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Rigidity of the canonical isometric imbedding of the quaternion projective plane $P^2(\boldsymbol{H})$

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Abstract. In this paper, we investigate isometric immersions of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$ and prove that the canonical isometric imbedding $\boldsymbol{f}_0$ of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$, which is defined in Kobayashi [11], is rigid in the following strongest sense: Any isometric immersion $\boldsymbol{f}_1$ of a connected open set U ($\subset P^2(\boldsymbol{H})$) into $\boldsymbol{R}^{14}$ coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\boldsymbol{R}^{14}$, i.e., there is a euclidean transformation a of $\boldsymbol{R}^{14}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.

Key words: Curvature invariant, isometric immersion, quaternion projective plane, rigidity, root space decomposition.

1. Introduction

In our previous paper [8], we proved the rigidity of the canonical isometric imbedding of the Cayley projective plane $P^2(\boldsymbol{Cay})$. The purpose of this paper is to investigate a similar problem for (local) isometric immersions of the quaternion projective plane $P^2(\boldsymbol{H})$. As we have proved in [7], any open set of the quaternion projective plane $P^2(\boldsymbol{H})$ cannot be isometrically immersed into $\boldsymbol{R}^{13}$. On the other hand, there is an isometric immersion $\boldsymbol{f}_0$ of $P^2(\boldsymbol{H})$ into the euclidean space $\boldsymbol{R}^{14}$, which is called the canonical isometric imbedding of $P^2(\boldsymbol{H})$ (see Kobayashi [11]). Therefore, it follows that $\boldsymbol{R}^{14}$ is the least dimensional euclidean space into which $P^2(\boldsymbol{H})$ can be (locally) isometrically immersed.

In the present paper, we will show that the canonical isometric imbedding $\boldsymbol{f}_0$ is rigid in the following strongest sense:

Theorem 1 *Let $\boldsymbol{f}_0$ be the canonical isometric imbedding of $P^2(\boldsymbol{H})$ into the euclidean space $\boldsymbol{R}^{14}$. Then, for any isometric immersion $\boldsymbol{f}_1$ defined on a connected open set U of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$, there exists a euclidean transformation a of $\boldsymbol{R}^{14}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.*

The proof of this theorem will be given by solving the Gauss equation

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associated with the isometric imbeddings (immersions) of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$ in the same line of [8] (see Theorem 7). We use the same notations and terminology as those of the previous papers [6], [7] and [8].

2. The quaternion projective plane $P^2(\boldsymbol{H})$

In this section we review the structure of the quaternion projective plane $P^2(\boldsymbol{H})$ and prepare several formulas concerning the bracket operation.

As is well-known, $P^2(\boldsymbol{H})$ can be represented by $P^2(\boldsymbol{H}) = G/K$, where $G = Sp(3)$ and $K = Sp(2) \times Sp(1)$. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K) and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ associated with the symmetric pair (G, K). We denote by $(\ ,\ )$ the inner product of $\mathfrak{g}$ given by the (-1)-multiple of the Killing form of $\mathfrak{g}$. As usual, we can identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\}$. We assume that the G-invariant Riemannian metric g of G/K satisfies

$$g_o(X, Y) = (X, Y), \quad X, Y \in \mathfrak{m}.$$

Then, it is well-known that at the origin o the Riemannian curvature tensor R of type $(1, 3)$ is given by

$$R_o(X, Y)Z = -\big[[X, Y], Z\big], \qquad \forall X,\ Y,\ Z \in \mathfrak{m}.$$

We now take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ and fix it in the following discussions. We note that since $\operatorname{rank}(P^2(\boldsymbol{H})) = 1$, we have $\dim \mathfrak{a} = 1$.

For each element $\lambda \in \mathfrak{a}$ we define two subspaces $\mathfrak{k}(\lambda)$ ($\subset \mathfrak{k}$) and $\mathfrak{m}(\lambda)$ ($\subset \mathfrak{m}$) by

$$\mathfrak{k}(\lambda) = \Big\{ X \in \mathfrak{k} \,\Big|\, \big[H, [H, X]\big] = -(\lambda, H)^2 X, \quad \forall H \in \mathfrak{a} \Big\},$$
$$\mathfrak{m}(\lambda) = \Big\{ Y \in \mathfrak{m} \,\Big|\, \big[H, [H, Y]\big] = -(\lambda, H)^2 Y, \quad \forall H \in \mathfrak{a} \Big\}.$$

Let Σ be the set of all non-zero restricted roots. (An element $\lambda \in \mathfrak{a}$ is called a *restricted root* if $\mathfrak{m}(\lambda) \neq 0$.) As is known, there is a restricted root μ such that $\Sigma = \{\pm\mu, \pm 2\mu\}$. We take and fix such a restricted root μ. For each integer i we set $\mathfrak{k}_i = \mathfrak{k}(|i|\mu)$, $\mathfrak{m}_i = \mathfrak{m}(|i|\mu)$ ($|i| \leq 2$), $\mathfrak{k}_i = \mathfrak{m}_i = 0$ ($|i| > 2$). Then, we have $\mathfrak{m}_0 = \mathfrak{a} = \boldsymbol{R}\mu$ and

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_1 + \mathfrak{k}_2 \quad (\textit{orthogonal direct sum}),$$
$$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1 + \mathfrak{m}_2 \quad (\textit{orthogonal direct sum}).$$

The dimensions of the factors are given by $\dim \mathfrak{k}_0 = 6$, $\dim \mathfrak{k}_1 = \dim \mathfrak{m}_1 = 4$ and $\dim \mathfrak{k}_2 = \dim \mathfrak{m}_2 = 3$ (precisely, see [7]).

We now show several formulas concerning the bracket operation of $\mathfrak{g}$. By the definition of the subspaces $\mathfrak{k}_i$ and $\mathfrak{m}_i$ we easily have

$$\big[\mathfrak{k}_i, \mathfrak{k}_j\big] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad \big[\mathfrak{m}_i, \mathfrak{m}_j\big] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad \big[\mathfrak{k}_i, \mathfrak{m}_j\big] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}. \tag{2.1}$$

Moreover, we have

Proposition 2 *Let Y_0, $Y_0' \in \mathfrak{a} + \mathfrak{m}_2$, Y_1, $Y_1' \in \mathfrak{m}_1$. Then:*

$$\big[Y_i, \big[Y_i, Y_j'\big]\big] = -(1 + 3\delta_{ij})(\mu, \mu)\big\{(Y_i, Y_i)Y_j' - (Y_i, Y_j')Y_i\big\}, \quad (i, j = 0, 1), \tag{2.2}$$

$$\big[Y_i, \big[Y_i', Y_j\big]\big] + \big[Y_i', \big[Y_i, Y_j\big]\big] = -2(\mu, \mu)(Y_i, Y_i')Y_j, \quad (i, j = 0, 1,\ i \neq j), \tag{2.3}$$

$$\big[Y_i, \big[Y_i, X_1\big]\big] = -(\mu, \mu)(Y_i, Y_i)X_1, \quad \forall X_1 \in \mathfrak{k}_1 \quad (i = 0, 1), \tag{2.4}$$

where δ_{ij} denotes the Kronecker delta.

Proof. We first prove (2.2). Assume that $i = j$ and $Y_i \neq 0$. Set $Y_i'' = Y_i' - (Y_i', Y_i)/(Y_i, Y_i) \cdot Y_i$. Then, we know that $(Y_i, Y_i'') = 0$ and that $Y_i'' \in \mathfrak{a} + \mathfrak{m}_2$ if $i = 0$ and $Y_i'' \in \mathfrak{m}_1$ if $i = 1$. Hence, by Proposition 10 of [7], we have $\big[Y_i, \big[Y_i, Y_i''\big]\big] = -4(\mu, \mu)(Y_i, Y_i)Y_i''$. Therefore, we can easily obtain (2.2) in the case $i = j$. In the case $i \neq j$, (2.2) directly follows from Proposition 10 of [7].

We next prove (2.3). Since $i \neq j$, it follows that $(Y_i, Y_j) = (Y_i', Y_j) = 0$. Hence, by (2.2) we have $\big[Y_i + Y_i', \big[Y_i + Y_i', Y_j\big]\big] = -(\mu, \mu)(Y_i + Y_i', Y_i + Y_i')Y_j$. This, together with $\big[Y_i, \big[Y_i, Y_j\big]\big] = -(\mu, \mu)(Y_i, Y_i)Y_j$ and $\big[Y_i', \big[Y_i', Y_j\big]\big] = -(\mu, \mu)(Y_i', Y_i')Y_j$, proves (2.3).

We finally prove (2.4). We note that $\big[Y_1, \mathfrak{a} + \mathfrak{m}_2\big] = \mathfrak{k}_1$ holds for any $Y_1 \in \mathfrak{m}_1$ ($\neq 0$). In fact, it is easy to see $\big[Y_1, \mathfrak{a} + \mathfrak{m}_2\big] \subset \mathfrak{k}_1$ (see (2.1)). Moreover, the map $\mathfrak{a} + \mathfrak{m}_2 \ni Y_0' \longmapsto \big[Y_1, Y_0'\big] \in \mathfrak{k}_1$ is bijective, because $\big[Y_1, Y_0'\big] \neq 0$ if $Y_0' \in \mathfrak{a} + \mathfrak{m}_2$ ($Y_0' \neq 0$) (recall that $\mathrm{rank}(P^2(\boldsymbol{H})) = 1$) and because $\dim(\mathfrak{a} + \mathfrak{m}_2) = \dim \mathfrak{k}_1$. Let $X_1 \in \mathfrak{k}_1$. Then, by $\big[Y_1, \mathfrak{a} + \mathfrak{m}_2\big] = \mathfrak{k}_1$ we can take an element $Y_0' \in \mathfrak{a} + \mathfrak{m}_2$ such that $\big[Y_1, Y_0'\big] = X_1$. Now, applying $\operatorname{ad} Y_1$ to the equality $\big[Y_1, \big[Y_1, Y_0'\big]\big] = -(\mu, \mu)(Y_1, Y_1)Y_0'$ (see (2.2)), we have $\big[Y_1, \big[Y_1, X_1\big]\big] = -(\mu, \mu)(Y_1, Y_1)X_1$, proving (2.4) for the case $i = 1$. Similarly, we can prove (2.4) for the case $i = 0$. $\square$

Let Y_0, $Y_0' \in \mathfrak{a} + \mathfrak{m}_2$. Define a linear mapping $L(Y_0, Y_0')$ of $\mathfrak{m}_1$ to $\mathfrak{m}$ by

$$L(Y_0, Y_0')Y_1 = \big[Y_0, \big[Y_0', Y_1\big]\big], \qquad Y_1 \in \mathfrak{m}_1.$$

Then, we have

Proposition 3 *Let Y_0, $Y_0' \in \mathfrak{a} + \mathfrak{m}_2$. Then:*
(1) $L(Y_0, Y_0')\mathfrak{m}_1 \subset \mathfrak{m}_1$. *The transpose of $L(Y_0, Y_0')$ with respect to* (,) *is given by $L(Y_0', Y_0)$, i.e., ${}^tL(Y_0, Y_0') = L(Y_0', Y_0)$.*
(2) *Let $\mathbf{1}_{\mathfrak{m}_1}$ be the identity map of $\mathfrak{m}_1$. Then:*
(2*a*) $L(Y_0, Y_0') + L(Y_0', Y_0) = -2(\mu, \mu)(Y_0, Y_0')\, \mathbf{1}_{\mathfrak{m}_1}$;
(2*b*) $L(Y_0, Y_0') \cdot L(Y_0', Y_0) = (\mu, \mu)^2 (Y_0, Y_0)(Y_0', Y_0')\, \mathbf{1}_{\mathfrak{m}_1}$.

Proof. The assertion (1) is clear from (2.1) and the $\mathrm{ad}\,\mathfrak{g}$-invariance of (,). Let $Y_1 \in \mathfrak{m}_1$. Since $\big[Y_0, Y_1\big] \in \mathfrak{k}_1$, we have $\big[Y_0', \big[Y_0', \big[Y_0, Y_1\big]\big]\big] = -(\mu, \mu)(Y_0', Y_0')\big[Y_0, Y_1\big]$ (see (2.4)). Hence, by applying $\mathrm{ad}\, Y_0$ to this equality, we easily have (2*b*). The equality (2*a*) directly follows from (2.3). □

Here, we recall the notion of pseudo-abelian subspace of $\mathfrak{m}$. Let Q be a subspace of $\mathfrak{m}$. Q is called *pseudo-abelian* if it satisfies $\big[Q, Q\big] \subset \mathfrak{k}_0$ (see [6]).

Proposition 4 (1) *Any subspace Q of $\mathfrak{m}_2$ is pseudo-abelian.*
(2) *Let Q be a pseudo-abelian subspace satisfying $Q \not\subset \mathfrak{m}_2$. Then,* $\dim Q \leq 2$.

Accordingly, the inequality $\dim Q \leq 3$ *holds for any pseudo-abelian subspace Q, and the equality holds when and only when $Q = \mathfrak{m}_2$.*

Proof. Since $\big[\mathfrak{m}_2, \mathfrak{m}_2\big] \subset \mathfrak{k}_0$ (see (2.1)), it follows that any subspace of $\mathfrak{m}_2$ is pseudo-abelian. On the contrary, we already proved in Lemma 5.4 of [6] that for a pseudo-abelian subspace Q with $Q \not\subset \mathfrak{m}_2$ it holds $\dim Q \leq 1 + n(\mu)$, where $n(\mu)$ means the local pseudo-nullity of the restricted root μ. (For the definition of the local pseudo-nullity, see §3 of [6].) In the case $G/K = P^2(\boldsymbol{H})$, we have $n(\mu) = 1$ (see Theorem 3.2 and Table 3 of [6]). Hence, we have $\dim Q \leq 2$. □

For later use, we obtain the normal form of a 2-dimensional pseudo-abelian subspace Q with $Q \not\subset \mathfrak{m}_2$.

Proposition 5 *Let ξ_1 and η_1 be elements of $\mathfrak{m}_1$ satisfying $(\xi_1, \xi_1) = 2(\mu, \mu)$, $\eta_1 \neq 0$ and $(\xi_1, \eta_1) = 0$. Then, the 2-dimensional subspace Q ($\subset \mathfrak{m}$) defined by*

$$Q = \boldsymbol{R}(\mu+\xi_1)+\boldsymbol{R}\left(\eta_1+\frac{1}{4(\mu,\mu)^2}\bigl[\mu,\,[\xi_1,\eta_1]\bigr]\right) \tag{2.5}$$

is pseudo-abelian and $Q \not\subset \mathfrak{m}_2$.

Conversely, if Q is a pseudo-abelian subspace of $\mathfrak{m}$ with $Q \not\subset \mathfrak{m}_2$ and $\dim Q = 2$*, then Q can be written in the form* (2.5) *by utilizing suitable elements ξ_1 and $\eta_1 \in \mathfrak{m}_1$ satisfying $(\xi_1,\xi_1) = 2(\mu,\mu)$, $\eta_1 \neq 0$ and $(\xi_1,\eta_1) = 0$.*

Proof. Let ξ_1 and η_1 be elements of $\mathfrak{m}_1$ satisfying $(\xi_1,\xi_1) = 2(\mu,\mu)$, $\eta_1 \neq 0$ and $(\xi_1,\eta_1) = 0$. Then, the subspace Q defined by (2.5) satisfies $Q \not\subset \mathfrak{m}_2$ and $\dim Q = 2$. Set $\eta_2 = (1/4(\mu,\mu)^2)\bigl[\mu,\,[\xi_1,\eta_1]\bigr]$. Then, it is easily verified that $\eta_2 \in \mathfrak{m}_2$. We now show that Q is pseudo-abelian. By (2.3) and $(\xi_1,\eta_1) = 0$, we have $\bigl[\xi_1,\,[\eta_1,\mu]\bigr] = -\bigl[\eta_1,\,[\xi_1,\mu]\bigr]$. Hence, by the Jacobi identity we have

$$\bigl[\mu,\,[\xi_1,\eta_1]\bigr] = \bigl[[\mu,\xi_1],\eta_1\bigr]+\bigl[\xi_1,\,[\mu,\eta_1]\bigr] = -2\bigl[\xi_1,\,[\eta_1,\mu]\bigr].$$

Consequently, we have $\eta_2 = -(1/2(\mu,\mu)^2)\bigl[\xi_1,\,[\eta_1,\mu]\bigr]$. Note that $[\eta_1,\mu] \in \mathfrak{k}_1$. Then, by the formula (2.4) and the assumption $(\xi_1,\xi_1) = 2(\mu,\mu)$ we have

$$[\xi_1,\eta_2] = -\frac{1}{2(\mu,\mu)^2}\bigl[\xi_1,\,[\xi_1,\,[\eta_1,\mu]]\bigr] = \frac{(\xi_1,\xi_1)}{2(\mu,\mu)}[\eta_1,\mu] = -[\mu,\eta_1].$$

Moreover, since $[\mu,\eta_2] + [\xi_1,\eta_1] \in \mathfrak{k}$ and since

$$\bigl[\mu,\,[\mu,\eta_2]+[\xi_1,\eta_1]\bigr] = -4(\mu,\mu)^2\eta_2+\bigl[\mu,\,[\xi_1,\eta_1]\bigr] = 0,$$

it follows that $[\mu,\eta_2] + [\xi_1,\eta_1] \in \mathfrak{k}_0$. (Note that an element $X \in \mathfrak{k}$ belongs to $\mathfrak{k}_0$ if and only if $[\mu,X] = 0$.) By these relations we have

$$\begin{aligned}[\mu+\xi_1,\eta_1+\eta_2] &= [\mu,\eta_1] + [\xi_1,\eta_2] + [\mu,\eta_2] + [\xi_1,\eta_1] \\ &= 0 + [\mu,\eta_2] + [\xi_1,\eta_1] \in \mathfrak{k}_0.\end{aligned}$$

Since $Q = \boldsymbol{R}(\mu+\xi_1) + \boldsymbol{R}(\eta_1+\eta_2)$, this implies that Q is a pseudo-abelian subspace.

We next prove the converse. Let Q be a pseudo-abelian subspace with $Q \not\subset \mathfrak{m}_2$ and $\dim Q = 2$. Then, viewing the proof of Lemma 5.4 of [6], we know that $Q \cap \mathfrak{m}_2 = 0$ and $\dim(Q \cap (\mathfrak{m}_1+\mathfrak{m}_2)) \leq n(\mu) = 1$. Consequently, we have $Q \not\subset \mathfrak{m}_1+\mathfrak{m}_2$, because $\dim Q = 2$. Therefore, there is a basis $\{\xi,\eta\}$ of Q written in the form $\xi = \mu+\xi_1+\xi_2$, $\eta = \eta_1+\eta_2$, where ξ_1, $\eta_1 \in \mathfrak{m}_1$, ξ_2, $\eta_2 \in \mathfrak{m}_2$. Here, we note that $\eta_1 \neq 0$, because $Q \cap \mathfrak{m}_2 = 0$. Subtracting a constant multiple of η from ξ if necessary, we may assume that $(\xi_1,\eta_1) = 0$.

Since

$$\big[\xi,\eta\big] = \big[\mu+\xi_2,\eta_1\big]+\big[\xi_1,\eta_2\big]+\big[\mu+\xi_2,\eta_2\big]+\big[\xi_1,\eta_1\big] \in \mathfrak{k}_0$$

and since $\big[\mu+\xi_2,\eta_1\big]+\big[\xi_1,\eta_2\big] \in \mathfrak{k}_1$, $\big[\mu+\xi_2,\eta_2\big]+\big[\xi_1,\eta_1\big] \in \mathfrak{k}_0+\mathfrak{k}_2$ and $\big[\xi_2,\eta_2\big] \in \mathfrak{k}_0$, it follows that

$$\big[\mu+\xi_2,\eta_1\big]+\big[\xi_1,\eta_2\big] = 0, \tag{2.6}$$

$$\big[\mu,\eta_2\big]+\big[\xi_1,\eta_1\big] \in \mathfrak{k}_0. \tag{2.7}$$

Applying $\operatorname{ad}\mu$ to (2.7), we have $\eta_2 = (1/4(\mu,\mu)^2)\big[\mu,\big[\xi_1,\eta_1\big]\big]$. By this equality and the assumption $(\xi_1,\eta_1)=0$, we can deduce $\big[\xi_1,\eta_2\big] = \big((\xi_1,\xi_1)/2(\mu,\mu)\big)\big[\eta_1,\mu\big]$ (see the arguments stated above). Putting this into (2.6), we have

$$\left[\left(1-\frac{(\xi_1,\xi_1)}{2(\mu,\mu)}\right)\mu+\xi_2,\eta_1\right] = 0.$$

Since $\eta_1 \neq 0$ and $\operatorname{rank}(P^2(\boldsymbol{H}))=1$, we have $\big(1-(\xi_1,\xi_1)/2(\mu,\mu)\big)\mu+\xi_2=0$. This proves $(\xi_1,\xi_1)=2(\mu,\mu)$ and $\xi_2=0$, completing the proof of the converse. □

3. The Gauss equation

Let $\boldsymbol{N}$ be a euclidean vector space, i.e., $\boldsymbol{N}$ is a vector space over $\boldsymbol{R}$ endowed with an inner product $\langle\ ,\ \rangle$. Let $S^2\mathfrak{m}^*\otimes\boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on $\mathfrak{m}$. We call the following equation on $\boldsymbol{\Psi} \in S^2\mathfrak{m}^*\otimes\boldsymbol{N}$ the *Gauss equation* associated with $\boldsymbol{N}$:

$$\big(\big[\big[X,Y\big],Z\big],W\big) = \big\langle\boldsymbol{\Psi}(X,Z),\boldsymbol{\Psi}(Y,W)\big\rangle-\big\langle\boldsymbol{\Psi}(X,W),\boldsymbol{\Psi}(Y,Z)\big\rangle, \tag{3.1}$$

where X, Y, Z, $W \in \mathfrak{m}$. We denote by $\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$ the set of all solutions of (3.1), which is called the *Gaussian variety* associated with $\boldsymbol{N}$.

As in the case of $P^2(\boldsymbol{Cay})$ (Theorem 11 of [8]), we can prove the following

Theorem 6 *Let $\boldsymbol{N}$ be a euclidean vector space with* $\dim\boldsymbol{N}=6$. *Let $\boldsymbol{\Psi} \in S^2\mathfrak{m}^*\otimes\boldsymbol{N}$ be a solution of the Gauss equation* (3.1), *i.e.,* $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$. *Then:*

(1) *There are linearly independent vectors* $\mathbf{A}$ *and* $\mathbf{B} \in \boldsymbol{N}$ *satisfying*

(i) $\langle\mathbf{A},\mathbf{A}\rangle = \langle\mathbf{B},\mathbf{B}\rangle = 4(\mu,\mu)$ *and* $\langle\mathbf{A},\mathbf{B}\rangle = 2(\mu,\mu)$;

(ii) $\boldsymbol{\Psi}(Y_0, Y_0') = (Y_0, Y_0')\mathbf{A}, \quad \forall Y_0,\, Y_0' \in \mathfrak{a} + \mathfrak{m}_2$;
(iii) $\boldsymbol{\Psi}(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}, \quad \forall Y_1,\, Y_1' \in \mathfrak{m}_1$;
(iv) $\big\langle \mathbf{A}, \boldsymbol{\Psi}(\mu, \mathfrak{m}_1)\big\rangle = \big\langle \mathbf{B}, \boldsymbol{\Psi}(\mu, \mathfrak{m}_1)\big\rangle = 0$.

(2) $\boldsymbol{\Psi}(Y_1, Y_2) = -\dfrac{1}{(\mu,\mu)^2}\boldsymbol{\Psi}(\mu, L(\mu, Y_2)Y_1), \qquad \forall Y_1 \in \mathfrak{m}_1,\ \forall Y_2 \in \mathfrak{m}_2.$

(3) $\big\langle \boldsymbol{\Psi}(\mu, Y_1), \boldsymbol{\Psi}(\mu, Y_1')\big\rangle = (\mu,\mu)^2 (Y_1, Y_1'), \qquad \forall Y_1,\, Y_1' \in \mathfrak{m}_1.$

Let $O(\boldsymbol{N})$ be the orthogonal transformation group of $\boldsymbol{N}$. We define an action of $O(\boldsymbol{N})$ on $S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ by

$$(h\boldsymbol{\Psi})(X, Y) = h(\boldsymbol{\Psi}(X, Y)),$$

where $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$, $h \in O(\boldsymbol{N})$. It is easily seen that $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ is invariant under this action, i.e., $h\,\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N}) = \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ for any $h \in O(\boldsymbol{N})$. We say that the Gaussian variety $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ is *EOS* if $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N}) \neq \emptyset$ and if $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ is consisting of essentially one solution, i.e., for any solutions $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}' \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$, there is an element $h \in O(\boldsymbol{N})$ satisfying $\boldsymbol{\Psi}' = h\boldsymbol{\Psi}$ (see [8]).

By Theorem 6 we can show

Theorem 7 *Let $\boldsymbol{N}$ be a euclidean vector space with* $\dim \boldsymbol{N} = 6$. *Then, $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ is EOS.*

Proof. The proof of this theorem is quite similar to that of Theorem 10 in [8].

First we note that $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N}) \neq \emptyset$, because the second fundamental form of the canonical isometric imbedding $\boldsymbol{f}_0$ at the origin $o \in P^2(\boldsymbol{H})$ satisfies (3.1).

Let $\{E_i\ (1 \le i \le 4)\}$ be an orthonormal basis of $\mathfrak{m}_1$. (Note that $\dim \mathfrak{m}_1 = 4$.) Let $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ and let $\mathbf{A}, \mathbf{B}$ be the vectors of $\boldsymbol{N}$ stated in Theorem 6. We define vectors $\{\mathbf{F}_i\ (1 \le i \le 6)\}$ of $\boldsymbol{N}$ by setting $\mathbf{F}_i = \boldsymbol{\Psi}(\mu, E_i)/(\mu,\mu)$ $(1 \le i \le 4)$, $\mathbf{F}_5 = (\mathbf{A}+\mathbf{B})/2\sqrt{3}\,|\mu|$ and $\mathbf{F}_6 = (\mathbf{A}-\mathbf{B})/2|\mu|$. By Theorem 6 we can show that $\{\mathbf{F}_i\ (1 \le i \le 6)\}$ forms an orthonormal basis of $\boldsymbol{N}$. Now let $\boldsymbol{\Psi}'$ be another element of $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. Let $\mathbf{A}'$ and $\mathbf{B}'$ be the vectors stated in Theorem 6 for $\boldsymbol{\Psi}'$. As in the case of $\boldsymbol{\Psi}$ we can also define an orthonormal basis $\{\mathbf{F}_i'\ (1 \le i \le 6)\}$ of $\boldsymbol{N}$. Then, there is an element $h \in O(6)$ satisfying $\mathbf{F}_i' = h\mathbf{F}_i$ $(1 \le i \le 6)$. Here, we note that $\mathbf{A}' = h\mathbf{A}$, $\mathbf{B}' = h\mathbf{B}$ and $\boldsymbol{\Psi}'(\mu, E_i) = h\boldsymbol{\Psi}(\mu, E_i)$ $(1 \le i \le 4)$. Set $\boldsymbol{\Phi} =$

$\mathbf{\Psi}' - h\mathbf{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$. Then, by Theorem 6 (1) we have

$$\mathbf{\Phi}(\mathfrak{a}+\mathfrak{m}_2, \mathfrak{a}+\mathfrak{m}_2) = \mathbf{\Phi}(\mathfrak{m}_1, \mathfrak{m}_1) = \mathbf{\Phi}(\mathfrak{a}, \mathfrak{m}_1) = 0.$$

By Theorem 6 (2) and by the fact $L(\mu, \mathfrak{m}_2)\mathfrak{m}_1 \subset \mathfrak{m}_1$ we have

$$\mathbf{\Phi}(\mathfrak{m}_2, \mathfrak{m}_1) \subset \mathbf{\Phi}(\mu, L(\mu, \mathfrak{m}_2)\mathfrak{m}_1) \subset \mathbf{\Phi}(\mathfrak{a}, \mathfrak{m}_1) = 0,$$

which proves $\mathbf{\Phi}(\mathfrak{m}_2, \mathfrak{m}_1) = 0$. Therefore, we have $\mathbf{\Phi} = 0$, i.e., $\mathbf{\Psi}' = h\mathbf{\Psi}$, completing the proof of Theorem 7. □

By Theorem 7 we know that $P^2(\boldsymbol{H})$ is formally rigid in codimension 6 in the sense of Agaoka-Kaneda [8]. Therefore, Theorem 1 can be obtained by Theorem 7 and the rigidity theorem (Theorem 5 of [8]).

Before proceeding to the proof of Theorem 6, we make several preparations.

Let $\boldsymbol{N}$ be a euclidean vector space. In what follows we assume $\dim \boldsymbol{N} = 6$. Let $S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on $\mathfrak{m}$. Let $\mathbf{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ and $Y \in \mathfrak{m}$. We define a linear map $\mathbf{\Psi}_Y$ of $\mathfrak{m}$ to $\boldsymbol{N}$ by

$$\mathbf{\Psi}_Y : \mathfrak{m} \ni Y' \longmapsto \mathbf{\Psi}(Y, Y') \in \boldsymbol{N},$$

and denote by $\mathbf{Ker}(\mathbf{\Psi}_Y)$ the kernel of $\mathbf{\Psi}_Y$. We call an element $Y \in \mathfrak{m}$ *singular* (resp. *non-singular*) with respect to $\mathbf{\Psi}$ if $\mathbf{\Psi}_Y(\mathfrak{m}) \neq \boldsymbol{N}$ (resp. $\mathbf{\Psi}_Y(\mathfrak{m}) = \boldsymbol{N}$).

Let $\mathbf{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ and let $Y \in \mathfrak{m}$ ($Y \neq 0$). Take an element $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Then, as shown in the proof of Proposition 5 of [7], the subspace $Q_Y = \mathrm{Ad}(k)^{-1}\,\mathbf{Ker}(\mathbf{\Psi}_Y)$ is a pseudo-abelian subspace of $\mathfrak{m}$.

Proposition 8 *Let* $\mathbf{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ *and let* $Y \in \mathfrak{m}$ ($Y \neq 0$). *Then:*
(1) $\dim \mathbf{Ker}(\mathbf{\Psi}_Y) = 2$ *or* 3. *Moreover,* Y *is non-singular* (*resp. singular*) *with respect to* $\mathbf{\Psi}$ *if and only if* $\dim \mathbf{Ker}(\mathbf{\Psi}_Y) = 2$ (*resp.* $\dim \mathbf{Ker}(\mathbf{\Psi}_Y) = 3$).
(2) *Let* $k \in K$ *satisfy* $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. *Then,* $\mathbf{Ker}(\mathbf{\Psi}_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$. *Consequently,* Y *is non-singular* (*resp. singular*) *with respect to* $\mathbf{\Psi}$ *if and only if* $\mathbf{Ker}(\mathbf{\Psi}_Y) \subsetneq \mathrm{Ad}(k)\mathfrak{m}_2$ (*resp.* $\mathbf{Ker}(\mathbf{\Psi}_Y) = \mathrm{Ad}(k)\mathfrak{m}_2$).

Remark 1 Recall that in the case of the Cayley projective plane $P^2(\boldsymbol{Cay})$ the inclusion $\mathbf{Ker}(\mathbf{\Psi}_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$ in Proposition 8 (2) can be proved by a simple discussion. There, the inclusion automatically follows from the fact that any high-dimensional pseudo-abelian subspace must be contained in $\mathfrak{m}_2$ (see Propositions 8 and 12 of [8]). In contrast, it is not a simple task to show the inclusion $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$ in our case $P^2(\boldsymbol{H})$. We will prove this inclusion by making use of the normal form of the pseudo-abelian subspaces not contained in $\mathfrak{m}_2$ (see Proposition 5).

Proof of Proposition 8. Let $Y \in \mathfrak{m}$ ($Y \neq 0$). Set $Q_Y = \mathrm{Ad}(k)^{-1}\,\mathbf{Ker}(\boldsymbol{\Psi}_Y)$, where $k \in K$ is an element satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Since Q_Y is pseudo-abelian, it follows that $\dim Q_Y \leq 3$ (see Proposition 4). Hence, $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \leq 3$. On the other hand, since $\dim \boldsymbol{N} = 6$ and $\dim \mathfrak{m} = 8$, it follows that $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \geq 2$. Therefore, Y is non-singular (resp. singular) with respect to $\boldsymbol{\Psi}$ if and only if $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 2$ (resp. $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 3$). This proves (1).

To show the first statement of (2) it suffices to prove $Q_Y \subset \mathfrak{m}_2$. Now, let us suppose the contrary, i.e., $Q_Y \not\subset \mathfrak{m}_2$. Then, we have $\dim Q_Y = 2$ (see (1) and Proposition 4 (2)). Hence, there is a basis $\{\xi, \eta\}$ of Q_Y written in the form $\xi = \mu + \xi_1$, $\eta = \eta_1 + (1/4(\mu,\mu)^2)\big[\mu, [\xi_1, \eta_1]\big]$, where ξ_1 and η_1 are elements of $\mathfrak{m}_1$ satisfying $(\xi_1, \xi_1) = 2(\mu, \mu)$, $\eta_1 \neq 0$, $(\xi_1, \eta_1) = 0$ (see Proposition 5). Let $\{\zeta_1^1, \zeta_1^2\}$ be a basis of the orthogonal complement of $\boldsymbol{R}\xi_1 + \boldsymbol{R}\eta_1$ in $\mathfrak{m}_1$. Set $\zeta^i = \zeta_1^i + (1/4(\mu,\mu)^2)\big[\mu, [\xi_1, \zeta_1^i]\big]$ ($i = 1, 2$). Since $\big[\mu, [\xi_1, \zeta_1^i]\big] \in \mathfrak{m}_2$ ($i = 1, 2$), we know that the vectors ζ^1 and ζ^2 are linearly independent. More strongly, they are linearly independent modulo Q_Y, i.e., $Q_Y \cap (\boldsymbol{R}\zeta^1 + \boldsymbol{R}\zeta^2) = 0$. Moreover, by Proposition 5 we know that the subspace $Q^i = \boldsymbol{R}\xi + \boldsymbol{R}\zeta^i$ ($i = 1, 2$) is also pseudo-abelian, because $(\xi_1, \zeta_1^i) = 0$. Consequently, we have $\big[[\xi, \zeta^i], \mu\big] = 0$ ($i = 1, 2$).

Set $X = \mathrm{Ad}(k)\xi$, $Z^i = \mathrm{Ad}(k)\zeta^i$ ($i = 1, 2$). Then, we have $X \in \mathbf{Ker}(\boldsymbol{\Psi}_Y)$ ($X \neq 0$), $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \cap (\boldsymbol{R}Z^1 + \boldsymbol{R}Z^2) = 0$ and $\big[[X, Z^i], Y\big] = 0$ ($i = 1, 2$). By the Gauss equation (3.1) we have

$$\begin{aligned} 0 &= \big(\big[[X, Z^i], Y\big], W\big) \\ &= \big\langle \boldsymbol{\Psi}(X, Y), \boldsymbol{\Psi}(Z^i, W)\big\rangle - \big\langle \boldsymbol{\Psi}(X, W), \boldsymbol{\Psi}(Z^i, Y)\big\rangle, \quad (i = 1, 2), \end{aligned}$$

where W is an arbitrary element of $\mathfrak{m}$. Since $\boldsymbol{\Psi}_Y(X) = 0$, we obtain by this equality $\big\langle \boldsymbol{\Psi}_X(W), \boldsymbol{\Psi}(Z^i, Y)\big\rangle = 0$, i.e., $\big\langle \boldsymbol{\Psi}_X(\mathfrak{m}), \boldsymbol{\Psi}(Z^i, Y)\big\rangle = 0$ ($i = 1, 2$). We note that the vectors $\boldsymbol{\Psi}(Z^1, Y)$ and $\boldsymbol{\Psi}(Z^2, Y)$ are linearly independent, because $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \cap (\boldsymbol{R}Z^1 + \boldsymbol{R}Z^2) = 0$. Hence, we have $\dim \boldsymbol{\Psi}_X(\mathfrak{m}) \leq \dim \boldsymbol{N} - 2 = 4$, implying $\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \geq 4$. This contradicts the assertion (1). Thus, we have $Q_Y \subset \mathfrak{m}_2$, proving the first statement of (2). The last statement of (2) is now clear. □

As a corollary of Proposition 8 we obtain

Proposition 9 *Let* $\mathbf{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. *Then:*
(1) *Let* $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ $(Y_0 \neq 0)$. *Then,* $\mathbf{Ker}(\mathbf{\Psi}_{Y_0}) \subset \{\xi \in \mathfrak{a} + \mathfrak{m}_2 \,|\, (\xi, Y_0) = 0\}$. *If* Y_0 *is singular with respect to* $\mathbf{\Psi}$, *then* $\mathbf{Ker}(\mathbf{\Psi}_{Y_0}) = \{\xi \in \mathfrak{a} + \mathfrak{m}_2 \,|\, (\xi, Y_0) = 0\}$.
(2) *Let* $Y_1 \in \mathfrak{m}_1$ $(Y_1 \neq 0)$. *Then,* $\mathbf{Ker}(\mathbf{\Psi}_{Y_1}) \subset \{\eta \in \mathfrak{m}_1 \,|\, (\eta, Y_1) = 0\}$. *If* Y_1 *is singular with respect to* $\mathbf{\Psi}$, *then* $\mathbf{Ker}(\mathbf{\Psi}_{Y_1}) = \{\eta \in \mathfrak{m}_1 \,|\, (\eta, Y_1) = 0\}$.

Proof. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ $(Y_0 \neq 0)$. Then, we can take an element $k_0 \in K$ such that $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$ and $\mathrm{Ad}(k_0)(\mathfrak{m}_2) = \{\xi \in \mathfrak{a} + \mathfrak{m}_2 \,|\, (\xi, Y_0) = 0\}$ (see Proposition 7 of [7]). This proves (1). Similarly, for $Y_1 \in \mathfrak{m}_1$ $(Y_1 \neq 0)$, we can easily show (2). □

Let $\mathbf{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$. We call a subspace U of $\mathfrak{m}$ *singular* with respect to $\mathbf{\Psi}$ if each element of U is singular with respect to $\mathbf{\Psi}$.

Proposition 10 *Let* $\mathbf{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. *Assume that* $Y \in \mathfrak{m}$ $(Y \neq 0)$ *is non-singular with respect to* $\mathbf{\Psi}$. *Then, there is a non-zero vector* $\mathbf{E} \in \boldsymbol{N}$ *such that*

$$\boldsymbol{N} = \boldsymbol{R}\mathbf{E} \dot{+} \mathbf{\Psi}_\xi(\mathfrak{m}) \qquad (\textit{orthogonal direct sum}) \tag{3.2}$$

holds for any $\xi \in \mathbf{Ker}(\mathbf{\Psi}_Y)$ $(\xi \neq 0)$. *Consequently,* $\mathbf{Ker}(\mathbf{\Psi}_Y)$ *is a singular subspace with respect to* $\mathbf{\Psi}$.

Proof. Take an element $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Then, since Y is non-singular, we have $\mathbf{Ker}(\mathbf{\Psi}_Y) \subsetneq \mathrm{Ad}(k)\mathfrak{m}_2$. Take a non-zero element satisfying $Y' \in \mathrm{Ad}(k)\mathfrak{m}_2$ and $Y' \notin \mathbf{Ker}(\mathbf{\Psi}_Y)$ and set $\mathbf{E} = \mathbf{\Psi}(Y, Y')$ $(\neq 0)$. Let $\xi \in \mathbf{Ker}(\mathbf{\Psi}_Y)$ $(\xi \neq 0)$. Then, by the Gauss equation (3.1) we have

$$\big(\big[[\xi, Y'], Y\big], W\big) = \big\langle \mathbf{\Psi}(\xi, Y), \mathbf{\Psi}(Y', W)\big\rangle - \big\langle \mathbf{\Psi}(\xi, W), \mathbf{\Psi}(Y', Y)\big\rangle,$$

where W is an arbitrary element of $\mathfrak{m}$. Here, we note that $\big[[\xi, Y'], Y\big] = 0$, because $\big[[\xi, Y'], Y\big] \in \mathrm{Ad}(k)\big[[\mathfrak{m}_2, \mathfrak{m}_2], \mu\big] = 0$. Since $\mathbf{\Psi}(\xi, Y) = 0$, we obtain by the above equality $\big\langle \mathbf{E}, \mathbf{\Psi}(\xi, W)\big\rangle = 0$. This shows $\big\langle \mathbf{E}, \mathbf{\Psi}_\xi(\mathfrak{m})\big\rangle = 0$ and hence $\mathbf{\Psi}_\xi(\mathfrak{m}) \neq \boldsymbol{N}$. Consequently, ξ is singular with respect to $\mathbf{\Psi}$. Since $\dim \mathbf{Ker}(\mathbf{\Psi}_\xi) = 3$ (see Proposition 8), we have $\dim \mathbf{\Psi}_\xi(\mathfrak{m}) = 5$, which proves the decomposition (3.2). □

4. Proof of Theorem 6

In this section, with the preparations in the previous sections, we will prove Theorem 6. We first show

Proposition 11 *Let $\mathbf{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. Then, there are singular subspaces U $(\subset \mathfrak{a} + \mathfrak{m}_2)$ and V $(\subset \mathfrak{m}_1)$ with respect to $\mathbf{\Psi}$ satisfying* $\dim U \geq 2$ *and* $\dim V \geq 2$.

Proof. If $\mathfrak{a} + \mathfrak{m}_2$ contains no non-singular element with respect to $\mathbf{\Psi}$, then set $U = \mathfrak{a} + \mathfrak{m}_2$. On the contrary, if there is a non-singular element $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$, then set $U = \mathbf{Ker}(\mathbf{\Psi}_{Y_0})$. In this case we know that $\dim U = 2$, $U \subset \mathfrak{a} + \mathfrak{m}_2$ and that U is a singular subspace with respect to $\mathbf{\Psi}$ (see Proposition 8, Proposition 9 and Proposition 10).

Similarly, we can show that there is a singular subspace V of $\mathfrak{m}_1$ with respect to $\mathbf{\Psi}$ satisfying the desired properties. □

Proposition 12 *Let $\mathbf{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. Let U $(\subset \mathfrak{a}+\mathfrak{m}_2)$ and V $(\subset \mathfrak{m}_1)$ be singular subspaces with respect to $\mathbf{\Psi}$ satisfying* $\dim U \geq 2$ *and* $\dim V \geq 2$. *Then, there are vectors $\mathbf{A}$, $\mathbf{B} \in \boldsymbol{N}$ such that:*

(1) $\langle \mathbf{A}, \mathbf{A} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$.

(2) *Let $\xi \in U$ and $\eta \in V$. Then:*

(2a) $\mathbf{\Psi}(\xi, Y_0) = (\xi, Y_0)\mathbf{A}, \quad \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2$;

(2b) $\mathbf{\Psi}(\eta, Y_1) = (\eta, Y_1)\mathbf{B}, \quad \forall Y_1 \in \mathfrak{m}_1$.

(3) *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Then:*

(3a) $\langle \mathbf{A}, \mathbf{\Psi}_{Y_0}(\mathfrak{m}_1) \rangle = \langle \mathbf{B}, \mathbf{\Psi}_{Y_0}(\mathfrak{m}_1) \rangle = 0$;

(3b) $\langle \mathbf{A}, \mathbf{\Psi}_{Y_1}(\mathfrak{a} + \mathfrak{m}_2) \rangle = \langle \mathbf{B}, \mathbf{\Psi}_{Y_1}(\mathfrak{a} + \mathfrak{m}_2) \rangle = 0$.

(4) *Let $\xi \in U$ $(\xi \neq 0)$ and $\eta \in V$ $(\eta \neq 0)$. Then:*

(4a) $\mathbf{\Psi}_\xi(\mathfrak{m}) = \boldsymbol{R}\mathbf{A} + \mathbf{\Psi}_\xi(\mathfrak{m}_1)$ *(orthogonal direct sum)*;

(4b) $\mathbf{\Psi}_\eta(\mathfrak{m}) = \boldsymbol{R}\mathbf{B} + \mathbf{\Psi}_\eta(\mathfrak{a} + \mathfrak{m}_2)$ *(orthogonal direct sum)*.

(5) *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Then:*

(5a) $\langle \mathbf{\Psi}(Y_0, Y_0), \mathbf{A} \rangle = 4(\mu, \mu)(Y_0, Y_0)$;

(5b) $\langle \mathbf{\Psi}(Y_1, Y_1), \mathbf{B} \rangle = 4(\mu, \mu)(Y_1, Y_1)$.

(6) *Let $\xi \in U$, $\eta \in V$, $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Assume that $(\xi, Y_0) = (\eta, Y_1) = 0$. Then:*

(6a) $\langle \mathbf{\Psi}(Y_0, Y_0), \mathbf{\Psi}_\xi(\mathfrak{m}_1) \rangle = 0$;

(6b) $\langle \mathbf{\Psi}(Y_1, Y_1), \mathbf{\Psi}_\eta(\mathfrak{a} + \mathfrak{m}_2) \rangle = 0$.

Proof. The assertions (1), (2) and (3) can be proved in the same manner as in the proof of Proposition 16 of [8]. Hence, we omit their proofs.

Let $\xi \in U$ ($\xi \neq 0$). By (2a) we easily get $\boldsymbol{\Psi}_\xi(\mathfrak{a}+\mathfrak{m}_2) = \boldsymbol{R}\mathbf{A}$ and hence $\boldsymbol{\Psi}_\xi(\mathfrak{m}) = \boldsymbol{R}\mathbf{A} + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$. Since $\langle \mathbf{A}, \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)\rangle = 0$ (see (3a)), we have the decomposition (4a). Similarly, we can show (4b).

The assertions (5a) and (6a) are proved as follows: Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$. Take $\xi \in U$ ($\xi \neq 0$) such that $(\xi, Y_0) = 0$. Then, we have $\big[[Y_0,\xi],Y_0\big] = 4(\mu,\mu)(Y_0,Y_0)\xi$ (see (2.2)) and $\boldsymbol{\Psi}(\xi, Y_0) = 0$ (see (2a)). By the Gauss equation (3.1) we have

$$\big(\big[[Y_0,\xi],Y_0\big],\xi\big) = \big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(\xi,\xi)\big\rangle - \big\langle \boldsymbol{\Psi}(Y_0,\xi), \boldsymbol{\Psi}(\xi,Y_0)\big\rangle,$$
$$\big(\big[[Y_0,\xi],Y_0\big],Y_1'\big) = \big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(\xi,Y_1')\big\rangle - \big\langle \boldsymbol{\Psi}(Y_0,Y_1'), \boldsymbol{\Psi}(\xi,Y_0)\big\rangle,$$

where Y_1' is an arbitrary element of $\mathfrak{m}_1$. By these equalities we have $\big\langle \boldsymbol{\Psi}(Y_0,Y_0), \mathbf{A}\big\rangle = 4(\mu,\mu)(Y_0,Y_0)$ and $\big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(\xi,Y_1')\big\rangle = 0$. Therefore, we obtain (5a) and (6a). The assertions (5b) and (6b) can be proved in a similar way. □

Remark 2 As seen in the proof of Proposition 11, singular subspaces U and V may not be uniquely determined. However, the vectors $\mathbf{A}$ and $\mathbf{B}$ in Proposition 8 do not depend on the choice of singular subspaces U and V, which will be clarified at the last part of this section (see Lemma 20).

In the following argument, we take and fix an element $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$. We denote by U and V singular subspaces with respect to $\boldsymbol{\Psi}$ satisfying U ($\subset \mathfrak{a}+\mathfrak{m}_2$), V ($\subset \mathfrak{m}_1$), $\dim U \geq 2$ and $\dim V \geq 2$. We also denote by $\mathbf{A}$, $\mathbf{B}$ the vectors of $\boldsymbol{N}$ obtained by applying Proposition 12 to the pair of singular subspaces U and V.

Lemma 13 (1) *Let* $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$. *Then:*

$$\begin{aligned}&\big\langle \boldsymbol{\Psi}_{Y_0}(Y_1), \boldsymbol{\Psi}_{Y_0}(Y_1')\big\rangle \\ &\quad = \big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(Y_1,Y_1')\big\rangle - (\mu,\mu)(Y_0,Y_0)(Y_1,Y_1'), \quad \forall Y_1,\, Y_1' \in \mathfrak{m}_1.\end{aligned}$$

(2) *Let* $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ *and* $\xi \in U$ *satisfy* $(\xi, Y_0) = 0$. *Then:*

$$\big\langle \boldsymbol{\Psi}_{Y_0}(Y_1), \boldsymbol{\Psi}_\xi(Y_1')\big\rangle = (L(Y_0,\xi)Y_1, Y_1'), \quad \forall Y_1,\, Y_1' \in \mathfrak{m}_1.$$

Proof. Putting $X = Y_0$, $Y = Y_1$, $Z = Y_0$, $W = Y_1'$ into (3.1), we have

$$\big(\big[[Y_0,Y_1],Y_0\big],Y_1'\big) = \big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(Y_1,Y_1')\big\rangle - \big\langle \boldsymbol{\Psi}(Y_0,Y_1'), \boldsymbol{\Psi}(Y_1,Y_0)\big\rangle.$$

Since $\big[Y_0,\big[Y_0,Y_1\big]\big] = -(\mu,\mu)(Y_0,Y_0)Y_1$ (see (2.2)), we easily get (1).

Similarly, putting $X = \xi$, $Y = Y_1$, $Z = Y_0$ and $W = Y_1'$ into (3.1), we have

$$\begin{aligned}\big(\big[[\xi, Y_1], Y_0\big], Y_1'\big) &= \big\langle \boldsymbol{\Psi}(\xi, Y_0), \boldsymbol{\Psi}(Y_1, Y_1')\big\rangle - \big\langle \boldsymbol{\Psi}(\xi, Y_1'), \boldsymbol{\Psi}(Y_1, Y_0)\big\rangle \\ &= \big\langle \mathbf{A}, \boldsymbol{\Psi}(Y_1, Y_1')\big\rangle(\xi, Y_0) - \big\langle \boldsymbol{\Psi}_\xi(Y_1'), \boldsymbol{\Psi}_{Y_0}(Y_1)\big\rangle.\end{aligned}$$

Since $(\xi, Y_0) = 0$, we have

$$\big\langle \boldsymbol{\Psi}_\xi(Y_1'), \boldsymbol{\Psi}_{Y_0}(Y_1)\big\rangle = -\big(\big[[\xi, Y_1], Y_0\big], Y_1'\big) = (L(Y_0, \xi)Y_1, Y_1'),$$

proving (2). □

Let $\xi \in U$ ($\xi \neq 0$). Since $\dim \mathbf{Ker}(\boldsymbol{\Psi}_\xi) = 3$ (see Proposition 8) and since $\dim \mathfrak{m} = 8$, we have $\dim \boldsymbol{\Psi}_\xi(\mathfrak{m}) = 5$. Let us denote by $\boldsymbol{E}_\xi$ the one dimensional orthogonal complement of $\boldsymbol{\Psi}_\xi(\mathfrak{m})$ in $\boldsymbol{N}$.

Proposition 14 *Set $C = \langle \mathbf{A}, \mathbf{B}\rangle - (\mu, \mu)$. Then:*
(1) *Let $\xi \in U$. Then:*

$$\big\langle \boldsymbol{\Psi}_\xi(Y_1), \boldsymbol{\Psi}_\xi(\eta)\big\rangle = C(\xi, \xi)(Y_1, \eta), \qquad \forall Y_1 \in \mathfrak{m}_1,\ \forall \eta \in V. \tag{4.1}$$

(2) *The inequality $0 < C \leq 3(\mu, \mu)$ holds. The vectors $\mathbf{A}$ and $\mathbf{B}$ are linearly independent if $C \neq 3(\mu, \mu)$ and $\mathbf{A} = \mathbf{B}$ if $C = 3(\mu, \mu)$.*
(3) *Let $\xi \in U$ ($\xi \neq 0$). Then, $\boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) \subset \boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$, $\quad \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2$.*
(4) *If $C \neq 3(\mu, \mu)$, then:*

$$\boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) = \boldsymbol{\Psi}_\xi(\mathfrak{m}_1), \qquad \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2\ (Y_0 \neq 0),\ \forall \xi \in U\ (\xi \neq 0); \tag{4.2}$$

$$\boldsymbol{\Psi}(Y_0, Y_0) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}, \qquad \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2; \tag{4.3}$$

$$\boldsymbol{\Psi}(Y_1, Y_1) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}, \qquad \forall Y_1 \in \mathfrak{m}_1. \tag{4.4}$$

Proof. Put $Y_0 = \xi$ and $Y_1' = \eta$ into Lemma 13 (1). Then, since $\boldsymbol{\Psi}(\xi, \xi) = (\xi, \xi)\mathbf{A}$ and $\boldsymbol{\Psi}(Y_1, \eta) = (Y_1, \eta)\mathbf{B}$, we get (4.1).

In view of Proposition 12 (1), we easily have $\langle \mathbf{A}, \mathbf{B}\rangle \leq 4(\mu, \mu)$ and hence $C \leq 3(\mu, \mu)$. Further, by putting $Y_1 = \eta$ ($\neq 0$) into (4.1) we know $C > 0$, because $\boldsymbol{\Psi}_\xi(\eta) \neq 0$ (see Proposition 9). This shows $\langle \mathbf{A}, \mathbf{B}\rangle > (\mu, \mu)$. Therefore, $\mathbf{A}$ and $\mathbf{B}$ are linearly independent if $\langle \mathbf{A}, \mathbf{B}\rangle \neq 4(\mu, \mu)$, i.e., $C \neq 3(\mu, \mu)$. It is easy to see that if $C = 3(\mu, \mu)$, i.e., $\langle \mathbf{A}, \mathbf{B}\rangle = 4(\mu, \mu)$, then $\mathbf{A} = \mathbf{B}$.

We next prove (3). Let $\xi \in U$ ($\xi \neq 0$). By Proposition 12 ($4a$) we know that the orthogonal complement of $\boldsymbol{R}\mathbf{A}$ in $\boldsymbol{N}$ is given by $\boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$.

Hence, by Proposition 12 (3a), we have $\boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) \subset \boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$ for any $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$.

Finally, we prove (4). Since $C \neq 3(\mu, \mu)$, the subspace $\boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ forms a 2-dimensional subspace of $\boldsymbol{N}$. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ ($Y_0 \neq 0$). Then, by Proposition 12 (3a) we know that $\boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1)$ coincides with the orthogonal complement of $\boldsymbol{R}\mathbf{A}+\boldsymbol{R}\mathbf{B}$ in $\boldsymbol{N}$. (Recall that $\dim \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) = 4$ and $\dim \boldsymbol{N} = 6$.) Let $\xi \in U$ ($\xi \neq 0$). Since $\boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$ is also an orthogonal complement of $\boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$, it follows that $\boldsymbol{\Psi}_\xi(\mathfrak{m}_1) = \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1)$. If we take $\xi \in U$ ($\xi \neq 0$) satisfying $(\xi, Y_0) = 0$, then by Proposition 12 (6a) we obtain $\boldsymbol{\Psi}(Y_0, Y_0) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$. Similarly, we can prove $\boldsymbol{\Psi}(Y_1, Y_1) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ for any $Y_1 \in \mathfrak{m}_1$, completing the proof of (4). □

Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $\xi \in U$ ($\xi \neq 0$). Define a linear mapping $\boldsymbol{\Theta}_{Y_0,\xi}\colon \mathfrak{m}_1 \longrightarrow \boldsymbol{N}$ by

$$\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) = \boldsymbol{\Psi}_{Y_0}(Y_1) + \frac{1}{C(\xi,\xi)}\boldsymbol{\Psi}_\xi(L(\xi, Y_0)Y_1), \quad Y_1 \in \mathfrak{m}_1. \tag{4.5}$$

Then, we have

Proposition 15 *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$, $\xi \in U$ ($\xi \neq 0$) and $Y_1 \in \mathfrak{m}_1$. Assume that $(\xi, Y_0) = 0$ and $L(\xi, Y_0)Y_1 \in V$. Then:*
(1) $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{E}_\xi$. *More strongly, if $C \neq 3(\mu, \mu)$, then $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) = 0$.*
(2) $|\boldsymbol{\Theta}_{Y_0,\xi}(Y_1)|^2 = \big\langle \boldsymbol{\Psi}(Y_0, Y_0), \boldsymbol{\Psi}(Y_1, Y_1)\big\rangle - (\mu, \mu)\big\{1 + (\mu, \mu)/C\big\}(Y_0, Y_0)(Y_1, Y_1)$.

Proof. By Proposition 14 (3) we know that $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$. Here, we note that $\langle \boldsymbol{E}_\xi, \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)\rangle = 0$, because $\boldsymbol{E}_\xi$ is orthogonal to $\boldsymbol{\Psi}_\xi(\mathfrak{m})$. Let $Y_1' \in \mathfrak{m}_1$. Then, by Lemma 13 (2), Proposition 14 (1) and Proposition 3 (2) we have

$$\begin{aligned}
&\big\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_\xi(Y_1')\big\rangle \\
&\quad = \big\langle \boldsymbol{\Psi}_{Y_0}(Y_1), \boldsymbol{\Psi}_\xi(Y_1')\big\rangle + \frac{1}{C(\xi,\xi)}\big\langle \boldsymbol{\Psi}_\xi(L(\xi, Y_0)Y_1), \boldsymbol{\Psi}_\xi(Y_1')\big\rangle \\
&\quad = (L(Y_0, \xi)Y_1, Y_1') + (L(\xi, Y_0)Y_1, Y_1') \\
&\quad = 0,
\end{aligned}$$

proving $\big\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)\big\rangle = 0$. This implies that $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{E}_\xi$. In the case where $C \neq 3(\mu, \mu)$, we have $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1) = \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$ (see (4.2)), which proves $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) = 0$.

Next, we show (2). By Lemma 13 and by the equality $\big\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)\big\rangle$

$= 0$, we have

$$\begin{aligned}
&\big\langle \mathbf{\Theta}_{Y_0,\xi}(Y_1), \mathbf{\Theta}_{Y_0,\xi}(Y_1)\big\rangle \\
&\quad = \big\langle \mathbf{\Theta}_{Y_0,\xi}(Y_1), \mathbf{\Psi}_{Y_0}(Y_1)\big\rangle \\
&\quad = \big\langle \mathbf{\Psi}_{Y_0}(Y_1), \mathbf{\Psi}_{Y_0}(Y_1)\big\rangle + \frac{1}{C(\xi,\xi)}\big\langle \mathbf{\Psi}_{\xi}(L(\xi,Y_0)Y_1), \mathbf{\Psi}_{Y_0}(Y_1)\big\rangle \\
&\quad = \big\langle \mathbf{\Psi}(Y_0,Y_0), \mathbf{\Psi}(Y_1,Y_1)\big\rangle - (\mu,\mu)(Y_0,Y_0)(Y_1,Y_1) \\
&\qquad\qquad + \frac{1}{C(\xi,\xi)}\big(L(\xi,Y_0)Y_1, L(Y_0,\xi)Y\big).
\end{aligned}$$

On the other hand, by Proposition 3 we have

$$\begin{aligned}
\big(L(\xi,Y_0)Y_1, L(Y_0,\xi)Y_1\big) &= \big(L(\xi,Y_0)L(\xi,Y_0)Y_1, Y_1\big) \\
&= -\big(L(Y_0,\xi)L(\xi,Y_0)Y_1, Y_1\big) \\
&= -(\mu,\mu)^2(\xi,\xi)(Y_0,Y_0)(Y_1,Y_1).
\end{aligned}$$

Therefore, we get the assertion (2). □

With these preparations we begin with the proof Theorem 6. First, we consider the case $\dim V = 2$.

Lemma 16 *Assume that* $\dim V = 2$*. Then,* $C \neq 3(\mu,\mu)$*. Accordingly, the vectors* $\mathbf{A}$ *and* $\mathbf{B} \in \boldsymbol{N}$ *are linearly independent.*

Proof. Take non-zero elements ξ, $\xi' \in U$ satisfying $(\xi,\xi') = 0$. Then, by Proposition 3 (2) it follows that $L(\xi,\xi') = -L(\xi',\xi)$ and $L(\xi,\xi')$ gives an isomorphism of $\mathfrak{m}_1$ onto itself. Let $Y_1 \in L(\xi,\xi')V$. Then, by Proposition 3 (2b) we have $L(\xi,\xi')Y_1 \in V$. Hence, by Proposition 15 (1) we have $\mathbf{\Theta}_{\xi',\xi}(Y_1) \in \boldsymbol{E}_\xi$. Since $\dim L(\xi,\xi')V = \dim V = 2$ and $\dim \boldsymbol{E}_\xi = 1$, it is possible to take a non-zero element $Y_1 \in L(\xi,\xi')V$ satisfying $\mathbf{\Theta}_{\xi',\xi}(Y_1) = 0$. Therefore, by Proposition 15 (2) and Proposition 12 (2a) we have

$$\begin{aligned}
0 &= |\mathbf{\Theta}_{\xi',\xi}(Y_1)|^2 \\
&= \big[\big\langle \mathbf{\Psi}(Y_1,Y_1), \mathbf{A}\big\rangle - (\mu,\mu)\{1 + (\mu,\mu)/C\}(Y_1,Y_1)\big](\xi',\xi').
\end{aligned}$$

Since $(\xi',\xi') \neq 0$, we have

$$\big\langle \mathbf{\Psi}(Y_1,Y_1), \mathbf{A}\big\rangle = (\mu,\mu)\{1+(\mu,\mu)/C\}(Y_1,Y_1). \tag{4.6}$$

Now, we suppose the case $C = 3(\mu,\mu)$. Then, by (4.6) we have $\big\langle \mathbf{\Psi}(Y_1,Y_1), \mathbf{A}\big\rangle = \frac{4}{3}(\mu,\mu)(Y_1,Y_1)$. On the other hand, by Proposition 12 (5b)

we have $\big\langle \mathbf{\Psi}(Y_1, Y_1), \mathbf{A} \big\rangle = 4(\mu,\mu)(Y_1, Y_1)$, because $\mathbf{A} = \mathbf{B}$ in case $C = 3(\mu,\mu)$ (see Proposition 14 (2)). Hence, we have $(Y_1, Y_1) = 0$, which contradicts the assumption $Y_1 \neq 0$. Therefore, we have $C \neq 3(\mu,\mu)$ and hence $\mathbf{A}$ and $\mathbf{B}$ are linearly independent. □

Lemma 17 *Assume that* $\dim V = 2$. *Then,* V *can be extended to a* 3*-dimensional singular subspace contained in* $\mathfrak{m}_1$*, i.e., there is a singular subspace* $\widehat{V}$ $(\subset \mathfrak{m}_1)$ *such that* $V \subset \widehat{V}$ *and* $\dim \widehat{V} = 3$*.*

Proof. Let $\mathbf{F} \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ be a unit vector which is orthogonal to $\mathbf{B}$. Then, for any $\eta \in V$ we have $\big\langle \mathbf{F}, \mathbf{\Psi}_\eta(\mathfrak{m}) \big\rangle = 0$, because $\big\langle \mathbf{F}, \mathbf{\Psi}_\eta(\mathfrak{m}) \big\rangle = \big\langle \mathbf{F}, \boldsymbol{R}\mathbf{B} + \mathbf{\Psi}_\eta(\mathfrak{a} + \mathfrak{m}_2) \big\rangle = 0$ (see Proposition 12 (4*b*) and (3*b*)).

Now, define a symmetric bilinear form χ on $\mathfrak{m}_1$ by setting

$$\chi(Y_1, Y_1') = \big\langle \mathbf{\Psi}(Y_1, Y_1'), \mathbf{F} \big\rangle, \qquad Y_1,\, Y_1' \in \mathfrak{m}_1.$$

Since $\mathbf{\Psi}(Y_1, Y_1') \in \boldsymbol{R}\mathbf{B} + \boldsymbol{R}\mathbf{F}$ (see Proposition 14 (4)) and $\big\langle \mathbf{\Psi}(Y_1, Y_1'), \mathbf{B} \big\rangle = \langle \mathbf{B}, \mathbf{B} \rangle (Y_1, Y_1')$ for $Y_1,\, Y_1' \in \mathfrak{m}_1$ (see Proposition 12 (5)), we have

$$\mathbf{\Psi}(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B} + \chi(Y_1, Y_1')\mathbf{F}, \qquad Y_1,\, Y_1' \in \mathfrak{m}_1. \tag{4.7}$$

Let $V^\perp$ be the orthogonal complement of V in $\mathfrak{m}_1$. Then, we have $\dim V^\perp = 2$. (Recall that $\dim \mathfrak{m}_1 = 4$ and $\dim V = 2$.) Let $\{Y_1, Y_1'\}$ be an orthonormal basis of $V^\perp$. Then, putting $X = Z = Y_1$ and $Y = W = Y_1'$ into the Gauss equation (3.1), we have

$$\begin{aligned} \big(\big[[Y_1, Y_1'], Y_1\big], Y_1'\big) &= \langle \mathbf{B}, \mathbf{B} \rangle (Y_1, Y_1)(Y_1', Y_1') \\ &\quad + \chi(Y_1, Y_1)\chi(Y_1', Y_1') - \chi(Y_1, Y_1')\chi(Y_1', Y_1). \end{aligned}$$

Since $\big(\big[[Y_1, Y_1'], Y_1\big], Y_1'\big) = \langle \mathbf{B}, \mathbf{B} \rangle (Y_1, Y_1)(Y_1', Y_1')$ (see (2.2)), we have

$$\chi(Y_1, Y_1)\chi(Y_1', Y_1') - \chi(Y_1, Y_1')\chi(Y_1', Y_1) = 0.$$

This implies that χ is degenerate on $V^\perp$. Therefore, there is a non-zero vector $\zeta \in V^\perp$ such that $\chi(\zeta, V^\perp) = 0$, i.e., $\big\langle \mathbf{F}, \mathbf{\Psi}_\zeta(V^\perp) \big\rangle = 0$.

Let us show that the subspace $\widehat{V} = \boldsymbol{R}\zeta + V$ $(\subset \mathfrak{m}_1)$ is singular with respect to $\mathbf{\Psi}$. Note that $\big\langle \mathbf{F}, \mathbf{\Psi}_\zeta(\mathfrak{a} + \mathfrak{m}_2) \big\rangle = 0$ (see Proposition 12 (3*b*)). Then, since $\mathfrak{m} = \mathfrak{a} + \mathfrak{m}_2 + V + V^\perp$ and $\mathbf{\Psi}_\zeta(V) \subset \boldsymbol{R}\mathbf{B}$, it follows that

$$\begin{aligned} \big\langle \mathbf{F}, \mathbf{\Psi}_\zeta(\mathfrak{m}) \big\rangle &= \big\langle \mathbf{F}, \mathbf{\Psi}_\zeta(\mathfrak{a} + \mathfrak{m}_2) + \mathbf{\Psi}_\zeta(V) + \mathbf{\Psi}_\zeta(V^\perp) \big\rangle \\ &\subset 0 + \langle \mathbf{F}, \boldsymbol{R}\mathbf{B} \rangle + 0 = 0. \end{aligned}$$

Hence, we have $\big\langle \mathbf{F}, \boldsymbol{\Psi}_{a\zeta+\eta}(\mathfrak{m})\big\rangle = 0$ for any $a \in \boldsymbol{R}$ and $\eta \in V$. Consequently, $\boldsymbol{\Psi}_{a\zeta+\eta}(\mathfrak{m}) \neq \boldsymbol{N}$, which implies that $a\zeta+\eta \in \widehat{V}$ is singular with respect to $\boldsymbol{\Psi}$. □

Now, we assume that $\dim V = 2$ and denote by $\widehat{V}$ be the singular subspace stated in the above lemma. Let $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ be the vectors obtained by applying Proposition 12 to the pair of singular subspaces U and $\widehat{V}$. Then, by Proposition 12 (2) we can easily see that $\widehat{\mathbf{A}} = \mathbf{A}$ and $\widehat{\mathbf{B}} = \mathbf{B}$. Therefore, we know that all the statements in Proposition 12 and hence the arguments developed after Proposition 12 are also true if we simply replace V by $\widehat{V}$. Accordingly, without loss of generality we can assume that $\dim V \geq 3$.

Lemma 18 $\big\langle \boldsymbol{\Psi}(Y_0, Y_0), \mathbf{B}\big\rangle = (\mu,\mu)\{1+(\mu,\mu)/C\}(Y_0, Y_0), \quad \forall Y_0 \in \mathfrak{a}+\mathfrak{m}_2.$

Proof. As in the proof of Lemma 16, we can prove that $C \neq 3(\mu,\mu)$. Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ ($Y_0 \neq 0$). Take $\xi \in U$ ($\xi \neq 0$) such that $(\xi, Y_0) = 0$, which is possible because $\dim U \geq 2$. Then, by Proposition 3 (2) it follows that $L(\xi, Y_0) = -L(Y_0, \xi)$ and that the map $L(\xi, Y_0)$ gives an isomorphism of $\mathfrak{m}_1$ onto itself. Now, take $\eta \in V$ ($\eta \neq 0$) such that $L(\xi, Y_0)\eta \in V$. This is also possible because $\dim L(\xi, Y_0)V = \dim V \geq 3$ and $\dim(V \cap L(\xi, Y_0)V) \geq 2$. (Note that $\dim \mathfrak{m}_1 = 4$.) Then, by Proposition 15 and Proposition 12 ($2b$) we have

$$\begin{aligned} 0 &= |\boldsymbol{\Theta}_{Y_0,\xi}(\eta)|^2 \\ &= \big[\big\langle \boldsymbol{\Psi}(Y_0, Y_0), \mathbf{B}\big\rangle - (\mu,\mu)\{1+(\mu,\mu)/C\}(Y_0, Y_0)\big](\eta,\eta). \end{aligned}$$

Since $(\eta,\eta) \neq 0$, we get the lemma. □

Lemma 19 *$C = (\mu,\mu)$, i.e., $\langle \mathbf{A}, \mathbf{B}\rangle = 2(\mu,\mu)$.*

Proof. Take $\xi \in U$ ($\xi \neq 0$). Then, by Lemma 18 and $\boldsymbol{\Psi}(\xi,\xi) = (\xi,\xi)\mathbf{A}$ (see Proposition 12 ($2a$)), we have $\langle \mathbf{A}, \mathbf{B}\rangle = (\mu,\mu)\{1+(\mu,\mu)/C\}$. Since $C = \langle \mathbf{A}, \mathbf{B}\rangle - (\mu,\mu)$, we easily have $C^2 = (\mu,\mu)^2$. Moreover, since $C > 0$ (see Proposition 14 (2)), it follows that $C = (\mu,\mu)$, i.e., $\langle \mathbf{A}, \mathbf{B}\rangle = 2(\mu,\mu)$. □

Now, we show

Lemma 20 (1) $\boldsymbol{\Psi}(Y_0, Y_0') = (Y_0, Y_0')\mathbf{A}, \quad \forall Y_0,\ Y_0' \in \mathfrak{a}+\mathfrak{m}_2.$
(2) $\boldsymbol{\Psi}(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}, \quad \forall Y_1,\ Y_1' \in \mathfrak{m}_1.$

Proof. On account of an elementary fact concerning symmetric bilinear forms, we have only to show $\mathbf{\Psi}(Y_0,Y_0)=(Y_0,Y_0)\mathbf{A}$ and $\mathbf{\Psi}(Y_1,Y_1)=(Y_1,Y_1)\mathbf{B}$ for any $Y_0\in\mathfrak{a}+\mathfrak{m}_2$ and $Y_1\in\mathfrak{m}_1$.

Let $Y_0\in\mathfrak{a}+\mathfrak{m}_2$. Then, by Lemma 18 and Lemma 19 we have $\big\langle\mathbf{\Psi}(Y_0,Y_0),\mathbf{B}\big\rangle=\langle\mathbf{A},\mathbf{B}\rangle(Y_0,Y_0)$. Moreover, by Proposition 12 (1) and (5a) we have $\big\langle\mathbf{\Psi}(Y_0,Y_0),\mathbf{A}\big\rangle=\langle\mathbf{A},\mathbf{A}\rangle(Y_0,Y_0)$. Since $\mathbf{\Psi}(Y_0,Y_0)\in\boldsymbol{R}\mathbf{A}+\boldsymbol{R}\mathbf{B}$ (see (4.3)), it follows that $\mathbf{\Psi}(Y_0,Y_0)=(Y_0,Y_0)\mathbf{A}$, which proves (1).

We next prove (2). Let $Y_1\in\mathfrak{m}_1$ $(Y_1\neq 0)$. Take elements $\xi\in U$ $(\xi\neq 0)$ and $\eta\in V$ $(\eta\neq 0)$ such that $(\eta,Y_1)=0$. Set $Y_0=\big[Y_1,[\xi,\eta]\big]$. Then, it is easy to see that $\big[\xi,\eta\big]\in\mathfrak{k}_1$ and $Y_0\in\mathfrak{a}+\mathfrak{m}_2$ (see (2.1)). Further, we have $(\xi,Y_0)=0$ and $L(\xi,Y_0)Y_1\in V$, because

$$\begin{aligned}(\xi,Y_0)&=\big(\xi,\big[Y_1,[\xi,\eta]\big]\big)=-\big(\big[\xi,[\xi,\eta]\big],Y_1\big)\\&=(\mu,\mu)(\xi,\xi)(\eta,Y_1)=0,\\L(\xi,Y_0)Y_1&=\big[\xi,\big[\big[Y_1,[\xi,\eta]\big],Y_1\big]\big]=(\mu,\mu)(Y_1,Y_1)\big[\xi,[\xi,\eta]\big]\\&\qquad\qquad=-(\mu,\mu)^2(\xi,\xi)(Y_1,Y_1)\eta\in V\end{aligned}$$

(see (2.2) and (2.4)). Thus, by Proposition 15 (2), Lemma 19 and $\mathbf{\Psi}(Y_0,Y_0)=(Y_0,Y_0)\mathbf{A}$ (see (1)), we have

$$0=|\mathbf{\Theta}_{Y_0,\xi}(Y_1)|^2=\big[\big\langle\mathbf{A},\mathbf{\Psi}(Y_1,Y_1)\big\rangle-2(\mu,\mu)(Y_1,Y_1)\big](Y_0,Y_0).$$

Here, we note that $Y_0\neq 0$, because $L(\xi,Y_0)Y_1\neq 0$. Hence, by the above equality and Lemma 19, we have $\big\langle\mathbf{\Psi}(Y_1,Y_1),\mathbf{A}\big\rangle=\langle\mathbf{B},\mathbf{A}\rangle(Y_1,Y_1)$. On the other hand, by Proposition 12 (1) and (5b) we have $\big\langle\mathbf{\Psi}(Y_1,Y_1),\mathbf{B}\big\rangle=\langle\mathbf{B},\mathbf{B}\rangle(Y_1,Y_1)$. Consequently, it follows that $\mathbf{\Psi}(Y_1,Y_1)=(Y_1,Y_1)\mathbf{B}$, because $\mathbf{\Psi}(Y_1,Y_1)\in\boldsymbol{R}\mathbf{A}+\boldsymbol{R}\mathbf{B}$ (see (4.4)). This proves (2). □

We are now in a final position of the proof of Theorem 6. Let $Y_0\in\mathfrak{a}+\mathfrak{m}_2$ $(Y_0\neq 0)$. Then, by Lemma 20 (1) we have $\mathbf{Ker}(\mathbf{\Psi}_{Y_0})\supset\{Y_0'\in\mathfrak{a}+\mathfrak{m}_2\,|\,(Y_0,Y_0')=0\}$. This shows $\dim\mathbf{Ker}(\mathbf{\Psi}_{Y_0})\geq 3$ and hence Y_0 is singular with respect to $\mathbf{\Psi}$ (see Proposition 9 (1)). Accordingly, $\mathfrak{a}+\mathfrak{m}_2$ is a singular subspace. Similarly, by Lemma 20 (2) we can show that $\mathfrak{m}_1$ is also a singular subspace.

Now, let us put into Proposition 12 $U=\mathfrak{a}+\mathfrak{m}_2$ and $V=\mathfrak{m}_1$. Then, by Lemma 20 we know that the vectors $\mathbf{A}$ and $\mathbf{B}$ are not altered by this change of singular subspaces. Therefore, all the statements in Proposition 12 and the arguments developed after Proposition 12 are also true under our setting $U=\mathfrak{a}+\mathfrak{m}_2$ and $V=\mathfrak{m}_1$. Consequently, by Proposition 12 (1), (2), (3) and Lemma 19 we get the assertion (1) of Theorem 6. We also obtain by Proposition 14 and $C = (\mu, \mu)$ (see Lemma 19) the assertion (3) of Theorem 6.

Finally, we prove the assertion (2) of Theorem 6. Let $Y_2 \in \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Then, since $C \neq 3(\mu, \mu)$ and $(\mu, Y_2) = 0$, we have

$$\boldsymbol{\Theta}_{Y_2,\mu}(Y_1) = \boldsymbol{\Psi}_{Y_2}(Y_1) + \frac{1}{(\mu,\mu)^2}\boldsymbol{\Psi}_{\mu}(L(\mu, Y_2)Y_1) = 0$$

(see Proposition 15). Here we note that the conditions $\mu \in U$ and $L(\mu, Y_2)Y_1 \in V$ in Proposition 15 have no significance, because $U = \mathfrak{a} + \mathfrak{m}_2$ and $V = \mathfrak{m}_1$. Accordingly, we obtain the assertion (2). This completes the proof of Theorem 6. □

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