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Affine semiparallel surfaces with constant Pick invariant

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Abstract. In this paper, we classify all affine semiparallel surfaces in $\mathbb{R}^3$ with constant Pick invariant. As its application, we can characterize affine semiparallel surfaces of constant Pick invariant but of which shape operators are not parallel.

Key words: affine shape operator, Pick invariant, affine sphere, semiparallel.

1. Introduction

Let M be a non-degenerate affine surface of $\mathbb{R}^3$ with induced connection ∇ and let R be the curvature tensor and S the affine shape operator of M, respectively. Surfaces with parallel shape operator $\nabla S = 0$ were classified in [5], [6] and [11] (see Theorem 2.2).

In 1946, E. Cartan [2] introduced the spaces with $R(X, Y) \cdot R = 0$ for all vector fields X and Y, where the linear endomorphism $R(X, Y)$ acts on R as a derivation. These spaces including locally symmetric Riemannian spaces were classified by Z.I. Szabó [15]. It would be natural to define for an affine surface M in $\mathbb{R}^3$ to be *semiparallel* if $R(X, Y) \cdot S = 0$. Clearly $\nabla S = 0$ implies $R(X, Y) \cdot S = 0$, and the converse is not true (see Example 4.3, 4.4, and 4.5). W. Jelonek [5] shows that any hypersurface is semiparallel if and only if the shape operator is either a constant multiple of the identity or $S^2 = 0$ and $\mathrm{rank}(S) \leq 1$. But he classifies only all affine surfaces with parallel shape operator.

In this paper, we classify all affine semiparallel surfaces in $\mathbb{R}^3$ with constant Pick invariant. As its application, we can characterize affine semiparallel surfaces of constant Pick invariant but not parallel shape operator.

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2. Definitions and statement of results

Let M^n be a smooth manifold and $f\colon M \to \mathbb{R}^{n+1}$, a smooth immersion. We can choose a smooth vector field ξ along to f which is transversal to M, i.e. for all $x \in M$,

$$T_{f(x)}\mathbb{R}^{n+1} = f_*(T_xM) \oplus \mathbb{R}\xi_x.$$

Let $\mathfrak{X}(M)$ be the set of all smooth vector fields on M. The standard connection D on $\mathbb{R}^{n+1}$, induces the torsion free affine connection ∇ and the symmetric (0, 2)-tensor field h on M with Gauss' formula:

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)\xi,$$

and the (1, 1)-tensor field S and 1-form τ on M with Weingarten's formula:

$$D_X\xi = -f_*(SX) + \tau(X)\xi,$$

for arbitrary $X, Y \in \mathfrak{X}(M)$.

If h is non-degenerate, f is called a *non-degenerate immersion*.

If f is non-degenerate, there is a transversal vector field ξ, which is unique up to sign, and satisfies $\tau = 0$ and $\theta = \omega_h$, where ω_h is the volume form of h, and θ is the volume form on M defined by:

$$\theta(X_1, \ldots, X_n) := \det[f_*(X_1), \ldots, f_*(X_n), \xi].$$

In this case, ξ is called an *affine normal vector field*, f with ξ is called a *Blaschke immersion*, h is called the *affine metric*, and (∇, h, S) is called the *Blaschke structure* of f. S is called the *affine shape operator* for ξ. If S in Blaschke structure is a constant multiple of the identity, $f\colon M \to \mathbb{R}^{n+1}$ is called an *affine hypersphere*. On a Blaschke immersion, $H \overset{\text{def}}{=} (1/n)\operatorname{tr} S$ is called the *affine mean curvature*. Clearly, on an affine hypersphere, $S = HI$. And if $H \equiv 0$, $f\colon M \to \mathbb{R}^{n+1}$ is called an *affine minimal hypersurface*.

On a Blaschke immersion, the tensor field C on M defined by $C(X, Y, Z) := (\nabla_X h)(Y, Z)$, called the *cubic form*, which is known to be totally symmetric. And it is well known as Pick-Berwald's theorem (cf. [13, p. 53, Theorem 4. 5]) that if $C \equiv 0$, M is a quadratic hypersurface.

The function J on M defined by $J := (1/(4n(n-1)))h(C, C)$ is called the *Pick invariant*. It is known that $\hat{\rho} = H + J$, where $\hat{\rho}$ is the scalar curvature of the affine metric h (cf. [13, p. 78, Proposition 9. 3]). And in the case $n = 2$, it is known that the immersion $f\colon M \to \mathbb{R}^3$ is a ruled surface if and only if h is indefinite metric and $J \equiv 0$ (cf. [13, pp. 89, 90, Definition 11. 1, and Theorems 11. 3, 11. 4]). Hence the surface with $J \equiv 0$ is quadratic or ruled, because that definiteness of h and $J \equiv 0$ mean that $C \equiv 0$.

Definition 2.1 Let $f\colon M \to \mathbb{R}^{n+1}$ be a Blaschke immersion with affine normal vector field ξ and Blaschke structure (∇, h, S). We define the curvature tensor of S with respect to ∇ as

$$\begin{aligned}\big(R(X, Y)\cdot S\big)Z :=& \big(\nabla_X\nabla_Y S - \nabla_Y\nabla_X S - \nabla_{[X,Y]}S\big)(Z)\\ =& R(X, Y)(SZ) - S\big(R(X, Y)Z\big)\end{aligned}$$

where $X, Y, Z \in \mathfrak{X}(M)$. If $R(X, Y)\cdot S$ vanishes everywhere on M for all $X, Y \in \mathfrak{X}(M)$, then we call f an *affine semiparallel hypersurface.*

Classification of affine surfaces with parallel shape operator see also R. Niebergall and P. Ryan [11].

W. Jelonek [5] shows that any hypersurface is semiparallel if and only if the shape operator is either a constant multiple of the identity or $S^2 = 0$ and $\operatorname{rank}(S) \le 1$.

The classification of affine surfaces with parallel shape operator and constant Pick invariant, recovers the classification of the affine spheres with constant curvature metrics shown by M.A. Magid and P.J. Ryan [10] and U. Simon [14]:

Theorem 2.2 [6] *Assume that a Blaschke surface in* $\mathbb{R}^3$ *satisfies* $\nabla S = 0$ *and* J *is constant. Then it is affinely congruent to one of the following surfaces*

$$xyz = 1 \tag{1}$$
$$(x^2 + y^2)z = 1 \tag{2}$$
$$z = x^2 + y^2 \tag{3}$$
$$z = xy + \Phi(x) \tag{4}$$

for some smooth function $\Phi(x)$ *in* x*,*

$$x^2 + y^2 + z^2 = 1 \tag{5}$$
$$x^2 - y^2 - z^2 = 1 \tag{6}$$
$$f(u, v) = u\alpha(v) + \alpha'(v) \tag{7}$$

where $\alpha(v)$ *is an* $\mathbb{R}^3$*-valued function in* v *satisfying that* $\det[\alpha, \alpha', \alpha'']$ *is a non-zero constant,*

$$z = ye^x + \Phi(x) \tag{8}$$

$$z = y \tan x + \Phi(x) \tag{9}$$

for some smooth function $\Phi(x)$ *in* x.

Now we state our main theorem.

Theorem 2.3 (i) *Any affine semiparallel surface with constant Pick invariant in* $\mathbb{R}^3$ *is either an affine sphere with constant curvature metric or an affine minimal ruled surface.*
(ii) *Any affine minimal ruled surface can be written as*:

$$z = yA(x) + B(x),$$

where $A(x)$ *is a non-constant smooth function in* x *and* $B(x)$ *is a smooth function in* x. *Conversely, any surface which can be written as above is an affine minimal ruled surface.*

Remark 2.4 All surface with $\nabla S = 0$ is affine semiparallel, so all the surfaces in Theorem 2.2 are in Theorem 2.3(i). In fact, from (1) to (7) in the list of Theorem 2.2 are affine spheres with constant curvature metrics, and (8) and (9) in Theorem 2.2 have the expression as in Theorem 2.3.

3. Preliminary

We need the following lemmas to prove Theorem 2.3.

Lemma 3.1 *Let* (∇, h, S) *be the Blaschke structure on* M, C *the cubic form and* J *the Pick invariant. If* $J = constant \neq 0$, *we can choose two smooth vector fields* X_1, X_2 *defined locally on a neighborhood* U *in* M *satisfying that*

$$h(X_i, X_j) = \varepsilon_i \delta_{ij}, \tag{10}$$

$$\begin{cases} C(X_1, X_1, X_1) = -\varepsilon_1\sqrt{2\varepsilon_1 J}, \ C(X_1, X_2, X_2) = \varepsilon_2\sqrt{2\varepsilon_1 J}, \\ \qquad C(X_1, X_1, X_2) = C(X_2, X_2, X_2) = 0, \end{cases} \tag{11}$$

where $\varepsilon_i = \pm 1$, *respectively. And there exist three smooth functions* a, b, H *such that*

$$\begin{cases} \nabla_{X_1} X_1 = \sqrt{\varepsilon_1 J/2} X_1 + \varepsilon_1 a X_2, \\ \nabla_{X_1} X_2 = -\varepsilon_2 a X_1 - \sqrt{\varepsilon_1 J/2} X_2, \\ \nabla_{X_2} X_1 = -(b + \sqrt{\varepsilon_1 J/2}) X_2, \\ \nabla_{X_2} X_2 = \varepsilon_1 \varepsilon_2 (b - \sqrt{\varepsilon_1 J/2}) X_1, \end{cases} \tag{12}$$

$$\begin{cases} SX_1 = (H + 3\varepsilon_1 b \sqrt{\varepsilon_1 J/2}) X_1 - 3a \sqrt{\varepsilon_1 J/2} X_2, \\ SX_2 = -3\varepsilon_1 \varepsilon_2 a \sqrt{\varepsilon_1 J/2} X_1 + (H - 3\varepsilon_1 b \sqrt{\varepsilon_1 J/2}) X_2, \end{cases} \tag{13}$$

and

$$H = \varepsilon_1 X_1 b + X_2 a - \varepsilon_2 a^2 - \varepsilon_1 b^2 - J, \tag{14}$$

$$X_2 H = 3\varepsilon_1 \varepsilon_2 \sqrt{\varepsilon_1 J/2} (-\varepsilon_2 X_2 b - X_1 a + 4ab - 2a \sqrt{\varepsilon_1 J/2}), \tag{15}$$

$$\begin{aligned} X_1 H = {} & 3\sqrt{\varepsilon_1 J/2} \\ & \times (\varepsilon_1 X_1 b - X_2 a + 2\varepsilon_2 a^2 - 2\varepsilon_1 b^2 - 2\varepsilon_1 b \sqrt{\varepsilon_1 J/2}). \end{aligned} \tag{16}$$

Clearly, in the case of Lemma 3.1, H is the affine mean curvature, and the surface is an affine sphere if and only if $a = b = 0$.

Remark 3.2 $\varepsilon_1 J$ is always positive for (∇, h, S) satisfying the assumptions in Lemma 3.1.

Proof of Lemma 3.1. It is known that the difference tensor K defined by $K_X Y := \nabla_X Y - \hat{\nabla}_X Y$ where $\hat{\nabla}$ is the Levi-Civita connection of the affine metric h satisfies that $C(X, Y, Z) = -2h(K_X Y, Z)$ and $\operatorname{tr} K_X = 0$ for arbitrary $X, Y, Z \in \mathfrak{X}(M)$ (cf. [13, pp. 50, 51, Proposition 4.1, Theorem 4.3]).

Since $J \neq 0$ (cf. [13, pp. 87, 88, Propositions 11.1, 11.2]), there exists a null direction $X \in T_x M$ of the cubic form C, i.e. $C(X, X, X) = 0$, $X \neq 0$ for each $x \in U$. Because of the fact that $h(X, X) \neq 0$, we can take X_1 and $X_2 \in \mathfrak{X}(U)$ such that $\{(X_1)_x, (X_2)_x\}$ is an orthonormal basis of $T_x M$ with respect to h and that $(X_2)_x = |h(X, X)|^{-1/2} X$ for each $x \in U$, i.e.

$$h(X_i, X_j) = \varepsilon_i \delta_{ij}, \ \varepsilon_i = \pm 1, \ C(X_2, X_2, X_2) = 0.$$

Then, we obtain

$$\begin{aligned} & \varepsilon_1 C(X_1, X_1, X_1) + \varepsilon_2 C(X_1, X_2, X_2) \\ = & -2\varepsilon_1 h(K_{X_1} X_1, X_1) - 2\varepsilon_2 h(K_{X_1} X_2, X_2) \\ = & -2 \operatorname{tr} K_{X_1} = 0 \end{aligned}$$

and

$$\begin{aligned} C(X_1,\, X_1,\, X_2) =& -2h(K_{X_2}X_1,\, X_1) \\ =& -2\varepsilon_1\{\operatorname{tr} K_{X_2} - \varepsilon_2 h(K_{X_2}X_2,\, X_2)\} \\ =& -\varepsilon_1\varepsilon_2 C(X_2,\, X_2,\, X_2) = 0. \end{aligned}$$

So we have

$$\begin{aligned} J = \frac{1}{8}h(C,\, C) =& \frac{1}{8}\varepsilon_1\big(C(X_1,\, X_1,\, X_1)^2 + 3C(X_1,\, X_2,\, X_2)^2\big) \\ =& \frac{1}{2}\varepsilon_1 C(X_1,\, X_1,\, X_1)^2. \end{aligned}$$

Thus we have X_1 and X_2 satisfying (10) and (11), by replacing X_1 and X_2 into $-X_1$ and $-X_2$ if $\varepsilon_1 C(X_1,\, X_1,\, X_1) > 0$.

By our choice of $\{X_1,\, X_2\}$, we have

$$\begin{aligned} \varepsilon_1 h(\nabla_{X_1}X_1,\, X_1) =& -\frac{1}{2}\varepsilon_1\{X_1 h(X_1,\, X_1) - 2h(\nabla_{X_1}X_1,\, X_1)\} \\ =& -\frac{1}{2}\varepsilon_1 C(X_1,\, X_1,\, X_1) = \sqrt{\varepsilon_1 J/2}, \\ \varepsilon_2 h(\nabla_{X_1}X_2,\, X_2) =& -\frac{1}{2}\varepsilon_2\{X_1 h(X_2,\, X_2) - 2h(\nabla_{X_1}X_2,\, X_2)\} \\ =& -\frac{1}{2}\varepsilon_2 C(X_1,\, X_2,\, X_2) = -\sqrt{\varepsilon_1 J/2}, \\ \varepsilon_1 h(\nabla_{X_2}X_1,\, X_1) =& -\frac{1}{2}\varepsilon_1\{X_2 h(X_1,\, X_1) - 2h(\nabla_{X_2}X_1,\, X_1)\} \\ =& -\frac{1}{2}\varepsilon_1 C(X_1\,,X_1,\, X_2) = 0, \\ \varepsilon_2 h(\nabla_{X_2}X_2,\, X_2) =& -\frac{1}{2}\varepsilon_2\{X_2 h(X_2,\, X_2) - 2h(\nabla_{X_2}X_2,\, X_2)\} \\ =& -\frac{1}{2}\varepsilon_2 C(X_2,\, X_2,\, X_2) = 0, \\ h(\nabla_{X_1}X_1,\, X_2) + h(\nabla_{X_1}X_2, X_1) =& X_1 h(X_1,\, X_2) - C(X_1,\, X_1,\, X_2) \\ =& 0, \end{aligned}$$

and

$$\begin{aligned} h(\nabla_{X_2}X_1,\, X_2) + h(\nabla_{X_2}X_2,\, X_1) =& X_2 h(X_1,\, X_2) - C(X_1,\, X_2,\, X_2) \\ =& -\varepsilon_2\sqrt{2\varepsilon_1 J}. \end{aligned}$$

Then, we can define two smooth functions a and b on U by

$$a := \varepsilon_1\varepsilon_2 h(\nabla_{X_1}X_1,\, X_2) = -\varepsilon_1\varepsilon_2 h(\nabla_{X_1}X_2,\, X_1),$$
$$b := -\varepsilon_2 h(\nabla_{X_2}X_1,\, X_2) - \sqrt{\varepsilon_1 J/2} = \varepsilon_2 h(\nabla_{X_2}X_2,\, X_1) + \sqrt{\varepsilon_1 J/2}.$$

So, we obtain that

$$\nabla_{X_1}X_1 = \sqrt{\varepsilon_1 J/2}X_1 + \varepsilon_1 a X_2,\ \nabla_{X_1}X_2 = -\varepsilon_2 a X_1 - \sqrt{\varepsilon_1 J/2}X_2,$$
$$\nabla_{X_2}X_1 = -(b + \sqrt{\varepsilon_1 J/2})X_2,\ \nabla_{X_2}X_2 = \varepsilon_1\varepsilon_2(b - \sqrt{\varepsilon_1 J/2})X_1.$$

Therefore, we have

$$R(X_1,\, X_2)X_1 = 3\varepsilon_2 a\sqrt{\varepsilon_1 J/2}X_1 + (3b\sqrt{\varepsilon_1 J/2} - \varepsilon_1 H)X_2,$$
$$R(X_1,\, X_2)X_2 = (\varepsilon_2 H + 3\varepsilon_1\varepsilon_2 b\sqrt{\varepsilon_1 J/2})X_1 - 3\varepsilon_2 a\sqrt{\varepsilon_1 J/2}X_2$$

where H is a smooth function on U defined by (14).

By Gauss' equation, we obtain

$$\begin{aligned} SX_1 &= \varepsilon_2\big(R(X_1,\, X_2)X_2 + h(X_1,\, X_2)SX_2\big) \\ &= (H + 3\varepsilon_1 b\sqrt{\varepsilon_1 J/2})X_1 - 3a\sqrt{\varepsilon_1 J/2}X_2, \\ SX_2 &= \varepsilon_1\big(-R(X_1,\, X_2)X_1 + h(X_2,\, X_1)SX_1\big) \\ &= -3\varepsilon_1\varepsilon_2 a\sqrt{\varepsilon_1 J/2}X_1 + (H - 3\varepsilon_1 b\sqrt{\varepsilon_1 J/2})X_2. \end{aligned}$$

Since Codazzi's equation,

$$\begin{aligned} (\nabla_{X_1}S)X_2 &= 3\varepsilon_1\varepsilon_2\sqrt{\varepsilon_1 J/2}\big\{-X_1 a + 2a(b - \sqrt{\varepsilon_1 J/2})\big\}X_1 \\ &\quad + \big\{X_1 H - 3\sqrt{\varepsilon_1 J/2}(2\varepsilon_2 a^2 + \varepsilon_1 X_1 b)\big\}X_2 \end{aligned}$$

equals to

$$\begin{aligned} (\nabla_{X_2}S)X_1 &= \big\{X_2 H + 3\varepsilon_1\sqrt{\varepsilon_1 J/2}(X_2 b - 2\varepsilon_2 ab)\big\}X_1 \\ &\quad + 3\sqrt{\varepsilon_1 J/2}\big\{-X_2 a - 2\varepsilon_1 b(b + \sqrt{\varepsilon_1 J/2})\big\}X_2, \end{aligned}$$

so we have (15) and (16). □

Lemma 3.3 *Let (∇, h, S) be the Blaschke structure on M, C the cubic form and J the Pick invariant. If $J \equiv 0$ and $C \neq 0$, we can choose two smooth vector fields X_1, X_2 defined locally on a neighborhood U in M sat-*

isfying that

$$h(X_i, X_j) = 1 - \delta_{ij}, \tag{17}$$

$$\begin{cases} C(X_1, X_1, X_1) = -2, \\ C(X_1, X_1, X_2) = C(X_1, X_2, X_2) = C(X_2, X_2, X_2) = 0. \end{cases} \tag{18}$$

And there exist three smooth functions a, b, H *such that*

$$\begin{cases} \nabla_{X_1} X_1 = aX_1 + X_2, & \nabla_{X_1} X_2 = -aX_2, \\ \nabla_{X_2} X_1 = -bX_1, & \nabla_{X_2} X_2 = bX_2, \end{cases} \tag{19}$$

$$SX_1 = HX_1 - 3bX_2, \ SX_2 = HX_2, \tag{20}$$

and

$$H = -X_1 b - X_2 a - 2ab, \tag{21}$$

$$X_2 H = 0, \tag{22}$$

$$X_1 H = -3(X_2 b + 2b^2). \tag{23}$$

Clearly, in the case of Lemma 3.3, H is the affine mean curvature, and the surface is an affine sphere if and only if $b = 0$.

Proof. Since $J = 0$ (cf. [13, p. 88, Propositions 11.2]), there exist two tangent vectors X, $Y \in T_x M$ such that

$$\begin{aligned} &C(X, V, W) = 0 \quad \text{for all} \quad V, W \in T_x M, \\ &h(X, X) = h(Y, Y) = 0, \\ &h(X, Y) = C(Y, Y, Y) = 1 \end{aligned}$$

for each $x \in U$. Then, we take X_1, X_2 such that $(X_1)_x = -2^{1/3} Y$, $(X_2)_x = 2^{-1/3} X$ for each $x \in U$, respectivery, and obviously they satisfy (17) and (18).

By our choice of $\{X_1, X_2\}$, we have

$$\begin{aligned} h(\nabla_{X_1} X_1, X_1) &= -\frac{1}{2}\{X_1 h(X_1, X_1) - 2h(\nabla_{X_1} X_1, X_1)\} \\ &= -\frac{1}{2} C(X_1, X_1, X_1) = 1, \\ h(\nabla_{X_1} X_2, X_2) &= -\frac{1}{2}\{X_1 h(X_2, X_2) - 2h(\nabla_{X_1} X_2, X_2)\} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}C(X_1,\, X_2,\, X_2) = 0,\\
h(\nabla_{X_2}X_1,\, X_1) &= -\frac{1}{2}\big\{X_2 h(X_1,\, X_1) - 2h(\nabla_{X_2}X_1,\, X_1)\big\}\\
&= -\frac{1}{2}C(X_1,\, X_1,\, X_2) = 0,\\
h(\nabla_{X_2}X_2,\, X_2) &= -\frac{1}{2}\big\{X_2 h(X_2,\, X_2) - 2h(\nabla_{X_2}X_2,\, X_2)\big\}\\
&= -\frac{1}{2}C(X_2,\, X_2,\, X_2) = 0,\\
h(\nabla_{X_1}X_1,\, X_2) + h(\nabla_{X_1}X_2,\, X_1) &= X_1 h(X_1,\, X_2) - C(X_1,\, X_1,\, X_2)\\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
h(\nabla_{X_2}X_1,\, X_2) + h(\nabla_{X_2}X_2,\, X_1) &= X_2 h(X_1,\, X_2) - C(X_1,\, X_2,\, X_2)\\
&= 0.
\end{aligned}
$$

Then, we can define two smooth functions a and b on M by

$$
\begin{aligned}
&a := h(\nabla_{X_1}X_1,\, X_2) = -h(\nabla_{X_1}X_2,\, X_1),\\
&b := -h(\nabla_{X_2}X_1,\, X_2) = h(\nabla_{X_2}X_2,\, X_1).
\end{aligned}
$$

So, we obtain that

$$
\begin{aligned}
&\nabla_{X_1}X_1 = aX_1 + X_2,\ \nabla_{X_1}X_2 = -aX_2,\\
&\nabla_{X_2}X_1 = -bX_1,\ \nabla_{X_2}X_2 = bX_2.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&R(X_1,\, X_2)X_1 = HX_1 - 3bX_2,\\
&R(X_1,\, X_2)X_2 = -HX_2
\end{aligned}
$$

where H is a smooth function on U defined by (21).

By Gauss' equation, we obtain

$$
\begin{aligned}
&SX_1 = R(X_1,\, X_2)X_1 + h(X_1,\, X_1)SX_2 = HX_1 - 3bX_2,\\
&SX_2 = -R(X_1,\, X_2)X_2 + h(X_2,\, X_2)SX_1 = HX_2.
\end{aligned}
$$

And since Codazzi's equation,

$$
(\nabla_{X_1}S)X_2 = (X_1 H)X_2
$$

equals to

$$(\nabla_{X_2}S)X_1 = (X_2H)X_1 - 3(X_2b + 2b^2)X_2,$$

so we have (22) and (23). □

Proposition 3.4 (Due to T. Konno) *For an arbitrary smooth vector field X on $U \subset M^2$, there exists a non-zero function ψ which satisfies $X\psi = \psi$.*

Proof. Let $\{\varphi_u\}$ be a 1-parameter group of local transformations for X. And let $c\colon \mathbb{R} \ni v \mapsto c(v) \in U$ be a smooth curve transversal to X, i.e. $\dot{c}(v_0)$ and $X_{c(v_0)}$ are linear independent for all $v_0 \in \mathbb{R}$ where $\dot{c}(v)$ means differentiation by the parameter v. Then $\Phi(u, v) := \varphi_u\big(c(v)\big)$ gives a local coodinate system on U, because $\Phi_*(\partial/\partial u) = (\varphi_u)_*(X_{c(v)})$ and $\Phi_*(\partial/\partial v) = (\varphi_u)_*\big(\dot{c}(v)\big)$ are linearly independent. By taking $\psi := (\Phi^{-1})^*(e^u)$, we have for all $(u_0, v_0) \in \Phi^{-1}(U) \subset \mathbb{R}^2$,

$$\begin{aligned} X_{\Phi(u_0, v_0)}\psi &= \frac{\partial}{\partial u}\bigg|_{u=0} \psi \circ \varphi_u\big(\Phi(u_0, v_0)\big) \\ &= \frac{\partial}{\partial u}\bigg|_{u=0} \psi \circ \Phi(u_0 + u, v_0) \\ &= \frac{\partial}{\partial u}\bigg|_{u=0} e^{u_0+u} = e^{u_0} \\ &= \psi\big(\Phi(u_0, v_0)\big). \end{aligned}$$

Hence we have $X\psi = \psi$. □

4. Proof of theorem

Now, we are in position to show the main theorem. The proof of (i) in Theorem 2.3 is divided into two parts whether the Pick invariant vanishes or not. And the proof of (ii) is also divided into two parts and we will also show the last statement of Theorem 2.3, and finally give examples.

4.1. The case the Pick invariant is non-zero constant.

If the Pick invariant J of M is non-zero constant, we can choose X_1, $X_2 \in \mathfrak{X}(M)$ and three smooth functions a, b and H on M satisfying from (10) to (16) by Lemma 3.1. Since the equations (12) and (13), we obtain

$$\begin{aligned} \big(R(X_1, X_2) \cdot S\big)X_1 = &-6\varepsilon_2 aH\sqrt{\varepsilon_1 J/2}X_1 \\ &+ 3\big(3J(b^2 + \varepsilon_1\varepsilon_2 a^2) - 2bH\sqrt{\varepsilon_1 J/2}\big)X_2, \end{aligned}$$

and

$$\big(R(X_1,\,X_2)\cdot S\big)X_2 = -3\varepsilon_1\varepsilon_2\big(3J(b^2+\varepsilon_1\varepsilon_2 a^2)+2bH\sqrt{\varepsilon_1 J/2}\big)X_1 \\ +6\varepsilon_2 aH\sqrt{\varepsilon_1 J/2}X_2.$$

Assume that M is semiparallel. Then from the above, we get

$$aH = 0, \tag{24}$$

$$bH = 0, \tag{25}$$

$$b^2+\varepsilon_1\varepsilon_2 a^2 = 0. \tag{26}$$

If the surface is an affine sphere, since $\hat{\rho} = H+J$ is constant, it has been completely classified by M.A. Magid and P.J. Ryan [10] and U. Simon [14]. So we consider the case that it is not an affine sphere, i.e. we assume that $a^2+b^2 \neq 0$. Then from (24) and (25) we have $H = 0$, and from (26) we have $\varepsilon_1\varepsilon_2 = -1$ and $a = \pm b$. So, from (15) we have

$$\varepsilon_1 X_2 b \mp X_1 b \pm 4b^2 \mp 2b\sqrt{\varepsilon_1 J/2} = 0 \tag{27}$$

and from (16) we have

$$\varepsilon_1 X_1 b \mp X_2 b - 4\varepsilon_1 b^2 - 2\varepsilon_1 b\sqrt{\varepsilon_1 J/2} = 0. \tag{28}$$

But (27) and (28) imply that $b = 0$, so these contradict the assumption $ab \neq 0$. Hence the surface is an affine sphere. □

4.2. The case the Pick invariant vanishes.

We consider the case that the Pick invariant J of M vanishes. If the cubic form C vanishes, then the surface is a quadratic surface. So we assume that $C \neq 0$. Here, the surface is a ruled as in Section 2.

And by $J \equiv 0$ and $C \neq 0$, we can choose $X_1,\,X_2 \in \mathfrak{X}(M)$ and three smooth functions a, b and H on M satisfying from (17) to (23) since Lemma 3.3. By (19) and (20), we obtain

$$\big(R(X_1,\,X_2)\cdot S\big)X_1 = 6bHX_2,$$

and

$$\big(R(X_1,\,X_2)\cdot S\big)X_2 = 0.$$

Assume that M is semiparallel. Then from the above, we get

$$bH = 0. \tag{29}$$

If the surface is an affine sphere, since $\hat{\rho} = H$ is constant, it has been completely classified by M.A. Magid and P.J. Ryan [10] and U. Simon [14]. So we consider the case that it is not an affine sphere, i.e. we assume that $b \neq 0$. By (29), we have $H = 0$. Hence the surface is an affine minimal ruled surface. Therefore, we have (i) in Theorem 2.3. □

4.3. Affine minimal ruled surface which is not affine sphere.

It is known that any ruled affine sphere which is affine minimal can be written as (4) in Theorem 2.2. So we only have to show that any affine minimal ruled surface which is not an affine sphere can be written as

$$z = yA(x) + B(x),$$

which completes the proof of (ii) in Theorem 2.3.

Now, we retain the notation in the above subsection 4.2. By (23), we have

$$X_2 b + 2b^2 = 0. \tag{30}$$

If there exist two non-zero functions ψ_1, ψ_2 on U such that

$$[\psi_1 X_1,\, \psi_2 X_2] = 0, \tag{31}$$

then we can take local coordinate system $\{t,\, s\}$ satisfying that $\partial/\partial t = \psi_1 X_1$ and $\partial/\partial s = \psi_2 X_2$. Since $[X_1,\, X_2] = bX_1 - aX_2$, the condition (31) is equivalent to (32) and (33):

$$X_2\psi_1 = b\psi_1, \tag{32}$$

$$X_1\psi_2 = a\psi_2. \tag{33}$$

The existence of ψ_1 and ψ_2 is guaranteed by Proposition 3.4. Then, we can take the above $\{t,\, s\}$. By (30), we can write ψ_2 as $\psi_2 = -(1/2)b^{-2}(\partial b/\partial s)$.

Now, we choose a function ψ_1 satisfying (32). Here we put $\psi_1 = (\pm b)^k$ for some $k \in \mathbb{R}$. Then the left hand side of (32) coincides with

$$\begin{aligned} X_2\left((\pm b)^k\right) &= \pm k(\pm b)^{k-1} X_2 b \\ &= \mp 2k(\pm b)^{k+1}, \end{aligned}$$

and the right hand side is also $b(\pm b)^k = \pm(\pm b)^{k+1}$. We have $k = -1/2$.

Then, we can take $\psi_1 = (\pm b)^{-1/2}$. By (33), we have

$$a = {\psi_1}^{-1}{\psi_2}^{-1}\frac{\partial}{\partial t}\psi_2 = (\pm b)^{1/2}\frac{\partial^2 b}{\partial s\partial t}\Big(\frac{\partial b}{\partial s}\Big)^{-1} \mp 2(\pm b)^{-1/2}\frac{\partial b}{\partial t}.$$

Therefore, by (21) and $H = 0$,

$$\begin{aligned}
0 = -H =& 2ab + X_1 b + X_2 a\\
=& 2b\left((\pm b)^{1/2}\frac{\partial^2 b}{\partial s\partial t}\left(\frac{\partial b}{\partial s}\right)^{-1} \mp 2(\pm b)^{-1/2}\frac{\partial b}{\partial t}\right) + (\pm b)^{1/2}\frac{\partial b}{\partial t}\\
&- 2b^2\left(\frac{\partial b}{\partial s}\right)^{-1}\frac{\partial}{\partial s}\left((\pm b)^{1/2}\frac{\partial^2 b}{\partial s\partial t}\left(\frac{\partial b}{\partial s}\right)^{-1} \mp 2(\pm b)^{-1/2}\frac{\partial b}{\partial t}\right)\\
=& -2(\pm b)^{5/2}\left(\frac{\partial b}{\partial s}\right)^{-1}\frac{\partial^2}{\partial s\partial t}\left(\log\left(\frac{\partial}{\partial s}\big((\pm b)^{-3/2}\big)\right)\right).
\end{aligned}$$

Hence there exist two functions $\phi_1 \neq 0$ and ϕ_2 in t, and a non-constant function η in s satisfying that

$$b = \pm\big(\phi_1(t)\eta(s) + \phi_2(t)\big)^{-2/3}.$$

Therefore, we can derive into the following five differential equations on U:

$$\begin{cases}
\dfrac{\partial^2 f}{\partial t^2} = D_{\partial/\partial t} f_*\Big(\dfrac{\partial}{\partial t}\Big) = {\phi_1}'(t)\phi_1(t)^{-1}\dfrac{\partial f}{\partial t}\\
\qquad\qquad \pm 3\phi_1(t)^{-1}\eta'(s)^{-1}\big(\phi_1(t)\eta(s) + \phi_2(t)\big)\dfrac{\partial f}{\partial s},\\
\dfrac{\partial^2 f}{\partial t\partial s} = D_{\partial/\partial t} f_*\Big(\dfrac{\partial}{\partial s}\Big) = \pm\dfrac{1}{3}\phi_1(t)\eta'(s)\xi,\\
\dfrac{\partial^2 f}{\partial s^2} = D_{\partial/\partial s} f_*\Big(\dfrac{\partial}{\partial s}\Big) = \eta''(s)\eta'(s)^{-1}\dfrac{\partial f}{\partial s},\\
\dfrac{\partial \xi}{\partial t} = D_{\partial/\partial t}\xi = 9\phi_1(t)^{-1}\eta'(s)^{-1}\dfrac{\partial f}{\partial s},\\
\dfrac{\partial \xi}{\partial s} = D_{\partial/\partial s}\xi = 0.
\end{cases} \tag{34}$$

Since the third equation in (34) coincides with

$$\frac{\partial}{\partial s}\Big(\eta'(s)^{-1}\frac{\partial f}{\partial s}\Big) = \eta'(s)^{-1}\frac{\partial^2 f}{\partial s^2} - \eta''(s)\eta'(s)^{-2}\frac{\partial f}{\partial s} = 0,$$

there exists a function μ in t such that

$$\frac{\partial f}{\partial s} = \eta'(s)\mu(t). \tag{35}$$

By the last equation in (34), $\xi = \xi(t)$ is a function only in t. By the second equation in (34), we have

$$\xi(t) = \pm 3\phi_1(t)^{-1}\mu'(t).$$

By the fourth equation in (34), we have

$$\mp 3{\phi_1}'(t)\phi_1(t)^{-2}\mu'(t) \pm 3\phi_1(t)^{-1}\mu''(t) = 9\phi_1(t)^{-1}\mu(t).$$

Therefore, μ is a solution of

$$\mu''(t) - {\phi_1}'(t)\phi_1(t)^{-1}\mu'(t) \mp 3\mu(t) = 0. \tag{36}$$

Since this is a second order linear differential equation, any solution of it can be written as a linear combination of two linear independental solutions, i.e.

$$\mu(t) = c_1\phi_3(t) + c_2\phi_4(t), \tag{37}$$

where c_1 and c_2 are constants and ϕ_3, ϕ_4 are two linear independental solutions of (36).

On the other hand, by (35), there exists a function α in t,

$$f(t, s) = \eta(s)\mu(t) + \alpha(t).$$

By the first equation in (34),

$$\begin{aligned}\eta(s)\mu''(t) + \alpha''(t) =& {\phi_1}'(t)\phi_1(t)^{-1}\big(\eta(s)\mu'(t) + \alpha'(t)\big)\\ &\pm 3\phi_1(t)^{-1}\mu(t)\big(\phi_1(t)\eta(s) + \phi_2(t)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\big(\alpha'(t)\phi_1(t)^{-1}\big)' =& \alpha''(t)\phi_1(t)^{-1} - \alpha'(t){\phi_1}'(t)\phi_1(t)^{-2}\\ =& \pm 3\mu(t)\phi_2(t)\phi_1(t)^{-2}.\end{aligned}$$

By (37), we have

$$\big(\alpha'(t)\phi_1(t)^{-1}\big)' = \pm 3c_1\phi_3(t)\phi_2(t)\phi_1(t)^{-2} \pm 3c_2\phi_4(t)\phi_2(t)\phi_1(t)^{-2}. \tag{38}$$

Let ϕ_5 and ϕ_6 be functions in t which satisfy that

$$\big({\phi_5}'(t)\phi_1(t)^{-1}\big)' = \pm 3\phi_3(t)\phi_2(t)\phi_1(t)^{-2},$$

$$\left(\phi_6{}'(t)\phi_1(t)^{-1}\right)' = \pm 3\phi_4(t)\phi_2(t)\phi_1(t)^{-2}.$$

Then, we obtain

$$\alpha'(t)\phi_1(t)^{-1} = c_1\phi_5{}'(t)\phi_1(t)^{-1} + c_2\phi_6{}'(t)\phi_1(t)^{-1} + c_3$$

where c_3 is a constant. And let ϕ_7 be a function in t which satisfies that $\phi_7{}'(t) = \phi_1(t)$. Then we obtain

$$\alpha(t) = c_1\phi_5(t) + c_2\phi_6(t) + c_3\phi_7(t) + c_4$$

where c_4 is a constant. All solutions for (38) are given as these α. Hence, the general solution of all equations in (34) can be represented as

$$\begin{aligned} f(t,\, s) = c_1&\big(\phi_3(t)\eta(s) + \phi_5(t)\big) \\ &+c_2\big(\phi_4(t)\eta(s) + \phi_6(t)\big) + c_3\phi_7(t) + c_4 \end{aligned} \tag{39}$$

with constant c_1, c_2, c_3 and c_4.

Now, any $\mathbb{R}^3$-valued function satisfying (34) can be written as (39) with replacing the constants c_1, c_2, c_3 and c_4 into some constant vectors C_1, C_2, C_3 and C_4. Here we may assume that $C_4 = 0$ and $\{C_3, C_2, C_1\}$ is linearly independent. Let $\{x, y, z\}$ be the local coordinate system on $\mathbb{R}^3$ with respect to these three vectors. Then all points $p \in f(M)$ can be uniquely written as $p = xC_3 + yC_2 + zC_1$, with real coefficients x, y and z. Then, we have

$$x = \phi_7(t),\ y = \phi_4(t)\eta(s) + \phi_6(t),\ z = \phi_3(t)\eta(s) + \phi_5(t).$$

Thus, the surface can be written as

$$z = y\frac{\phi_3 \circ \phi_7{}^{-1}(x)}{\phi_4 \circ \phi_7{}^{-1}(x)} + \phi_5 \circ \phi_7{}^{-1}(x) - \phi_6 \circ \phi_7{}^{-1}(x)\frac{\phi_3 \circ \phi_7{}^{-1}(x)}{\phi_4 \circ \phi_7{}^{-1}(x)}.$$

With taking

$$A(x) = \frac{\phi_3 \circ \phi_7{}^{-1}(x)}{\phi_4 \circ \phi_7{}^{-1}(x)}$$

and

$$B(x) = \phi_5 \circ \phi_7{}^{-1}(x) - \phi_6 \circ \phi_7{}^{-1}(x)\frac{\phi_3 \circ \phi_7{}^{-1}(x)}{\phi_4 \circ \phi_7{}^{-1}(x)},$$

we obtain the expression $z = yA(x) + B(x)$. □

4.4. The simplest expression of affine minimal ruled surface.
Conversely, we consider the surface given by

$$z = yA(x) + B(x)$$

where A and B are two arbitrary smooth functions in x.

Non-degeneracy of the surface is equivalent to $A'(x) \neq 0$ where $'$ means the differentiation by x. Since the surface is ruled, we have $J \equiv 0$. For the immersion:

$$f\colon (x,\, y) \mapsto \begin{pmatrix} x \\ y \\ yA(x) + B(x) \end{pmatrix},$$

the affine normal vector field is given by

$$\begin{pmatrix} 0 \\ \left(\left|A'(x)\right|^{-1/2}\right)' \\ \left(\left|A'(x)\right|^{-1/2} A(x)\right)' \end{pmatrix},$$

and the shape operator S is given as follows.

$$S\frac{\partial}{\partial x} = -\left(\left|A'(x)\right|^{-1/2}\right)''\frac{\partial}{\partial y},\ S\frac{\partial}{\partial y} = 0.$$

Hence our surface is affine minimal. We are done. □

Remark 4.1 W. Blaschke [1, §79–81] have shown that any affine minimal ruled surface can be written as:

$$f(u,\, v) = \begin{pmatrix} \displaystyle\int (V_1{V_2}' - V_2{V_1}')du \\ \displaystyle vV_1 + \int (V_3{V_1}' - V_1{V_3}')du \\ \displaystyle -vV_2 + \int (V_2{V_3}' - V_3{V_2}')du \end{pmatrix}$$

where V_1, V_2 and V_3 are functions in u. Here by this expression, we can show (ii) of Theorem 2.3 by a different way as follows: By putting

$$x = \int (V_1{V_2}' - V_2{V_1}')du, \quad y = v + \frac{1}{V_1}\int (V_3{V_1}' - V_1{V_3}')du$$

and taking $A\big(x(u)\big) = -V_2(u)/V_1(u)$ and

$$B\big(x(u)\big) = \int (V_2{V_3}' - V_3{V_2}')du + \frac{V_2}{V_1}\int (V_3{V_1}' - V_1{V_3}')du,$$

we have also $z = yA(x) + B(x)$. But our expression in Theorem 2.3 is better than his, since in his expression one needs three equations but in ours one needs only one, his requires three functions and ours requires only two, and our expression is much simpler than his.

Proposition 4.2 *On an affine minimal ruled surface, if the shape operator is parallel then the surface is affinely congruent to one of* (4), (8) *or* (9) *in the list of* Theorem 2.2.

Proof. On the surface $z = yA(x) + B(x)$, we have

$$\big(\nabla_{\partial/\partial y}S\big)\frac{\partial}{\partial y} = \big(\nabla_{\partial/\partial x}S\big)\frac{\partial}{\partial y} = \big(\nabla_{\partial/\partial y}S\big)\frac{\partial}{\partial x} = 0$$

and

$$\big(\nabla_{\partial/\partial x}S\big)\frac{\partial}{\partial x} = \left(\frac{\big(\big|A'(x)\big|^{-1/2}\big)''}{\big|A'(x)\big|^{-1/2}}\right)' \big|A'(x)\big|^{-1/2}\frac{\partial}{\partial y}.$$

Therefore the shape operator is parallel if and only if $\big(\big|A'(x)\big|^{-1/2}\big)''$ is a constant multiple of $\big|A'(x)\big|^{-1/2}$. If the constant is positive, $\big|A'(x)\big|^{-1/2}$ is an exponential function, and it is easy to show that the surface is affinely congruent to (8). If the constant is negative, $\big|A'(x)\big|^{-1/2}$ is a trigonometric function, and the surface is affinely congruent to (9). And if the constant vanishes, $\big|A'(x)\big|^{-1/2}$ is a linear function, and the surface is affinely congruent to (4). □

Example 4.3 The affine surface of $\mathbb{R}^3$ given by the graph of $z = ye^x + xy$, is semiparallel with shape operator is not parallel. In fact, the second derivative of $\big|(e^x + x)'\big|^{-1/2} = (1 + e^x)^{-1/2}$ is $(1/4)e^x(e^x - 2)(1 + e^x)^{-5/2}$, which is not a constant multiple of $(1 + e^x)^{-1/2}$.

Example 4.4 The affine surface of $\mathbb{R}^3$ given by the graph of $z = y\log x$, is semiparallel with shape operator is not parallel. In fact, the second derivative of $\big|(\log x)'\big|^{-1/2} = x^{1/2}$ is $-(1/4)x^{-3/2}$, which is not a constant multiple of $x^{1/2}$.

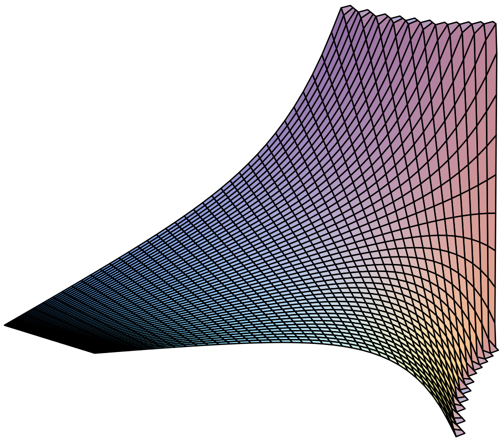

Fig. 1. The graph of $z = ye^x + xy$.

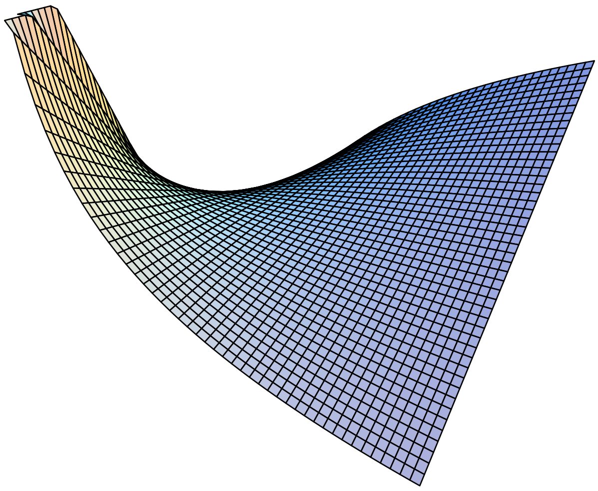

Fig. 2. The graph of $z = y \log x$.

Example 4.5 The affine surface of $\mathbb{R}^3$ given by the graph of $z = y \cos x$, is semiparallel with shape operator is not parallel. In fact, the second derivative of $\left|(\cos x)'\right|^{-1/2} = (\sin x)^{-1/2}$ is $(1/4)(2 + 3/\tan^2 x)(\sin x)^{-1/2}$, which is not a constant multiple of $(\sin x)^{-1/2}$.

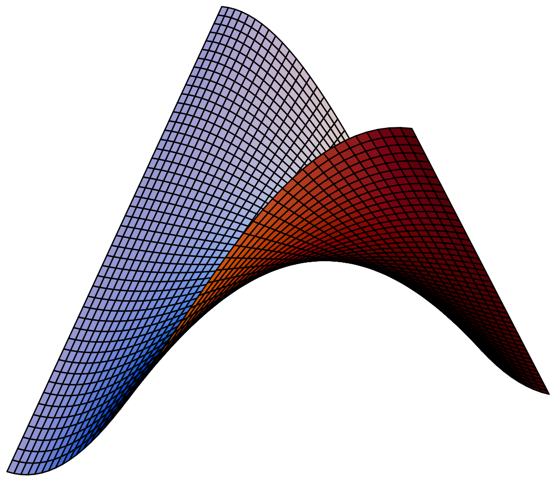

Fig. 3. The graph of $z = y \cos x$.

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