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First order extensions of holomorphic foliations

(Dedicated to professor Tatsuo Suwa for his 60^{th} birthday)

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Abstract. Let S be a subvariety of a complex manifold M. Let $\mathcal{F}$ be a holomorphic foliation on S and $\mathcal{E}$ a coherent sheaf on S. We give a definition of first order tangency extension of $\mathcal{F}$ to M with respect to $\mathcal{E}$ and prove that, under some suitable hypotheses, the existence of extensions give rise to localization of certain characteristic classes on S. This point of view includes both the classical Camacho-Sad index theorem, variation and the newer indices theorems for holomorphic self-maps along fixed points sets.

Key words: holomorphic foliations, localization of characteristic classes, index theorems.

Introduction

The theory of holomorphic foliations has been studied since the time of Poincaré [16] and Dulac [9]. One of the main question was that of the existence of separatrices through a singular point for a (germ of) one-dimensional foliation in $\mathbb{C}^2$. It has been known since the early years of the past century that "generically" the answer is affirmative. But a final positive answer was obtained only in 1982 by Camacho and Sad [8] who exploited an "index theorem" to reduce the non-generic cases to a known ones. The work of Camacho and Sad gave rise to many studies on those "indices (or residues) theorems". After preliminary works of Lins Neto [15] and Suwa [18], a general comprehension of this phenomenon, together with general principles, is, at least in the opinion of the author, due to Lehmann and Suwa (see, *e.g.*, [12], [13], [14] and [19]) who understood that the Camacho-Sad index theorem and its further generalizations were essentially examples of localizations of characteristic classes of a particular vector bundle due to the existence of a so-called "holomorphic action" on such a bundle outside some closed subsets.

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Referring the reader to [19] or to section 3 for a precise definition of holomorphic actions, here we content ourselves to state more precisely Camacho-Sad's type theorems in terms of the Lehmann-Suwa theory. Let M be a complex manifold of dimension n and $S \subset M$ a submanifold of dimension m. Suppose $\mathcal{F}$ is a holomorphic foliation on M of dimension r for which S is invariant. Then, outside the singular locus of $\mathcal{F}$ (see section 1), $\mathcal{F}$ holomorphically acts on the normal bundle N_S of S in M, and there exists a "special connection" for N_S so that, as a consequence of the *Bott vanishing theorem* (see section 3), the associated characteristic forms of degree greater than $2(m-r)$ vanish on such an open subset of S. The Čech-de Rham cohomology allows then to localize the characteristic classes of N_S of degree greater than $2(m-r)$ near the singular locus of $\mathcal{F}$. If S is compact the Poincaré and Alexander dualities give then the corresponding localization at homology level, that is, the residues theorem (see section 4). A similar localization, called *variation*, is done for the virtual bundle $TM|_S - \mathcal{F}$ (see [19]).

If the foliation $\mathcal{F}$ were defined only on S one indeed could interpret residues theorems as obstructions to the existence of an extension of $\mathcal{F}$ to a foliation of M (or at least of an open neighborhood of S in M).

On the other hand, in recent works on discrete holomorphic dynamics by Abate, Tovena and the author (see [1], [6], [5] and [2]) it turned out that a Camacho-Sad type theorem holds (generically) even when $S \subset M$ is the fixed points set of a holomorphic self-map f of M. In such a case indeed it is possible to define a natural holomorphic one-dimensional foliation on S and from this a holomorphic action on N_S outside some "singular points" of f and then apply the Lehmann-Suwa machinery to produce residues theorems (which, as in the foliations case, can be used to get information about the dynamics of f near S). However we remark that the natural holomorphic foliation on S coming from f is not extendable to M, and thus the holomorphic action is not coming from a holomorphic foliation of M having S invariant, but only from a sort of "first order extension" of such a foliation.

In the present paper, using the sheaves language, we propose a general framework which in particular encompasses the Camacho-Sad and variation type theorems coming both from holomorphic foliations and from holomorphic mappings. In other terms the idea we try to formalize and generalize in here is that a holomorphic action on the normal bundle of a submanifold

S of M is only determined by its first jet extension along the tangential directions to S.

To be more precise, let Θ_S be the holomorphic tangent sheaf of S and let $\mathcal{E} \subset \Theta_S$ be a coherent subsheaf. Given a foliation $\mathcal{F}$ of S we define a *first order tangency extension with respect to* $\mathcal{E}$ to be a family of local extensions of $\mathcal{F}$ in M which glue together in a suitable way, that is, in such a way that two different extensions of the same element coincide up to order two in the "normal directions" to $\mathcal{E}$ (see Definition 2.5). Assume $\mathcal{F}$ and $\mathcal{E}$ are locally free and let F, E be the associated bundle. If E is *compatible with* F—which is always the case if E is involutive and $F \subset E$, (see Definition 3.2)—and $\mathcal{F}$ has a first order tangency extension with respect to $\mathcal{E}$, then there is a natural holomorphic action of F on $TM|_S/E$ (see Theorem 3.3). Thus one has localization of characteristic classes, that is residues theorems. The case S is singular (but satisfies some generic suitable hypothesis) is also included in the theory.

Aside the already cited examples of first order tangency extensions provided by restrictions of ambient foliations and by holomorphic self-maps of the ambient, our picture includes the case S is foliated by a foliation $\mathcal{E}$ whose leaves are themselves foliated by another foliation $\mathcal{F}$ coming from the restriction of an ambient foliation (see Corollary 3.5).

The plan of the paper is the following. In the first section we recall the basics about foliations. In the second section we do some commutative algebra in order to obtain a natural definition of "first order tangency with respect to some sheaf" and extensions and provide some examples. In section 3 we discuss holomorphic actions and show how a first order tangency extension with respect to a compatible subbundle gives one. In the last section we recall briefly the Lehmann-Suwa theory and determine the residues theorem for our setting.

1. Holomorphic foliations basics

Let M be a connected complex manifold of dimension n. As a matter of notations, $\mathcal{C}^\infty$ will denote the sheaf of C^∞ functions on M and $\mathcal{O}_M$ the sheaf of holomorphic functions on M. For a bundle E on M, we indicate by $\mathcal{C}^\infty(E)$ the sheaf of C^∞ sections of E on M, while we reserve the italic symbol $\mathcal{E}$ to the sheaf of *holomorphic* sections of E.

Let Θ_M denote the sheaf of germs of holomorphic vector fields on M, that is the sheaf of holomorphic sections of the holomorphic vector bundle TM of M. Let $\mathcal{F}$ be a coherent subsheaf of Θ_M. We say that $\mathcal{F}$ is *involutive* if $[\mathcal{F}_p, \mathcal{F}_p] \subset \mathcal{F}_p$ for any $p \in M$. Let $\mathcal{Q} = \Theta_M/\mathcal{F}$ be the quotient sheaf. Let

$$\mathrm{Sing}(\mathcal{F}) = \{p \in M : \mathcal{Q}_p \text{ is not } \mathcal{O}_{M,p}\text{-free}\}.$$

Definition 1.1 We call an involutive coherent subsheaf $\mathcal{F} \subset \Theta_M$ a (*singular*) *holomorphic foliation* of M. The *dimension* of $\mathcal{F}$, denoted by $\dim \mathcal{F}$, is the rank of $\mathcal{F}_p$ for some (and hence any) $p \in M \setminus \mathrm{Sing}(\mathcal{F})$. The closed complex subvariety $\mathrm{Sing}(\mathcal{F}) \subset M$ is called the *singular locus* of $\mathcal{F}$.

Remark 1.2 1. In [3] a foliation $\mathcal{F}$ is required also to be *full*, *i.e.*, if for any open set $U \subset M$ and section $s \in \Gamma(\Theta_M, U)$ such that $s_p \in \mathcal{F}_p$ for any $p \in U \setminus \mathrm{Sing}(\mathcal{F})$ it follows that actually $s \in \Gamma(\mathcal{F}, U)$. A foliation (defined as we did) which is full is sometimes called *reduced*. There is a canonical way to reduce a non-reduced foliation ([3], [17]). Note also that if a coherent subsheaf $\mathcal{F} \subset \Theta_M$ is full and involutive on $M \setminus \mathrm{Sing}(\mathcal{F})$ then it is actually a foliation.

2. If $\mathcal{Q}_p$ is $\mathcal{O}_{M,p}$-free so is $\mathcal{F}_p$. On the contrary $\mathcal{F}_p$ might be $\mathcal{O}_{M,p}$-free while $\mathcal{Q}_p$ might not (for instance consider the foliation $\mathcal{F}$ on M generated by a single holomorphic vector field v. Then $\mathcal{F}$ is $\mathcal{O}_M$-free but $\mathcal{Q}$ is not $\mathcal{O}_M$-free on the zero set of v.)

3. At each $p \in M$ the foliation $\mathcal{F}$ naturally defines a $\mathbb{C}$-vector subspace F_p of T_pM. It is easy to see that $\dim_{\mathbb{C}} F_p = \dim \mathcal{F}$ if and only if $p \notin \mathrm{Sing}(\mathcal{F})$. Therefore $F = \cup_{p \in M \setminus \mathrm{Sing}(\mathcal{F})} F_p$ is a vector subbundle of TM on $M \setminus \mathrm{Sing}(\mathcal{F})$, whose associated sheaf of holomorphic sections is $\mathcal{F}$. Thus a nonsingular foliation is exactly an involutive distribution of TM.

4. Let $(z_1, \ldots, z_n)$ be local coordinates near $p \in M$ and assume $\mathcal{F}$ is generated there by $X_1, \ldots, X_s$. Then $X_j = \sum_{l=1}^n a_{jl} \partial/\partial z_l$ for some $a_{jl} \in \mathcal{O}_M$. Let $A = (a_{jl})$. By the previous remark it follows that $p \in \mathrm{Sing}(\mathcal{F})$ if and only if $\mathrm{rank}\, A(p) < \dim \mathcal{F}$.

Let $S \subset M$ be an m-dimensional globally irreducible complex subvariety of M. Let $\mathcal{I}_S \subset \mathcal{O}_M$ denote the sheaf of germs of holomorphic functions identically vanishing on S. Thus

$$0 \to \mathcal{I}_S \to \mathcal{O}_M \to \mathcal{O}_S \to 0 \tag{1.1}$$

is an exact sequence of sheaves. One can define the sheaf Θ_S of germs of holomorphic vector fields on S as follows. Let Ω_M be the sheaf of germs of holomorphic 1-forms on M. One first defines the sheaf Ω_S of holomorphic forms on S by means of the following exact sequence of $\mathcal{O}_S$-modules:

$$\mathcal{I}_S/\mathcal{I}_S^2 \to \Omega_M \otimes \mathcal{O}_S \to \Omega_S \to 0,$$

where $\mathcal{I}_S/\mathcal{I}_S^2 \ni [f] \mapsto df \otimes 1 \in \Omega_M \otimes \mathcal{O}_S$. Applying the functor $\mathcal{H}om_{\mathcal{O}_S}(\cdot, \mathcal{O}_S)$ to the previous exact sequence and denoting by $\Theta_S = \mathcal{H}om_{\mathcal{O}_S}(\Omega_S, \mathcal{O}_S)$ and $\mathcal{N}_S = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{I}_S/\mathcal{I}_S^2, \mathcal{O}_S)$ we have the exact sequence

$$0 \to \Theta_S \to \Theta_M \otimes \mathcal{O}_S \to \mathcal{N}_S.$$

If $S' = S \setminus \mathrm{Sing}(S)$ then $\Theta_S|_{S'}$ is the sheaf of germs of sections of the holomorphic tangent vector bundle TS'. Similarly $\mathcal{N}_S$ coincides on S' with the sheaf of sections of the normal bundle $N_{S'} = TM|_{S'}/TS'$.

Definition 1.3 We say that a coherent subsheaf $\mathcal{F} \subset \Theta_S$ is a *foliation* of S if $\mathcal{F}$ is a holomorphic foliation of $S' = S \setminus \mathrm{Sing}(S)$.

As we shall see, the behavior of $\mathcal{F}$ on $\mathrm{Sing}(S)$ is not important and for our aim one could define the foliation only on the nonsingular part of S.

2. First order tangency extensions of foliations

Let $S \subset M$ be a complex subvariety of M of dimension $m < n$ and codimension $k = n - m$. Let $S' = S \setminus \mathrm{Sing}(S)$. Let

$$\varpi\colon \Theta_M \approx \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_M \longrightarrow \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S,$$
$$\varpi\colon v = v \otimes 1 \mapsto v \otimes 1.$$

Note that ϖ is a surjective morphism of $\mathcal{O}_M$-modules but $\Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$ has a natural structure of $\mathcal{O}_S$-module. As an $\mathcal{O}_S$-module on S' one can regard $\Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$ as the sheaf of holomorphic sections of the restriction of the holomorphic tangent bundle TM to S.

One can also think of Θ_S as an $\mathcal{O}_M$-module with the restriction of scalars coming from the surjection $\mathcal{O}_M \to \mathcal{O}_S$. Let $\mathcal{E}$ be a coherent $\mathcal{O}_S$-submodule of Θ_S of rank $s \leq m$ (possibly $\mathcal{E} = \Theta_S$). We have the following diagram of $\mathcal{O}_M$-modules with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{E} & \xrightarrow{\gamma} & \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S & \longrightarrow & \mathcal{N}_{\mathcal{E}} & \longrightarrow & 0 \\
 & & & & \varpi \uparrow & & & & \\
 & & & & \Theta_M & & & & \\
 & & & & j \uparrow & & & & \\
0 & \longrightarrow & \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S^2 & \longrightarrow & \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S & \xrightarrow{\psi} & \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2 & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}
\tag{2.1}
$$

We start with the following lemma:

Lemma 2.1 *The following morphism of $\mathcal{O}_M$-modules*

$$
\begin{array}{rcl}
\Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2 & \longrightarrow & \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2 \otimes_{\mathcal{O}_M} \mathcal{O}_S, \\
v \otimes [f] & \mapsto & v \otimes [f] \otimes 1
\end{array}
\tag{2.2}
$$

is an isomorphism.

Proof. Let $\mathcal{T} := \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2$. Apply the functor $\mathcal{T} \otimes_{\mathcal{O}_M} \cdot$ to the exact sequence (1.1). Thus we obtain the following exact sequence

$$
\mathcal{T} \otimes_{\mathcal{O}_M} \mathcal{I}_S \xrightarrow{A} \mathcal{T} \otimes_{\mathcal{O}_M} \mathcal{O}_M \xrightarrow{B} \mathcal{T} \otimes_{\mathcal{O}_M} \mathcal{O}_S \longrightarrow 0. \tag{2.3}
$$

Hence B is an isomorphism if and only if $\operatorname{Im} A = 0$. Denote by $Y : \mathcal{I}_S \to \mathcal{O}_M$. Let $w = v \otimes [f] \otimes g \in \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2 \otimes_{\mathcal{O}_M} \mathcal{I}_S$. Then

$$
A(w) = v \otimes [f] \otimes Y(g) = v \otimes [gf] \otimes 1 = 0,
$$

and $\operatorname{Im} A = 0$ has wanted. □

By Lemma 2.1 we can well define the following $\mathcal{O}_M$-morphism:

$$
\chi^{\mathcal{E}} : \mathcal{E} \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2 \to \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2. \tag{2.4}
$$

The map $\chi^{\mathcal{E}}$ is given by composing the map (γ is given by (2.1))

$$
\gamma \otimes \mathrm{id} : \mathcal{E} \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2 \longrightarrow (\Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S) \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2, \tag{2.5}
$$

with the inverse of the isomorphism given by Lemma 2.1.

Proposition 2.2 *If S is locally complete intersection then the morphism* (2.4) *is injective.*

Proof. We have only to show that the map (2.5) is injective. If $\mathcal{A}$, $\mathcal{B}$ are $\mathcal{O}_S$-modules and we indicate by $\mathcal{A}'$, $\mathcal{B}'$ the $\mathcal{O}_M$-modules defined by $\mathcal{A}$, $\mathcal{B}$ by restriction of scalars then (as $\mathcal{O}_M$-modules)

$$(\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{B})' \approx \mathcal{A}' \otimes_{\mathcal{O}_M} \mathcal{B}'.$$

Since S is locally complete intersection then $\mathcal{I}_S/\mathcal{I}_S^2$ is locally $\mathcal{O}_S$-free. Thus we have the following commuting diagram of $\mathcal{O}_M$-modules with exact rows and columns:

$$\begin{array}{ccc} \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{I}_S/\mathcal{I}_S^2 & = & \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2 \\ \uparrow & & \uparrow {\scriptstyle \gamma\otimes\mathrm{id}} \\ \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{I}_S/\mathcal{I}_S^2 & = & \mathcal{E} \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2 \\ \uparrow & & \\ 0 & & \end{array} \qquad (2.6)$$

Therefore $\chi^{\mathcal{E}}$ is injective. □

Definition 2.3 Let $v \in \Theta_M$. We say that v is *tangentially vanishing at the first order with respect to $\mathcal{E}$* if $\varpi(v) = 0$ and, if $w \in \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S$ is the (only) element such that $j(w) = v$, then

$$\psi(w) \in \chi^{\mathcal{E}}\big(\mathcal{E} \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2\big),$$

where ϖ, j and ψ are defined in (2.1) and $\chi^{\mathcal{E}}$ is defined in (2.4).

Remark 2.4 Let $p \in S'$ and let $\{z_1, \ldots, z_n\}$ be local coordinates centered at p such that $S' = \{z_{m+1} = \cdots = z_n = 0\}$. Assume that $X_1, \ldots, X_t \in \Theta_S$ generate $\mathcal{E}$ near p and let $X_i = \sum_{j=1}^m a_{ij}\partial/\partial z_j$. Let $v \in \Theta_{M,p}$. Then v is tangentially vanishing at the first order with respect to $\mathcal{E}$ if and only if there exist $\tilde{a}_{ij}$, f_i, $g_k \in \mathcal{O}_M$, $i = 1, \ldots, t$, $j = 1, \ldots, m$, $k = 1, \ldots, n$ such that $\tilde{a}_{ij}|_S = a_{ij}$, $f_i \in \mathcal{I}_S$, $g_i \in \mathcal{I}_S^2$ and

$$v = \sum_{\substack{i=1,\ldots,t \\ j=1,\ldots,m}} f_i \tilde{a}_{ij} \frac{\partial}{\partial z_j} + \sum_{i=1}^{n} g_i \frac{\partial}{\partial z_i}.$$

Now let $\mathcal{F} \subset \Theta_S$ be a dimension r foliation of S. Let $\{U_\alpha\}$ be a covering of S made of open subsets of M. For any α let $\mathcal{G}_\alpha \subset \Theta_M|_{U_\alpha}$ be an involutive subsheaf of Θ_M restricted to U_α. Thus we have the following commuting diagram of $\mathcal{O}_M$-modules with exact rows and columns:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{G}_\alpha & \xrightarrow{\iota_\alpha} & \Theta_M|_{U_\alpha} \\ & & \varpi_\alpha \downarrow & & \downarrow \varpi \\ & & \mathcal{G}_\alpha \otimes_{\mathcal{O}_M} \mathcal{O}_S & \xrightarrow{\kappa_\alpha} & \Theta_M|_{U_\alpha} \otimes_{\mathcal{O}_M} \mathcal{O}_S \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array} \tag{2.7}$$

Note that κ_α is not injective in general.

Definition 2.5 We say that $\{U_\alpha, \mathcal{G}_\alpha\}$ is a *first order tangency extension of $\mathcal{F}$ with respect to $\mathcal{E}$* if

1. $\kappa_\alpha(\mathcal{G}_\alpha \otimes_{\mathcal{O}_M} \mathcal{O}_S) = \mathcal{F}|_{U_\alpha \cap S}$ for any α.
2. Let $p \in U_\alpha \cap U_\beta \cap S$. If $f_\alpha \in \mathcal{G}_{\alpha,p}$ and $f_\beta \in \mathcal{G}_{\beta,p}$ are such that $\varpi(\iota_\alpha(f_\alpha)) = \varpi(\iota_\beta(f_\beta))$ then $\iota_\alpha(f_\alpha) - \iota_\beta(f_\beta)$ is tangentially vanishing at the first order with respect to $\mathcal{E}$.

Remark 2.6 1. We remark explicitly that condition (2) must hold also for $\alpha = \beta$.

2. Assume $p \in S \cap U_\alpha$. Let $v \in \mathcal{F}_p$ and let $\tilde{v} \in \mathcal{G}_\alpha$ such that $\varpi(\iota_\alpha(\tilde{v})) = v$. In the sequel we refer to such a $\tilde{v}$ as an *extension* of v and sometimes we simply write $\tilde{v}|_S = v$.

Let $p \in S'$ and let $\{z_1, \ldots, z_n\}$ be local coordinates centered at p such that $S' = \{z_{m+1} = \cdots = z_n = 0\}$. Assume that $X_1, \ldots, X_t \in \Theta_S$ generate $\mathcal{E}$ at p. Let $X_i = \sum_{j=1}^m a_{ij}\partial/\partial z_j$, with $a_{ij} \in \mathcal{O}_S$. Let $\tilde{a}_{ij} \in \mathcal{O}_M$ be such that $\tilde{a}_{ij}|_S = a_{ij}$ and set $\tilde{X}_i = \sum \tilde{a}_{ij}\partial/\partial z_j$. Let $v = \sum_{j=1}^m h_j \partial/\partial z_j \in \mathcal{F}$ for $h_j \in \mathcal{O}_S$. Then condition (1) means that there exist α and $\tilde{v} \in \mathcal{G}_\alpha$ given by $\tilde{v} = \sum_{l=1}^n \tilde{h}_l \partial/\partial z_l$ with $\tilde{h}_j \in \mathcal{O}_M$ such that

$$\begin{aligned} &\tilde{h}_j(z_1, \ldots, z_m, 0, \ldots, 0) \equiv h_j(z_1, \ldots, z_m), \quad j = 1, \ldots, m \\ &\tilde{h}_j(z_1, \ldots, z_m, 0, \ldots, 0) \equiv 0, \quad j = m+1, \ldots, n. \end{aligned} \tag{2.8}$$

Also, for any other extension $\tilde{w} \in \mathcal{G}_\beta$ of v (possibly $\alpha = \beta$) it follows that

$$\tilde{v} - \tilde{w} = \sum_{l=}^{t} a_l \tilde{X}_l + \sum_{l=1}^{n} b_l \frac{\partial}{\partial z_l}$$

with $a_l(z_1, \ldots, z_m, 0, \ldots, 0) \equiv 0$ for $l = 1, \ldots, t$ and

$$b_l(z_1, \ldots, z_n) = \sum_{m+1 \le j,\, k \le n} \tilde{b}^l_{jk} z_j z_k \quad \text{for} \quad l = 1, \ldots, n, \quad \tilde{b}^l_{jk} \in \mathcal{O}_M.$$

Before providing some examples of first order tangency extension, we state the following simple fact.

Proposition 2.7 *Let M be a complex manifold and $S \subset M$ a subvariety. Let $\mathcal{E} \subset \mathcal{E}' \subset \Theta_S$ be two coherent $\mathcal{O}_S$-submodules of Θ_S. If $\mathcal{F} \subset \Theta_S$ is a holomorphic foliation of S which has a first order tangency extension with respect to $\mathcal{E}$ then it has also a first order tangency extension with respect to $\mathcal{E}'$.*

Proposition 2.7 means in particular that each time a foliation $\mathcal{F}$ of S has a first order tangency extension with respect to some submodule of Θ_S then it has indeed a first order tangency extension with respect to Θ_S.

If S is an invariant set of a holomorphic foliation $\mathcal{F}$ on M then the foliation $\mathcal{F}|_S$ of S defined as the image of $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_S$ into Θ_S, is a foliation in S. It is not obvious—and indeed it is not true in general—that extension is the inverse operation of restriction. That is to say $\mathcal{F}|_S$ might not have $\mathcal{F}$ as a first order tangency extension with respect to $\mathcal{F}|_S$ (or with respect to Θ_S) as the following example shows.

Example 2.8 Consider the foliation $\mathcal{F}$ of $M = \mathbb{C}^3$ with coordinates (z_1, z_2, z_3) generated by $X_1 = z_1\partial/\partial z_1$ and $X_2 = \partial/\partial z_2$. Let $S = \{z_1 = 0\}$. Then S is invariant by $\mathcal{F}$ and $\mathcal{F}|_S$ is generated (on $\mathcal{O}_S$) by X_2. However X_2 has the following two extensions in $\mathcal{F}$: $v_1 = X_2$ and $v_2 = X_1 + X_2$. But $v_1 - v_2 \in \Theta_M$ comes from $\partial/\partial z_1 \otimes z_1 \in \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S$ and thus projects to $\partial/\partial z_1 \otimes [z_1] \in \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2$. Since the image of $\mathcal{F}|_S \otimes \mathcal{I}_S/\mathcal{I}_S^2$ into $\Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2$ is generated by $\partial/\partial z_2 \otimes [z_1]$ then $\mathcal{F}$ is *not* a first order tangency extension of $\mathcal{F}|_S$ with respect to $\mathcal{F}|_S$ or with respect to Θ_S. Note however that $S = \operatorname{Sing}\mathcal{F}$, and the foliation on M generated by $\partial/\partial z_2$ provides a first order tangency extension of $\mathcal{F}|_S$ with respect to $\mathcal{F}|_S$ and with respect to Θ_S.

The problem with the previous example is that the map $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_S \to \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$ is not injective on $\operatorname{Sing}(\mathcal{F})$. However when $\mathcal{F}$ is locally free everything works. Indeed we have

Proposition 2.9 *Let $\mathcal{F} \subset \Theta_M$ be a holomorphic foliation of M. Let $S \subset M$ be a complex subvariety of M which is invariant by $\mathcal{F}$. Let $\mathcal{F}|_S$ be the image of $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_S$ into Θ_S. Let $S^0 := S \setminus \mathrm{Sing}(\mathcal{F})$. Then $\mathcal{F}$ is a first order tangency extension of $\mathcal{F}|_{S^0}$ with respect to $\mathcal{F}|_{S^0}$.*

Proof. On S^0 the sheaf $\mathcal{F}$ is locally $\mathcal{O}_M$-free. Let

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

be an exact sequence of $\mathcal{O}_M$-modules on S^0. Suppose $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{B} \to \Theta_M \otimes_{\mathcal{O}_M} \mathcal{B}$ is injective. Thus we have the following commuting diagram (on S^0):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \xrightarrow{(4)} & \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{C} & \longrightarrow & \Theta_M \otimes_{\mathcal{O}_M} \mathcal{C} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{B} & \longrightarrow & \Theta_M \otimes_{\mathcal{O}_M} \mathcal{B} & \longrightarrow & \mathcal{N}_{\mathcal{B}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \xrightarrow{(2)} & \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{A} & \longrightarrow & \Theta_M \otimes_{\mathcal{O}_M} \mathcal{A} & \longrightarrow & \mathcal{N}_{\mathcal{A}} & \longrightarrow & 0 \\
 & & {\scriptstyle (1)}\uparrow & & \uparrow & & \uparrow{\scriptstyle (3)} & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2.9}
$$

Here $\mathcal{N}_{\mathcal{A}}$ (and similarly $\mathcal{N}_{\mathcal{B}}$) is defined to be the quotient of the modules on the same row. The exactness of the diagram is clear except, maybe, at points (1), (2), (3) and (4). Exactness at point (1) comes from being $\mathcal{F}$ an $\mathcal{O}_M$-free and thus $\mathcal{O}_M$-flat module. From this and from a simple diagram chasing, exactness at point (2) follows. Once we have this, we can define a natural injective map at (3) and using this, another diagram chasing gives exactness at point (4).

Now let $\mathcal{A} = \mathcal{I}_S$, $\mathcal{B} = \mathcal{O}_M$ and $\mathcal{C} = \mathcal{O}_S$. The previous argument shows that

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{I}_S \to \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S \tag{2.10}$$

and

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_S \to \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S \tag{2.11}$$

are exact. In particular by (2.10) we can repeat the argument of (2.9) for $\mathcal{A} = \mathcal{I}_S^2$, $\mathcal{B} = \mathcal{I}_S$ and $\mathcal{C} = \mathcal{I}_S/\mathcal{I}_S^2$ to get the exact sequence

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2 \to \Theta_M \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2. \tag{2.12}$$

Thus if $v \in \mathcal{F}$ is such that $v \otimes 1 = 0$ in $\Theta_M \otimes \mathcal{O}_S$ then $v \otimes 1 = 0$ in $\mathcal{F} \otimes \mathcal{O}_S$ by (2.11). Therefore there exists $w \otimes f \in \mathcal{F} \otimes \mathcal{I}_S$ which lifts $v \otimes 1$. Thanks to (2.12) it follows that $v \otimes [f] \in \mathcal{F} \otimes \mathcal{I}_S/\mathcal{I}_S^2 \subset \Theta_M \otimes \mathcal{I}_S/\mathcal{I}_S^2$. Therefore any two extensions of the same element of $\mathcal{F}|_S$ differ by an element which is in $\mathcal{F} \otimes \mathcal{I}_S/\mathcal{I}_S^2$. Arguing as in Lemma 2.1 it follows that $\mathcal{F} \otimes \mathcal{I}_S/\mathcal{I}_S^2 \approx \mathcal{F} \otimes \mathcal{O}_S \otimes \mathcal{I}_S/\mathcal{I}_S^2 = \mathcal{F}|_S \otimes \mathcal{I}_S/\mathcal{I}_S^2$ and thus the difference of two extensions of the same element of $\mathcal{F}|_S$ is tangentially vanishing at the first order with respect to $\mathcal{F}|_S$. □

Example 2.10 If S has codimension one in M and it is the fixed points set of a holomorphic self-map f of M which is *tangential* (or *nondegenerate*) to S or if S is *comfortably embedded into M* (see [2], [5]) then it is possible to define a natural one-dimensional foliation on S which has a natural first order tangency extension (but not a true extension) with respect to Θ_S and (in some cases) with respect to $\mathcal{F}$ on S'. For the reader convenience we briefly sketch here such a construction. Let $p \in S$. First for $H \in \mathcal{O}_{M,p}$ we define $T_p(H) := \max\{l \in \mathbb{N} \colon H \circ f - H \in \mathcal{I}_S^l\}$. Then we define $\nu_f(p) = \min\{T_p(H) \colon H \in \mathcal{O}_{M,p}\}$. If S is globally irreducible (as we suppose), then $\nu_f(p)$ is independent of $p \in S$ and we simply denote it by ν_f. Then on each local chart $\{U, (z_1, \ldots, z_n)\}$ we consider the (local) section

$$\sum_{j=1}^{n} [z_j \circ f - z_j] \otimes \frac{\partial}{\partial z_j} \in (\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes \nu_f} \otimes_{\mathcal{O}_S} (\Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S),$$

where $[z_j \circ f - z_j]$ is the class of $z_j \circ f - z_j$ in $\mathcal{I}_S^{\nu_f}/\mathcal{I}_S^{\nu_f+1} \simeq (\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes \nu_f}$. It turns out that those local sections glue together to form a global section X_f of $(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes \nu_f} \otimes_{\mathcal{O}_S} (\Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S)$. We say that f is *tangential* (or *nondegenerate* in the terminology of [1], [6], [7]) if actually X_f is a section of $(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes \nu_f} \otimes_{\mathcal{O}_S} \Theta_S$ (note that being $\mathcal{I}_S/\mathcal{I}_S^2$ a rank one $\mathcal{O}_S$-free module there is a natural injection of $(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes \nu_f} \otimes_{\mathcal{O}_S} \Theta_S$ into $(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes \nu_f} \otimes_{\mathcal{O}_S} (\Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S)$). Also, if f is non-tangential but S satisfies some cohomological condition (for instance if S is the zero section of a line bundle M on S) then there are natural projections from $(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes \nu_f} \otimes_{\mathcal{O}_S} (\Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S)$ to $(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes \nu_f} \otimes_{\mathcal{O}_S} \Theta_S$ and still one can consider X_f as a section of

$(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes\nu_f}\otimes_{\mathcal{O}_S}\Theta_S$. In the latter case we say that S is *comfortably embedded* into M (see [2]). Now, since $\mathcal{I}_S/\mathcal{I}_S^2$ is $\mathcal{O}_S$-free, there is a natural injective morphism from $(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes\nu_f}\otimes_{\mathcal{O}_S}\Theta_S$ to $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{H}om_{\mathcal{O}_S}((\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes\nu_f},\mathcal{O}_S),\Theta_S)$.

Thus if f is tangential or S is comfortably embedded into M one has a natural one dimensional foliation $\mathcal{F}$ of S given by the image of the morphism from $\mathcal{N}_S^{\otimes\nu_f} = \mathcal{H}om_{\mathcal{O}_S}((\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes\nu_f},\, \mathcal{O}_S)$ to Θ_S induced by X_f. There is a first order tangency extension of $\mathcal{F}$ with respect to Θ_S restricted to the nonsingular part $S' = S\setminus \mathrm{Sing}(S)$. For instance, in case f is tangential and $\{U_\alpha,\, (z_1,\, \ldots,\, z_n)\}$ is a local coordinates system such that $S'\cap U = \{z_n = 0\}$ we let $\mathcal{G}_\alpha$ be the sheaf of $\mathcal{O}_M$-modules generated by $\sum(z_j\circ f - z_j)/z_n^{\nu_f}\partial/\partial z_j$. Then one can show that $\{U_\alpha,\, \mathcal{G}_\alpha\}$ is a first order tangency extension of $\mathcal{F}$ with respect to Θ_S. Similarly for the case S is comfortably embedded (see [2]). Moreover in case $\nu_f > 1$, S is comfortably embedded and f is tangential there is a first order tangency extension of $\mathcal{F}$ with respect to $\mathcal{F}$ itself (see *Theorem* 5.3 in [2]).

3. Holomorphic actions for first order tangency extensions

Let M be an n-dimensional complex manifold and let TM be its holomorphic tangent bundle. First we recall the definition of holomorphic action (see, *e.g.*, [19], p. 75).

Definition 3.1 Let F be an involutive subbundle of TM. A *holomorphic action* of F on a holomorphic vector bundle L over M is a $\mathbb{C}$-bilinear map $\theta\colon \mathcal{C}^\infty(F)\times\mathcal{C}^\infty(L)\to\mathcal{C}^\infty(L)$ such that

1. $\theta([u,\, v],\, s) = \theta(u,\, \theta(v,\, s)) - \theta(v,\, \theta(u,\, s))$ for $u,\, v\in\mathcal{C}^\infty(F)$ and $s\in\mathcal{C}^\infty(L)$;
2. $\theta(hu,\, s) = h\theta(u,\, s)$ for $h\in\mathcal{C}^\infty$, $u\in\mathcal{C}^\infty(F)$ and $s\in\mathcal{C}^\infty(L)$;
3. $\theta(u,\, hs) = h\theta(u,\, s) + u(h)s$ for $h\in\mathcal{C}^\infty$, $u\in\mathcal{C}^\infty(F)$ and $s\in\mathcal{C}^\infty(L)$;
4. $\theta(u,\, s)\in\mathcal{L}$ for $u\in\mathcal{F}$ and $s\in\mathcal{L}$.

Holomorphic actions were introduced by Bott [4] in case of a holomorphic vector field. We need another definition:

Definition 3.2 Let S be a complex manifold. Let $L,\, F\subset TS$ be two vector bundles. We say that L is *compatible* with F if $[L,\, F]\subseteq L$.

Note that TS is compatible with any of its subbundle and each involutive bundle is compatible with itself. Moreover generally if L is involutive and $F\subset L$ then L is compatible with F.

Theorem 3.3 *Let M be an n-dimensional complex manifold and $S \subset M$ a (nonsingular) submanifold of dimension $m < n$. Let $\mathcal{F}$ be an r-dimensional nonsingular holomorphic foliation on S and let $F \subset TS$ be the associated subbundle. Let $L \subset TS$ be a subbundle compatible with F and let $\mathcal{L}$ be the sheaf of its holomorphic sections. If $\mathcal{F}$ admits a first order tangency extension with respect to $\mathcal{L}$ then there exists a holomorphic action of F on $N_L := TM|_S/L$.*

Proof. We want to define a holomorphic action $\theta\colon \mathcal{C}^\infty(F) \times \mathcal{C}^\infty(N_L) \to \mathcal{C}^\infty(N_L)$. Since $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{C}^\infty = \mathcal{C}^\infty(F)$, it is enough to define θ for $u \in \mathcal{F}$ and $s \in \mathcal{C}^\infty(N_L)$. Let $\tilde{u} \in \mathcal{G}_\alpha$ be an extension of u for some α (see Definition 2.5). Let $\tilde{s} \in \mathcal{C}^\infty(TM)$ be such that $\pi(\tilde{s}|_S) = s$, where $\pi\colon TM|_S \to N_L$ is the canonical projection. Define

$$\theta(u, s) = \pi([\tilde{u}, \tilde{s}]|_S),$$

where the Lie bracket $[\cdot, \cdot]$ has obviously to be thought of in Θ_M. First we show that θ is well defined, that is, it is independent of the extensions $\tilde{s}$ and $\tilde{u}$ chosen. Let $\tilde{s}' \in \mathcal{C}^\infty(N_L)$ be such that $\pi(\tilde{s}'|_S) = s$. Since $\tilde{u}|_S \in \mathcal{F}$ and $(\tilde{s}-\tilde{s}')|_S \in \mathcal{C}^\infty(L)$, and since $[L, F] \subset L$, it follows that $[\tilde{u}, \tilde{s}'-\tilde{s}]|_S \in \mathcal{C}^\infty(L)$ and thus it goes to 0 once applying π, hence θ is independent of $\tilde{s}$. As for the independence from the extension $\tilde{u}$, let $\tilde{u}'$ be another such an extension for u. By definition $\tilde{u} - \tilde{u}' = \sum g_j v_j + \sum h_j w_j$ with $g_j|_S \equiv 0$ and $v_j \in \Theta_M$ such that $v_j|_S \in \mathcal{L}$, and $h_j \in \mathcal{I}_S^2$ and $w_j \in \Theta_M$. Thus

$$\begin{aligned} [\tilde{u} - \tilde{u}', \tilde{s}]\,|_S &= \left[\sum g_j v_j + \sum h_j w_j, \tilde{s}\right]\Big|_S \\ &= -\left(\sum \tilde{s}(g_j) v_j + \sum \tilde{s}(h_j) w_j\right)\Big|_S \\ &= -\sum \tilde{s}(g_j)\,|_S\, v_j\,|_S \in \mathcal{C}^\infty(L). \end{aligned}$$

Applying π even this term goes to zero and thus θ is well defined.

It is straightforward to see that θ satisfies properties (2) to (4) of Definition 3.1. As for property (1), let $u, v \in \mathcal{F}$ and let $\tilde{u}, \tilde{v}$ be local extensions of u and v respectively, belonging to the same $\mathcal{G}_\alpha$. Since this latter is involutive by hypothesis, it is easy to see that $[\tilde{u}, \tilde{v}] \in \mathcal{G}_\alpha$ (Lie bracket made in Θ_M) is a local extension of $[u, v]$ (where this time the Lie bracket as to be thought in Θ_S). Therefore by the Jacobi identity

$$\theta(u, \theta(v,s)) = \pi([[\tilde{u}, \tilde{v}], \tilde{s}]|_S) = \pi([\tilde{u}, [\tilde{v}, \tilde{s}]]|_S) - \pi([\tilde{v}, [\tilde{u}, \tilde{s}]]|_S).$$

Now

$$\theta(u, \theta(v, s)) = \theta(u, \pi([\tilde{v}, \tilde{s}]|_S)) = \pi([\tilde{u}, \tilde{w}]|_S),$$

where $\tilde{w} \in \mathcal{C}^\infty(TM)$ is any vector field such that $\pi(\tilde{w}|_S) = \pi([\tilde{v}, \tilde{s}]|_S)$. Since we can certainly take $\tilde{w} = [\tilde{v}, \tilde{s}]$, and similarly we can argue for $\theta(v, \theta(u, s))$, it follows that even property (4) is satisfied. □

Remark 3.4 1. In case $L = TS$ we call the holomorphic action given by Theorem 3.3 a *Camacho-Sad* action for the first example of such an action in the case $n = 2$, $m = 1$ and $\mathcal{F}$ is the restriction of a holomorphic foliation on M, is due to Camacho and Sad [8].

2. In case $L = F$ we call the holomorphic action given by Theorem 3.3 a *Lehmann-Suwa* action (see [14] and [11]).

3. The bundle L in Theorem 3.3 needs not to be involutive.

Aside the previous examples, a new typical setting where Theorem 3.3 applies is provided by the following situation.

Corollary 3.5 *Let M be a complex manifold of dimension n. Let $S \subset M$ be a submanifold of dimension $m < n$. Assume that $\mathcal{L} \subset \Theta_S$ is a nonsingular foliation of dimension s of S. Let $\mathcal{F} \subset \Theta_M$ be a nonsingular foliation of M of dimension $r < s$ leaving invariant the leaves of $\mathcal{L}$, i.e., $\mathcal{F}|_S \subset \mathcal{L}$. Let L, $F \subset TS$ denote the bundle associated to $\mathcal{L}$ and $\mathcal{F}|_S$ respectively. Then there exists a holomorphic action of F on $N_L := TM|_S/L$.*

Proof. The bundle L is compatible with F. Moreover, since $\mathcal{F}|_S$ has a first order tangency extension with respect to $\mathcal{F}|_S$ (see Proposition 2.9) and $\mathcal{F}|_S \subset \mathcal{L}$ then, by Proposition 2.7, Theorem 3.3 applies and one has a holomorphic action of F on $TM|_S/L$. □

4. Residues Theorems

Existence of holomorphic actions on a holomorphic bundle are obstructed by characteristic classes. Indeed we have

Theorem 4.1 (Bott vanishing theorem) *Let S be a complex manifold of dimension m. Let $F \subset TS$ be an involutive subbundle of rank r. Suppose that F holomorphically acts on the holomorphic subbundle $L \subset TS$. Then there exists a connection ∇ for L such that for any homogeneous symmetric* *polynomial φ of degree $d > m - r$ it follows*

$$\varphi(\nabla) = 0.$$

In particular any characteristic class of L of degree $> m - r$ is zero.

For the reader convenience we sketch here a proof of this result and refer to [19], p. 76, or [3] for details.

Proof of Theorem 4.1. Let $\theta\colon \mathcal{C}^\infty(F) \times \mathcal{C}^\infty(L) \to \mathcal{C}^\infty(L)$ denote the holomorphic action of F on L. We let $\mathbb{C} \otimes TS = F \oplus F' \oplus T^{0,1}S$ for some $\mathcal{C}^\infty$ complement F' of F. Then we define the connection ∇ for L such that:

$$\nabla_v s = \theta(v, s) \quad \text{for } v \in \mathcal{C}^\infty(F), \quad s \in \mathcal{C}^\infty(L),$$
$$\nabla_v s = \overline{\partial}_v s \quad \text{for } v \in \mathcal{C}^\infty(T^{0,1}S), \quad s \in \mathcal{C}^\infty(L).$$

Thus from property (1) in Definition 3.1 and by the very definition of ∇ it follows that if K is the curvature of ∇ then $K(x, y) = 0$ whenever either $x, y \in \mathcal{C}^\infty(F)$, or $x \in \mathcal{C}^\infty(F)$, $y \in \mathcal{C}^\infty(T^{0,1}S)$ or $x, y \in \mathcal{C}^\infty(T^{0,1}S)$. Therefore in a basis which respects the decomposition $\mathbb{C} \otimes TS^* = F^* \oplus F'^* \oplus (T^{0,1}S)^*$ it follows that the matrix of K is made of forms which belong to the ideal generated by the basis of F'^*, from which the result follows. □

Now we recall briefly the general Lehmann-Suwa philosophy for localization of characteristic classes (see, *e.g.*, [19]). Assume S is a subvariety of dimension m of the complex n-dimensional manifold M. Suppose $\tilde{W}$ is a $\mathcal{C}^\infty$ complex vector bundle on M. Let S^0 be an open subset of $S \setminus \mathrm{Sing}(S)$ and assume $W = \tilde{W}|_{S^0}$ is holomorphic. Moreover suppose that for some reason (like existence of holomorphic actions) there exists a connection ∇ for W on S^0 such that $\varphi(\nabla) = 0$ for any homogeneous symmetric polynomial of a given degree d. Then we denote by $\tilde{U}_0$ a tubular neighborhood of S^0 in M. Also we denote by $\tilde{U}_1$ a regular neighborhood of $\Sigma := S \setminus S^0$ (we are assuming such a regular neighborhood does exist, which is always the case if Σ is an analytic set, as in our setting). For the forthcoming considerations we may assume without loss of generality that $\tilde{U}_0 \cup \tilde{U}_1$ is a regular neighborhood of S in M. Let ∇_0 be the pull back to $\tilde{U}_0$ of ∇. Let ∇_1 be any connection for $\tilde{W}$ on $\tilde{U}_1$. Let φ be a homogenoeus symmetric polynomial of degree d. Let $\varphi(\nabla_0, \nabla_1)$ denote the Bott difference form of $\varphi(\nabla_0)$, $\varphi(\nabla_1)$ relative to the covering $\tilde{U}_0$, $\tilde{U}_1$. The cocycle

$$(\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1))$$

represents in the Čech-de Rham cohomology relative to $\{\tilde{U}_0, \tilde{U}_1\}$ the class $\varphi(\tilde{W}) \in H^{2d}(S, \mathbb{C})$. Since $\varphi(\nabla_0) = 0$ it follows that actually the cocycle represents a class in the relative cohomology $H^{2d}(S, S \setminus \Sigma, \mathbb{C})$ which we indicate by $\varphi(\tilde{W}, \Sigma)$. If Σ is compact, the Alexander homomorphism $A\colon H^{2d}(S, S \setminus \Sigma, \mathbb{C}) \to H_{2m-2d}(\Sigma, \mathbb{C})$ (see, *e.g.*, [19], VI.4) sends $\varphi(\tilde{W}, \Sigma)$ to a "residual class"

$$\operatorname{Res}(\tilde{W}, \Sigma) = A(\varphi(\tilde{W}, \Sigma)) \in H_{2m-2d}(\Sigma, \mathbb{C}).$$

Now, let $\Sigma = \cup \Sigma_\lambda$ be the decomposition in connected components of Σ and let $i_\lambda\colon H_*(\Sigma_\lambda, \mathbb{C}) \to H_*(S, \mathbb{C})$ be the morphism coming from the inclusion $\Sigma_\lambda \hookrightarrow S$. if S is compact then by the Poincaré homomorphism $P\colon H^*(S,\mathbb{C}) \to H_{2m-*}(S, \mathbb{C})$ we have the following "residue theorem" in $H_{2n-2d}(S, \mathbb{C})$:

$$\sum_\lambda i_\lambda \operatorname{Res}(\tilde{W}, \Sigma_\lambda) = [S] \cap \varphi(\tilde{W}).$$

In case $\mathcal{W}$ is a $\mathcal{C}^\infty_M$-module such that $\mathcal{W} \otimes_{\mathcal{O}_M} \mathcal{O}_S$ is locally free on S^0, one can argue similarly as before, considering a finite resolution of $\mathcal{W} \otimes_{\mathcal{O}_M} \mathcal{A}_M$ (where $\mathcal{A}_M$ is the sheaf of real analytic functions on M) made of real analytic locally free sheaves (see [3] or [19], p. 184).

Now we go back to our situation. Let $\mathcal{F} \subset \Theta_S$ be a holomorphic foliation of S of dimension r. Let $\mathcal{L} \subset \Theta_S$ be a coherent sheaf and denote by

$$\operatorname{Sing}(\mathcal{L}) = \{p \in S \setminus \operatorname{Sing}(S) : \Theta_{S,p}/\mathcal{L}_p \text{ is not locally } \mathcal{O}_{S,p}\text{-free}\}.$$

Moreover let

$$\Sigma(S, \mathcal{F}, \mathcal{L}) := \operatorname{Sing}(S) \cup \operatorname{Sing}(\mathcal{F}) \cup \operatorname{Sing}(\mathcal{L}),$$

and $\cup_\lambda \Sigma_\lambda = \Sigma(S, \mathcal{F}, \mathcal{L})$ be the connected components decomposition. Let F, $L \subset TS^0$ be the holomorphic bundle associated respectively to $\mathcal{F}$ and $\mathcal{L}$ on S^0. Assume the following:

(a) L is compatible with F and $\mathcal{F}$ has a first order tangency extension with respect to $\mathcal{L}$ on S^0.

(b) There exists a $\mathcal{C}^\infty_M$-module $\mathcal{Q}$ such that $\mathcal{Q} \otimes_{\mathcal{O}_M} \mathcal{O}_S = (\Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S)/\mathcal{L}$, where $j\colon \mathcal{Q} \otimes_{\mathcal{O}_M} \mathcal{O}_S \to \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$.

The previous recalled Lehmann-Suwa theory gives us:

Theorem 4.2 *Let φ be a homogeneous symmetric polynomial of degree $d > m - r$. In the above situation, for any λ such that Σ_λ is compact, there* *exists a residue* $\mathrm{Res}_\varphi(\mathcal{F}, \mathcal{Q}; \Sigma_\lambda) \in H_{2m-2d}(\Sigma_\lambda, \mathbb{C})$ *determined only by the local behavior of the first order tangency extension of* $\mathcal{F}$ *near* Σ_λ.

If moreover S *is compact then*

$$\sum_\lambda i_\lambda \mathrm{Res}_\varphi(\mathcal{F}, \mathcal{Q}; \Sigma_\lambda) = [S] \cap \varphi(\mathcal{Q}) \ \textit{in} \ H_{2m-2d}(S, \mathbb{C}).$$

Some final remarks are in order.

1. The previous condition (b) means that the sheaf $\Theta_M \otimes \mathcal{O}_S/\mathcal{L}$ has some relation with the ambient M. Actually one needs only an extension of $TM|_S - L$ to M in the K-theory. In particular if S is nonsingular and $\mathcal{L}$ is locally free on S then one can take $\mathcal{Q}$ to be the $\mathcal{C}^\infty$-sheaf of sections of the pull back of $TM|_S/L$ to a tubular neighborhood of S. In the singular cases the existence of $\mathcal{Q}$ depends on $\mathcal{L}$. For instance if $\mathcal{L}$ is the restriction to S of an ambient foliation than $\mathcal{Q}$ naturally exists. If $\mathcal{L} = \Theta_S$ then $\mathcal{Q}$ exists in case S is a so-called *strongly locally complete intersection* (see [13]).
2. As remarked several times at various places of the paper, the existence of a foliation $\mathcal{F}$ extending to the first order tangency which acts on L needs only to have a vanishing of certain forms on S^0. Therefore Theorem 4.2 would apply even if $\mathcal{F}$ were defined only outside $\mathrm{Sing}(S) \cup \mathrm{Sing}(\mathcal{L})$.
3. The explicit calculation of residues is a very important and usually very involved part of a useful residue theorem. However, we do not pursue this task here, and refer the reader to [19], [13], [2] for effective calculations. With regard to the previous comment we only remark that if $\mathcal{F}$ does not exist on $\mathrm{Sing}(S) \cup \mathrm{Sing}(\mathcal{L})$ than it could be impossible (or at least, not yet done) to calculate explicitly the residue.

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