Then $g_{ij}(x) > 0$ for all $x \in F'_{ip} \setminus \{p\}$ and, for all $x \in F_p \setminus \{p\}$, there exist $j = 1, \ldots, n$ with $g_{ij}(x) < 0$. Consider the open semianalytic germ

$$G_{ip} := \bigcap_{j=1}^{n} \{g_{ij} > 0\}.$$

Then $F'_{ip} \subset G_{ip} \cup \{p\}$ and $F_p \cap \overline{G_{ip}} = \{p\}$.

We can construct G_{ip} in the same way for another $i = 1, \ldots, m$. Set $G_p := \bigcup_{i=1}^{m} G_{ip}$. Then G_p satisfies the requirement. □

Lemma 2.5 *Let M, F_p, F'_p and G_p be the same as in* Lemma 2.4*. Let*

$$G_p := \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} \{g_{ij} > 0\},$$

where $g_{ij} \in \mathcal{O}_{M,p}$ are the germs constructed in the proof of Lemma 2.4*. Then there exists a natural number μ such that, if $g'_{ij} - g_{ij} \in m_p^{\mu}$, then G'_p satisfies the conditions on G_p in* Lemma 2.4*, where m_p is the maximal ideal of $\mathcal{O}_{M,p}$ and*

$$G'_p := \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} \{g'_{ij} > 0\}.$$

Proof. We define a real analytic function d as in the proof of Lemma 2.4. Consider the subanalytic function $\phi\colon [0, r) \to \mathbb{R}$ defined by

$$\phi(t) := \min\Big\{ \max_{i=1,\ldots m} \min_{j=1,\ldots,n} g_{ij}(x);\ x \in F'_p,\ d(x) = t \Big\},$$

where r is a small positive number. Then $\phi(0) = 0$ and $\phi(t) > 0$ for all $0 < t < r$. In the same way as the proof of Lemma 2.4, there exists $C_1 > 0$ and $q_1 \in \mathbb{N}$ such that $\phi(t) > C_1 t^{q_1}$ for any sufficiently small $t > 0$. Remark that, for any $f \in m_p^{2q_1+1}$, $C_1 d(x)^{q_1} > |f|$ on a small neighborhood of p. Hence, if $g'_{ij} - g_{ij} \in m_p^{2q_1+1}$, $F'_p \subset G'_p \cup \{p\}$.

Remember that, for all $i = 1, \ldots, m$ and for all $x \in F_p \setminus \{p\}$, there exist $j = 1, \ldots, n$ with $g_{ij}(x) < 0$. Consider the subanalytic function $\psi\colon [0, r) \to \mathbb{R}$ defined by

$$= \psi(t) : \min\Big\{ \min_{i=1,\ldots m} \max_{j=1,\ldots,n} -g_{ij}(x);\ x \in F_p,\ d(x) = t \Big\},$$

where r is a small positive number. Then $\psi(0) = 0$ and $\psi(t) > 0$ for all $0 < t < r$. In the similar way, we can find $q_2 \in \mathbb{N}$ such that, if $g'_{ij} - g_{ij} \in m_p^{2q_2+1}$, for all $i = 1, \ldots, m$ and for all $x \in F_p \setminus \{p\}$, there exist $j = 1, \ldots, n$ with $g_{ij}(x) < 0$. Hence, if $g'_{ij} - g_{ij} \in m_p^{2q_2+1}$, $F'_p \cap \overline{G'_p} = \{p\}$.

Set $\mu = \max\{2q_1+1, 2q_2+1\}$, then μ satisfies the requirement. □

By generalizing the argument of [2], we can show that a union of connected components of a global semianalytic set is again global semianalytic when the intersection of the closure of disjoint two components is of dimension 0 or less.

Proposition 2.6 *Let M be a connected paracompact real analytic manifold and X be a global semianalytic subset of M. Let W be a union of connected components of X. Assume that* $\dim(\overline{W} \cap \overline{X \setminus W}) \leq 0$. *Then W is again a global semianalytic subset of M.*

Moreover, there exist natural numbers q_1, q_2 depending only on the dimension of the Zariski closure of X and real analytic functions f_{ij} on M such that

$$W = X \cap \bigcup_{i=1}^{q_1} \bigcap_{j=1}^{q_2} \{x \in M;\ f_{ij}(x) > 0\}.$$

Proof. Let $\{x_n\}_{n\in\mathbb{N}} = \overline{W} \cap \overline{X \setminus W}$. There exist open semianalytic germs G_n at x_n with $\overline{W} \subset G_n \cup \{x_n\}$ and $\overline{X \setminus W} \cap \overline{G_n} = \{x_n\}$ by Lemma 2.4. There exist p, $q \in \mathbb{N}$ depending only on the dimension of the Zariski closure of X satisfying the following condition by [1, Theorem VIII.2.12, Corollary VIII.3.2]. There exist real analytic germs g_{ijn} at x_n for $i = 1, \ldots, p$ and $j = 1, \ldots, q$ with

$$G_n = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{g_{ijn} > 0\}.$$

Applying Lemma 2.5 in the case where $F'_p = \overline{W}$, $F_p = \overline{X \setminus W}$ and $p = x_n$, there exists $\mu_n \in \mathbb{N}$ such that, if $g_{ijn} - g'_{ijn} \in m_{x_n}^{\mu_n}$, then $\overline{W} \subset G'_n \cup \{x_n\}$ and $\overline{X \setminus W} \cap \overline{G'_n} = \{x_n\}$, where m_{x_n} is the maximal ideal of $\mathcal{O}_{M,x_n}$ and

$$G'_n = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{g'_{ijn} > 0\}.$$

Let $\mathcal{M}$ be the coherent sheaf of ideals of $\mathcal{O}_M$ defined by

$$\mathcal{M}_x := \begin{cases} m_{x_n}^{\mu_n} & \text{if } x = x_n, \\ \mathcal{O}_x & \text{otherwise.} \end{cases}$$

We define global sections s_{ij} of $\mathcal{O}_M/\mathcal{M}$ by

$$s_{ij,x} := \begin{cases} g_{ijn} + m_{x_n}^{\mu_n} & \text{if } x = x_n, \\ \mathcal{O}_x & \text{otherwise.} \end{cases}$$

Since $H^0(M, \mathcal{O}_M) \to H^0(M, \mathcal{O}_M/\mathcal{M})$ is surjective by Cartan's theorem B, there exist real analytic functions h_{ij} on M with $h_{ij} - g_{ijn} \in m_{x_n}^{\mu_n}$ for all $n \in \mathbb{N}$. Consider the global semianalytic set

$$H := \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in M;\, h_{ij}(x) > 0\}.$$

Then $\overline{W \cap H} \cap \overline{X \cap H \setminus W} = \emptyset$ and $\overline{W \setminus H} \cap \overline{X \setminus W} = \emptyset$. There exist real analytic functions h_1 and h_2 on M with $h_1 > 0$ on $W \cap H$, $h_1 < 0$ on $X \cap H \setminus W$, $h_2 > 0$ on $W \setminus H$ and $h_2 < 0$ on $X \setminus W$ by Lemma 2.1. Then $W = X \cap (H \cap \{x \in M;\, h_1(x) > 0\} \cup \{x \in M;\, h_2(x) > 0\})$. We have finished the proof of this proposition. □

3. Proof of Theorem 1.1

We will show Theorem 1.1 in the present section. We first show the following technical lemma.

Lemma 3.1 *Let M be a connected paracompact real analytic manifold and Y be a coherent real analytic subset of M. Let X be a global semianalytic subset of M with $X \subset Y$ and V be a union of connected components of X. Set $W := X \setminus V$. Assume that* $\mathrm{Sing}(Y) \cap (\overline{\overline{V} \cap \overline{W}}^{Z})$ *is of dimension* $< \dim(Y) - 1$. *Here* $\mathrm{Sing}(Y)$ *denotes the singular locus of Y. Then there exists a real analytic function f on M such that*

$$\dim(\overline{V \cap \{f > 0\}} \cap \overline{W \cap \{f > 0\}}) \leq \dim(Y) - 2 \quad \textit{and}$$
$$\dim(\overline{V \cap \{f < 0\}} \cap \overline{W \cap \{f < 0\}}) \leq \dim(Y) - 2.$$

Proof. Set $Z := (\overline{\overline{V} \cap \overline{W}}^{Z})$ and let T be the union of all analytically irreducible components of Z of dimension $\leq \dim(Y) - 2$. The constant function $f \equiv 1$ satisfies the inequalities of this lemma in the case where $\dim(Z) < \dim(Y) - 1$. Hence we only consider the case where $\dim(Z) = \dim(Y) - 1$. Set $Y' := Y \setminus (\mathrm{Sing}(Y) \cup T \cup \mathrm{Sing}(Z))$, then Y' is a real analytic manifold. Here $\mathrm{Sing}(Z)$ denotes the singular locus of Z. We set

$\mathcal{O}_Y := \mathcal{O}_M/\mathcal{I}_Y$ and $\mathcal{I}_Z^Y := \mathcal{I}_Z/\mathcal{I}_Y$. Remark that $\mathcal{O}_{Y'}$ is isomorphic to $\mathcal{O}_Y|_{Y'}$. Hence the stalk $\mathcal{I}_{Z,x}^Y$ at $x \in Y'$ is a principal ideal of $\mathcal{O}_{Y,x}$ because $\mathcal{O}_{Y',x}$ is an unique factorization domain. By [12], $\mathcal{I}_Z^Y(Y')$ is finitely generated. Since $\mathcal{I}_Z^Y(Y')$ is generated by $\mathcal{I}_Z^Y(Y) := H^0(M, \mathcal{I}_Z/\mathcal{I}_Y)$ by Cartan's theorem A, we can choose generators $f_1, \dots, f_k \in H^0(M, \mathcal{O}_Y)$ of $\mathcal{I}_Z^Y(Y')$. Consider the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_M \to \mathcal{O}_Y \to 0.$$

There exist real analytic functions g_i on M such that $f_i - g_i \in \mathcal{I}_Y(M)$ by Cartan's theorem B. Replacing f_i with g_i, we may assume that the generators $f_1, \dots, f_k$ are all real analytic functions on M.

We will choose $f \in C^\omega(M)$ and a real analytic subset D of Z with $\dim(D) \leq \dim(Y) - 2$ such that $\mathcal{I}_{Z,x}^Y$ is generated by f for all $x \in Z \setminus D$. Let $\{Z_i\}_{i\in\mathbb{N}}$ be the analytically irreducible components of Z of dimension $\dim(Y) - 1$. Let $x_n \in Y' \cap \left(Z_n \setminus \bigcup_{i\neq n} Z_i\right)$. Consider the set

$$F_n = \bigcup_{n=1}^{\infty} \left\{ (a_1, \dots, a_k) \in \mathbb{R}^k;\ \left(\sum_{j=1}^{k} a_j f_j\right) \mathcal{O}_{Y,x_n} \neq \mathcal{I}_{Z,x_n}^Y \right\}.$$

Write $F = \bigcup_n F_n$. As $\mathcal{I}_{Z,x_n} = (f_1, \dots, f_k)_{x_n}$, using Nakayama lemma, we see that F_n is a proper analytic set, and then by the Baire theorem, F is proper. Since F is proper, there exists $(a_1, \dots, a_k) \in \mathbb{R}^k$ such that $\sum_{i=1}^k a_i f_i$ generates $\mathcal{I}_{Z,x_n}^Y$ for all $n \in \mathbb{N}$. Set $f := \sum_{i=1}^k a_i f_i$. For any analytically irreducible component Z_i, the set

$$D_i := \{x \in Z_i;\ f\mathcal{O}_{Y,x} \neq \mathcal{I}_{Z,x}^Y\}$$

is a real analytic subset of Z_i. Since $x_i \notin D_i$ by the definition, D_i is of smaller dimension than Z_i. Set

$$D := (\mathrm{Sing}(Y) \cap Z) \cup T \cup \mathrm{Sing}(Z) \cup \left(\bigcup_{n\in\mathbb{N}} D_n\right) \cup \left(\bigcup_{i\neq j} Z_i \cap Z_j\right).$$

Then f and D satisfy the requirement of the claim.

We next show that $E := \overline{V \cap \{f > 0\}} \cap \overline{W \cap \{f > 0\}} \subset D$. It is obvious that $E \subset Z$ by the definition. Let $x \in Z \setminus D$. Remark that Y is a real analytic submanifold of M and Z is that of Y in a small neighborhood of x. When $x \notin \partial V \cap \partial W$, it is obvious that $x \notin E$. Consider the case where $x \in \partial V \cap \partial W$. Choose a small neighborhood U of x in M, then all subsets

$\{y \in Y \cap U;\ f(y) > 0\}$, $Z \cap U$ and $\{y \in Y \cap U;\ f(y) < 0\}$ are connected and are not empty. Since $\overline{V} \cap \overline{W} \subset Z$, either $V \cap \{f > 0\} \cap U$ or $W \cap \{f > 0\} \cap U$ is empty. Hence $x \notin E$.

We can also show that $\overline{V \cap \{f < 0\}} \cap \overline{W \cap \{f < 0\}} \subset D$ in the same way. We have finished the proof of this lemma. □

Recall the normalization of a complex analytic set [14], [17]. It is possible to define the normalization of a real analytic set in a similar way. The real normalization is compatible with the complex normalization of its complexification [15].

Proposition 3.2 *Let M be a paracompact real analytic manifold. Let $Y \subset M$ be an analytically irreducible analytic set and $\pi\colon Y' \to Y$ be its normalization. Then the quotient fields $\mathcal{K}(Y)$ of $C^\omega(Y)$ and $\mathcal{K}(Y')$ of $C^\omega(Y')$, respectively, are isomorphic.*

Proof. The normalization $\pi\colon Y' \to Y$ induces an injection $\pi^*\colon C^\omega(Y) \to C^\omega(Y')$. This can be extended to $\pi^*\colon \mathcal{K}(Y) \to \mathcal{K}(Y')$. We show that π^* is an isomorphism.

Let $M^{\mathbb{C}}$ and $Y^{\mathbb{C}}$ be the complexifications. Fix $f' \in \mathcal{O}(Y')$. For any $p \in Y$, let $Y_{1,p}, \dots, Y_{r,p}$ be the complex analytic germs of $Y_p^{\mathbb{C}}$ at p.

The notation $\mathcal{O}_{M^{\mathbb{C}},p}$ denotes the ring of all complex analytic function germs at p. For any complex analytic germ Z_p at p, $\mathcal{I}_{Z_p}^{\mathbb{C}}$ denotes the ideal of complex analytic functions vanishing on Z_p. The notation $\mathcal{I}_{Z_p,p}$ denotes the ideal of real analytic functions whose complexification vanishes on Z_p. Remark that $\mathcal{I}_{Z_p}^{\mathbb{C}} = \mathcal{I}_{Z_p} \otimes \mathbb{C}$. Remember that $\widehat{\mathcal{O}_{M^{\mathbb{C}},p}/\mathcal{I}_{Z,p}^{\mathbb{C}}} = \widehat{\mathcal{O}_{M,p}/\mathcal{I}_Y} \otimes \mathbb{C}$ and $\widehat{\mathcal{O}_{M,p}/\mathcal{I}_{Y,p}} = \widehat{\mathcal{O}_{M,p}/\mathcal{I}_{Y_{1,p},p}} \times \cdots \times \widehat{\mathcal{O}_{M,p}/\mathcal{I}_{Y_{r,p},p}}$, where $\widehat{\ }$ denotes the normalization ([18, Proposition III.2.7]). Hence there exist complex analytic set germs W_i which are the normalizations of $Y_{i,p}$ and $\bigcup_{q\in\pi^{-1}(p)}(Y')_q^{\mathbb{C}} = \bigcup_{i=1}^r W_i$. Remark that $\mathcal{I}_{Y,p} = \mathcal{I}_{Y_{1,p}} \cap \cdots \cap \mathcal{I}_{Y_{r,p}}$. Hence the quotient ring $\mathcal{K}(Y_p)$ of $\mathcal{O}_{M,p}/\mathcal{I}_{Y,p}$ is isomorphic to the direct product $\prod \mathcal{K}(Y_{i,p})$. The intersection of W_i with $\pi^{-1}(p)$ consists of one point, say p_i. Since $\mathcal{O}_{W_i,p_i}$ is the integral closure of $\mathcal{O}_{M,p}/\mathcal{I}_{Y_{i,p},p}$ in $\mathcal{K}(Y_{i,p})$, $\mathcal{K}(W_{i,p_i}) = \mathcal{K}(Y_{i,p})$. Hence $\mathcal{K}(Y_p)$ is isomorphic to $\prod \mathcal{K}(W_{i,p_i})$. We may consider naturally that the germ f'_p is an element of $\mathcal{K}(Y_p)$. Consider the sheaf $\mathcal{M}(Y)$ of germs of meromorphic functions. Then it is coherent by [14, Proposition 6.3.1] and $\mathcal{M}(Y)_p = \mathcal{K}(Y_p)$ by the definition. Then $f' \in H^0(M, \mathcal{M}(Y)) = \mathcal{K}(Y)$. We have shown that $\pi^*\colon \mathcal{K}(Y) \to \mathcal{K}(Y')$ is an isomorphism. □

Corollary 3.3 (Artin-Lang Property for Surface) *Let M be a connected paracompact real analytic manifold and $X \subset M$ be an irreducible analytic subset of M of dimension 2. Assume that the normalization of X is affine. Let $f_1, \dots, f_m$ be real analytic functions on M and $\alpha \in \mathrm{Spec}_r(\mathcal{K}(X))$. Assume that $f_1(\alpha) > 0, \dots, f_m(\alpha) > 0$, then the set $U := \{x \in X; f_1(x) > 0, \dots, f_m(x) > 0\}$ is not empty. Here $f(\alpha) > 0$ denotes the conditions $f \in \alpha$ and $-f \notin \alpha$.*

Proof. Let $\pi\colon X' \to X$ be the normalization. By Proposition 3.2, $\mathcal{K}(X) = \mathcal{K}(X')$. Hence we may view α as a point in $\mathrm{Spec}_r(\mathcal{K}(X'))$, naturally. Since Artin-Lang Property for affine normal surfaces holds true [5], the set $U' := \{x \in X';\ f_1 \circ \pi(x) > 0, \dots, f_m \circ \pi(x) > 0\}$ is not empty. Consider the set $U := \{x \in X; f_1(x) > 0, \dots, f_m(x) > 0\}$. Let $y \in U'$, then $\pi(y) \in U$. Especially $U \neq \emptyset$. □

We recall the definition of the tilde operator.

Definition 3.4 Let A be an irreducible real analytic set of dimension 2 whose normalization is affine. Let X be a global semianalytic subset of the form

$$X = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} \{x \in A;\ f_{ij}(x) *_{ij} 0\},$$

where $*_{ij} \in \{=, >\}$. Set

$$\widetilde{X} = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} \{\alpha \in \mathrm{Spec}_r(\mathcal{K}(A));\ f_{ij}(\alpha) *_{ij} 0\}.$$

Then the map $\widetilde{\cdot}$ defines a surjective homomorphism from the lattice of global semianalytic subsets of A onto the lattice of constructible subsets of $\mathrm{Spec}_r(\mathcal{K}(A))$ and $\widetilde{X} \neq \emptyset$ if and only if the interior of X in $\mathrm{Reg}(A)$ is not empty by [2, Proposition 2.4] and Corollary 3.3.

We have finished the preparation of the proof of Theorem 1.1.

Proof of Theorem 1.1.

We first consider the case where A is irreducible. Let $\pi\colon A' \to A$ be the normalization map. Consider the sets $X' := \pi^{-1}(X)$ and $W' := \pi^{-1}(W)$. Remark that X' is a global semianalytic subset of A'. Since $\dim(\mathrm{Sing}(A')) \leq 0$, we can apply Lemma 3.1 to X' and W'. Then there exists a real analytic function f on A' such that

$$\dim\big(\overline{W'\cap\{f>0\}}\cap\overline{(X'\setminus W')\cap\{f>0\}}\big)\le 0 \quad \text{and}$$
$$\dim\big(\overline{W'\cap\{f<0\}}\cap\overline{(X'\setminus W')\cap\{f<0\}}\big)\le 0.$$

Apply Proposition 2.6 to $X'\cap\{f>0\}$ and $X'\cap\{f<0\}$, then there exist natural numbers p, q depending only on $\dim(A')$ and $f'_{ij}\in\mathcal{O}(A')$ with $W'\setminus\{f=0\}=X'\cap U'$, where

$$U':=\bigcup_{i=1}^{p}\bigcap_{j=1}^{q}\{x\in A';\ f'_{ij}(x)>0\}.$$

Since $\mathcal{K}(A')=\mathcal{K}(A)$ by Proposition 3.2, there exists $f_{ij}\in\mathcal{K}(A)$ corresponding to f'_{ij}. Multiplying non-negative real analytic functions on A, we may assume that all f_{ij} are real analytic functions on A and f_{ij} equals to f'_{ij} multiplied by some non-negative real analytic function. Set

$$U:=\bigcup_{i=1}^{p}\bigcap_{j=1}^{q}\{x\in A;\ f_{ij}(x)>0\}\ \text{ and }\ V:=X\cap U.$$

Then $\dim\big((W'\setminus\pi^{-1}(V))\cup(\pi^{-1}(V)\setminus W')\big)\le 1$ by [2, Proposition 2.4] because $\widetilde{W'}=\widetilde{\pi^{-1}(W)}$. Hence $\dim\big((W\setminus V)\cup(V\setminus W)\big)\le 1$. We have succeeded in constructing a global semianalytic set V of the form

$$V=X\cap\bigcup_{i=1}^{p}\bigcap_{j=1}^{q}\{x\in A;\ f_{ij}(x)>0\}$$

with $\dim\big((W\setminus V)\cup(V\setminus W)\big)\le 1$.

We next consider the case where A is not irreducible. Let $A_1, A_2, \dots$ be the irreducible components of A. For any l, there exists $f_{lij}\in\mathcal{O}(A_l)$ with $\dim\big((W\cap A_l\setminus V_l)\cup(V_l\setminus W\cap A_l)\big)\le 1$, where

$$U_l:=\bigcup_{i=1}^{p}\bigcap_{j=1}^{q}\{x\in A;\ f_{lij}(x)>0\}\ \text{ and }\ V_l:=X\cap U_l.$$

By Cartan's theorem B, we may assume that f_{lij} are real analytic functions on M. We fix real analytic functions $c_i\colon M\to\mathbb{R}$ with $c_i\equiv 0$ on $\bigcup_{j\ne i}A_j$ and $c_i>0$ on $M\setminus\bigcup_{j\ne i}A_j$. Set $I_x:=\{l;\ x\in A_l\}$. The function germ $f_{ij,x}:=\sum_{l\in I_x}c_{l,x}f_{lij,x}$ is well defined and a global section of $\mathcal{O}_M/\mathcal{I}_A$. Hence there exists a real analytic extension of f_{ij} to M by Cartan's theorem B. We also denote this extension f_{ij}. Set $U := \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in X;\ f_{ij}(x) > 0\}$ and $V := X \cap U$. One easily checks that

$$V \cap \left(A_m \setminus \bigcup_{n \neq m} A_n \right) = V_m \cap \left(A_m \setminus \bigcup_{n \neq m} A_n \right),$$

hence off the curve $\bigcup_{n \neq m}(A_n \cap A_m)$, the set V works as the V_m's, and in the end $V \setminus W$ and $W \setminus V$ have dimension ≤ 1. We have succeeded in constructing a global semianalytic set V of the form

$$V = X \cap \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in A;\ f_{ij}(x) > 0\}$$

with $\dim\big((W \setminus V) \cup (V \setminus W)\big) \leq 1$.

Since both $W \setminus V$ and $V \setminus W$ are of dimension ≤ 1, they are global semianalytic by Lemma 2.2. Hence $W = V \cup (W \setminus V) \setminus (V \setminus W)$ is global semianalytic. $\square$

Using Łojasiewicz inequality for global semianalytic sets [3, Theorem], we can generalize Theorem 1.1.

Remark 3.5 Let M be a paracompact real analytic manifold and A be an analytic subset of M of dimension 2. Let S be the singular locus of A and $S_1, S_2, \ldots$ be its connected components. Assume that there exists an open neighborhood U_i of S_i in A such that the normalization of $U_i \cap A$ is affine. Let $X \subset A$ be a global semianalytic subset of M and W be a union of connected components of X. Then W is again global semianalytic.

Proof. Remark that we can choose U_i to be subanalytic.

Define Z as the Zariski closure of $\overline{W} \cap \overline{X \setminus W}$. We first show that Z is of dimension ≤ 1. Let $\{A_\lambda\}$ be irreducible components of A. When $\overline{W} \cap \overline{X \setminus W} \cap A_\lambda$ is empty, Z is contained in the union of the irreducible components A_μ other than A_λ. Hence we will consider the other case. Assume that $X \cap A_\lambda$ is of the form: $\bigcup_i \bigcap_j \{f_{ij} > 0\} \cap \{g_i = 0\}$, where $f_{ij}, g_i \in C^\omega(M)$ and $f_{ij} \not\equiv 0$ on A_λ. Then $\overline{W} \cap \overline{X \setminus W} \cap A_\lambda \subset \partial X \subset \bigcup_i \bigcup_j \{f_{ij} = 0\} \neq A_\lambda$. Since A_λ is irreducible, $\dim(Z \cap A_\lambda) \leq 1$. We have shown that $\dim(Z) \leq 1$.

Consider the family $\mathcal{C}$ of all analytically irreducible components of Z of dimension 1. Set $\mathcal{C}_C := \mathcal{C} \setminus \{C\}$. Define $\mathcal{S}_C$ as the family of all analytically irreducible components S' of S with $S' \neq C$.

Claim There exist natural numbers p, q satisfying the following condition. For any $C \in \mathcal{C}$, there exist real analytic functions f_{ijC} on M and there exists an analytic subset $D_C \subset C$ of dimension 0 and an open subanalytic neighborhood U_C of $C \setminus D_C$ with $\dim\big(\overline{W_C} \cap \overline{X \setminus W_C} \cap U_C\big) \leq 0$ and $W \cap U_C = W_C \cap U_C$. Here $W_C := \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in X;\ f_{ijC}(x) > 0\}$.

Proof of Claim. We first consider the case where $C \not\subset S$. We will show that there exists $f_C \in C^\omega(M)$ with

$$(*)\quad \dim\big(\overline{W \cap \{f_C > 0\}} \cap \overline{(X \setminus W) \cap \{f_C > 0\}} \cap C\big) \leq 0 \quad \text{and}$$
$$\dim\big(\overline{W \cap \{f_C < 0\}} \cap \overline{(X \setminus W) \cap \{f_C < 0\}} \cap C\big) \leq 0.$$

Set $E_C := \bigcup_{C' \in \mathcal{C}, C' \neq C} C \cap C'$. Choose a subanalytic open neighborhood T_C of $C \setminus E_C$ with $\overline{T_C} \cap Z = C$. We may assume that T_C is global semianalytic by Lemma 4.4 which we will show later. Apply Lemma 3.1 to $T_C \cap X$, $T_C \cap W$ and $T_C \cap (X \setminus W)$, then there exists $f_C \in C^\omega(M)$ satisfying the condition $(*)$.

Set

$$D_C^+ := \overline{W \cap \{f_C > 0\}} \cap \overline{(X \setminus W) \cap \{f_C > 0\}} \cap C,$$
$$D_C^- := \overline{W \cap \{f_C < 0\}} \cap \overline{(X \setminus W) \cap \{f_C < 0\}} \cap C$$

and

$$D_C := \bigcup_{C' \in \mathcal{C}_C} (C' \cap C) \cup \bigcup_{S' \in \mathcal{S}_C} (C \cap S') \cup D_C^+ \cup D_C^-.$$

Then there exists an open subanalytic neighborhood U_C of $C \setminus D_C$ such that

$$\dim\big(\overline{W \cap \{f_C > 0\}} \cap \overline{(X \setminus W) \cap \{f_C > 0\}} \cap U_C\big) \leq 0 \quad \text{and}$$
$$\dim\big(\overline{W \cap \{f_C < 0\}} \cap \overline{(X \setminus W) \cap \{f_C < 0\}} \cap U_C\big) \leq 0$$

by the condition $(*)$. Let C^+ be the interior of $\overline{W \cap \{f_C > 0\} \cap C \setminus D_C}$ in $C \setminus D_C$. We define C^- in the same way. There exists $g_C^+ \in C^\omega(M)$ with $C^+ = \{x \in C;\ g_C^+(x) > 0\}$ by [4, Theorem 4.4]. Construct g_C^- in the same way. Set $W_C := \{x \in X;\ f_C > 0,\ g_C^+ > 0\} \cup \{x \in X; f_C < 0,\ g_C^- > 0\}$ and shrink U_C if necessary. Then D_C, U_C and W_C satisfy the requirement.

We next consider the case where $C \subset S$. We may assume without loss of generality that $C \subset S_1$. By the proof of Theorem 1.1, there exist natural numbers p, q depending only on $\dim(M)$, $f'_{ijC} \in C^\omega(U_1)$ with $\dim((W \cap U_1 \setminus W'_C) \cup (W'_C \setminus W)) \leq 1$. Here $W'_C := \bigcup_{i=1}^p \bigcap_{j=1}^q \{x \in U_1;\ f'_{ijC}(x) > 0\}$. Consider the family $\mathcal{F}_C$ of analytically irreducible components B of $A \cap \bigcup_{i=1}^p \bigcup_{j=1}^q (f'_{ij})^{-1}(0)$ of dimension 1 with $B \neq C$. Set D_C as the intersection of C with the union of all elements of $\mathcal{F}_C$, $\mathcal{C}_C$ and $\mathcal{S}_C$. There exists an open subanalytic neighborhood U_C of $C \setminus D_C$ such that $\overline{U_C}$ does not intersect with any elements of $\mathcal{F}_C$, $\mathcal{C}_C$ and $\mathcal{S}_C$ except at D_C. We may assume that U_C is global semianalytic by Lemma 4.4. Set $W'_{iC} := U_C \cap \{x \in U_1;\ f'_{i1C}(x) > 0, \dots, f'_{iqC}(x) > 0\}$. Remark that $\overline{W'_{iC}} \cap (f'_{ij})^{-1}(0) \subset C$ for any $j = 1, \dots, q$. There exists a positive equation $d_C \in C^\omega(M)$ of C with $|f'_{ijC}| \geq d_C$ on $W'_{iC} \cap V_C$, where V_C is an open neighborhood of C, by [3, Theorem]. By Cartan's theorem B, there exists $f_{ijC} \in C^\omega(M)$ with $f'_{ijC} - d_C^2 - f_{ijC} \in H^0(M, d_C^3 \mathcal{O}_M)$. Consider the set $W_C := \bigcup_{i=1}^p \bigcap_{j=1}^q \{x \in X;\ f_{ijC}(x) > 0\}$. By the definition of W_C, there exists an open subanalytic neighborhood U'_C of $C \setminus D_C$ such that $\dim\big(\overline{W_C} \cap \overline{X \setminus W} \cap U'_C\big) \leq 0$ and $W_C \cap U'_C = W'_C \cap U'_C$. Replace U_C with U'_C, then D_C, U_C and W_C satisfy the requirement. We have finished the proof of Claim.

Enlarging D_C if necessary, we may assume that $C \cap C' \subset D_C$ and $C \cap S' \subset D_C$ for any $C' \in \mathcal{C}_C$ and $S' \in \mathcal{S}_C$. Shrinking U_C if necessary, we may assume that $\overline{U_C} \cap Z = C$ and $U_C \cap U_{C'} = \emptyset$ for any $C, C' \in \mathcal{C}$ with $C \neq C'$.

Consider the set $W''_{i,C} := W_C \cap U_C \cap \{x \in X_C;\ f_{i1C}(x) > 0, \dots, f_{iqC}(x) > 0\}$, then $f_{ijC}^{-1}(0) \cap \overline{W''_{i,C}} \subset C$. Hence there exist positive equations ϵ_{ijC} of C such that $|\epsilon_{ijC}| \leq |f_{ijC}|$ on $W''_{i,C} \setminus C$ by [3, Theorem]. Let $b \in D_C$ and $d_b\colon M \to \mathbb{R}$ be a positive equation of $\{b\}$. Consider the subanalytic function $\rho_{ijC,b}\colon (0,r) \to \mathbb{R}$ defined by

$$\rho_{ijC,b}(t) := \max\left\{ \frac{|f_{ijC}(x)|}{\epsilon_{ijC}(x)^2};\ x \notin U_C,\ d_b(x) = t \right\},$$

where r is a small positive integer. By Łojasiewicz inequality, there exist $m(i,j,C,b) \in \mathbb{N}$ and real numbers $B(i,j,C,b) > 0$ with $\rho_{ijC,b}(t) > 1/\big(B(i,j,C,b) t^{m(i,j,C,b)}\big)$. We can construct real analytic functions h_{ijC} with $h_{ijC} \geq 0$ on M, $h_{ijC}^{-1}(0) = D_C$ and $h_{ijC} \in m_b^{2m(i,j,C,b)}$ in the same way as the proof of Lemma 4.4. Then there exist positive continuous functions ϕ_{ijC} with $\phi_{ijC} > (h_{ijC}|f_{ijC}|)/\epsilon_{ijC}^2 + 2$ outside of U_C. There exist real analytic functions ψ_{ijC} on M with $|\psi_{ijC} - \phi_{ijC}| < 1$ by [13] and [21]. Set $g_{ijC} :=$

$h_{ijC}f_{ijC} - \psi_{ijC}\epsilon^2_{ijC}$. It is obvious that $g^{-1}_{ijC}(0) \subset U_C \cup C$. Furthermore, there exists an open neighborhood V'_C of $C \setminus D_C$ with

$$W \cap U_C \cap V'_C = V'_C \cap \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in X;\, g_{ijC}(x) > 0\}$$

by the definition of g_{ijC}.

Fix $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Let $\mathcal{A}_{ij}$ be the family of analytically irreducible components of $\bigcup_{C\in\mathcal{C}} g^{-1}_{ijC}(0)$ of dimension $= \dim(M) - 1$. Set $A_{ij} := \bigcup_{Y\in\mathcal{A}_{ij}} Y$. Since $U_C \cap U_{C'} = \emptyset$ and $\overline{U_C} \cap \overline{U_{C'}} = C \cap C'$ for any $C, C' \in \mathcal{C}$, A_{ij} is a coherent real analytic set. Remember that $Y \in \mathcal{A}_{ij}$ is an irreducible component of $g^{-1}_{ijC}(0)$, $g^{-1}_{ijC}(0) \subset U_C \cup C$ and $\overline{U_C} \cap Z = C$. Hence any $Y \in \mathcal{A}_{ij}$ contains only one curve $C \in \mathcal{C}$. Define m_Y as the minimum number satisfying that $g_{ijC} \in H^0(M, \mathcal{I}_Y)^{m_Y}$, where C is the unique irreducible component of Z contained in Y. Consider the sheaf $\mathcal{I}_{ij}$ defined by

$$\mathcal{I}_{ij,x} := \prod_{Y\in\mathcal{A}_{ij}, x\in Y} \mathcal{I}^{m_Y}_{Y,x}.$$

Since A_{ij} is coherent, $\mathcal{I}_{ij}$ is coherent. Since $\mathcal{I}_{ij,x}$ is a principal ideal of $\mathcal{O}_{M,x}$ for any $x \in M$, the ideal $\mathcal{I}_{ij}(M)$ is finitely generated by [12]. Let $G_{ij1}, \ldots, G_{ijm} \in C^\omega(M)$ be its generators. For any $l = 1, \ldots, m$, we set

$$\Phi_{ijl} := \{x \in Z;\, G_{ijl}\mathcal{O}_{M,x} = \mathcal{I}_{ij,x}\}.$$

We choose a discrete points set $D \subset Z$ such that

- $\mathrm{Sing}(Z) \subset D$,
- $\Phi_{ijl} \setminus D$ are open for all $l = 1, \ldots, m$,
- Z does not intersect with any other analytically irreducible components of $g^{-1}_{ijC}(0)$ except $Y \in \mathcal{A}_{ij}$ at $x \in C \setminus D \subset Z$, where $C \in \mathcal{C}$, and
- $\{\Phi_{ijl} \setminus D\}_{l=1,\ldots,m}$ covers $Z \setminus D$.

Here $\mathrm{Sing}(Z)$ denotes the singular locus of Z. By the definition of $\mathcal{I}_{ij}$ and Φ_{ijl}, for any $x \in \Phi_{ijl}\setminus D$, there exists a unit $u_{ijl,x}$ with $g_{ijC} = u_{ijl,x}G_{ijl} \in \mathcal{O}_{A,x}$. We define the set $\Phi_{ijl}(1) \subset \Phi_{ijl} \setminus D$ consisting of points x with $u_{ijl,x} > 0$ near x. The set $\Phi_{ijl}(-1) \subset \Phi_{ijl} \setminus D$ is the set of all points $x \in Z$ with $u_{ijl,x} < 0$ near x. Remark that $\Phi_{ijl} \setminus D = \Phi_{ijl}(1) \cup \Phi_{ijl}(-1)$. By Lemma 4.4, there exist open global semianalytic neighborhoods $T_{ijl}(k)$ of

$\Phi_{ijl}(k)$ in M such that $\overline{T_{ijl}(k)} \cap (Z \setminus \Phi_{ijl}(k)) \subset D \cap \overline{\Phi_{ijl}(k)}$ for any $k = -1, 1$. Consider the global semianalytic set

$$W_l(\overline{k}) := \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} T_{ijl}(k_{ij}) \cap \{x \in X;\, \mathrm{sign}(G_{ijl}(x)) = k_{ij}\}$$

for any $l = 1, \dots, m$ and tuples $\overline{k} = (k_{11}, k_{12}, k_{21}, k_{22}) \in \{\pm 1\}^4$.

Fix $l = 1, \dots, m$ and $\overline{k} = (k_{11}, k_{12}, k_{21}, k_{22}) \in \{\pm 1\}^4$. We show that, for any $x \in (Z \setminus D) \cap \overline{W_l(\overline{k})}$, there exists a neighborhood U of M with $W \cap U = W_l(\overline{k}) \cap U$. By the definition of $T_{ijl}(k_{ij})$, there exists a unit $u_{ij,x}$ with $G_{ijl} = u_{ij,x} g_{ijC} \in \mathcal{O}_{A,x}$ and $\mathrm{sign}(u_{ij,x}) = k_{ij}$ near x, where $C \in \mathcal{C}$ and $x \in C$. Therefore, there exists an open neighborhood U of x with

$$U \cap \{x \in M;\, g_{ijC}(x) > 0\} = U \cap \{x \in M;\, k_{ij} G_{ijl}(x) > 0\}.$$

Since $W \cap U = U \cap \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in X; g_{ijC}(x) > 0\}$, $W \cap U = W_l(\overline{k}) \cap U$.

As we have shown above, $\dim\big(\big(\overline{W_l(\overline{k}) \cap W}\big) \cap \big(\overline{W_l(\overline{k}) \setminus W}\big)\big) \le 0$. Hence, $W_l(\overline{k}) \cap W$ is global semianalytic by Proposition 2.6. Set $X_\infty := X \setminus \bigcup_{l,\overline{k}} W_l(\overline{k})$. Then $\dim\big(\overline{W \cap X_\infty} \cap \overline{X_\infty \setminus W}\big) \le 0$ because $\dim\big(C \cap \overline{W \cap X_\infty}\big) \le 0$ for all $C \in \mathcal{C}$. Therefore, $W \cap X_\infty$ is global semianalytic by Proposition 2.6. Since $W = (W \cap X_\infty) \cup \bigcup_{l,\overline{k}} W \cap W_l(\overline{k})$, W is also global semianalytic. □

4. Proof of Theorem 1.2

We prove Theorem 1.2 in the present section. For that, we first show that the localization of the ring of real analytic functions is regular.

Proposition 4.1 *Let M be a connected paracompact real analytic manifold and X be an analytically irreducible coherent real analytic subset of M. We set $\mathcal{O}_X := \mathcal{O}_M/\mathcal{I}_X$. Let Y be an irreducible coherent real analytic subset of X with* $\dim(Y \cap \mathrm{Sing}(X)) < \dim(Y)$ *and $\mathcal{I}_Y \subset \mathcal{O}_X$ denote the ideal sheaf of Y. Here* $\mathrm{Sing}(X)$ *denotes the singular locus of X. Then the ring $H^0(X, \mathcal{O}_X)_{H^0(X,\mathcal{I}_Y)}$ is a regular local ring of Krull dimension* $\dim X - \dim Y$.

Proof. Set $A := H^0(X, \mathcal{O}_X)$ and $P := H^0(X, \mathcal{I}_Y)$. Let Q be a prime ideal of A_P. There exists a prime ideal Q_1 of A with $Q_1 A_P = Q$. Since Q_1 is contained in P, the complex common zero set Z of Q_1 contains the complexification of Y. Set $r(Y) := \dim X - \dim Y$. We first show the following claim.

Claim There exist $f_1, \dots, f_r \in Q_1$ which generate Q. Moreover, we can choose $r(Y)$ generators $f_1, \dots, f_{r(Y)}$ when $Z = Y$.

Proof of Claim. We first need to define an ideal sheaf for the proof of this claim. For any $\mathcal{O}_X$-modules $\mathcal{A}$ and $\mathcal{B}$, $(\mathcal{A} : \mathcal{B})$ denote the sheaf generated by $\{f \in \mathcal{O}_X;\, f \cdot \mathcal{A} \subset \mathcal{B}\}$. It is well-known that $(\mathcal{A} : \mathcal{B})$ is coherent if so are both $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{I}_Z \subset \mathcal{O}_X$ denote the ideal sheaf of Z. We first show that $H^0(X, \mathcal{I}_Z) = Q_1$. Let x be a regular point of Y, namely, $x \in Y \setminus (\mathrm{Sing}(Y) \cup \mathrm{Sing}(X))$. Remark that the common complex zero set germs of $H^0(X, I_Z)\mathcal{O}_{X,x}$ and $Q_1\mathcal{O}_{X,x}$ coincide. Hence $\sqrt{Q_1\mathcal{O}_{X,x}} = \mathcal{I}_{Z,x}$ by Hilbert's Nullstellensatz. For any $f \in H^0(X, \mathcal{I}_Z)$, there exists $n \in \mathbb{N}$ with $f^n \in Q_1\mathcal{O}_{X,x}$. On the other hand, finitely many generators $g_1, \dots, g_k \in Q$ of $Q_1\mathcal{O}_{X,x}$ exist because $\mathcal{O}_{X,x}$ is Noetherian. Consider the coherent ideal sheaf $\mathcal{D} := (f^n\mathcal{O}_X : (g_1, \dots, g_k)\mathcal{O}_X)$. Since $\mathcal{D}_x = \mathcal{O}_{X,x}$, there exists $h \in H^0(X, \mathcal{D})$ such that $Y \not\subset h^{-1}(0)$, namely, $h \notin P$. Remark that $H^0(X, (g_1, \dots, g_k)\mathcal{O}_X) = (g_1, \dots, g_k)$ by Cartan's theorem B. Hence, $hf^n \in (g_1, \dots, g_k) \subset Q_1$. Since Q_1 is prime, $f \in Q_1$. We have shown that $H^0(X, \mathcal{I}_Z) = Q_1$.

The coherent sheaf is generated by its global sections by Cartan's Theorem A and $\mathcal{I}_{Z,x}$ is generated by $f_1, \dots, f_r \in H^0(X, \mathcal{I}_Z) = Q_1$. Moreover, we can choose $r(Y)$ generators $f_1, \dots, f_{r(Y)} \in Q_1$ of $\mathcal{I}_{Y,x}$ when $Z = Y$. We will show that these $f_1, \dots, f_r$ generate Q. Let $g \in Q_1$. Consider the coherent sheaf $(g\mathcal{O}_X : (f_1, \dots, f_r)\mathcal{O}_X)$. Applying the same argument as above, there exists $h' \in A$ with $h' \notin P$ and $hg' \in (f_1, \dots, f_r)$. We have shown that $f_1, \dots, f_r$ generate Q and have finished the proof of Claim.

We first show that A_P is Noetherian. We have only to show that all prime ideals Q of A_P are finitely generated. Hence A_P is Noetherian by Claim.

Since PA_P is generated by $r(Y)$ elements, $\dim A_P \leq r(Y)$. Let $f_1, \dots, f_{r(Y)} \in P$ be the generators given in Claim. Choose $y \in Y \setminus (\mathrm{Sing}(X) \cup \mathrm{Sing}(Y))$. Let $(x_1, \dots, x_{\dim(Y)})$ be an analytic local coordinate of Y at y. Since the maximal ideal of $\mathcal{O}_{X,y}$ is generated by $x_1, \dots, x_{\dim(Y)}, f_1, \dots, f_{r(Y)}$, $(x_1, \dots, x_{\dim(Y)}, f_1, \dots, f_{r(Y)})$ is a local coordinate of X at y. Define Y_i as the irreducible component of $\bigcup_{j=1}^{i} f_j^{-1}(0)$ containing a point y. Then $Y_1 \supsetneq Y_2 \supsetneq \dots \supsetneq Y_{r(Y)} = Y$ because $(x_1, \dots, x_{\dim(Y)}, f_1, \dots, f_{r(Y)})$ is a local coordinate. Hence we can construct the strictly ascending chain

$$\mathcal{I}(Y_1)A_P \subsetneq \mathcal{I}(Y_2)A_P \subsetneq \cdots \subsetneq \mathcal{I}(Y_{r(Y)})A_P = PA_P,$$

where $\mathcal{I}(Y_j)$ denotes the ideal of real analytic functions on Z vanishing on Y_j. Hence $\dim A_P = r(Y)$. By Claim, the only one maximal ideal is generated by $\dim A_P$ $(= r(Y))$ elements, namely, A_P is a regular local ring of dimension $r(Y)$. □

We prove Thom's Lemma for convergent power series. The original Thom's Lemma says that, for any polynomial $f(x)$ with one valuable x of degree n, the set $\{x \in \mathbb{R};\ \mathrm{sign}(f(x)) = i_0, \mathrm{sign}(f'(x)) = i_1, \ldots, \mathrm{sign}(f^{(n)}(x)) = i_n\}$ is empty, a point or an open connected interval, where $i_j = -1, 0, 1$ and, for any $a \in \mathbb{R}$, the notation $\mathrm{sign}(a) = -1, 0, 1$ represent the conditions $a < 0$, $a = 0$ and $a > 0$, respectively.

Lemma 4.2 *We first fix $n \geq 1$. Let m_0 be the maximal ideal (x_1, x_2) of the ring $\mathbb{R}\{x_1, x_2\}$ of convergent power series. Fix $f \in m_0^n \setminus m_0^{n+1}$. Then there exists a map $\sigma\colon \{1, \ldots, n\} \to \{1, 2\}$ satisfying the following condition. Set $f_0 := f$ and $f_k := \partial f_{k-1}/\partial x_{\sigma(k)}$ for $1 \leq k \leq n$. Then $f_i \notin m_0^{n-i+1}$ for all $i = 0, \ldots, n$.*

Proof. We prove this lemma by the induction on n. When $n = 1$, $(\partial f/\partial x_1)(0) \neq 0$ or $(\partial f/\partial x_2)(0) \neq 0$. Set $\sigma(1)$ such that $(\partial f/\partial x_{\sigma(1)})(0) \neq 0$.

We next consider the case where $n > 1$. By the assumption of the induction, $\partial f/\partial x_1 \notin m_0^n$ or $\partial f/\partial x_2 \notin m_0^n$. We define the number $\sigma(1)$ such that $\partial f/\partial x_{\sigma(1)} \notin m_0^n$. Apply this lemma to $\partial f/\partial x_{\sigma(1)}$. There exists a map $\sigma'\colon \{1, \ldots, n-1\} \to \{1, 2\}$ satisfying the conditions of this lemma for $\partial f/\partial x_{\sigma(1)}$. Set $\sigma(k) := \sigma'(k-1)$ for all $2 \leq k \leq n$. Then σ satisfies the requirement. □

Lemma 4.3 (Thom's Lemma for convergent power series) *Let n, m_0, f be the same as in* Lemma 4.2. *Let $\sigma\colon \{1, \ldots, n\} \to \{1, 2\}$ be a map. We define $f_0, f_1, \ldots, f_n$ in the same way as* Lemma 4.2. *Assume that $f_i \notin m_0^{n-i+1}$ for all $i = 0, \ldots, n$. Then the semianalytic set germ*

$$\begin{aligned} X_{IJ} := \{&\mathrm{sign}(f_0(x)) = I(0),\ \mathrm{sign}(f_1(x)) = I(1), \ldots, \\ &\mathrm{sign}(f_n(x)) = I(n),\ \mathrm{sign}(x_1) = J(1),\ \mathrm{sign}(x_2) = J(2)\} \end{aligned}$$

is a connected germ for any maps $I\colon \{0,1,\dots,n\} \to \{-1,0,1\}$ *and* $J\colon \{1,2\} \to \{-1,0,1\}$.

Proof. We prove this lemma by the induction on n. When $n = 1$, the set germ $\{f = 0\}$ is a germ of a real analytic submanifold of dimension 1, and $\{f > 0\}$ and $\{f < 0\}$ is connected. Since $(\partial f/\partial x_{\sigma(1)})(0) \neq 0$, X_{IJ} is obviously connected.

We next consider the case where $n > 1$. We fix maps $I\colon \{0,1,\dots,n\} \to \{-1,0,1\}$ and $J\colon \{1,2\} \to \{-1,0,1\}$. We show that X_{IJ} is connected.

The semianalytic set germ

$$\begin{aligned} X'_{IJ} := \{&\mathrm{sign}(f_1(x)) = I(1), \dots, \mathrm{sign}(f_n(x)) = I(n),\\ &\mathrm{sign}(x_1) = J(1),\, \mathrm{sign}(x_2) = J(2)\} \end{aligned}$$

is connected. Hence we may consider X'_{IJ} is a global semianalytic subset of an open neighborhood U of $[-r,r]^2$ for any sufficiently small positive number $r > 0$. When $\dim(X'_{IJ}) \leq 1$, X_{IJ} is connected because $X_{IJ} \subset X'_{IJ}$.

Consider the case where $\dim(X'_{IJ}) = 2$. We may assume without loss of generality that $\sigma(1) = 1$ and $J(1) = J(2) = 1$. We first consider the case where $\dim(X_{IJ}) = 1$. Assume that X_{IJ} is not connected. Shrinking U if necessary, we may assume that there exist two strictly increasing continuous subanalytic functions h_1, $h_2\colon [0,r] \to [0,\infty)$ such that $h_1|_{(0,r]} < h_2|_{(0,r]}$ are real analytic functions and

$$\begin{aligned} &\{(x_1,x_2);\ 0 \leq x_1 \leq r,\ x_2 = h_1(x_1)\}\\ &\cup \{(x_1,x_2);\ 0 \leq x_1 \leq r,\ x_2 = h_2(x_1)\} \subset X_{IJ}. \end{aligned}$$

The cardinality of the set $\{x_2 = a\} \cap X_{IJ}$ is more than 1. On the other hand, $f(\,\cdot\,,a)\colon (0,r] \to \mathbb{R}$ is constant, strictly increasing or strictly decreasing because $\mathrm{sign}\big((\partial f/\partial x_1)(\,\cdot\,,a)\big)$ is constant on X'_{IJ}. Contradiction. Hence X_{IJ} is connected.

We next consider the case where $\dim(X_{IJ}) = 2$. Shrink U if necessary, then the boundary $\overline{X_{IJ}} \setminus X_{IJ}$ is a finite union of graphs of continuous subanalytic functions $h_1,\dots,h_m$ with $h_1 < h_2 < \cdots < h_m$ and h_j are real analytic on $(0,r]$. As we showed before, the set $X'_{IJ} \cap \{f = 0\} \cap \{x_2 = a\}$ consists of only one element for any sufficiently small $a > 0$. Assume that X_{IJ} is not connected. The analytic function $f(\cdot,a)$ attains a minimal value at a point in $(0,r]$. Since $f(\cdot,a)$ is constant, strictly increasing or strictly decreasing, $f(\cdot,a)$ is constant. Hence $f_1 \equiv 0$, which contradicts the assumption $\partial f/\partial x_1 \notin m_0^n$. □

We give a technical lemma on separation of analytic subsets.

Lemma 4.4 *Let M be a paracompact real analytic manifold and $C \subset M$ be an analytic set of dimension 1. Let $D \subset C$ be an analytic subset of dimension 0. Consider an open subanalytic neighborhood U of $C \setminus D$. Then there exists an open global semianalytic neighborhood V of $C \setminus D$ with $V \subset U$.*

Proof. There exists a real analytic function h on M with $h^{-1}(0) = C$ and $h \leq 0$ on M. For any point $b \in D$, let d_b be a real analytic function $d_b^{-1}(0) = \{b\}$ and $d_b \geq 0$ on M. Consider the function $\rho_b : [0, r) \to [0, \infty)$ given by

$$\rho_b(t) := \min\{|h(y)|;\ y \in M \setminus U,\ d_b(y) = t\},$$

where r is a small positive number. Since ρ_b is subanalytic, there exists $m_b \in \mathbb{N}$ and $C_b > 0$ with $\rho_b(t) \geq C_b t^{m_b}$ for all $0 \leq t < r$ by Łojasiewicz's inequality. Consider the locally principal sheaf of ideals $\mathcal{J}$, whose nontrivial stalks are $d_b^{m_b+1}\mathcal{O}_b$. Then, by [12], this sheaf has finitely many (in fact, three) global generators α_i. Then $\alpha = \sum_i \alpha_i^2$ generates $\mathcal{J}^2$, so that $\alpha = \text{unit} \cdot d_b^{n_b}$ for some $n_b > m_b$. Let U_b be a sufficiently small neighborhood of b. Then $C_b d_b(y)^{m_b} > \alpha(y)$ for all $y \in U_b \setminus \{b\}$. Set $G = \alpha + h$. Since $h \equiv 0$ on C, $G > 0$ on $C \setminus D$. By the definition of ρ_b, $|h(y)| \geq C_b d_b(y)^{m_b}$ for any $y \in U_b$. Hence, $G < 0$ on $U_b \setminus U$. Consider a sufficiently small closed neighborhood S of C. Then,

$$S \cap \left(\overline{\{x \in M; G(x) > 0\} \setminus U}\right) = \emptyset.$$

There exists a real analytic function G' on M such that $G' > 0$ on S and $G' < 0$ on $\{x \in M;\ G(x) > 0\} \setminus U$ by Lemma 2.1. Set $V := \{x \in M;\ G(x) > 0,\ G'(x) > 0\}$, then V satisfies the requirement. □

The following theorem is the generalization of Theorem 1.2.

Theorem 4.5 *Let M be a connected paracompact real analytic manifold of dimension 3. Let X be a global semianalytic subset of M and W be a union of connected components of X. Let S be an analytically irreducible component of the Zariski closure $\overline{\overline{W} \cap \overline{X \setminus W}}^Z$ of dimension 1. Then there exists a proper analytic subset $D \subset S$ and a global semianalytic neighborhood U of $S \setminus D$ such that $W \cap U$ is global semianalytic and $\overline{W \cap U} \cap \overline{X \setminus W}$*

is of dimension ≤ 0.

Proof. Let B be the Zariski closure of $\overline{X} \setminus \mathrm{Int}(X)$, where $\mathrm{Int}(X)$ is the interior of X. Fix a point $x \in S \setminus \mathrm{Sing}(S)$ such that S does not intersect at x with any irreducible component of $\overline{\overline{W} \cap \overline{X \setminus W}}^Z$ except S and any irreducible component of B which does not contain S. Since S is regular at x, there exist h_1, $h_2 \in C^\omega(M)$ which generate the ideal $\mathcal{I}_{Y,x} \subset \mathcal{O}_{M,x}$ by Proposition 4.1. Choose a real analytic function $h_3 \in C^\omega(M)$ such that (h_1, h_2, h_3) is a local coordinate on an open neighborhood of x. We may assume that M is a closed real analytic submanifold of a Euclidean space $\mathbb{R}^N$. We may assume that x is the origin and linearly independent linear functions x_1, x_2, x_3 are also a local coordinate of M in an open neighborhood of x. Consider Hessian $H = \partial(h_1, h_2, h_3)/\partial(x_1, x_2, x_3)$. Then $\det(H)(x) \neq 0$. Let Δ_{ij} be the (i, j)-minor of H. We define partials

$$\partial_j \colon C^\omega(M)_{(\det(H))} \to C^\omega(M)_{(\det(H))} \quad \text{by} \quad \partial_j(f) := \frac{\displaystyle\sum_{i=1}^{3} \Delta_{ij} \frac{\partial f}{\partial x_i}}{\det(H)}$$

for all $j = 1, 2, 3$. There exists a distinct points set $D \subset Z$ such that, for any $y \in Z \setminus D$, $\det(H)(y) \neq 0$, $(h_1, h_2, h_3 - h_3(y))$ and $(x_1 - x_1(y), x_2 - x_2(y), x_3 - x_3(y))$ are local coordinates of M at y. Furthermore, we may assume that, at any point $y \in S \setminus D$, S does not intersect with any irreducible component of B of dimension < 2. There exists an analytic function $f \in C^\omega(M)$ with $f^{-1}(0) = B$. Since $\bigcap_{l=1}^\infty \mathcal{I}_S^l = (0)$, there exists $n \in \mathbb{N}$ with $f \in \mathcal{I}_S^n \setminus \mathcal{I}_S^{n+1}$. Since the sheaf $(\mathcal{I}_S^{n+1} \colon \mathcal{I}_S^n)$ is coherent, the set $\{y \in S;\, f \in \mathcal{I}_{S,y}^{n+1}\}$ is an analytic set of dimension ≤ 0. Enlarging D if necessary, we may assume that $f \in \mathcal{I}_{S,y}^n \setminus \mathcal{I}_{S,y}^{n+1}$ for all $y \in S \setminus D$. Choose a point $x' \in S \setminus D$, then the surface $h_3^{-1}(h_3(x'))$ is a real analytic manifold near x' and (h_1, h_2) is a local coordinate of this surface. Hence, by Lemma 4.2, there exists $\sigma \colon \{1, \ldots, n\} \to \{1, 2\}$ with $\partial_{\sigma(i)}(\cdots(\partial_{\sigma(1)}(f|_{h_3^{-1}(h_3(x'))}))\cdots) \notin (h_1, h_2)^{n+1-i} \subset \mathcal{O}_{M,x'}/(h_3 - h_3(x'))$ for all $i = 1, \ldots, n$. Set $P := \mathcal{I}_Y(M)$ and $R := C^\omega(M)_P$. Then R is a regular local ring by Proposition 4.1. Hence there exists a monomorphism $\tau \colon R \to (R/PR)[[h_1, h_2]]$. Therefore there exists $\tau_{\alpha_1,\alpha_2} \in R/PR$ with $\tau(f) := \sum_{\alpha_1,\alpha_2} \tau_{\alpha_1,\alpha_2} h_1^{\alpha_1} h_2^{\alpha_2}$. We may consider the element of R/PR as the quotient of $C^\omega(Y)$, namely, the fraction of real analytic functions on Y. For any $i = 1, \ldots, n$, we define $\alpha_1(\sigma)$ as the cardinality of the set $\{k;\, 1 \le k \le n,\ \sigma(k) = 1\}$ and set $\alpha_2(\sigma) := n - \alpha_1(i)$. Set $\tau_\sigma := \tau_{\alpha_1(\sigma),\alpha_2(\sigma)}$. Then $\partial_{\sigma(n)}(\cdots(\partial_{\sigma(1)}(f))\cdots)(x') = \alpha_1(\sigma)!\,\alpha_2(\sigma)!\,\tau_\sigma(x')$ by the definition. Hence $\tau_\sigma \not\equiv 0$. Enlarging D, we may assume that, for all $y \in S \setminus D$, τ_σ is defined near y and $\tau_\sigma(y) \neq 0$. We define $f_i := \partial_{\sigma(i)}(\cdots(\partial_{\sigma(1)}(f))\cdots)$ for all $i = 1, \ldots, n$, then $f_i|_{h_3^{-1}(h_3(y))} \in \mathcal{O}_{h_3^{-1}(h_3(y)),y}$ coincides with f_i constructed from $f|_{h_3^{-1}(h_3(y))}$ in Lemma 4.2 under the real analytic diffeomorphism between an open neighborhood of the origin in $\mathbb{R}^2$ and $h_3^{-1}(h_3(y))$ for all $i = 0, \ldots, n$.

Consider the family $\mathcal{M}$ of all pairs of maps $I\colon \{0,1,\ldots,n\} \to \{-1,0,1\}$ and $J\colon \{1,2\} \to \{-1,0,1\}$. For any $(I,J) \in \mathcal{M}$, set

$$\begin{aligned} X_{IJ} := \{y \in M;\ &\mathrm{sign}(f_i) = I(i) \text{ for all } i = 0,\,1,\ldots,\,n,\\ &\mathrm{sign}(h_1) = J(1),\ \mathrm{sign}(h_2) = J(2),\ \det(H) \neq 0\}. \end{aligned}$$

Then X_{IJ} is a global semianalytic set. Remark that the germ of $f_3^{-1}(y) \cap X_{IJ}$ at $y \in S \setminus D$ is connected by Lemma 4.3.

It is well known that the class $\mathbb{R}_{\mathrm{an}}$ of sets (in $\mathbb{R}^n$, for $n = 0,\,1,\,2,\ldots$) which is subanalytic in the projective space $\mathbb{P}^m(\mathbb{R})$ forms an o-minimal structure. (Here $\mathbb{R}^m$ is identified with an open subset of $\mathbb{P}^m(\mathbb{R})$ via the map $(x_1,\ldots,x_m) \mapsto (1 : x_1 : \cdots : x_m)$.) See [20] for the theory of an o-minimal structure. Applying the theory of an o-minimal structure, we modify D and construct a subanalytic open neighborhood U of $S \setminus D$ satisfying the following conditions:

(a) $\overline{U} \cap \overline{\overline{W} \cap \overline{X \setminus W}}^Z = S$.

(b) Any connected component V of U contains at most one connected component C of $S \setminus D$, $\overline{U} \cap S = \overline{C}$ if U contains a connected component C of $S \setminus D$, and $\overline{U} \cap S = \emptyset$ otherwise.

(c) For any connected component V of U, $V \cap X_{IJ}$ is connected.

We will begin with the construction of U.

Choose a subanalytic open neighborhood U' of $S \setminus D$ with $\overline{U'} \cap \overline{\overline{W} \cap \overline{X \setminus W}}^Z = S$. Let $\{K_\lambda\}$ be a locally finite subanalytic open covering of S in M such that $\overline{K_\lambda}$ is compact and $\bigcup_\lambda(\overline{K_\lambda} \setminus K_\lambda) \cap S$ is a discrete points set. Such a covering exists because M is a closed real analytic submanifold of a Euclidean space $\mathbb{R}^m$ by [13]. Enlarging D if necessary, we may assume that $\bigcup_\lambda(\overline{K_\lambda} \setminus K_\lambda) \cap S \subset D$. Remark that K_λ, $K_\lambda \cap S$, $K_\lambda \cap U'$ and $K_\lambda \cap X_{IJ}$ are all definable in $\mathbb{R}_{\mathrm{an}}$ as a subset of $\mathbb{R}^m$. Apply the cell decomposition theorem to K_λ, $K_\lambda \cap D$, $K_\lambda \cap S$, $K_\lambda \cap U'$ and $K_\lambda \cap X_{IJ}$.

There exist finite connected definable sets $\{N_{\lambda j}\}_j$, which are called *cells*, satisfying the following conditions:

(i) $N_{\lambda j} \cap N_{\lambda k} = \emptyset$ if $j \neq k$ and $K_\lambda = \bigcup_j N_{\lambda j}$.

(ii) If $\overline{N_{\lambda j}} \cap N_{\lambda k} \neq \emptyset$, then $N_{\lambda k} \subset \overline{N_{\lambda j}}$.

(iii) Let E be one of the sets $K_\lambda \cap D$, $K_\lambda \cap S$, $K_\lambda \cap U'$ and $K_\lambda \cap X_{IJ}$. If $N_{\lambda j} \cap E \neq \emptyset$, then $N_{\lambda j} \subset E$.

Define E_λ as the union of all 0-dimensional cells contained in S. Remark that $\bigcup_\lambda E_\lambda \cup D$ is a discrete points set. Enlarging D if necessary, we may assume that $E_\lambda \subset D$. Define U_λ as the union of all 1-dimensional cells contained in S and all cells whose closure contain 1-dimensional cells contained in S. It is an open definable set, especially, it is subanalytic. Set $U := \bigcup_\lambda U_\lambda$, then it is a subanalytic open neighborhood of $S \setminus D$ because $\{U_\lambda\}_\lambda$ is locally finite.

Since U is contained in U', U satisfies the condition (a). The condition (b) is also satisfied because $\bigcup_\lambda(\overline{K_\lambda} \setminus K_\lambda) \cap S \subset D$ and U is contained in U'. We show that U satisfies the condition (c). Let V be a connected component of U. The set V is a finite union of connected components V_λ of U_λ. We first show that $V_\lambda \cap X_{IJ}$ is connected. Assume otherwise. Let $\{N_{\lambda j(i)}\}_i$ be cells contained in $V_\lambda \cap X_{IJ}$ and set $C := V_\lambda \cap S$. Remark that C coincides with a cell, hence, $\overline{N_{\lambda j(i)}}$ contains C. Choose $y \in C$. Since $C \subset \overline{N_{\lambda j(i)}}$ and $f_3^{-1}(y)$ is transversal to C, the germ of $N_{\lambda j(i)} \cap f_3^{-1}(y)$ at y is not the empty set germ. It is obvious that the germ $\bigcup_i N_{\lambda j(i)} \cap f_3^{-1}(y)$ at y is not connected because $V_\lambda \cap X_{IJ} = \bigcup_i N_{\lambda j(i)}$ is not connected. However, the germ of $\bigcup_i N_{\lambda j(i)} \cap f_3^{-1}(y)$ coincides with the germ of $X_{IJ} \cap f_3^{-1}(y)$ and the latter is connected by Lemma 4.3. Contradiction. We have shown that $V_\lambda \cap X_{IJ}$ is connected. For any λ and λ', $V_\lambda \cap V_{\lambda'}$ is a neighborhood of C. Hence, $V_\lambda \cap V_{\lambda'} \cap X_{IJ}$ is not empty. Therefore, $V \cap X_{IJ} = \bigcup_\lambda V_\lambda \cap X_{IJ}$ is connected. We have shown that U satisfies the condition (c).

Since U satisfies the condition (c), there exists only one subfamily $\mathcal{M}_C \subset \mathcal{M}$ with $(*)$: $W \cap V = \bigcup_{(I,J)\in\mathcal{M}_C} X_{IJ} \cap V$ for any connected component C of $S \setminus D$. There exists a global semianalytic open neighborhood of $S \setminus D$ contained in U by Lemma 4.4. Replace U with this global semianalytic neighborhood. Then U is a global semianalytic open neighborhood of $S \setminus D$ satisfying the conditions (a), (b) and $(*)$. For any (I,J), set $S_{IJ} := \bigcup C$, where C runs all connected components of $S \setminus D$ with $(I,J) \in \mathcal{M}_C$. There exists a real analytic function f_{IJ} on M with $S_{IJ} = \{x \in S; f_{IJ} > 0\}$ by [4, Theorem 4.4]. Let g_{IJ} be a real analytic function on M with $C \setminus (D \cup S_{IJ}) =$

$\{x \in C; g_{IJ} > 0\}$. Replace U with

$$U \cap \bigcup_{(I,J)\in\mathcal{M}} \{x \in M;\ f_{IJ}(x) > 0\} \cup \{x \in M; g_{IJ}(x) > 0\}.$$

Then

$$U \cap W = U \cap \bigcup_{(I,J)\in\mathcal{M}} \{x \in M;\ f_{IJ}(x) > 0\} \cap X_{IJ} \quad \text{and}$$

$$U \cap (X \setminus W) = U \cap \bigcup_{(I,J)\in\mathcal{M}} \{x \in M;\ g_{IJ}(x) > 0\} \cap X_{IJ}.$$

We have shown that $U \cap W$ and $U \cap (X \setminus W)$ are global semianalytic. □

Proof of Theorem 1.2.

Let $S_1, \dots, S_m$ be analytically irreducible components of $\overline{\overline{W} \cap \overline{X \setminus W}}^Z$ of dimension 1. By Theorem 4.5, there exist analytic subsets $D_i \subset S_i$ of dimension 0 and open global semianalytic neighborhoods U_i of $S_i \setminus D_i$ with $\dim\big(\overline{W \cap U_i} \cap \overline{X \setminus W}\big) \le 0$ for any $i = 1, \dots, m$. Consider the global semianalytic subset $X \setminus \bigcup_{i=1}^m U_i$ of M. Then $\dim\big(\overline{W \setminus \bigcup_{i=1}^m U_i} \cap \overline{X \setminus (\bigcup_{i=1}^m U_i \cup W)}\big) \le 0$ by the definition of S_i and U_i. Hence $W \setminus \bigcup_{i=1}^m U_i$ is global semianalytic by Proposition 2.6. Therefore $W = \bigcup_{i=1}^m (W \cap U_i) \cup \big(W \setminus \bigcup_{i=1}^m U_i\big)$ is global semianalytic. □

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