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Purifiable Subgroups II

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Abstract. Let G be an arbitrary group. A subgroup A of G is purifiable in G if, among the pure subgroups of G containing A, there exists a minimal one. We studied purifiable subgroups of abelian groups in [4]. In this note, we give simple proofs of [4, Theorem 4.6], [4, Theorem 4.7], and [4, Theorem 4.8].

Key words: almost-dense subgroup, p-purifiable subgroup, purifiable subgroup, p-overhang set, torsion-complete.

1. Introduction

In [4], we studied purifiable subgroups of an arbitrary abelian group. In Section 3 of [4], we introduced maximal p-vertical subgroups and used the concept of maximal p-vertical subgroups to prove [4, Theorem 4.6], [4, Theorem 4.7], and [4, Theorem 4.8].

In this note, we give proofs for [4, Theorem 4.6], [4, theorem 4.7], and [4, Theorem 4.8] without the use of maximal p-vertical subgroups.

All groups considered are arbitrary abelian groups. The terminologies and notations not expressly introduced follow the usage of [1]. Throughout this note, $\mathbf{Z}$ denotes the ring of integers, $\mathbf{P}$ the set of all prime integers, p always denotes a prime, T the maximal torsion subgroup, and G_p the p-component of an abelian group G.

2. Notation and basics

In this section, we recall definitions and properties in [4] that are used frequently in this note.

From the definition [4, Definition 1.1] of p-almost-dense subgroups and its characterization [4, Proposition 1.3, Proposition 1.4], we can define p-almost-dense and almost-dense subgroups as follows.

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Definition 2.1 A subgroup A of a group G is said to be p-almost-dense in G if, for all integers $n \geqq 0$,

$$p^n G[p] \subseteq A + p^{n+1} G.$$

Moreover, the subgroup A is said to be almost-dense in G if A is p-almost-dense in G for every $p \in \mathbf{P}$.

Recall the definition of p-purifiable [purifiable] in a group G.

Definition 2.2 Let G be a group. A subgroup A of G is said to be p-purifiable [purifiable] in G if, among the p-pure [pure] subgroups of G containing A, there exists a minimal one. Such a minimal p-pure [pure] subgroup is called a p-pure [pure] hull of A.

Proposition 2.3 [4, Theorem 1.8, Theorem 1.11] *Let G be a group and A a subgroup of G. Let H be a p-pure* [*pure*] *subgroup of G containing A. Then H is a p-pure* [*pure*] *hull of A in G if and only if the following three conditions are satisfied*:

1. *for all integers $n \geqq 0$* [*and all $p \in \mathbf{P}$*],

$$p^n H[p] \subseteq A + p^{n+1} H;$$

2. *H/A is p-primary* [*torsion*];
3. [*for every $p \in \mathbf{P}$,*] *there exists a nonnegative integer m_p such that*

$$p^{m_p} H[p] \subseteq A.$$

The following is a relationship between purifiability and p-purifiability.

Proposition 2.4 [4, Theorem 1.12] *Let G be a group. A subgroup A of G is purifiable in G if and only if, for every $p \in \mathbf{P}$, A is p-purifiable in G.*

Definition 2.5 Let G be a group and A a subgroup of G. For every nonnegative integer n, we define the nth p-overhang of A in G to be the vector space

$$V_{p,n}(G, A) = \frac{(A + p^{n+1}G) \cap p^n G[p]}{(A \cap p^n G)[p] + p^{n+1} G[p]}.$$

Moreover, the set

$$O_A^G(p) = \{t \mid V_{p,t}(G, A) \neq 0\}$$

is called the p-overhang set of A in G.

In [4] the subgroup A of a group G is said to be *eventually p-vertical* in G if the set $O_A^G(p)$ is finite and A is said to be *p-vertical* in G if the set $O_A^G(p)$ is empty.

It is convenient to use the following notations for the numerator and the denominator of $V_{p,n}(G, A)$:

$$A_G^n(p) = (A + p^{n+1}G) \cap p^n G[p] = ((A \cap p^n G) + p^{n+1}G)[p],$$
$$A_n^G(p) = (A \cap p^n G)[p] + p^{n+1}G[p].$$

We immediately obtain the following results.

Proposition 2.6 *Let G be a group and A a subgroup of G. Then the following hold.*

1. *For any $x \in A_G^n(p) \setminus A_n^G(p)$, we have*

$$h_p(x) = n.$$

2. *If $x \in A_G^n(p)$, then $h_p^{G/A}(x + A) > n$.*
3. *If A is p-almost-dense in G, then $A + p^{n+1}G \supseteq p^n G[p]$, so $A_G^n(p) = p^n G[p]$.*
4. [4, Lemma 4.2(1)] $V_{p,m+n}(G, A) = V_{p,n}(p^m G, A \cap p^m G)$ *for all n, $m \geqq 0$.*

Proposition 2.7 [4, Proposition 2.2] *Let G be a group and A a subgroup of G. For a p-pure subgroup K of G containing A,*

$$V_{p,n}(G, A) \cong V_{p,n}(K, A)$$

for all $n \geqq 0$. Hence $O_A^G(p) = O_A^K(p)$.

Proposition 2.7 leads to the following intrinsic necessary condition for p-purifiability.

Proposition 2.8 [4, Theorem 2.3] *If a subgroup A of a group G is p-purifiable in G, then there exists a nonnegative integer m such that $V_{p,n}(G, A) = 0$ for all $n \geqq m$, and hence the set $O_A^G(p)$ is finite.*

Proposition 2.9 *Let G be a group and A a subgroup of G. Suppose that A is p-purifiable in G. Let H be a p-pure hull of A in G. Then the following are equivalent*:

1. $O_A^G(p) = \emptyset$;
2. $H[p] = A[p]$.

Proof. (1) ⇒ (2) Suppose that (1) is satisfied. By Proposition 2.3(3), there exists a nonnegative integer m such that $p^m H[p] \subseteq A$. Suppose that m is the least integer such that $p^m H[p] \subseteq A$ and $m \geqq 1$. By Proposition 2.6(3),

$$p^{m-1}H[p] = A_H^{m-1}(p) = A_{m-1}^H(p) = (A \cap p^{m-1}H)[p] + p^m H[p] \subseteq A.$$

This contradicts the choice of m. Thus $m = 0$ and $H[p] \subseteq A[p]$. Hence $H[p] = A[p]$.
(2) ⇒ (1) Suppose that (2) is satisfied. By Proposition 2.6(3), for all $n \geqq 0$, we have

$$A_H^n(p) = p^n H[p] \subseteq (A \cap p^n H)[p] \subseteq A_n^H(p).$$

Thus $O_A^H(p) = \emptyset$. By Proposition 2.7, the assertion is clear. □

Proposition 2.10 [4, Proposition 2.7] *Let G be a group and A a subgroup of G. Then the following properties are equivalent*:
1. $O_A^G(p) = \emptyset$;
2. $(A + p^n G)[p] = A[p] + p^n G[p]$ *for all* $n \geqq 0$.

Proposition 2.11 [4, Theorem 4.1] *Let G be a group and A a subgroup of G. If $A \cap p^m G$ is p-purifiable in $p^m G$ for some $m \geqq 0$, then A is p-purifiable in G.*

It is well-known that, for every subgroup A of a group G, there exists a minimal neat subgroup N of G containing A. Such a subgroup N is called a *neat hull* of A in G. Neat hulls are investigated in [3] and [5] independently. In this note, we use their results in the following form.

Proposition 2.12 *Let G be a group and A subgroup of G. Then there exists a neat hull N of A in G. Moreover, for all $p \in \mathbf{P}$, $N[p] = A[p]$.*

3. Purifiable subgroups

The following is a technical lemma.

Lemma 3.1 *Let G be a group and A a subgroup of G. Write $G/A = \bigoplus_p G^{(p)}/A$ where $G^{(p)}/A = (G/A)_p$. Then $G^{(p)}$ is p-pure in G.*

Proof. Let $p^n g \in G^{(p)}$ with $g \in G$ and $G^{(p')} = \sum_{q \neq p} G^{(q)}$. Then we can write $g = x + y$ for some $x \in G^{(p)}$ and $y \in G^{(p')}$. Since $p^n y \in G^{(p)} \cap G^{(p')} = A$, we have $y \in A$. Hence $G^{(p)}$ is p-pure in G. □

Now we can characterize the groups all of whose subgroups are p-purifiable.

Theorem 3.2 [4, Theorem 4.6] *Let G be a group. Then all subgroups of G are p-purifiable in G if and only if G_p is the direct sum of a divisible and a bounded subgroup.*

Proof. ($\Rightarrow$) Suppose that all subgroups of G are p-purifiable in G. Let A be any subgroup of G_p. By hypothesis, A is p-purifiable in G. Let L be a p-pure hull of A in G. By Proposition 2.3(2), L/A is a p-group. Thus $L \subseteq G_p$ and so A is p-purifiable in G_p. Since G_p is a p-group, A is purifiable in G_p. Hence all subgroups of G_p are purifiable in G_p and by [2, Theorem 1], G_p is the direct sum of a divisible and a bounded subgroup.
($\Leftarrow$) By hypothesis, G_p is a direct summand of G and $G_p = B \oplus D$ where B is a bounded subgroup of G_p and D is a divisible subgroup of G_p. Hence $G = G' \oplus G_p = G' \oplus B \oplus D$ for some subgroup G' of G. Since B is a bounded p-group, $p^m G = p^m G' \oplus D$ for some integer $m \geqq 0$. Let A be a subgroup of G. If $A \cap p^m G$ is p-purifiable in $p^m G$, then, by Proposition 2.11, A is p-purifiable in G. Hence, without loss of generality, we may assume that G_p is divisible. By Proposition 2.12, there exists a neat hull H of A_p in G_p and

$$H[p] = A[p]. \tag{3.3}$$

Since $H \cap pG_p = pH$ and $pG_p = G_p$, $H = pH$. Thus

$$G_p = H \oplus K \quad \text{for some subgroup } K \text{ of } \quad G_p. \tag{3.4}$$

Then K is also divisible. Let $G^{(p)}/A = (G/A)_p$. By Lemma 3.1, $G^{(p)}$ is p-pure in G. Since $(K \oplus A)/A \cong K$ is divisible, there exists a subgroup L of $G^{(p)}$ such that $G^{(p)}/A = L/A \oplus (K \oplus A)/A$. Hence

$$G^{(p)} = L \oplus K, \quad A \subseteq L. \tag{3.5}$$

By the definition of L, (3.3), (3.4), and (3.5), we have $L[p] = A[p]$. By Definition 2.1, A is p-almost-dense in L. By the definition of L, $L \subseteq G^{(p)}$ and L/A is a p-group. Therefore, by Proposition 2.3, L is a p-pure hull of A in $G^{(p)}$. Hence A is p-purifiable in G. $\square$

Proposition 2.4 and Theorem 3.2 combined lead to a characterization of the groups all of whose subgroups are purifiable.

Corollary 3.6 [4, Theorem 5.2] *All subgroups of a group G are purifiable in G if and only if, for every $p \in \mathbf{P}$, G_p is the direct sum of a bounded and a divisible subgroup.*

In Proposition 2.8 we established a necessary condition for a subgroup to be p-purifiable in a given group. We use Theorem 3.2 to prove that if a subgroup A of a group is p-almost-dense in G, then the condition also becomes sufficient.

Theorem 3.7 [4, Theorem 4.7] *Let G be a group and A a subgroup of G. Suppose that A is p-almost-dense in G. Then A is p-purifiable in G if and only if $O_A^G(p)$ is finite.*

Proof. By Proposition 2.8, it suffices to prove sufficiency. Suppose that the $O_A^G(p)$ is finite. By Proposition 2.11 and Proposition 2.6(4), without loss of generality, we may assume that $O_A^G(p) = \emptyset$. Then, by Proposition 2.10,

$$p^n G[p] \subseteq (A + p^{n+1}G)[p] = A[p] + p^{n+1}G[p] \quad \text{for all} \quad n \geqq 0.$$

Hence $A[p]$ is dense in $G[p]$. By [1, Theorem 66.3], there exists a pure subgroup H of G_p such that $A[p] = H[p]$ and G_p/H is divisible. Since $(G/H)_p = G_p/H$ is divisible, by Theorem 3.2, $(A+H)/H$ is p-purifiable in G/H. Let M/H be a p-pure hull of $(A+H)/H$ in G/H. Then M is p-pure in G.

We will prove that M is a p-pure hull of A in G by verifying the conditions of Proposition 2.3. Since $M/(A+H)$ and $(H+A)/A$ are p-groups, M/A is a p-group. By hypothesis and Definition 2.1, it is easy to see that A is p-almost-dense in M. Since M/H is a p-pure hull of $(A+H)/H$ in G/H, by Proposition 2.3(3), $p^k(M/H)[p] \subseteq (A+H)/H$ for some integer $k \geqq 0$. Further, since H is p-pure in G, we have $(p^k M[p] + H)/H = p^k(M/H)[p] \subseteq (A+H)/H$ and $p^k M[p] \subseteq (A+H)[p]$. If $a+h \in (A+H)[p]$ where $a \in A$ and $h \in H$, then $pa = -ph$ and $a \in A_p \subseteq H$. Hence $p^k M[p] \subseteq (A+H)[p] = H[p] = A[p]$. Therefore, by Proposition 2.3, M is a p-pure hull of A in G. □

Proposition 2.4 and Theorem 3.7 combined lead to the following result.

Corollary 3.8 [4, Theorem 5.3] *Let G be a group and A a subgroup of G. Suppose that A is almost-dense in G. Then A is purifiable in G if and only if, for every $p \in \mathbf{P}$, $O_A^G(p)$ is finite.*

Next we consider groups for which the reduced parts of the maximal p-primary subgroups are torsion-complete. First we give a technical lemma.

Lemma 3.9 *Let G be a group and A a subgroup of G. Let $\overline{A} = \bigcap_n (A + p^n G)$. Suppose that $O_A^G(p) = \emptyset$. Then the following hold.*
1. $O_{\overline{A}}^G(p) = \emptyset$.
2. $\overline{A}[p] = \overline{A[p]}$.
3. *If $\overline{A}$ is p-purifiable in G, then A is p-purifiable in G.*

Proof. (1) Since $A \subseteq \overline{A} \subseteq A + p^{n+1}G$, $\overline{A} + p^{n+1}G = A + p^{n+1}G$. Then $\overline{A}_G^n(p) = A_G^n(p) = A_n^G(p) \subseteq (\overline{A} \cap p^n G)[p] + p^{n+1}G[p] = \overline{A}_n^G(p)$. Hence $O_{\overline{A}}^G(p) = \emptyset$.

(2) $\overline{A}[p] = \overline{A} \cap G[p] = \bigcap_n (A + p^n G)[p] \overset{(2.10)}{=} \bigcap_n (A[p] + p^n G[p]) = \overline{A[p]}$.
(3) Suppose that $\overline{A}$ is p-purifiable in G. Let H be a p-pure hull of $\overline{A}$ in G. By (1) and Proposition 2.9, $H[p] = \overline{A}[p]$. By (2),

$$H[p] = \overline{A}[p] = \overline{A[p]} = \bigcap_n (A[p] + p^n G[p]) \subset A + p^n G$$

for all $n \geqq 0$. Hence A is p-almost-dense in H. By Proposition 2.7, $O_A^H(p) = \emptyset$. Hence, by Theorem 3.7, A is p-purifiable in H and in G. □

Lemma 3.10 *Let G be a group and A a subgroup of G. Suppose that G_p is torsion-complete and $O_A^G(p) = \emptyset$. Then A is p-purifiable in G.*

Proof. Let $G^{(p)}/A = (G/A)_p$. By Lemma 3.1, $G^{(p)}$ is p-pure in G. Further, by Proposition 2.7, $O_A^{G^{(p)}}(p) = \emptyset$. Hence, without loss of generality, we may assume that $G = G^{(p)}$, because $(G^{(p)})_p = G_p$ is torsion-complete.

Let $\overline{A} = \bigcap_n (A + p^n G)$. By Lemma 3.9(2), $\overline{A}[p] = \overline{A[p]}$. Since G_p is torsion-complete, there exists a pure subgroup K of G_p such that $\overline{A}[p] = \overline{A[p]} = K[p]$. Moreover $\overline{K}[p] = \overline{K[p]} = \overline{\overline{A[p]}} = \overline{A[p]} = K[p]$. Since G_p is torsion-complete, by [1, Corollary 68.9],

$$G_p = \overline{K} \oplus L \quad \text{for some subgroup } L \text{ of } G. \tag{3.11}$$

Next we prove that $(L \oplus \overline{A})/\overline{A}$ is pure in $G/\overline{A}$. Note that $((L \oplus \overline{A})/\overline{A})[p] = (L[p] \oplus \overline{A})/\overline{A}$. Let $x + \overline{A} \in (L[p] \oplus \overline{A})/\overline{A}$ with $x \in L[p]$. Suppose that $h_p^{G/\overline{A}}(x + A) = n$. Then $x + \overline{A} = p^n g + \overline{A}$ for some $g \in G$. By Lemma 3.9(1), $O_{\overline{A}}^G(p) = \emptyset$. Further, by Proposition 2.10, $x \in (\overline{A} + p^n G)[p] = \overline{A}[p] + p^n G[p]$. Since $G_p = \overline{K} \oplus L$, we have $x = a + p^n k + p^n l$ for some $a \in \overline{A}[p]$, $p^n k \in$

$p^n\overline{K}[p]$, and $p^n l \in p^n L[p]$. Since $\overline{K}[p] = \overline{A}[p]$, $x + \overline{A} = p^n l + \overline{A}$. Hence $h_p^{(L\oplus\overline{A})/\overline{A}}(x+\overline{A}) = n$ and $(L\oplus\overline{A})/\overline{A}$ is pure in $G/\overline{A}$.

Since $(L\oplus\overline{A})/\overline{A} \cong L$ is torsion-complete, $G/\overline{A} = H/\overline{A} \oplus (L\oplus\overline{A})/\overline{A}$ for some subgroup H of G. Then $G = H \oplus L$ and $H[p] \supseteq \overline{A}[p] = \overline{K}[p]$. By (3.11), $H[p] = \overline{A}[p]$. Since $H/\overline{A}$ is a p-group, by Proposition 2.3, H is a p-pure hull of $\overline{A}$ and $\overline{A}$ is p-purifiable in G. Hence, by Lemma 3.9(3), A is p-purifiable in G. □

Theorem 3.12 [4, Theorem 4.8] *Let G be a group and A a subgroup of G. Suppose that the reduced part of G_p is torsion-complete and $O_A^G(p)$ is finite. Then A is p-purifiable in G.*

Proof. Let $G^{(p)}/A = (G/A)_p$. By Lemma 3.1, $G^{(p)}$ is p-pure in G. Further, by Proposition 2.7, $O_A^{G^{(p)}}(p)$ is finite and $G_p \subseteq G^{(p)}$. By Proposition 2.11 and Proposition 2.6(4), without loss of generality, we may assume that $G = G^{(p)}$, $O_A^G(p) = \emptyset$, and the reduced part of G_p is torsion-complete.

Let $\overline{A} = \bigcap_n (A + p^n G)$ and let D be the maximal divisible subgroup of G_p. Then $D \subseteq \overline{A}$ and $\overline{A} = A' \oplus D$ for some subgroup A' of A. Let G' be a D-high subgroup of G containing A'. Then $G = G' \oplus D$ and G'_p is torsion-complete.

We prove that $O_{A'}^{G'}(p) = \emptyset$. By Lemma 3.9(1), $O_{\overline{A}}^G(p) = \emptyset$ and by Proposition 2.10, $(\overline{A}+p^nG)[p] = \overline{A}[p]+p^nG[p]$ for all $n \geqq 0$. Then $(A'+p^nG)[p] \subseteq (\overline{A}+p^nG)[p] = \overline{A}[p]+p^nG[p] = A'[p]+p^nG[p]$, because $\overline{A}[p] = A'[p] \oplus D[p]$. Hence, by Proposition 2.10 and Proposition 2.7, $O_{A'}^{G'}(p) = \emptyset$.

By Lemma 3.10, A' is p-purifiable in G'. Let H be a pure hull of A' in G'. Then $H \oplus D$ is a pure hull of $\overline{A}$ in G. By Lemma 3.9(3), A is p-purifiable in G. □

Proposition 2.4 and Theorem 3.12 combined lead to the following result.

Corollary 3.13 [4, Theorem 5.4] *Let G be a group. Suppose that the reduced part of T is torsion-complete. Let A be any subgroup of G such that $O_A^G(p)$ is finite for every $p \in \mathbf{P}$. Then A is purifiable in G.*

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