

## Real moduli in local classification of Goursat flags

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(Received July 4, 2002; Revised November 4, 2003)

**Abstract.** Goursat distributions are subbundles (of codimension at least 2) in the tangent bundles to manifolds having the flag of consecutive Lie squares of ranks not depending on a point and growing — very slowly — always by 1. This defining condition is rather strong, implying local polynomial pseudo-normal forms for them (proposed in 1981 by Kumpera and Ruiz) featuring only real parameters of *a priori* unknown status, many of them reducible by further diffeomorphisms of the base manifold.

We show that in the local  $C^\infty$  and  $C^\omega$  classifications of Goursat distributions genuine continuous moduli appear already in codimension 2. First examples of such moduli were given in codimension 3; in codimensions 0 and 1 the local classification is known and discrete.

*Key words:* Goursat flag, singularity, local classification, module, geometric class, basic geometry.

### 1. Geometric classes of germs of Goursat flags and Main Theorem

Goursat flags are certain special nested sequences, say  $\mathcal{F}$ , of variable length  $r$  ( $2 \leq r \leq n - 2$ ) of subbundles in the tangent bundle  $TM$  to a smooth ( $C^\infty$ ) or analytic ( $C^\omega$ )  $n$ -dimensional manifold  $M$ :  $D^r \subset D^{r-1} \subset \dots \subset D^1 \subset D^0 = TM$ . Namely, one demands, for  $l = r, r - 1, \dots, 1$  that (a) cork  $D^l = l$ , and (b) the Lie square of  $D^l$  be  $D^{l-1}$ . Every member of  $\mathcal{F}$  save  $D^1$  is called *Goursat distribution*,  $r$  is called the length of  $\mathcal{F}$ . They naturally generalize the well-known Cartan's distributions on the jet spaces of functions  $\mathbb{R} \rightarrow \mathbb{R}$ . The latter (the smallest flag's member is then of rank 2) satisfy (a)–(b), but display no singularities. While these conditions do admit singularities, as it has been known since 1978 (Giario-Kumpera-Ruiz, see their Example 1).

This, very restricted, class of objects was being investigated (intermittently) over the last 110 years, with important contributions by E. von Weber [23] and E. Cartan [4]; the latter arriving at those systems in his

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2000 Mathematics Subject Classification : 58A17, 30.

\*Supported by Polish KBN Grant 2 P03A 041 15.

analysis of the least underdetermined ODEs (see more on that below). They proved independently that every corank  $r$  Goursat distribution  $D^r$  around a generic point of  $M$  locally behaves in a unique way visualised by the *chained model* — the germ at  $0 \in \mathbb{R}^n(x^1, \dots, x^{r+2}; x^{r+3}, \dots, x^n)$  of

$$(\partial_n, \dots, \partial_{r+3}; \partial_{r+2}, \partial_1 + x^3\partial_2 + x^4\partial_3 + \dots + x^{r+2}\partial_{r+1}) \quad (\text{C})$$

(these are vector fields generators; effectively used are only first  $r + 2$  coordinates). In such a local expression one easily retrieves the Lie growth assumed in the definition: the Lie square produces only one new and linearly independent generator  $\partial_{r+1}$ , the next Lie square likewise produces only  $\partial_r$ , etc. Chained models can be viewed as the simplest instance of a family of local writings (preliminary normal forms with real parameters of, in general, unknown status) of Goursat distributions, obtained much later by Kumpera and Ruiz in [10] and quoted in Theorem 1. We call them *KR pseudo-normal forms*. As already told, Kumpera and Ruiz discovered *singularities* hidden in flags, and pseudo-normal forms were merely a byproduct. Those forms, however, have been an important step in the (still open) problem of the local classification ( $C^\infty$  or  $C^\omega$ ) of flags; they feature only numerical parameters, and no functional moduli.

**Theorem 1** ([10]) *For any Goursat flag of length  $r$  on a smooth ( $C^\infty$  or  $C^\omega$ ) manifold  $M$  of dimension  $n \geq r + 2$ ,  $TM = D^0 \supset D^1 \supset D^2 \supset \dots \supset D^r$ , around any point  $p \in M$  there exist local coordinates  $x^1, x^2, \dots, x^{r+2}; x^{r+3}, \dots, x^n$  centered at  $p$ , of the same class as  $M$ , such that in these coordinates each  $D^j$  has around  $p$  a Pfaffian description  $\omega^1 = \omega^2 = \dots = \omega^j = 0$ ,  $j = 1, 2, \dots, r$ , where*

$$\begin{aligned} \omega^1 &= dx^{i_1} - x^3 dx^{j_1}, & (i_1, j_1) &= (2, 1) \\ \omega^2 &= dx^{i_2} - x^4 dx^{j_2}, & (i_2, j_2) &= (3, j_1) = (3, 1) \\ \omega^3 &= dx^{i_3} - x^5 dx^{j_3}, & (i_3, j_3) &\in \{(4, j_2), (j_2, 4)\} \\ \omega^4 &= dx^{i_4} - X^6 dx^{j_4}, & (i_4, j_4) &\in \{(5, j_3), (j_3, 5)\} \\ & * & * & * & * \\ \omega^r &= dx^{i_r} - X^{r+2} dx^{j_r}, & (i_r, j_r) &\in \{(r+1, j_{r-1}), (j_{r-1}, r+1)\}. \end{aligned}$$

*In this writing, for  $6 \leq l \leq r + 2$ ,  $X^l = c^l + x^l$  excepting the cases of inversions  $(i_{l-2}, j_{l-2}) = (j_{l-3}, l-1)$  when simply  $X^l = x^l$ . That is, each  $D^j$  becomes in these coordinates the germ at  $0 \in \mathbb{R}^n$  of the indicated corank- $j$*

polynomial Pfaffian system. The  $c^6, c^7, \dots, c^{r+2}$  are real constants that are not, in general, uniquely determined by the flag' germ.

Moreover, when  $j_3 = j_4 = \dots = j_l = 1$  for certain  $4 \leq l \leq r$  (i.e., when there is no inversions of differentials in  $\omega^3, \omega^4, \dots, \omega^l$ ), then, specifically,  $c^6, c^7, \dots, c^{l+2}$  are zero.

Conversely, all pairs of sequences  $\{i_l\}$  and  $\{j_l\}$  ( $l = 1, \dots, r$ ) fulfilling the conditions written above, and arbitrary real constants  $c^6, c^7, \dots, c^{r+2}$  (when applicable) are permitted and always give a Goursat flag.

We underline that in this theorem all members of the flag of  $D^r$  simultaneously get certain neat descriptions. The generic model (C) comes out when there is no inversion of differentials in all 1-forms  $\omega^3, \omega^4, \dots, \omega^r$ .

**Corollary 1** *It follows automatically from Theorem 1 that Goursat distributions of arbitrary rank locally are the direct sums of integrable distributions (foliations) and of Goursat distributions of rank 2 invariant with respect to those foliations. In Kumpera-Ruiz coordinates for any given Goursat germ, that integrable direct summand gets a clear description  $dx^1 = dx^2 = \dots = dx^{r+2} = 0$ .*

At a first glance it is hardly perceptible how the constants reflect different geometric behaviours of flag's members. Prior to more conclusive Proposition 1, here are two statements aimed at showing that in pseudo-normal forms certain constants are of key importance, while others are simply irrelevant. For  $r = 5$ , in the family of KR pseudo-normal forms

$$\begin{aligned} & (dx^2 - x^3 dx^1, dx^3 - x^4 dx^1, dx^1 - x^5 dx^4, \\ & dx^5 - (c^6 + x^6) dx^4, dx^6 - (c^7 + x^7) dx^4) \end{aligned} \quad (1)$$

around  $0 \in \mathbb{R}^7$ , the objects with  $c^6 = 0$  are non-equivalent to those with  $c^6 \neq 0$ . Among the *former*, the value of  $c^7$  can be reduced either to 0 or to 1, and these two normalized values are non-equivalent. Among the *latter*,  $c^6$  can be reduced to 1, and (quite unexpectedly; overlooked in [10], rectified in [7] and [5])  $c^7$  to 0. Thus the non-equivalent 'model' values of  $(c^6, c^7)$  are just  $(1, 0)$ ,  $(0, 1)$  and  $(0, 0)$ .

For  $r = 8$ , in the family

$$\begin{aligned} & (dx^2 - x^3 dx^1, dx^3 - x^4 dx^1, dx^1 - x^5 dx^4, \\ & dx^5 - (c^6 + x^6) dx^4, dx^4 - x^7 dx^6, dx^7 - (c^8 + x^8) dx^6, \end{aligned} \quad (2)$$

$$dx^6 - x^9 dx^8, dx^9 - (c^{10} + x^{10}) dx^8)$$

of germs at  $0 \in \mathbb{R}^{10}$  with, say,  $c^6, c^8 \neq 0$ , the quantity  $(c^6)^{-1}(c^8)^2 c^{10}$  turns out to be an invariant (*module*) of the local classification of Goursat distributions, see Remark 3 in [15]. Therefore, after normalizing the preceding constants  $c^6$  and  $c^8$  to 1, the value of  $c^{10}$  is uniquely determined in such a pseudo-normal form.

*Attention.* The germs of corank- $r$  Goursat distributions are very rare — of codimension  $\infty$  — among all germs of corank- $r$  distributions. The unique and important exception is corank 2 in the ambient dimension 4; this is the pioneering Engel situation [6] giving a huge open set in the space of *all* germs. (The Engel local model appears in Theorem 1, as well as in (C): for  $n = 4$  and  $r = 2$  one finds there  $dx^2 - x^3 dx^1 = dx^3 - x^4 dx^1 = 0$ .)

Although far from being generic, these distributions are important in applications. For instance, they locally possess nilpotent bases and underlie classical kinematic systems ‘car + [many] passive trailers’ (see more on that in section 1.1); the absence of functional moduli is also a big advantage.

Moreover, as is proved in [13], modulo a splittable codimension two integrable subdistribution (see Corollary 1), Goursat distributions have a very neat construction — are locally nothing more than series of quite elementary *Cartan prolongations*<sup>1</sup> started from the tangent bundle to a [piece of] 2-surface. Also the resulting universal manifolds for flags of length  $r$  (called sometimes Monster Goursat Manifolds) are explicitly described in [13].

Cartan used in [4] a prototype of the technique bearing nowadays his name to answer what underdetermined systems of ODEs (degree of underdetermination 1) admit a parametrization of their generic solutions by one free function of one variable. (Note parenthetically that that approach of 1914 is currently being commented and extended to more deeply underdetermined systems of ODEs, [11], prompting the use of *multiflags* along with Goursat flags (1-flags).)

### 1.1. Sandwich Diagram and the definition of geometric classes

An extended geometric clarification, summarized below in Proposition 1, is possible to the pseudo-normal forms of Kumpera & Ruiz. It has been due mainly to Jean and Montgomery & Zhitomirskii. Upon closer

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<sup>1</sup>as defined, in more general setting, in [3]

inspection there emerges, [8, 13, 14], a stratification of germs of flags into **canonically** defined *geometric classes*, with strata encoded by words (of length equal to flag's length) over the alphabet {G, S, T}: Generic, Singular, Tangent, subject to certain restrictions. We want to recall that definition and draw some natural corollaries.

The first ingredient is the classical notion, for any distribution  $D$ , of the module (or sheaf of modules) of Cauchy-characteristic vector fields  $v$  with values in  $D$  that preserve  $D$ ,  $[v, D] \subset D$ . And one of first observations is that for  $D$  — Goursat,  $L(D)$  is a regular corank *two* subdistribution of  $D$ ,  $\text{rk } L(D) = \text{rk } D - 2$ , enjoying one additional (and key) property. Namely,  $L([D, D]) \subset D$ .

**Remark 1**  $L(D)$  is that integrable direct summand in  $D$  mentioned in Corollary 1. Also the property  $L([D, D]) \subset D$  is clearly visible in the glasses offered by Theorem 1 which supplies local forms simultaneously for  $D$  and  $[D, D]$ , and allows to compute  $L([D, D])$  as well.

The second ingredient is putting this all together for a corank  $r$  Goursat distribution  $D^r$ , first done in [13], p. 464 under the form of the Sandwich Diagram.

$$\begin{array}{ccccccccccc} D^1 & \supset & D^2 & \supset & D^3 & \supset & \dots & \supset & D^{r-1} & \supset & D^r \\ & & \cup & & \cup & & & & \cup & & \cup \\ & & L(D^1) & \supset & L(D^2) & \supset & \dots & \supset & L(D^{r-2}) & \supset & L(D^{r-1}) & \supset & L(D^r). \end{array}$$

In view of the mentioned properties, all direct inclusions in this diagram are of codimension one. One gets here  $r - 2$  squares (indexed by the upper right vertices) built of inclusions, and in each  $j$ -th square ( $j = 3, 4, \dots, r$ ) the distributions  $D^j$  and  $L(D^{j-2})$  have the same rank. These spaces can be perceived as certain fillings in a sandwich with covers  $D^{j-1}$  and  $L(D^{j-1})$  (of not the same dimension). With this interpretation at hand, Montgomery & Zhitomirskii say that  $D^j$  is at  $p$  in singular position when it coincides at  $p$  with  $L(D^{j-2})$ :  $D^j(p) = L(D^{j-2})(p)$ . That is, when the fillings in the  $j$ -th sandwich coincide (coalesce) at  $p$ .

A straightforward check (Proposition 2 in [18]) shows that this happens iff in *any* pseudo-normal form for  $D^j$  around  $p$  there occurs the inversion of differentials in  $\omega^j$ . (The pioneers Kumpera and Ruiz were *not* aware of this interpretation. Its first mention ever was made, in a slightly veiled form, in [3], p. 455<sup>7-11</sup>.) As a consequence,  $D^3, D^4, \dots, D^r$  can be, at any

fixed point, in singular positions one independently of the others, giving rise to  $2^{r-2}$  invariant classes of local flag's geometries (behaviours), or of flag's germs. Now the unique generic behaviour of flags, best visible via (C), becomes more comprehensible (cf. elegant Proposition 2.1 in [13]): a rather dry absence of inversions is nothing but the geometric absence of singular positions.

Let us encode the coalescences happening at  $p$  by a word of length  $r$ , with a  $j$ -th letter ( $3 \leq j \leq r$ ) being S iff  $D^j$  is in singular position at  $p$ , and with all letters before (i.e., to the left from) the first appearing letter S (if any) being G. This implies that the two first letters are always G — the two biggest flag's members  $D^1$  and  $D^2$  are never in singular positions. Note that, at this stage, a code may well have **many blank spaces**. The only codes without blank spaces *now* are the code of the generic behaviour (all letters G) and the codes starting with a number of G's followed uniquely by a number of S's, like, for inst., in GGSSSS. Before going further, let us watch more closely a simple case of this last type — the local geometry GGS.

**Example 1** Let  $D^3$  be a corank 3 Goursat germ already in the [Giaro-Kumpera-Ruiz] normal form

$$dx^2 - x^3 dx^1 = dx^3 - x^4 dx^1 = dx^1 - x^5 dx^4 = 0$$

(the inversion of differentials  $dx^1$  and  $dx^4$  in the last Pfaffian equation  $\omega^3 = 0$ ). Its flag's member  $D^1$  is given by (cf. Theorem 1)  $dx^2 - x^3 dx^1 = 0$  and the Cauchy characteristics  $L(D^1)$  of that member are  $(\partial_4, \partial_5)$ . This 2-plane coincides with  $D^3$  at 0, and — more generally — at all points of the hypersurface  $\{x^5 = 0\}$ . Therefore,  $D^3$  is in singular position not at isolated points, but **in codimension 1**. It is likewise with materializations in flags of any single singularity S, see for inst. Lemma 1 in [18].

Heading towards geometric classes and labels over  $\{G, S, T\}$ , we have to fill in strings of blank spaces standing behind, or past, letters S. In the many justifications that are omitted in this long definition, consequently used are pseudo-normal forms of Theorem 1, serving in the guise of 'night glasses' in the Goursat world. Let now the  $j$ -th letter in the actual word be S followed by one or several blank spaces. In the eventual label this S is followed by T when  $D^{j+1}(p)$  is tangent to the locus  $H$  — always a regular hypersurface in  $M$  — of the previous singularity ' $D^j$  in singular position'. ( $D^j$  is never tangent to 'its' singularity locus  $H$ , but  $D^{j+1} \subset D^j$  may, at

some more singular points, be so.)

*Attention.* The only thing to be checked with this definition of T going directly after an S is that such tangent space  $D^{j+1}(p)$  be not ... the first order singular position  $L(D^{j-1})(p)$  (prohibited when the  $(j+1)$ -th place in the preliminary word has been assumed blank). And it is straightforward, for in KR coordinates  $L(D^{j-1}) = (\partial_{j+2}, \partial_{j+3}, \dots, \partial_n)$  is nowhere tangent to  $H = \{x^{j+2} = 0\}$ .

If  $D^{j+1}(p)$  is not tangent to  $H$ , we insert a sequence of G's past that S at the  $j$ -th place until meeting the next S in the word (or till the end if there is no next S).

The alternative ST or SGG... is equally transparent on the level of pseudo-normal forms. Since there is no letter S at the  $(j+1)$ -th place, one discusses the position of  $D^{j+1}$  at 0 in the situation when the incoming Pfaffian equation  $\omega^{j+1} = 0$  has no inversion of differentials, hence features a constant  $c^{j+3}$ . And ST means just  $c^{j+3} = 0$ , while SG means  $c^{j+3} \neq 0$ , in whatever taken Kumpera-Ruiz coordinates.

If the  $(j+2)$ -th place is still blank, then one watches the locus, say  $N$ , of ' $D^j$  in singular position and  $D^{j+1}$  in tangent position' which is always an embedded codimension-two submanifold, cf. Proposition 3 in [18]. As could be expected,  $D^{j+1}$  is never tangent to 'its' double singularity locus  $N$ , and the same applies to  $L(D^j)$ , but  $D^{j+2}$  may (at some still more singular points) already be tangent. And fills in the  $(j+2)$ -th place with a letter T precisely when  $D^{j+2}(p)$  is tangent to  $N$ . (Checking, as a matter of record, that such tangent space  $D^{j+2}(p)$  does not coincide with ... the singular position  $L(D^j)(p)$ , prohibited when the  $(j+2)$ -th place in the word has been blank. Namely, in any KR glasses one sees immediately that  $L(D^j)(p)$  is not tangent to  $N$ .)

Otherwise Montgomery & Zhitomirskii fill in with G's the string of remaining blank spaces until the next S in the word, or till the end.

Needless to say, this alternative also reflects itself on the pseudo-normal level: no inversion of differentials in  $\omega^{j+2} = 0$ , hence  $c^{j+4}$  pops up and STT corresponds precisely to  $c^{j+4} = 0$ , while STGG... corresponds to  $c^{j+4} \neq 0$ , regardless of the Kumpera-Ruiz coordinates in use.

The construction of the label 'geometric class at  $p$ ' carries on as long as there remain blank spaces. The moment when a sequence of tangencies breaks down is ear-marked by starting a string of G's that goes until the

next S standing already in the word (or, in default, till the end). On the pseudo-normal level it is marked by the apparition of a non-zero constant after a suite of constants zero tied to a suite of tangencies. Alternatively, tangencies, accompanied by the zero values of the incoming Kumpera-Ruiz constants, may last till the next letter S in the word, or, in default, till flag's end.

**Example 2** The family of pseudo-normal forms (1) represents and visualises the geometric classes GGSGG (when  $c^6 \neq 0$ ), GGSTG (when  $c^6 = 0$ ,  $c^7 \neq 0$ ), GGSTT (when  $c^6 = c^7 = 0$ ).

The family (2) represents the classes GGS GSGSG (when  $c^{10} \neq 0$ ) and GGS GSGST (when  $c^{10} = 0$ ).

Let us watch back what are the only restrictions the obtained codes of geometric classes are subject to. They are just the two necessary G's in the beginning and that a letter T cannot go directly after a letter G. Therefore, for length 2 there is but one class GG, for length 3 — only GGG and GGS (the latter — historically the first singular class found — discussed in Example 1), for length 4 — GGGG, GGSG, GGST, GGSS, GGGG.<sup>2</sup>

Making a point and recapitulating the construction of Montgomery & Zhitomirskii,

**Proposition 1** *A geometric class  $\mathcal{C}$  encoded as a word over the alphabet 'Generic, Singular, Tangent' is represented by the pseudo-normal forms (glasses for Goursat distributions) subject to the following limitations:*

- *the inversions of differentials occur precisely and only in the Pfaffian equations corresponding to the letters S in  $\mathcal{C}$  (that is, corresponding to the flag members in singular positions at a reference point),*
- *if the  $j$ -th letter in  $\mathcal{C}$  is T, then  $c^{j+2} = 0$ , whatever the pseudo-normal form under consideration,*
- *if the  $j$ -th letter in  $\mathcal{C}$  is G going directly after a letter T, then  $c^{j+2} \neq 0$ , whatever the pseudo-normal form.*

*Also conversely, any pseudo-normal form satisfying these conditions sits in  $\mathcal{C}$ .*

These precisions are fundamental for the exposition that follows.

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<sup>2</sup>A straightforward recurrence yields that there exist  $u_{2r-3}$  (Fibonacci number) geometric classes of the germs of flags of length  $r$ .

Lastly in this section, we want to underline that basically (although not being explicitly spelled) the geometric classes, have been created already in [8]. Jean considered a kinematic model of a car drawing a given number  $n$  of attached passive trailers, representing (as has been well-known since the beginning of the 1990s) a rank 2 Goursat distribution,  $D$ , on the configuration space  $\Sigma = \mathbb{R}^2 \times (S^1)^{n+1}$ ; the length of the flag of  $D$  was then  $n + 1$ . He described (under certain obvious normalizing conditions on the car systems) — in terms of the sequence of critical angles  $a_1 = \pi/2$ ,  $a_{i+1} = \arctan(\sin a_i)$  — a stratification of  $\Sigma$  into ‘regions’. As a matter of fact, his strata can be encoded by the same words over  $\{G, S, T\}$  as introduced above and are nothing but the geometric classes of the germs of  $D$  at different points of  $\Sigma$ . In fact, when the trailers are indexed backwards (the last is number 1, one before last is number 2, etc., the car itself is ‘number  $n + 1$ ’), the right angle  $\pm a_1$  between trailers number  $j - 1$  and  $j$  ( $3 \leq j \leq n + 1$ ) exactly corresponds to the letter S at the  $j$ -th place. Then a very natural computation (assuredly felt but not done by Jean) shows that consecutive critical angles  $\pm a_2, \pm a_3, \dots$  appearing directly after (i.e., closer to the car!) a  $\pm a_1$  in an instantaneous configuration, mean precisely a string of consecutive tangencies: ST when the neighbouring angles are  $a_1, a_2$  (omitting a sign that should be, naturally from the kinematic point of view, one and the same in the whole string); STT when the neighbouring angles are  $a_1, a_2, a_3$ , etc.

In [5], chapter 6, Jean’s strata were encoded by words over  $\{1, 2, 3\}$  subject to the same limitations as the words of Montgomery & Zhitomirskii. The ‘GST’ code is obtained from an admissible word over  $\{1, 2, 3\}$  via the translation  $1 \rightarrow T, 2 \rightarrow G, 3 \rightarrow S$ , and adding two G’s on the left (they are written, as the reader may recall, just to keep track of the fact that flag’s members  $D^1$  and  $D^2$  are never in singular positions).

We note also that another way of constructing the higher order singularities of Goursat flags has been proposed by Pasillas & Respondek in [22]. That way turns out, despite much different language being used, equivalent to the above-outlined canonical definition based on consecutive tangencies. Geometric classes are called in [22] *singularity types* and are encoded by strings of:  $a_0$  and the letters  $a_1, a_2, \dots$  (recalling, these letters have meant in [8] the consecutive critical values of angles in the configurations of trailers. Just giving an example, a Jean stratum defined by angles  $**a_1 a_2 a_3** a_1 a_2 *$  has the singularity type  $a_0 a_0 a_1 a_2 a_3 a_0 a_0 a_1 a_2 a_0$ , while it is the geometric

class GGSTTGGSTG.)

Often, instead of ‘belongs to a class  $\mathcal{C}$ ’, we will (all the time after [14]) say ‘has the *basic geometry*  $\mathcal{C}$ ’.

Returning to Proposition 1, it also helps to make sure that, in any fixed length  $r$ , all  $u_{2r-3}$  ‘GST’ labels are geometrically realizable (in the language of [14]: all theoretically possible basic geometries of flags do materialize). As regards their materializations for concrete flags, and codimensions of appearing singularity loci, there holds a handy

**Proposition 2** *For any flag  $\mathcal{F}$  on  $M$ , the locus of points at which  $\mathcal{F}$  belongs to a fixed geometric class  $\mathcal{C}$  is an embedded submanifold of  $M$  of codimension equal to the number of S’s and T’s in the code of  $\mathcal{C}$ .*

*Therefore, only the locus of points materializing the class GG...G is open, and also dense, in  $M$ ; these are the generic points around which  $\mathcal{F}$  can be brought to the relevant chained model (C).*

This general and important statement can be quickly proved (locally) in any chosen KR glasses offered by Theorem 1, just as it has been the case during the construction of geometric classes. In particular we note that Proposition 2 holds for *any* flag, not only for flags being ‘generic among flags’. The explanation is that Goursat condition appears to be so stringent as to materialize *only* transversally to stratifications by geometric classes that are discussed in more detail in the next section. (The loci addressed in Proposition 2 are shadows, or counterimages, of those strata.) Surprisingly, then, all flags are generic in certain sense.

## 1.2. Geometric classes form a stratification

Needless to say, in any length, the geometric classes are pairwise disjoint and invariant under the action of local diffeomorphisms between manifolds. As a matter of fact, the geometric classes having codes of length  $r$  are defined in the space  $H^r$  of the  $r$ -jets of corank- $r$  Goursat distributions. It is so because such Goursat germs are  $r$ -determined<sup>3</sup> ([13]) and, reiterating, these classes are invariant w.r.t. diffeomorphisms. They do form a stratification of  $H^r$  in Thom’s sense — are embedded submanifolds of codimensions equal to the codimensions of their materializations for concrete distributions (and hence equal to the total numbers of letters S, T in their codes), and are

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<sup>3</sup>what is cardinal in this result is that they are  $r$ -determined only *within* the Goursat world

adjacent only to classes of smaller codimensions. (A disjoint from a submanifold  $A$  submanifold  $B$  is *adjacent* to  $A$ ,  $B \rightarrow A$ , when  $B \subset \overline{A}$ .)

Indeed, all counterimages by sections, or: counterimages by  $r$ -jets' prolongations of genuine Goursat distributions (materializations of geometric classes) are such submanifolds by Proposition 2, while the adjacencies in jets can be read off from the shadows of adjacencies visible in sections.

We want to be precise on this last statement. Focusing on a given class  $\mathcal{C}$  and on a representative  $D$  in pseudo-normal form of its *arbitrary* member, Proposition 1 allows to see explicitly what other classes materialize at points arbitrarily close to 0 (the reference point for  $D$ ). Putting it the other way round, to what other classes are objects in  $\mathcal{C}$  arbitrarily close (adjacent).

If  $\mathcal{C}$  is generic then it is adjacent to no other classes, because  $\mathcal{C}$  is open. If  $\mathcal{C}$  is of positive codimension, then at points close to 0 there is no room for other than those marked in  $\mathcal{C}$  flag's members to be in singular positions, as well as no room for longer strings of consecutive tangencies than those marked in  $\mathcal{C}$ . Those strings may only get shorter (and complementary to them strings of G's — longer) or disappear. Independently of that, also the singular positions S present in  $\mathcal{C}$  may get perturbed, hence disappear and so become G positions, and that together with their strings of possible tangencies! Some concrete instances of this are given in Example 3. More precisely (using explicitly the Kumpera-Ruiz glasses), even if it were an S position directly after another S, or directly after a string of T's, then in the vicinity of 0 that S could not jump to a T position: it could only shift to a non-singular position with a Kumpera-Ruiz constant in a neighbourhood of  $\infty$  ( $\infty$  corresponds to the original singular position), while the tangency would mean the constant zero.

Summing up, any geometric class to which  $\mathcal{C}$  is adjacent has smaller than  $\mathcal{C}$  the total number of letters S, T in its code. Hence is of smaller codimension, as required in stratifications.

### 1.3. The overview of Goursat singularities of small codimensions and main result

In the sequel, when speaking about singularities of Goursat distributions of a given codimension  $c$ , one may simply think about all geometric classes of codimension  $c$ . That is, on technical level, about all admissible words having in total  $c$  letters S, T. Prior to that we only need recall two general concepts.

One of the most important notions related to singularities in general is *modality*, as well as *simplicity* which means the modality zero. Modality can be defined for points of any manifold  $X$  having a Lie group  $G$  acting on it, and is equal to  $m$  for a point  $x \in X$  when a sufficiently small neighbourhood of  $x$  is covered by a finite number of  $m$ -parameter families of orbits of  $G$ , while  $m - 1$  does not have this property. Such, invariant for  $G$ , parameters are called *moduli*. A point  $x$  is simple when its modality is zero, that is when  $x$  possesses a neighbourhood covered by finitely many orbits of  $G$ . In other words,  $x$  is simple when the moduli of the action of  $G$  do not show up in the vicinity of  $x$ ; cf. [2] for all that framework.

In the context of singularity theory, when one deals with whatever objects (functions, vector fields, distributions, ...) that happen to be finitely determined, then modality, and in particular simplicity, can be defined for them, too.

In particular, in view of the facts evoked in the previous section, modality can be attached to every germ of Goursat distributions; for corank- $r$  objects it suffices to work on the manifold of  $r$ -jets of them, cf. Theorem 2 in [13]. Reiterating, then, a corank- $r$  Goursat germ  $D$  is *simple* when in the vicinity of the  $r$ -jet of  $D$  there occur only finitely many orbits of the  $r$ -jets of Goursat distributions. *Unimodal* corank- $r$  Goursat germs are those that are not simple and possess a neighbourhood in the  $r$ -jets covered by a finite number of 1-parameter families of orbits (viewed, without loss of generality, on the  $r$ -jets level).

**Example 3** (a) The Goursat germs (1) representing, after Example 2, the classes GGSTT, GGSTG, GGSGG, are all simple because the only existing adjacencies of these classes can be read off from the diagram

$$\text{GGSTT} \longrightarrow \text{GGSTG} \longrightarrow \text{GGSGG} \longrightarrow \text{GGGGG}.$$

Thus, the first listed class is adjacent only to GGSTG, GGSGG, GGGGG; the second — only to GGSGG and GGGGG; the third only to GGGGG. And all four classes appearing in this example are (modulo integrable factors, or: in a fixed dimension of the underlying manifold, see Corollary 1) single orbits.

(b) Whereas the germs in (2) representing, we recall, the classes GGSGSGST and  $\mathcal{C} = \text{GGSGSGSG}$ , are all *unimodal*. Indeed, GGSGSGST is a single orbit and is adjacent to:  $\mathcal{C}$ , whose orbits are parametrized by

exactly one invariant parameter visible in (2), and to all strata that are adjacent to  $\mathcal{C}$ , that is: GGSGSGGG, GGSGGGSG, GGGGSGSG (one letter S in  $\mathcal{C}$  perturbed to G), GGSGGGGG, GGGGSGGG, GGGGGGSG (two S's in  $\mathcal{C}$  perturbed to G), and GGGGGGGG (all three S's in  $\mathcal{C}$  perturbed to G); cf. the general explication in section 1.2. By a variety of arguments, including [15, 18] and the last statement in Theorem 2 below, all these seven neighbouring strata are, always modulo integrable factors, single orbits; the main Theorem 2 is needed (only in its easy part  $k = 3$ ) for the first listed GGSGSGGG.

(c) This unimodal example extends naturally to a family of (conjecturally) modality- $l$  examples,  $l \geq 1$ . In fact, it is rigorously shown in [15], p.111 that, for the germs in the geometric class  $\mathcal{C}_l = \text{GGSGSG} \dots \text{SG}$  with  $l + 2$  identical groups 'SG' going one after the other, their pseudo-normal forms with two first constants  $c^6, c^8$  normalized to 1 exactly parametrize — by the non-zero (cf. Proposition 1) values of the constants  $c^{10}, c^{12}, \dots, c^{2l+8}$  — the orbits of the local classification sitting in  $\mathcal{C}_l$ . That is, there are exactly  $l$  independent moduli in that class, while in the strata to which  $\mathcal{C}_l$  is adjacent there is, with all probability, less moduli. Unfortunately, this point is rigorously shown, in (b) above, only for  $l = 1$ .

*Notation.* In the present paper, in the context of words (codes, labels) a subscript will mean the number of repetitions of a letter in a word. Moreover, from now on, in the dotted places within the 'GST' codes there will only stand letters G; observe that it was not yet the case in Example 3(c).

What is known about the local classification of Goursat distributions beyond the generic strata  $G_r$ , dealt with in [23, 4], that are single orbits and display no singularities at all?

The singularities of codimension 1 (geometric classes with just one letter S in the codes) have been classified, for all lengths  $r$ , in [18]. On manifolds of fixed dimension, the germs in each class  $G_{k-1}\text{SG}_{r-k}$ ,  $3 \leq k \leq r$ , are all mutually equivalent. (The fact was used, for instance, in Example 3(a)–(b).) This together with the fact that each  $G_{k-1}\text{SG}_{r-k}$  is adjacent only to  $G_r$ , imply that the Goursat germs in classes of codimension 1 are all simple. In other words, the modality in the classes GGS..., GGGG..., GGGG..., ... is zero.

Singularities of codimension 2, or geometric classes with codes having *two* letters different from G, are more involved. These letters can be S,

S (possibly separated by a number of G's) or S, T. The classes with the sequence ST in the code have been the subject of [20]. It is shown there that: a) they are simple in pure codimension 2 (i.e., modulo other singularity loci of codimension 3; there is no module in their local classification and these 'ST' classes are only adjacent to codimension one and zero classes, well known to be single orbits); b) the first in this family class GGST... is simple without any excision.

Much less is known about classes having two letters S in their codes. The work on classes having the sequence SS in the code is not yet finished, and it keeps being plausible that the classification of these classes is discrete in any length.

In the present paper we show continuous invariants of the local classification in all classes, excepting a somehow simpler GGSGS..., having the sequence SGS in the code. The result is stated precisely below. Codimension 2 is, therefore, of a threshold nature, and is the codimension of the onset of moduli in the Goursat world.

For historical accuracy, first examples (1997, without the geometric classes language yet) of continuous invariants among germs of Goursat distributions were found in classes of codimension 3: GGGSTTGGG ([22]) and — [15] — in the family (2) discussed already in Examples 2–3. It is perhaps noteworthy that those examples, as well as the ones produced in Theorem 2 below, all realize the pattern 3 from an important systematization, of all thinkable one-step prolongations of Goursat germs, proposed in [13] (see also Remark 2 below). Whereas in [19] were given (in classes of codimension 4, the classes GGSTTT... among them) examples of moduli arising from a more involved pattern 2c.

**Theorem 2 (Main Theorem)** *Fix  $k \geq 4$ . Any germ  $D^{k+5}$  of a Goursat distribution of corank  $k + 5$ , on an  $n$ -dimensional  $C^\infty$  or  $C^\omega$  manifold, having the basic geometry  $G_{k-1}SGSGGG$  (that is, having in singular positions only its flag members of coranks  $k$  and  $k + 2$ , and having no tangencies) is locally equivalent to precisely one member in the family of germs at  $0 \in \mathbb{R}^n(x^1, \dots, x^{k+7}; x^{k+8}, \dots, x^n)$  of distributions described by the Pfaffian equations*

$$\begin{aligned} dx^2 - x^3 dx^1 &= 0, \\ dx^3 - x^4 dx^1 &= 0, \\ * & \quad * \end{aligned}$$

$$\begin{aligned}
 dx^k - x^{k+1}dx^1 &= 0, \\
 dx^1 - x^{k+2}dx^{k+1} &= 0, \\
 dx^{k+2} - (1 + x^{k+3})dx^{k+1} &= 0, \\
 dx^{k+1} - x^{k+4}dx^{k+3} &= 0, \\
 dx^{k+4} - (1 + x^{k+5})dx^{k+3} &= 0, \\
 dx^{k+5} - x^{k+6}dx^{k+3} &= 0, \\
 dx^{k+6} - (c + x^{k+7})dx^{k+3} &= 0,
 \end{aligned} \tag{M_c}$$

where  $c \in \mathbb{R}$ . All distributions  $(M_c)$  have at 0 the basic geometry  $G_{k-1}\text{SGSGGG}$ , and  $(M_c)$ ,  $(M_{\tilde{c}})$  are not equivalent when  $c \neq \tilde{c}$ .

When  $k = 3$ , all normal forms  $(M_c)$  are equivalent to  $(M_0)$ : there is no module of the local classification in the geometric class  $\text{GGSGSGGG}$ .

**Remark 2** The assertions of Theorem 2 can be equivalently stated in terms of the mentioned systematization of prolongations of Goursat germs from [13]. In the language of that reference work these reformulations go as follows.

For any  $k \geq 4$  and any germ, at a point  $p$ , of a Goursat flag  $D^{k+5} \subset D^{k+4} \subset D^{k+3} \subset \dots$  sitting in the class  $G_{k-1}\text{SGSGGG}$ , **firstly**, the prolongation pattern of the germ of  $D^{k+3}$  at  $p$  is 1: there is only one fixed point  $L(D^{k+2})(p)/L(D^{k+3})(p)$  on the circle  $S^1(D^{k+3})(p)$ , and only two orbits in it: this fixed point and all the remaining of the circle; the prolonged germ  $D^{k+4}$  sits in that second orbit. And **secondly** — the key property being established in the present paper — the prolongation pattern of the germ at  $p$  of  $D^{k+4}$  is 3: all points on the circle  $S^1(D^{k+4})(p)$  are fixed, thus giving rise to a module in the local classification of the one-step prolongations of  $D^{k+4}$ . The values of  $c$  in Theorem 2 parametrize all points of this circle except the vertical position point  $L(D^{k+3})(p)/L(D^{k+4})(p)$  (for which one would have, naturally, a different Kumpera-Ruiz normal form with an inversion in the bottommost Pfaffian equation).

The proof of Theorem 2 occupies most of the remaining of the paper. In separate Chapter 2 given is an overview of the proof: a sketch of our reducing and refining the relevant Kumpera-Ruiz pseudo-normal forms, computing infinitesimal symmetries of them, as well as a resulting (fundamental) reason for a module already in codimension two.

Then, starting the proper proof, one preliminary reduction of pseudo-

normal forms is explained in detail in Chapter 3. In turn, the invariant character, when  $k \geq 4$ , of the parameter  $c$  that remains after that simplification is justified in Chapter 4. The reduction is precisely aimed at making that (main) part of the proof as transparent as possible: to have to compare just two values of a *single* parameter in the relevant pseudo-normal form. One can note *post factum* (upon analyzing the entire proof) that **prior to** the reduction, a somehow obstruse combination (14) of parameters entering a more raw pseudo-normal form is an invariant of the local classification — see Corollary 3 below.

In the concluding part of the proof (section 4.2) we treat separately the exceptional case  $k = 3$ , explicitly driving then the parameter  $c$  to 0.

## 2. Overview of the proof

For certain time we work with any  $k \geq 3$ . To the germ  $D^{k+5}$  we apply Theorem 1, obtaining a KR pseudo-normal form for it with inversions of differentials only in  $k$ -th and  $(k+2)$ -th Pfaffian equations (Proposition 1). Within the range of the first  $k-1$  Pfaffian equations we can assume, with no loss of generality, all constants to be zero (cf. the remark before Corollary 1).  $D^{k+5}$  becomes thus the germ at  $0 \in \mathbb{R}^n(x^1, \dots, x^{k+7}; x^{k+8}, \dots, x^n)$  of  $(\omega^1, \omega^2, \dots, \omega^{k+4}, \omega^{k+5}) =$

$$\begin{aligned} & (dx^2 - x^3 dx^1, dx^3 - x^4 dx^1, \dots, dx^k - x^{k+1} dx^1, dx^1 - x^{k+2} dx^{k+1}, \\ & dx^{k+2} - (c^{k+3} + x^{k+3}) dx^{k+1}, dx^{k+1} - x^{k+4} dx^{k+3}, \\ & dx^{k+4} - (c^{k+5} + x^{k+5}) dx^{k+3}, dx^{k+5} - (c^{k+6} + x^{k+6}) dx^{k+3}, \\ & dx^{k+6} - (c^{k+7} + x^{k+7}) dx^{k+3}) \end{aligned} \quad (3)$$

with certain real constants  $c^{k+3} \neq 0$ ,  $c^{k+5} \neq 0$  (again Proposition 1),  $c^{k+6}$ ,  $c^{k+7}$ . For the remaining of the proof, without loss of generality we can assume that  $n = k+7$ . (When  $n > k+7$ , the Cauchy characteristic direct summand  $L(D^{k+5}) = (\partial_{k+8}, \dots, \partial_n)$  in  $D^{k+5}$  has no impact on the proof; for  $n = k+7$ ,  $\text{rk } D^{k+5} = 2$  and  $L(D^{k+5}) = 0$ .)

The task is now to simplify as far as possible the constants in (3). To begin with,  $c^{k+3}$  and  $c^{k+5}$  can be normalized to 1, at the expense of changing the values of  $c^{k+6}$ ,  $c^{k+7}$ . Indeed, to normalize  $c^{k+3}$  is straightforward; in doing so the remaining constants in (3) are changed. The normalization of  $c^{k+5}$  *preserving*  $c^{k+3} = 1$  (and changing again  $c^{k+6}$  and  $c^{k+7}$ ) is simple:

$$x^l = c^{k+5}\bar{x}^l \text{ for } l = k+5, k+4, k+1, k+2, \quad x^1 = (c^{k+5})^2\bar{x}^1, \quad x^{k-l} = (c^{k+5})^{2l+3}\bar{x}^{k-l} \text{ for } l = 0, 1, \dots, k-2.$$

Then comes  $c^{k+6}$  (wherever it causes no misunderstanding, after rescalings we tend to use the same letters). Because it is related to a letter G that is not first (but second) in a string following an S, Proposition 1 specifies nothing of it. At the same time the special case  $k = 3$  discussed in [15], see Lemma [32321] there, suggests that  $c^{k+6}$  is, possibly, reducible to 0. And indeed it is, even preserving  $c^{k+3} = c^{k+5} = 1$ ; simultaneously  $c^{k+7}$  changes again to certain new value, say  $c$ . At this moment  $D^{k+5}$  assumes the model form  $(M_c)$  from Theorem 2 and there ‘only’ remains to justify that: for  $k = 3$  all those  $(M_c)$  are equivalent, and for  $k > 3$  they are always pairwise nonequivalent.

In the annihilation of  $c^{k+6}$  (that is not immediate) used are *infinitesimal symmetries* of Goursat distributions. We will briefly recapitulate on them right now, because they furnish also, if indirect (or: rough), argument that — when  $k > 3$  — the  $(M_c)$ ’s are different.

Suppose a Goursat distribution  $D$  be already given in a pseudo-normal form of Theorem 1. Then it is basically visible that  $D$  is a sequence of certain ‘projective’ extensions of the differential system (a contact structure)  $\omega^1 = dx^2 - x^3 dx^1 = 0$  on  $\mathbb{R}^3$ . (In fact, those are local, coordinate manifestations of Cartan prolongations capable of locally producing any Goursat object, as has been recalled early in Chapter 1.) The infinitesimal symmetries of  $\omega^1 = 0$  are generated by all  $C^\infty$  (or analytic, depending on the chosen category) functions  $f(x^1, x^2, x^3)$  — a deep and basic thing observed long time ago by S. Lie. In modern expositions like [1, 12], those generating functions are called *contact hamiltonians*.

In view of the mentioned stepwise extensions yielding  $D$ , the i. s. ’s of  $D$  turn out to be sequences of relatively simple prolongations of the i. s. ’s of *that* Darboux structure. Consequently, they inherit the property of being locally 1–1 parametrized by  $C^\infty$  or  $C^\omega$  functions in three variables. As regards chronology, in small dimensions the relevant parametrization was used in [7] (with imprecisions, though), then in [9]; slightly later in [16] it was given explicit recurrence formulas with proofs.

However, the parametrization depends sensitively on the distribution of inversions of differentials in the pseudo-normal form for  $D$  (i.e., recalling, depends on which members of the flag of  $D$  are in singular positions at the reference point). Therefore, one has to deal in general with a long *binary*

tree of different parametrizations. This is a disadvantage, yet for  $D$  in a concrete pseudo-normal form one can advance rather far.

Having a  $D$  of corank  $r$  in a pseudo-normal form in the underlying dimension  $r + 2$ , one denotes by  $\mathcal{Y}_f$  its infinitesimal symmetry induced by a function  $f(x^1, x^2, x^3)$  and deliberately puts in relief in  $\mathcal{Y}_f$  the first three components,

$$\mathcal{Y}_f = A\partial_1 + B\partial_2 + C\partial_3 + \sum_{l=4}^{r+2} F^l \partial_l \quad (4)$$

— because the vector field  $A\partial_1 + B\partial_2 + C\partial_3$  is an i. s. of  $dx^2 - x^3 dx^1 = 0$ . Hence the classical expressions of Lie and his successors:  $A = -f_3$ ,  $B = f - x^3 f_3$ ,  $C = f_1 + x^3 f_2$  (the signs are different than in Appendix 4 of [1], because there the contact structure is written with the plus sign as  $dx^2 + x^3 dx^1 = 0$ ).

Continuing the overview, we consider now only  $k \geq 4$ ,  $D = (M_c)$  with  $n = k + 7$ , and write uniquely the results of recursive computations of the components of (4) for the object  $(M_c)$ , all of them evaluated at  $0 \in \mathbb{R}^{k+7}$ :

$$\begin{aligned} \mathcal{Y}_f | 0 = & -f_3 \partial_1 + f \partial_2 + \sum_{j=1}^{k-1} f_{1_j} \partial_{j+2} - (2f_2 + (2k-1)f_{13}) \partial_{k+3} \quad (5) \\ & + (5f_2 + (5k-3)f_{13}) \partial_{k+5} + 9f_{1_k} \partial_{k+6} \\ & + (63f_{1_k} + 9cf_2 + (9k-5)cf_{13}) \partial_{k+7} | 0 \end{aligned}$$

(remember that this expression comes already much simplified due to the absence of  $c^{k+6}$  in  $(M_c)$ , and due to the evaluation at 0 only).

Now explore the possibility of changing only the last constant  $c$  in the pseudo-normal form, keeping previously secured simplifications; because of that we assume that **all but the last** components of  $\mathcal{Y}_f | 0$  vanish.

Observe now the last component of  $\mathcal{Y}_f$ . Can it be non-zero, creating some room for the values of  $c$  in  $(M_c)$ ? It is visible that *not*, because the couples of coefficients standing by  $f_2$  and  $f_{13}$  in the  $\partial_{k+3}$  and  $\partial_{k+5}$  components in (5) are independent,

$$\begin{vmatrix} -2 & -(2k-1) \\ 5 & 5k-3 \end{vmatrix} = 1. \quad (6)$$

Hence the vanishing of these components implies  $f_2 | 0 = f_{13} | 0 = 0$ . When

additionally the component on  $\partial_{k+6}$  is zero, also  $f_{1_k} | 0 = 0$ . Then all terms in the component on  $\partial_{k+7}$  in (5) are zero. Therefore, when the description of the preceding part of the flag is frozen, it is impossible to perturb the value of  $c$  in  $(M_c)$  by means of the *embeddable* symmetries of  $(M_c)$  understood as a finite object.

This, thinking about possible symmetries *not* embeddable in flows, is a weaker statement than that in Theorem 2. Yet it gives at least an interpretation of the main result of the paper.

### 3. A simplifying reduction

We are going to reduce to 0 the constant  $c^{k+6}$  in the pseudo-normal form  $D^{k+5}$ , changing also — this is inevitable — the value of  $c^{k+7}$ , but preserving the (elementary) normalizations made in Chapter 2. The value of  $c^{k+6}$  will be moved to 0 gradually, using a flow of symmetries of the distribution  $D^{k+5}$  understood as an object defined on  $\mathbb{R}^{k+7}$  by the same equations (3), with  $c^{k+3}$  and  $c^{k+5}$  now normalized to 1.

For transparency reasons it is useful to work with a ‘universal’ distribution,  $D$ , that displays *no* constants shifting the last two variables, but keeps displaying the previously standardized constants,  $D =$

$$\begin{aligned} & (dx^2 - x^3 dx^1, dx^3 - x^4 dx^1, \dots, dx^k - x^{k+1} dx^1, dx^1 - x^{k+2} dx^{k+1}, \\ & dx^{k+2} - X^{k+3} dx^{k+1}, dx^{k+1} - x^{k+4} dx^{k+3}, dx^{k+4} - X^{k+5} dx^{k+3}, \\ & dx^{k+5} - x^{k+6} dx^{k+3}, dx^{k+6} - x^{k+7} dx^{k+3}), \end{aligned} \quad (7)$$

$X^{k+3} = 1 + x^{k+3}$ ,  $X^{k+5} = 1 + x^{k+5}$ . The reason for that is that the symmetries under consideration will keep all but the last two components of  $0 \in \mathbb{R}^{k+7}$ , while the two last ones will be moved. Prior to write the infinitesimal symmetries of  $D$ , using the formulas issuing from [16], we need the following three vector fields

$$\begin{aligned} y &= \partial_1 + x^3 \partial_2 + x^4 \partial_3 + \dots + x^{k+1} \partial_k, \\ Y &= x^{k+2} y + \partial_{k+1} + X^{k+3} \partial_{k+2}, \\ \widehat{Y} &= x^{k+4} Y + \partial_{k+3} + X^{k+5} \partial_{k+4} + x^{k+6} \partial_{k+5} + x^{k+7} \partial_{k+6}. \end{aligned} \quad (8)$$

With these notations, the first group of components of  $\mathcal{Y}_f$  contains, apart from functions  $A, B, C$ ,

$$F^4 = yC - x^4 yA, \quad F^l = yF^{l-1} - x^l yA \quad \text{for } 5 \leq l \leq k+1. \quad (9)$$

In the second group of components,

$$F^{k+2} = x^{k+2}(yA - YF^{k+1}), \quad F^{k+3} = YF^{k+2} - X^{k+3}YF^{k+1}; \quad (10)$$

$$F^{k+4} = x^{k+4}(YF^{k+1} - \widehat{Y}F^{k+3}), \quad F^{k+5} = \widehat{Y}F^{k+4} - X^{k+5}\widehat{Y}F^{k+3}; \quad (11)$$

$$F^{k+6} = \widehat{Y}F^{k+5} - x^{k+6}\widehat{Y}F^{k+3}, \quad F^{k+7} = \widehat{Y}F^{k+6} - x^{k+7}\widehat{Y}F^{k+3}. \quad (12)$$

Note parenthetically that (9)–(12) imply that, for  $4 \leq l \leq k+7$ , the function  $F^l$  depends only on  $x^1, x^2, x^3, \dots, x^l$  (in this respect, cf. a general Proposition 1 in [17]). These formulas will be used for a concrete contact hamiltonian prompted, reiterating, by the work [15].

**Lemma 1** *For any fixed  $k \geq 3$  and  $f = (x^1)^k$ , the associated infinitesimal symmetry  $\mathcal{Y}_f$  of  $D$  has all but the last two components vanishing at  $0 \in \mathbb{R}^{k+7}$ . Moreover,  $F^{k+6}(0, \dots, 0, x^{k+6}) = 9k!$ .*

Note that  $k = 3$  is allowed in this lemma. The particularity of  $k = 3$  resides in that, then, not only  $(x^1)^3$  works. Also the contact hamiltonian  $(x^3)^2$  generates then a symmetry with similar properties, see section 4.2.

*Proof.* For  $f = (x^1)^k$  a lot of calculations simplifies. Indeed,  $A \equiv 0$ ,  $B = f$ ,  $C = k(x^1)^{k-1}$  and, using many times (9),  $F^l = k(k-1) \cdots (k-l+3)(x^1)^{k-l+2}$  for  $l = 4, \dots, k+1$ . In particular,  $F^{k+1} = k!x^1$ . Then, by (10),  $F^{k+2} = -x^{k+2}YF^{k+1} = -k!(x^{k+2})^2$ . All these functions vanish at 0. The next component of  $\mathcal{Y}_f$ ,

$$\begin{aligned} F^{k+3} &= YF^{k+2} - X^{k+3}YF^{k+1} = Y(-k!(x^{k+2})^2) - X^{k+3}Y(k!x^1) \\ &= -2k!x^{k+2}X^{k+3} - X^{k+3}k!x^{k+2} = -3k!x^{k+2}X^{k+3}, \end{aligned}$$

also vanishes at 0, as well as  $F^{k+4}$  (the latter by its very expression (11)). We pass now to  $F^{k+5}|_0 =$

$$\begin{aligned} &\widehat{Y}F^{k+4} - X^{k+5}\widehat{Y}F^{k+3}|_0 \\ &= \widehat{Y}(x^{k+4}(YF^{k+1} - \widehat{Y}F^{k+3})) - X^{k+5}\widehat{Y}F^{k+3}|_0 \\ &= X^{k+5}(YF^{k+1} - 2\widehat{Y}F^{k+3})|_0 \\ &= X^{k+5}(k!x^{k+2} + 2\widehat{Y}(3k!x^{k+2}X^{k+3}))|_0 = 0 \end{aligned}$$

because  $\widehat{Y}x^{k+2} = X^{k+3}x^{k+4}$ . It remains to justify the ‘moreover’ part of the lemma. In the line of computation that follows we use the facts  $\widehat{Y}F^{k+3}|0 = -3k!((X^{k+3})^2x^{k+4} + x^{k+2})|0 = 0$ ,  $\widehat{Y}YF^{k+1}|0 = 0$ ,  $\widehat{Y}^2F^{k+3}|0 = -3k!\widehat{Y}((X^{k+3})^2x^{k+4} + x^{k+2})|0 = -3k!(3X^{k+3}x^{k+4} + (X^{k+3})^2X^{k+5})|0 = -3k!$ . Therefore,

$$\begin{aligned} F^{k+6}(0, \dots, 0, x^{k+6}) &= (\widehat{Y}F^{k+5})(0, \dots, 0, x^{k+6}) - x^{k+6}(\widehat{Y}F^{k+3})(0) \\ &= (\widehat{Y}^2F^{k+4})(0, \dots, 0, x^{k+6}) - (\widehat{Y}^2F^{k+3})(0) \\ &= 2(\widehat{Y}YF^{k+1})(0) - 3(\widehat{Y}^2F^{k+3})(0) \\ &= -3(-3k!) = 9k!. \end{aligned}$$

Lemma 1 is proved.  $\square$

**Lemma 2** *For the same distribution  $D$  and contact hamiltonian  $(x^1)^k$  there holds additionally  $F^{k+7}(0, \dots, 0, x^{k+6}, x^{k+7}) = 63k! + 30k!x^{k+6}$ .*

*Proof.* On top of the auxiliary computations used in the proof of Lemma 1, we will need also

$$\begin{aligned} (\widehat{Y}^2YF^{k+1})(0) &= \widehat{Y}^2(k!x^{k+2})(0) = k!\widehat{Y}(X^{k+3}x^{k+4})(0) \\ &= k!(x^{k+4} + X^{k+3}X^{k+5})(0) = k! \end{aligned}$$

and

$$\begin{aligned} (\widehat{Y}^3F^{k+3})(0, \dots, 0, x^{k+6}) &= -3k!\widehat{Y}(3X^{k+3}x^{k+4} + (X^{k+3})^2X^{k+5})(0, \dots, 0, x^{k+6}) \\ &= -3k!(3x^{k+4} + 5X^{k+3}X^{k+5} + (X^{k+3})^2x^{k+6})(0, \dots, 0, x^{k+6}) \\ &= -3k!(5 + x^{k+6}). \end{aligned}$$

Thus prepared, compute now

$$\begin{aligned} F^{k+7}(0, \dots, 0, x^{k+6}, x^{k+7}) &= (\widehat{Y}F^{k+6})(0, \dots, 0, x^{k+6}, x^{k+7}) - x^{k+7}(\widehat{Y}F^{k+3})(0) \\ &= (\widehat{Y}^2F^{k+5})(0, \dots, 0, x^{k+6}, x^{k+7}) - x^{k+6}(\widehat{Y}^2F^{k+3})(0) \\ &= \widehat{Y}^3[x^{k+4}(YF^{k+1} - \widehat{Y}F^{k+3})](0, \dots, 0, x^{k+6}, x^{k+7}) \\ &\quad - 3x^{k+6}(\widehat{Y}^2F^{k+3})(0) - (X^{k+5}\widehat{Y}^3F^{k+3})(0, \dots, 0, x^{k+6}) \\ &= 3(X^{k+5}\widehat{Y}^2YF^{k+1})(0) - 6x^{k+6}(\widehat{Y}^2F^{k+3})(0) \end{aligned}$$

$$\begin{aligned}
& -4(X^{k+5}\widehat{Y}^3F^{k+3})(0, \dots, 0, x^{k+6}) \\
& = 3k! - 6x^{k+6}(-3k!) - 4(-3k!(5 + x^{k+6})) \\
& = 63k! + 30k!x^{k+6}. \quad \square
\end{aligned}$$

Recalling,  $D^{k+5}$  is the germ of  $D$  at  $p = (0, \dots, 0, c^{k+6}, c^{k+7}) \in \mathbb{R}^{k+7}$ . So it may be instrumental to trace down the trajectory of  $\mathcal{Y}_{(x^1)^k}$  passing by  $p$  at, say,  $t = 0$ . In view of Lemma 1, that integral curve has non-zero only its two *last* coordinates:  $x^{k+6}(t) = c^{k+6} + 9k!t$  and  $x^{k+7}(t)$ , that can be explicitly computed as well.

Indeed,  $x^{k+7}(0) = c^{k+7}$  and, once  $x^{k+6}(t)$  known, a simple differential equation for  $x^{k+7}(t)$  is furnished by Lemma 2:

$$\frac{dx^{k+7}}{dt}(t) = 63k! + 30k!(c^{k+6} + 9k!t).$$

Therefore, and immediately,

$$x^{k+7}(t) = c^{k+7} + (63k! + 30k!c^{k+6})t + 135(k!)^2t^2. \quad (13)$$

**Corollary 2** *The integral curve of  $\mathcal{Y}_{(x^1)^k}$  through  $p$  is defined for all times  $t$ . Therefore, for any fixed time  $t$ , the time  $t$  flow of  $\mathcal{Y}_{(x^1)^k}$  is well defined in a small (depending on  $t$ ) neighbourhood of  $p$ .*

In our approach the time  $-c^{k+6}/9k!$  flow is needed, because it annihilates the  $x^{k+6}$  coordinate on the curve under consideration. Let us compute the corresponding value of  $x^{k+7}$  on that curve. Substituting to (13),

$$\begin{aligned}
x^{k+7}\left(-\frac{c^{k+6}}{9k!}\right) &= c^{k+7} - 7c^{k+6} - \frac{30}{9}(c^{k+6})^2 + \frac{135}{81}(c^{k+6})^2 \\
&= c^{k+7} - 7c^{k+6} - \frac{5}{3}(c^{k+6})^2.
\end{aligned}$$

Denoting

$$c = c^{k+7} - 7c^{k+6} - \frac{5}{3}(c^{k+6})^2, \quad (14)$$

we know, then, that an integral curve of an i. s. of  $D$  (given by the equations (7)) joins the point  $p$  and the point  $q = (0, \dots, 0, 0, c)$ . Thus the germs of  $D$  at  $p$  (i.e.,  $D^{k+5}$ ) and at  $q$  are equivalent, and the latter one is  $(M_c)$ . The simplification announced in the present chapter is achieved.

**Corollary 3** *When Goursat germs in the class  $G_{k-1}\text{SGSG}_3$ ,  $k \geq 4$ , are given in the pseudo-normal form*

$$\begin{aligned} & (dx^2 - x^3 dx^1, dx^3 - x^4 dx^1, \dots, dx^k - x^{k+1} dx^1, dx^1 - x^{k+2} dx^{k+1}, \\ & dx^{k+2} - (1 + x^{k+3}) dx^{k+1}, dx^{k+1} - x^{k+4} dx^{k+3}, \\ & dx^{k+4} - (1 + x^{k+5}) dx^{k+3}, dx^{k+5} - (c^{k+6} + x^{k+6}) dx^{k+3}, \\ & dx^{k+6} - (c^{k+7} + x^{k+7}) dx^{k+3}) \end{aligned}$$

(i.e., after normalizations of Chapter 2, but before the annihilation of  $c^{k+6}$  carried out in the present chapter), then the module evidenced in Theorem 2 assumes the form  $c^{k+7} - 7c^{k+6} - \frac{5}{3}(c^{k+6})^2$ .

## 4. Proof of Theorem 2

### 4.1. For $k \geq 4$ the germs $(M_c)$ are all nonequivalent

In this section  $k \geq 4$ . We will show that if  $(M_c)$  and  $(M_{\tilde{c}})$  are equivalent as germs at  $0 \in \mathbb{R}^{k+7}$  then  $c = \tilde{c}$ .

Let us introduce a particular vector field generator that is related to  $(M_c)$ ,  $\mathbf{Y}_c = \widehat{Y} + c \partial_{k+6}$ , with the vector field  $\widehat{Y}$  defined much earlier by (8).

In what follows we prove Theorem 2 in detail for  $k = 4$ . The proof, re-read after a purely formal change of indices  $5 \rightarrow k+1$ ,  $6 \rightarrow k+2$ ,  $\dots$ ,  $11 \rightarrow k+7$  ( $x^1$  remains  $x^1$ , the block of variables  $x^2, \dots, x^k$  is irrelevant for the proof) is valid for general  $k \geq 4$ .

Suppose that a local diffeomorphism  $g = (g^1, g^2, \dots, g^{11}): (\mathbb{R}^{11}, 0) \leftarrow$  sends  $D_c = (\partial_{11}, \mathbf{Y}_c)$  to  $D_{\tilde{c}} = (\partial_{11}, \mathbf{Y}_{\tilde{c}})$ . Notations and limitations are similar to those in [19]; for instance,  $g^1, \dots, g^{10}$  do not depend on  $x^{11}$ . In fact, for  $4 \leq l \leq 10$ ,  $g^l$  does not depend on  $x^{l+1}, \dots, x^{11}$ ;  $g^1, g^2, g^3$  depend only on  $x^1, x^2, x^3$ . Moreover, two of these coordinate functions must be of more special form,

$$g^6(x) = x^6 G^6(x^1, \dots, x^6), \quad g^8(x) = x^8 G^8(x^1, \dots, x^8),$$

because  $g$  preserves the loci (identical for  $D_c$  and  $D_{\tilde{c}}$ )  $\{x^6 = 0\}$  and  $\{x^8 = 0\}$  where the flags of  $D_c$  and  $D_{\tilde{c}}$  have members of coranks 4 and 6 in singular positions, S. There is much more limitations, in fact. The conjugacy  $g_* D_c = D_{\tilde{c}}$  implies immediately a basic set of relations

$$[Dg(x)]_1^{10} \mathbf{Y}_c(x) = f(x) \mathbf{Y}_{\tilde{c}}(g(x)) \tag{15}$$

(where  $[Dg(x)]_1^l$  always means the upper-left submatrix  $l \times l$  of the  $11 \times 11$  matrix  $Dg(x)$ ) holding for certain function  $f$ ,  $f|0 \neq 0$  (because  $g$  sends  $(\partial_{11})$  to itself). An auxiliary, also important, set of relations reads

$$[Dg(x)]_1^6 Y(x) = (fG^8)(x) Y(g(x)); \quad (16)$$

this system of equations is the first reduction of the system (15) in the terminology of [15], Chapter 5. Because there are two letters S in the code (GGSGSGGG) that we deal with in the proof, the second reduction of (15) (or: the first of (16)) exists, and holds, as well:

$$[Dg(x)]_1^4 y(x) = (fG^8G^6)(x) y(g(x)). \quad (17)$$

These three *sets* of equations form together a very powerful tool, reflecting certain rigidity of the flags of  $D_c$  and  $D_{\tilde{c}}$ ; rigidity so high that  $c = \tilde{c}$  will turn out inevitable. As in [19], we start to derive a long list of consequences of the three sets of relations, writing in “ ” the number of the actually evoked equation from a given set. To begin with,

$$g^7|0 = f|0 \quad (18)$$

by (15)-“7”, and

$$g^6|0 = fG^8|0 \quad (19)$$

from (16)-“6”. In fact, from (16)-“6” there follows more:

$$g^7 = -1 + (fG^8)^{-1}((*)x^6 + g^6|0(1 + x^7)), \quad (20)$$

implying

$$g^7|0 = (fG^8)^{-1}g^6|0 = 1 \quad (21)$$

by (19), because  $fG^8$  does not depend on  $x^7$ . In fact, as this function plays a key role in the sequel, here is its defining equation (16)-“5”:

$$fG^8 = x^6g^5|0 + x^3x^6g^5|0 + x^4x^6g^5|0 + x^5x^6g^5|0 + g^5|0. \quad (22)$$

Now, by (18) and (21),

$$f|0 = 1. \quad (23)$$

Knowing this, by (19),  $G^6|0 = g^6|0 = G^8|0$ . Simultaneously, from (15)-“8”,  $g^8|0 + g^8|0 = f(1 + g^9)|0 = 1$ , and the first summand on the LHS

vanishes, while the second is equal to  $G^8|0$ . Together there follows

$$G^6|0 = G^8|0 = 1. \quad (24)$$

Let us note now some consequences of the simplifying reduction done in Chapter 3, the first of them implied by (15)-“9”:

$$g^9_7 + g^9_8|0 = 0. \quad (25)$$

Because  $f = g^7_7 + (*)x^8$  by (15)-“7”, and  $g^7$  is affine in  $x^7$  (*cf.* (20)), there follows

$$f_7|0 = g^7_{77}|0 = 0, \quad (26)$$

and, further,

$$0 = (fG^8)_7|0 = f_7G^8 + fG^8_7|0 = G^8_7|0 \quad (27)$$

by (23) and (26). This and again (26) imply, after expressing  $g^9$  from (15)-“8”, that  $g^9_7|0 = 0$ . Thus (25) gets reduced to  $0 = g^9_8|0 = -f_8 + 2G^8_8|0$  (we have differentiated (15)-“8” sidewise w.r.t.  $x^8$  and used (23) and (27)). But, similarly,  $0 = (fG^8)_8|0 = f_8 + G^8_8|0$ , and, jointly,  $f_8|0 = 0$ . This last equality we write in terms of  $g^7$ , knowing by (15)-“7” that  $f = \widehat{Y}g^7$ :

$$g^7_5 + g^7_6|0 = 0. \quad (28)$$

There is a far-reaching analogy of (28) and (25), with (16) now to be used instead of (15). (This is precisely a result of two letters S in the geometric class’ code under consideration, and the source of certain, sufficient for the onset of moduli, rigidity binding the coordinate functions of all considered diffeomorphisms  $g$ .) The first summand on the LHS of (28) vanishes exactly as  $g^9_7|0$  has vanished there.

Indeed,  $g^5$  issuing from (17)-“4” is affine in  $x^5$ , just as is  $g^7$  in  $x^7$ :

$$g^5 = \frac{g^4_1 + x^3g^4_2 + x^4g^4_3 + x^5g^4_4}{fG^8G^6} = \frac{g^4_1 + x^3g^4_2 + x^4g^4_3 + x^5g^4_4}{g^1_1 + x^3g^1_2 + x^4g^1_3}.$$

Therefore, using also (22),

$$(fG^8)_5|0 = g^5_{55}|0 = 0, \quad (29)$$

which is the analogue of (26). The information (29), coupled with

$$0 = (g^1_1 + x^3g^1_2 + x^4g^1_3)_5|0 = (fG^8 \cdot G^6)_5|0$$

given by (17)-“1”, implies

$$G_5^6 | 0 = 0 \quad (30)$$

which is the analogue of (27). Differentiating now (20) sidewise w.r.t.  $x^5$  and using (29)–(30), the quantity  $fG^8 \cdot g_5^7 | 0$ , hence also  $g_5^7 | 0$ , vanishes as announced.

In consequence, (28), upon differentiating (20) sidewise w.r.t.  $x^6$ , becomes

$$0 = g_6^7 | 0 = -(fG^8)_6 + 2G_6^6 | 0.$$

Because, additionally,  $0 = (fG^8 \cdot G^6)_6 | 0 = (fG^8)_6 + G_6^6 | 0$ , there follow two statements. The first is of key importance, the other is just a formal equality:

$$0 = (fG^8)_6 | 0 = g_1^5 | 0 \quad (31)$$

(using (22) in the computation of  $(fG^8)_6 | 0$ ),

$$g_6^7 | 0 = -3(fG^8)_6 | 0 = -3g_1^5 | 0 \quad (32)$$

(in view of (31) all three parts have value 0). Remembering that  $f$  does not depend on  $x^9$ ,  $x^{10}$  and expressing first  $g^{10}$  from (15)-“9”, then  $g^9$  from (15)-“8”, there holds  $g_{10}^{10} | 0 = f^{-1}g_9^9 | 0 = f^{-2}g_8^8 | 0 = f^{-2}G^8 | 0 = 1$  by (23) and (24). Therefore, the evaluation at 0 of (15)-“10” reads

$$c + g_7^{10} + g_8^{10} | 0 = \tilde{c}. \quad (33)$$

**Lemma 3**  $g_7^{10} + g_8^{10} | 0 = 0$ .

*Proof of lemma.* We use for calculations (15)-“9”, taking into account (23), (26), and the important equality  $f_8 | 0 = 0$  — cf. (28), and get  $g_7^{10} | 0 = g_{77}^9 + g_{78}^9 | 0$ ,  $g_8^{10} | 0 = g_{78}^9 + g_{88}^9 + g_5^9 + g_6^9 | 0$ . This yields

$$g_7^{10} + g_8^{10} | 0 = g_5^9 + g_6^9 + g_{77}^9 + 2g_{78}^9 + g_{88}^9 | 0. \quad (34)$$

Now we use (15)-“8”, then “7”, taking into account (23), (24), (29), (31), and the easy facts  $f_{88} | 0 = G_{88}^8 | 0 = 0$

$$\begin{aligned} g_5^9 | 0 &= -f_5 + G_5^8 | 0 = -2f_5 | 0 = -2g_{57}^7 | 0; \\ g_6^9 | 0 &= -f_6 + G_6^8 | 0 = -f_6 + g_1^5 - f_6 | 0 = g_1^5 - 2g_{57}^7 | 0; \\ g_{77}^9 | 0 &= G_{77}^8 | 0 = -f_{77} | 0 = 0; \end{aligned}$$

$$\begin{aligned}
g_{78}^9 | 0 &= -f_{78} + G_{77}^8 + 2G_{78}^8 | 0 = -f_{78} - 2f_{78} | 0 \\
&= -3(g_6^7 + g_{57}^7 + g_{67}^7) | 0; \\
g_{88}^9 | 0 &= 2G_5^8 + 2G_6^8 + 2G_{78}^8 + 3G_{88}^8 | 0 \\
&= 2(-g_{57}^7 + g_1^5 - g_{67}^7 - f_{78}) | 0 \\
&= 2(g_1^5 - g_6^7 - 2g_{57}^7 - 2g_{67}^7) | 0.
\end{aligned}$$

After summing this up, a new form of (34) is

$$\begin{aligned}
g_7^{10} + g_8^{10} | 0 &= 3g_1^5 - 8g_6^7 - 12g_{57}^7 - 12g_{67}^7 | 0 \\
&= 27g_1^5 - 12g_{57}^7 - 12g_{67}^7 | 0,
\end{aligned} \tag{35}$$

where (32) has been used in the end. It remains to compute in (35) the two second order derivatives of  $g^7$  ( $g^7$  is obtained, we recall, from (16)-“6”). We have (29) and know that  $fG^8$  does not depend on  $x^7$ . Hence

$$\begin{aligned}
g_{57}^7 | 0 &= ((fG^8)^{-1}(G^6 + x^6G_6^6))_5 | 0 \\
&= (G^6 + x^6G_6^6)_5 | 0 = G_5^6 | 0 = 0
\end{aligned}$$

by (30). We have also (23), (24), (31) and remember that  $G_6^6 | 0 = -(fG^8)_6 | 0$ , so  $g_{67}^7 | 0 =$

$$\begin{aligned}
((fG^8)^{-1}(G^6 + x^6G_6^6))_6 | 0 &= -(fG^8)_6 + 2G_6^6 | 0 \\
&= -3(fG^8)_6 | 0 = -3g_1^5 | 0.
\end{aligned}$$

Equality (35) is now transformed into

$$g_7^{10} + g_8^{10} | 0 = 63g_1^5 | 0. \tag{36}$$

This quantity vanishes by (31). Lemma 3 is proved.  $\square$

Now (33), by Lemma 3, implies  $c = \tilde{c}$ . Theorem 2 is proved for  $k = 4$ .

What differences occur in the proof above when  $k > 4$ ? Constantly used are the sets of:  $k + 6$ ,  $k + 2$ ,  $k$  equations becoming, resp., (15), (16), and (17) when  $k = 4$ . They give the means of transforming expressions involving higher coordinates of a conjugating diffeo  $g$  to expressions involving lower coordinate functions of  $g$ . Changing the indices  $11 \rightarrow k + 7$ ,  $10 \rightarrow k + 6$ ,  $\dots$ ,  $5 \rightarrow k + 1$  and always making relevant reference to, instead of equation “ $l$ ”, equation “ $k + l - 4$ ” of the respective set, the proof remains valid word for word (the index 1 remains unchanged; the ranges of indices: 2 through 4 for  $k = 4$ , and 2 through  $k$  in general, are irrelevant for the proof). The key

quantity bound to be 0 by the preservation of  $c^{k+6} = 0$  is  $g_1^{k+1} | 0$  (cf. (31)), and the analogue of (36) is  $g_{k+3}^{k+6} + g_{k+4}^{k+6} | 0 = 63 g_1^{k+1} | 0 = 0$ . Therefore, Theorem 2 is proved in its part concerning  $k \geq 4$ .

#### 4.2. The annihilation of $c$ in the geometric class GGS $SGGG$

We will show that in this class ( $k$  is now equal to 3) all  $(M_c)$  are equivalent to  $(M_0)$ . Namely, at the expense of a more tricky contact hamiltonian, useful at the steps  $N^\circ k + 6$  and  $k + 7$  *only* for  $k = 3$ , one is able to proceed with  $c$  as with  $c^{k+6}$  in the simplifying reduction of Chapter 3.

As we know from that chapter, the hamiltonian  $(x^1)^3$  generates the i. s. of  $(M_0)$  (understood on  $\mathbb{R}^{10}$ ) having at all points of the  $x^{10}$ -axis the constant value  $54 \partial_9 + 378 \partial_{10}$  (Lemmata 1 and 2). At the same time, direct calculi using the formulas (9)–(12) show that the hamiltonian  $(x^3)^2$  generates the i. s. of  $(M_0)$  having at all points of the  $x^{10}$ -axis a constant value  $48 \partial_9 + 180 \partial_{10}$ . Therefore, a mixed hamiltonian  $48(x^1)^3 - 54(x^3)^2$  generates the i. s. of  $(M_0)$  having on that axis the constant value

$$(48 \cdot 54 - 54 \cdot 48) \partial_9 + (48 \cdot 378 - 54 \cdot 180) \partial_{10} = 8424 \partial_{10}.$$

Hence, in the pseudo-normal forms in question one can change the value of the last constant preserving all the remaining descriptions. And, thus, transform  $(M_0)$  into  $(M_c)$  for any value of  $c$  (in a very small, possibly depending on  $c$ , neighbourhood of  $0 \in \mathbb{R}^{10}$ ). The geometric class GGS $SGGG$  is now classified, and Theorem 2 fully proved.  $\square$

**Remark 3** The peculiarity of  $k = 3$  can be explained as follows. Preserving  $c^9 = 0$  is a condition of different kind than  $g_1^{k+1} | 0 = 0$ , responsible for keeping  $c^{k+6} = 0$  when  $k \geq 4$ . A separate computation gives this new condition explicitly:

$$3(g_1^4 + g_{44}^4) - 2g_3^1 | 0 = 0. \tag{37}$$

In the course of obtaining it there is no direct analogue of (29), because  $g^4$  is only rational in  $x^4$ , not affine. Another complication is that  $(fG^7G^5)_4 = g_3^1$  does not vanish automatically, in contrast to  $(fG^{k+4}G^{k+2})_{k+1}$ ,  $k \geq 4$ . The main question is whether the expression  $g_6^9 + g_7^9 | 0$  that is now responsible for hypothetical moves of  $c^{10}$ , is proportional to the LHS of (37). The answer is no. A sequel of that new computation gives it in the form

$$g_6^9 + g_7^9 | 0 = 63 g_1^4 + 35 g_{44}^4 - 20 g_3^1 | 0$$

$$= 33g_1^4 + 5g_{44}^4 \mid 0 \quad \text{mod} [3(g_1^4 + g_{44}^4) - 2g_3^1 \mid 0].$$

So in this case, restrictions to which the conjugating diffeomorphisms are subject, leave some freedom. This freedom is exploited in the construction of a pertinent i. s. given above.

### 5. Second independent module in $G_{k-1}\text{SGSG}_7$ for $k \geq 5$

The module  $c$  present in  $(M_c)$  in Theorem 2 (or the same module in more raw form (14) before reducing  $c^{k+6}$  to 0), is *not* the only module hidden in the geometric classes

$$G_{k-1}\text{SGS} \dots, \quad (38)$$

where now  $\dots$  stand for several, not just three, letters G. There is an entire eventail of moduli in these classes, and we know that modality of (38) is *at least*  $k - 3$  (for  $k \geq 4$  one module has already been shown in Theorem 2).<sup>4</sup>

In this chapter we will justify the existence of a second module, only when  $k \geq 5$ , in precise members

$$G_{k-1}\text{SGSG}_7 \quad (39)$$

of the classes (38), as well as outline the conjectural ‘triangle’ distribution of further moduli: a third one in (38) for  $k \geq 6$ , fourth for  $k \geq 7$ , and so on.

Coming back to Chapter 2, when dealing there with infinitesimal symmetries of each fixed model  $(M_c)$ , understood not as a germ but as a finite object, precise computations leading to the important formula (5) were made according to the recursive rules for the components of i. s.’s. And the outcome has been that the parameter  $c$  that sits, recalling, in the flag’s member  $D^{k+5}$  must have either been a module or belonged to a *discrete* orbit of values (keeping the description of the preceding member  $D^{k+4}$ ).

Indeed, thinking about conjugating its two different values by a symmetry: a) embeddable in the flow of an i. s. of  $D^{k+5}$ , and b) preserving the KR description of  $D^{k+4}$ , one has been seeking **in vain** a generating function  $f$  rendering all components that precede the component  $F^{k+7}$  — vanishing at 0, and  $F^{k+7}$  *not* zero at 0.

Now we are going to likewise compute further components of

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<sup>4</sup>Conjecturally, these estimations are sharp, giving precise modalities, but our methods give only the estimation from below.

infinitesimal symmetries of *farther* members of the flags from classes (38), obtaining at least strong hints as to the distribution of successive moduli in these classes.

### 5.1. Algebraic patterns in infinitesimal symmetries of $D^{k+9}$

For further use in the present section, we assume  $k \geq 5$ . By Theorem 2, the member  $D^{k+9}$  of a flag from the geometric class (39) is locally described, for a unique  $c \in \mathbb{R}$ , by the Pfaffian equations defining  $(M_c)$  in Theorem 2, and by four new equations with four unknown constants:

$$\begin{aligned} dx^{k+7} - (c^{k+8} + x^{k+8})dx^{k+3} &= 0, \\ dx^{k+8} - (c^{k+9} + x^{k+9})dx^{k+3} &= 0, \\ dx^{k+9} - (c^{k+10} + x^{k+10})dx^{k+3} &= 0, \\ dx^{k+10} - (c^{k+11} + x^{k+11})dx^{k+3} &= 0. \end{aligned} \tag{40}$$

We simplify first, as in Chapter 3. It is possible to get rid of  $c^{k+8}$ ,  $c^{k+9}$ , and  $c^{k+10}$  changing under way the value of  $c^{k+11}$ . In fact, a long computation shows that

$$\begin{aligned} F^{k+8} | 0 &= (120k + 60)f_{12} + \left(120 \binom{k}{2} + 60k - 15\right) f_{113} | 0 \\ &\quad \text{mod}(f_2, f_{13}, f_{1_k}) | 0. \end{aligned} \tag{41}$$

This and the previously gathered knowledge on lower components of  $\mathcal{Y}_f$  yield that  $F^{k+8} | 0 \neq 0$  is possible (and so even with the same value on the whole  $x^{k+8}$ -axis) simultaneously with all lower components vanishing at 0. Standard techniques make now possible to conjugate the values  $c^{k+8}$  and zero keeping the description  $(M_c)$  of  $D^{k+5}$ . After that the values of  $c^{k+9}$ ,  $c^{k+10}$ ,  $c^{k+11}$  are *new*, with no relation to the previous values. Then further lengthy calculi lead to the expression

$$\begin{aligned} F^{k+9} | 0 &= (1170k + 531)f_{12} + \left(1170 \binom{k}{2} + 531k - 153\right) f_{113} | 0 \\ &\quad \text{mod}(f_2, f_{13}, f_{1_k}) | 0. \end{aligned} \tag{42}$$

Is this time possible to have  $F^{k+9} | 0 \neq 0$  and all lower components, including (41), zero? Yes, because the coefficients in (41) and (42) are independent,

$$\left| \begin{array}{cc} 120k + 60 & 120\binom{k}{2} + 60k - 15 \\ 1170k + 531 & 1170\binom{k}{2} + 531k - 153 \end{array} \right| = -3240\left(k + \frac{3}{4}\right)\left(k + \frac{1}{2}\right), \quad (43)$$

evidently non-zero for  $k$  we are now interested in. (This should be *first* compared with (6), and is just one of many surprising regularities in the Goursat world, others being, for inst., those underlying the proofs in [20], see section 5.2 for more on that.) Therefore, in order to drive to 0 the constant  $c^{k+9}$  while not affecting the preceding simplifications, it suffices to take the contact hamiltonian  $f = (4k^2 - 1)x^1x^2 - (4k - 2)(x^1)^2x^3$  (for which (41) vanishes, hence (42) does not). After this step the constants  $c^{k+10}$ ,  $c^{k+11}$  are again new. Continuing computations under now crucial condition  $k \geq 5$ ,

$$F^{k+10} | 0 = 1575f_{1_{k+1}} \pmod{(f_2, f_{13}, f_{1_k}, f_{12}, f_{113})} | 0. \quad (44)$$

*Attention.* When  $k = 4$ , in  $F^{14} | 0$  along with  $f_{1_5} | 0$  there appears also a new partial  $f_{33} | 0$ , very much similarly to the situation for  $k = 3$  and  $F^{k+6}$ , discussed in section 4.2. This is a general pattern in the classes (38), cf. section 5.2.

So again  $F^{k+10} | 0 \neq 0$  can occur together with the vanishing of all lower components of certain i. s. at 0. This time the partial  $f_{1_{k+1}} | 0$ , appearing for the first time, allows — by standard techniques — to reduce  $c^{k+10}$  to 0 without affecting the preceding part of the normal form. This again brings in a completely new value  $e = c^{k+11}$ . We intend now to keep  $(M_c)$  **and** the achieved simple form

$$\begin{aligned} dx^{k+7} - x^{k+8}dx^{k+3} &= 0, \\ dx^{k+8} - x^{k+9}dx^{k+3} &= 0, \\ dx^{k+9} - x^{k+10}dx^{k+3} &= 0, \\ dx^{k+10} - (e + x^{k+11})dx^{k+3} &= 0 \end{aligned} \quad (45)$$

of (40). (The set of these equations is a KR pseudo-normal form for our  $D^{k+9}$  involving only two parameters: the module  $c$  and parameter  $e$ .) Any further simplification would now mean changing the value of  $e$ , in the first place by flows of pertinent infinitesimal symmetries. Is it possible at all?

## 5.2. Second module in $D^{k+9}$

Preserving all but the last equation means that we demand for the i. s.'s of  $D^{k+9}$  to have all starting components up to  $F^{k+10}$  vanishing at 0. In view of (5) and formulas (41)–(44), this implies, among others, the vanishing of partials

$$f_2, f_{13}, f_{1_k}, f_{12}, f_{113}, f_{1_{k+1}} \mid 0, \quad (46)$$

where  $f$ , as always, parametrizes the i. s.'s. However, one last line of computations shows that for  $k \geq 5$  (and not for 3 or 4) the component  $F^{k+11} \mid 0$  is a *real* combination of the values (46). Hence

**Proposition 3** *Fix any  $k \geq 5$ . For any Goursat germ  $D^{k+9}$  in the geometric class (39), let us choose and fix local KR coordinates in which  $D^{k+9}$  is normalized as above. Then for any i. s.  $\mathcal{Y}$  of  $D^{k+9}$ , the vanishing at 0 of the components of  $\mathcal{Y}$  lower than  $F^{k+11}$  automatically forces  $F^{k+11} \mid 0 = 0$ .*

**Corollary 4** *The value of  $e$  in (45) cannot be moved by those embeddable in flows symmetries of  $D^{k+9}$  which preserve the KR description of the square  $D^{k+8}$  of  $D^{k+9}$ .*

Thus the value of  $e$  is important in the classification of such  $D^{k+9}$  that are one-step prolongations of a **fixed**  $D^{k+8}$  from  $G_{k-1}\text{SGSG}_6$ . In fact, Corollary 4 implies that restricted in this sense orbits of equivalent values of  $D^{k+9}(0)$  are *discrete*. Looking now at the systematization of local prolongations proposed in [13] and recalled already in Remark 2 (Chapter 1), one observes that, out of five generally possible (and mutually exclusive) patterns 1, 2a, 2b, 2c, 3, the restricted orbits are discrete only in cases 2c and 3. So the prolongation  $D^{k+8} \rightarrow D^{k+9}$  is governed by either the pattern 2c or 3.

Then, on simply inspecting the cardinality of restricted orbits in 2c and 3, it follows that either  $e$  or  $|e|$  is a new module in the class (39), located four steps past the first module  $c$ .<sup>5</sup> We underline that the information issuing from the present work concerning that parameter  $e$  is not complete; in general it is not possible to tell the pattern 2c from 3 by the infinitesimal methods alone.

With the same degree of imprecision, we can indicate a conjectural distribution of moduli in each class of the type (38). Underlying this prediction,

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<sup>5</sup>*cf.* a similar situation (in codimension 3) in [21], Corollary 1, (ii)

as well as the above findings concerning the second module, is a series of arithmetic constructions in which we attach: *multiplicities* to KR variables (not to be confused with nonholonomic orders of variables as functions) and *abstract weights* to the components  $F^l$  of the infinitesimal symmetries. These tools were already proposed and used in [16], then in [20, 21].

For  $k \geq 6$ , a third module should appear in  $D^{k+13}$ , provided the coefficients at  $f_{112}|0$  and  $f_{1113}|0$  that appear for the first time in the i.s.'s of  $D^{k+10}$  and  $D^{k+11}$  are linearly independent, allowing similar reductions of KR constants as done in section 5.1. On the basis of surprisingly nice formulas for other determinants that show up in the basic geometry 'ST' and underlie the work [20]<sup>6</sup>, we extrapolate the expressions for determinants (6) and (43) and suppose this new determinant of coefficients to be

$$\text{const} \left( k + \frac{7}{4} \right) \left( k + \frac{3}{2} \right) \left( k + \frac{3}{4} \right) \left( k + \frac{1}{2} \right),$$

and, as such, clearly not vanish for the  $k$ 's in question. Then the prolongation  $D^{k+11} \rightarrow D^{k+12}$  is served by  $f_{1_{k+2}}$  alone (while for  $k = 5$  by  $f_{1_{k+2}} = f_{17}$  and  $f_{33}$ . This steady, only linearly retarded in function of  $k$ , appearance of  $f_{33}|0$  in the derivation of infinitesimal formulas at 0 is responsible for the triangle pattern of moduli we sketch.) After these simplifications, in order to respect them by the flows of i.s.'s, all partials (46) and  $f_{112}|0$ ,  $f_{1113}|0$ ,  $f_{1_{k+2}}|0$  are to be frozen to 0, and for the next prolongation  $D^{k+12} \rightarrow D^{k+13}$  there is no new partials available. Implying (as above) case 2c or 3 of [13] and discrete restricted orbits, hence a new module closely related with the geometric position of  $D^{k+13}(0)$  in  $D^{k+12}(0)$ .

This pattern should repeat itself periodically, only with linearly growing starting values of  $k$ : a fourth module sitting in the member  $D^{k+17}$  for  $k \geq 7$ , and so on. Summing up this conjectural mode: all classes of the type GGSGS... are, supposedly, simple. For any  $k \geq 4$ , in the classes of the type (38) the moduli appear in the following  $k - 3$  flag's members (if only the dimension of the underlying manifold allows for such flags' existence)

$$D^{k+5}, D^{k+9}, D^{k+13}, \dots, D^{5k-11}.$$

Only the first of them — we know it after the present work (see Remark 2) — is of Montgomery-Zhitomirskii' type 3. The remaining are either of type 2c

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<sup>6</sup>they are given with detailed proofs in author's preprint having No 39 at <http://www.mimuw.edu.pl/english/research/reports/tr-imat/>

or 3. An integral part of these conjectural statements would consist in computing involved coefficients, similar if worse than those in (41)–(42), in order to show that there is no module in the intermediate members of the flags in question.

*Added in revision.* When working on the revised version of the paper, we have obtained some generalizations of Theorem 2. They concern the geometric classes  $G_{k-1}SG_jSG_3$  for  $k \geq 4$  and  $j \geq 2$  (Theorem 2 deals with  $k \geq 4$  and  $j = 1$ ) and indicate that all these classes are unimodal, too. Therefore, a vast *majority* of geometric classes of codimension 2 appear to be not simple, whereas the ‘ST’ classes dealt with in [20] belong to the remaining minority. A proof of these generalizations will appear in author’s subsequent paper.

## References

- [1] Arnold V.I., *Mathematical Methods of Classical Mechanics*. Springer-Verlag, Berlin, 1978 (English translation).
- [2] Arnold V.I. Gusein-Zade S.M. and Varchenko A.N., *Singularities of Differentiable Maps*, I. Monographs in Math., vol. 82, Birkhäuser, Basel, 1985 (English translation).
- [3] Bryant R.L. and Hsu L., *Rigidity of integral curves of rank 2 distributions*. Invent. math. **114** (1993), 435–461.
- [4] Cartan E., *Sur l’équivalence absolue de certains systèmes d’équations différentielles et sur certaines familles de courbes*. Bull. Soc. Math. France **XLII** (1914), 12–48.
- [5] Cheaito M. and Mormul P., *Rank-2 distributions satisfying the Goursat condition: all their local models in dimension 7 and 8*. ESAIM: Control, Optimisation and Calculus of Variations (<http://www.edpsciences.org/cocv/>) **4** (1999), 137–158.
- [6] Engel F., *Zur Invariantentheorie der Systeme von Pfaff’schen Gleichungen*. Berichte Ges. Leipzig, Math-Phys. Classe **XLI** (1889), 157–176.
- [7] Gaspar M., *Sobre la clasificacion de sistemas de Pfaff en bandera*. Proceedings of 10th Spanish-Portuguese Conference on Math., University of Murcia, 1985, 67–74 (in Spanish).
- [8] Jean F., *The car with N trailers: characterisation of the singular configurations*. ESAIM: Control, Optimisation and Calculus of Variations (<http://www.edpsciences.org/cocv/>) **1** (1996), 241–266.
- [9] Kumpera A., *Toulouse Lectures*. Handwritten notes, Toulouse, 1998.
- [10] Kumpera A. and Ruiz C., *Sur l’équivalence locale des systèmes de Pfaff en drapeau*. Gherardelli F. (Ed), Monge-Ampère Equations and Related Topics, Florence 1980, Inst. Alta Math. F. Severi, Rome, 1982, pp. 201–248.
- [11] Kumpera A. and Rubin J.L., *Multi-flag systems and ordinary differential equations*. Nagoya Math. J. **166** (2002), 1–27.

- [12] Lychagin V., *Local classification of non-linear first order partial differential equations*. Russian Math. Surveys **30** (1975), 105–175 (English translation).
- [13] Montgomery R. and Zhitomirskii M., *Geometric approach to Goursat flags*. Ann. Inst. H. Poincaré — AN **18** (2001), 459–493.
- [14] Montgomery R. and Zhitomirskii M., *personal communication*. July 1999.
- [15] Mormul P., *Local classification of rank-2 distributions satisfying the Goursat condition in dimension 9*. Orro P. and Pelletier F. (Eds), Singularités et géométrie sous-riemannienne, Chambéry 1997, Travaux en cours vol. 62, Hermann, Paris, 2000, pp. 89–119.
- [16] Mormul P., *Goursat distributions with one singular hypersurface — constants important in their Kumpera-Ruiz pseudo-normal forms*. Preprint N° 185 Labo Topologie, Univ. of Dijon, 1999.
- [17] Mormul P., *Contact hamiltonians distinguishing locally certain Goursat systems*. Grabowski J. and Urbański P. (Eds), Poisson Geometry, Warsaw 1998, Banach Center Publications **51**, Inst. of Math., Polish Acad. Sci., Warsaw, 2000, pp. 219–230.
- [18] Mormul P., *Goursat flags: classification of codimension-one singularities*. J. Dynam. Control Syst. **6** (2000), 311–330.
- [19] Mormul P., *Examples of exotic moduli in local classification of Goursat flags*. Univ. Iagel. Acta Mathematica **XXXVIII** (2000), 17–28.
- [20] Mormul P., *Discrete models of codimension-two singularities of Goursat flags of arbitrary length with one flag's member in singular position*. Proc. Steklov Inst. Math. **236** (2002), 478–489.
- [21] Mormul P., *Superfixed positions in the geometry of Goursat flags*. Univ. Iagel. Acta Mathematica **XL** (2002), 183–196.
- [22] Pasillas-Lépine W. and Respondek W., *On the geometry of Goursat structures*. ESAIM: Control, Optimisation and Calculus of Variations (<http://www.edpsciences.org/cocv/>) **6** (2001), 119–181.
- [23] Von Weber E., *Zur Invariantentheorie der Systeme Pfaff'scher Gleichungen*. Berichte Ges. Leipzig, Math-Phys. Classe **L** (1898), 207–229.

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