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Local isometric imbeddings of $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$

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Abstract. We investigate local isometric imbeddings of the quaternion projective plane $P^2(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$ into the Euclidean spaces. We prove a non-existence theorem of local isometric imbeddings (see Theorem 2), by which we can conclude that the isometric imbeddings given in Kobayashi [8] are the least dimensional isometric imbeddings of $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$.

Key words: Pseudo-nullity, isometric imbedding, projective plane.

1. Introduction

In this paper we investigate local isometric imbeddings of the quaternion projective plane $P^2(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$ into the Euclidean spaces.

In [5], we determined the pseudo-nullity $p(G/K)$ for each compact rank one symmetric space G/K. (For the definition of the pseudo-nullity, see [5].) Utilizing $p(G/K)$, we have obtained the following result concerning the non-existence of isometric imbeddings of the complex projective spaces $P^n(\boldsymbol{C})\,(n \geq 2)$, the quaternion projective spaces $P^n(\boldsymbol{H})\,(n \geq 2)$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$ (see Theorem 5.6 of [5]).

Theorem 1 *Let G/K be one of the complex projective space $P^n(\boldsymbol{C})$ $(n \geq 2)$, the quaternion projective space $P^n(\boldsymbol{H})$ $(n \geq 2)$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$. Define an integer $q(G/K)$ by setting $q(G/K) = 2\dim G/K - p(G/K)$, i.e.,*

$$q(G/K) = \begin{cases} \min\{4n-2,\, 3n+1\}, & \text{if } \ G/K = P^n(\boldsymbol{C}) \ (n \geq 2), \\ \min\{8n-3,\, 7n+1\}, & \text{if } \ G/K = P^n(\boldsymbol{H}) \ (n \geq 2), \\ 25, & \text{if } \ G/K = P^2(\boldsymbol{Cay}). \end{cases}$$

Then, any open set of G/K cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^Q$ with $Q \leq q(G/K) - 1$.

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As is well known, $P^n(\boldsymbol{C})$ (resp. $P^n(\boldsymbol{H})$, resp. $P^2(\boldsymbol{Cay})$) can be globally isometrically imbedded into $\boldsymbol{R}^{n^2+2n}$ (resp. $\boldsymbol{R}^{2n^2+3n}$, resp. $\boldsymbol{R}^{26}$) (see Kobayashi [8]). By these facts, it follows that if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, then G/K can be isometrically imbedded into $\boldsymbol{R}^{q(G/K)+1}$. Then a natural question arises: Is there any isometric imbedding of $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$ into the Euclidean space $\boldsymbol{R}^{q(G/K)}$?

In this paper, we will solve this problem. The main result of this paper is the following

Theorem 2 *Let G/K be the quaternion projective plane $P^2(\boldsymbol{H})$ or the Cayley projective plane $P^2(\boldsymbol{Cay})$. Then any open set of G/K cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^{q(G/K)}$. Accordingly, $\boldsymbol{R}^{q(G/K)+1}$ is the least dimensional Euclidean space into which G/K can be locally isometrically imbedded.*

2. The Gauss equation

In the following G/K implies the quaternion projective plane $P^2(\boldsymbol{H}) = Sp(3)/Sp(2)\times Sp(1)$ or the Cayley projective plane $P^2(\boldsymbol{Cay}) = F_4/Spin(9)$.

Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ associated with the Riemannian symmetric pair (G, K). We denote by $(\ ,\ )$ the inner product of $\mathfrak{g}$ given by the (-1)-multiple of the Killing form of $\mathfrak{g}$. As usual we identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\} \in G/K$. We assume that the G-invariant Riemannian metric g of G/K satisfies $g(X, Y) = (X, Y)$ $(X, Y \in \mathfrak{m})$. Then the curvature tensor R at o is given by

$$R(X, Y)Z = -\big[[X, Y], Z\big], \quad \forall X, Y, Z \in \mathfrak{m}. \tag{2.1}$$

Suppose that there is a local isometric imbedding of G/K into the Euclidean space $\boldsymbol{R}^Q$, i.e., there is an open set U of G/K and an isometric imbedding $\boldsymbol{f}$ of U into $\boldsymbol{R}^Q$. Because of homogeneity, we may assume that U contains the origin $o \in G/K$. Let $\boldsymbol{N}$ be the normal space of $\boldsymbol{f}(U)$ at $\boldsymbol{f}(o)$ and let $\langle\ ,\ \rangle$ be the inner product of $\boldsymbol{N}$ induced from the canonical inner product of $\boldsymbol{R}^Q$. Then $\boldsymbol{N}$ is a vector space with $\dim \boldsymbol{N} = Q - \dim G/K$ and the second fundamental form $\boldsymbol{\Psi}$ of $\boldsymbol{f}$ at o, which is regarded as an $\boldsymbol{N}$-valued symmetric bilinear form on $\mathfrak{m}$, must satisfy the following Gauss equation:

$$\begin{aligned}-\big(R(X,Y)Z,W\big) = \big\langle \boldsymbol{\Psi}(X,Z), \boldsymbol{\Psi}(Y,W)\big\rangle \\ -\big\langle \boldsymbol{\Psi}(X,W), \boldsymbol{\Psi}(Y,Z)\big\rangle,\ \forall X,Y,Z,W \in \mathfrak{m}.\end{aligned} \tag{2.2}$$

On the contrary, we can prove

Theorem 3 *Let $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$. If $\dim \boldsymbol{N} \leq q(G/K) - \dim G/K$, then the Gauss equation (2.2) does not admit any solution, i.e., there is no $\boldsymbol{N}$-valued symmetric bilinear form Ψ on $\mathfrak{m}$ satisfying (2.2).*

Theorem 3 implies that if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, then there is no local isometric imbedding of G/K into $\boldsymbol{R}^{q(G/K)}$, proving Theorem 2.

We now make a preparatory discussion for the proof of Theorem 3. Take and fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$. Then we have $\dim \mathfrak{a} = 1$, because $\mathrm{rank}(G/K) = 1$. We consider the root space decompositions of $\mathfrak{k}$ and $\mathfrak{m}$ with respect to $\mathfrak{a}$. Let $\lambda \in \mathfrak{a}$. We define subspaces $\mathfrak{k}(\lambda)$ ($\subset \mathfrak{k}$) and $\mathfrak{m}(\lambda)$ ($\subset \mathfrak{m}$) by setting

$$\mathfrak{k}(\lambda) = \{X \in \mathfrak{k} \mid [H, [H, X]] = -(\lambda, H)^2 X, \quad \forall H \in \mathfrak{a}\},$$
$$\mathfrak{m}(\lambda) = \{Y \in \mathfrak{m} \mid [H, [H, Y]] = -(\lambda, H)^2 Y, \quad \forall H \in \mathfrak{a}\}.$$

λ is called a *restricted root* when $\mathfrak{m}(\lambda) \neq 0$. We denote by Σ the set of non-zero restricted roots. In the case where $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, it is well known that there is a restricted root μ satisfying $\Sigma = \{\pm\mu, \pm 2\mu\}$ and

$$\mathfrak{k} = \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \quad \text{(orthogonal direct sum)}, \tag{2.3}$$
$$\mathfrak{m} = \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \quad \text{(orthogonal direct sum)}, \tag{2.4}$$

where $\mathfrak{m}(0) = \mathfrak{a} = \boldsymbol{R}\mu$ (see § 5 of [5]). In the following discussions we fix this restricted root μ and the decompositions (2.3) and (2.4).

For convenience, for each integer i we set

$$\mathfrak{k}_i = \mathfrak{k}(|i|\mu),\ \mathfrak{m}_i = \mathfrak{m}(|i|\mu)\ (|i| \leq 2) \quad \text{and} \quad \mathfrak{k}_i = \mathfrak{m}_i = 0\ (|i| > 2).$$

Then we have

Proposition 4 (1) *Let $i, j = 0, 1, 2$. Then:*

$$\begin{aligned} &[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \\ &[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \\ &[\mathfrak{k}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}. \end{aligned} \tag{2.5}$$

(2) $\dim \mathfrak{k}_i = \dim \mathfrak{m}_i$ $(i = 1, 2)$.

(3) *The following table summarizes the basic data for $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$.*

G/K	$\dim G/K$	$\dim \mathfrak{m}_1$	$\dim \mathfrak{m}_2$	$q(G/K)$
$P^2(\boldsymbol{H})$	8	4	3	13
$P^2(\boldsymbol{Cay})$	16	8	7	25

Proof. (1) and (2) are well known (see Helgason [7], p. 335). (3) is obtained by Table 2 and Table 3 of [5]. □

3. Proof of Theorem 3

In this section we prove Theorem 3. Here we suppose that $\dim \boldsymbol{N} = q(G/K) - \dim G/K$ and that there is a solution Ψ of the Gauss equation (2.2).

Let $Y \in \mathfrak{m}$. We define a linear map Ψ_Y of $\mathfrak{m}$ to $\boldsymbol{N}$ by

$$\Psi_Y : \mathfrak{m} \ni Y' \longmapsto \Psi(Y, Y') \in \boldsymbol{N}.$$

By $\mathbf{Ker}(\Psi_Y)(\subset \mathfrak{m})$ we denote the kernel of the linear map Ψ_Y. We now show a key proposition, which plays an important role in the following discussion.

Proposition 5 *Let $Y \in \mathfrak{m}$ $(Y \neq 0)$ and let $k \in K$ satisfy* $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. *Then*

$$\mathbf{Ker}(\Psi_Y) = \mathrm{Ad}(k)\mathfrak{m}_2. \tag{3.1}$$

In particular, $\mathbf{Ker}(\Psi_\mu) = \mathfrak{m}_2$.

Before proceeding to the proof of Proposition 5, we recall the notion of pseudo-abelian subspaces of $\mathfrak{m}$ defined in [5]. Let V be a subspace of $\mathfrak{m}$. Then, V is called *pseudo-abelian* if it satisfies $[V, V] \subset \mathfrak{k}_0$ (or equivalently, $\big[[V, V], \mathfrak{a}\big] = 0$). By (2.5) we can easily verify that $\mathfrak{m}_2$ is pseudo-abelian. On the contrary, we have

Lemma 6 *Let $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$. Then, any pseudo-abelian subspace V of $\mathfrak{m}$ with $\dim V > 2$ must be contained in $\mathfrak{m}_2$.*

Proof. Let V be a pseudo-abelian subspace of $\mathfrak{m}$ satisfying $V \not\subset \mathfrak{m}_2$. Then by Lemma 5.4 of [5], we obtain $\dim V \leq 1 + n(\mu)$, where $n(\mu)$ is the local pseudo-nullity associated with μ. (For the definition of the local pseudo-nullity, see § 3 in [5].) Moreover, we have $n(\mu) = 1$ if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$ (see Table 2 of [5]). Therefore, we get $\dim V \leq 2$, proving the lemma. □

We now start the proof of Proposition 5.

Proof of Proposition 5. We first note that $\dim \mathbf{Ker}(\Psi_Y) \geq \dim \mathfrak{m}_2 > 2$. In fact, since $\dim \boldsymbol{N} = q(G/K) - \dim G/K = \dim G/K - \dim \mathfrak{m}_2$, we have $\dim \mathbf{Ker}(\Psi_Y) \geq \dim G/K - \dim \boldsymbol{N} = \dim \mathfrak{m}_2 > 2$ (see Proposition 4 (3)).

In § 1 of [2], by considering the Gauss equation (2.2), we have proved

$$R(\mathbf{Ker}(\Psi_Y),\, \mathbf{Ker}(\Psi_Y))Y = 0. \tag{3.2}$$

Because of (2.1), the equality (3.2) means

$$\big[[\mathbf{Ker}(\Psi_Y),\, \mathbf{Ker}(\Psi_Y)],\, Y\big] = 0. \tag{3.3}$$

Applying $\mathrm{Ad}(k^{-1})$ to the both sides of (3.3), we get

$$\big[[\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y),\, \mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y)],\, \mu\big] = 0.$$

(Note that $\mathrm{Ad}(k^{-1})Y$ can be written as $\mathrm{Ad}(k^{-1})Y = c\mu$ for some $c \in \boldsymbol{R}$ ($c \neq 0$).) Since $\mathfrak{a} = \boldsymbol{R}\mu$, we know that $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y)$ is a pseudo-abelian subspace of $\mathfrak{m}$ with $\dim \mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y) \geq \dim \mathfrak{m}_2 > 2$. Therefore, we have $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y) = \mathfrak{m}_2$ (see Lemma 6). This proves (3.1). □

Utilizing Proposition 5, we will characterize solutions Ψ of the Gauss equation (2.2). For this purpose we need more informations about the action of the isotropy group $\mathrm{Ad}(K)$.

As is well known, any element of $\mathfrak{m}$ is conjugate to an element of $\boldsymbol{R}\mu(= \mathfrak{a})$ under the action of $\mathrm{Ad}(K)$. More strongly, under our assumption $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, we have

Proposition 7 (1) *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ satisfy $Y_0 \neq 0$. Then there is an element $k_0 \in K$ satisfying $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$ and $\mathrm{Ad}(k_0)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{a} + \mathfrak{m}_2$. Consequently, $\mathrm{Ad}(k_0)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_0$ in $\mathfrak{a} + \mathfrak{m}_2$, i.e.,*

$$\mathrm{Ad}(k_0)\mathfrak{m}_2 = \big\{Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y_0',\, Y_0) = 0\big\}. \tag{3.4}$$

(2) *Let $Y_1 \in \mathfrak{m}_1$ satisfy $Y_1 \neq 0$. Then there is an element $k_1 \in K$ satisfying $\mathrm{Ad}(k_1)\mu \in \boldsymbol{R}Y_1$ and $\mathrm{Ad}(k_1)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{m}_1$. Consequently, $\mathrm{Ad}(k_1)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_1$ in $\mathfrak{m}_1$, i.e.,*

$$\mathrm{Ad}(k_1)\mathfrak{m}_2 = \big\{Y_1' \in \mathfrak{m}_1 \mid (Y_1',\, Y_1) = 0\big\}. \tag{3.5}$$

Under the same setting in Proposition 7 (2), we have

Proposition 8 *Let $Y_1 \in \mathfrak{m}_1$ satisfy $Y_1 \neq 0$. Then there is an element $k_1' \in K$ satisfying*

$$\mathrm{Ad}(k_1')\mu = \frac{1}{\sqrt{2}}\left\{\mu + \frac{|\mu|}{|Y_1|}Y_1\right\}, \tag{3.6}$$

$$\mathrm{Ad}(k_1')Y_2 = \frac{1}{\sqrt{2}}\left\{Y_2 + \frac{1}{|\mu|^3|Y_1|}[[\mu, Y_1], Y_2]\right\}, \quad \forall Y_2 \in \mathfrak{m}_2. \tag{3.7}$$

Here $|v|$ denotes the norm of $v \in \mathfrak{m}$, i.e., $|v| = (v, v)^{1/2}$.

The proofs of Proposition 7 and Proposition 8 will be given in §4.

Utilizing Propositions 5, 7 and 8 we first show the following:

Proposition 9 *Assume that $\dim \boldsymbol{N} = q(G/K) - \dim G/K$ and that there is a solution $\boldsymbol{\Psi}$ of the Gauss equation (2.2). Then there exist two vectors $\mathbf{A}$ and $\mathbf{B} \in \boldsymbol{N}$ satisfying*

$$\boldsymbol{\Psi}(Y_0, Y_0') = (Y_0, Y_0')\mathbf{A}, \qquad \forall Y_0, Y_0' \in \mathfrak{a} + \mathfrak{m}_2, \tag{3.8}$$

$$\boldsymbol{\Psi}(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}, \qquad \forall Y_1, Y_1' \in \mathfrak{m}_1, \tag{3.9}$$

$$\boldsymbol{\Psi}(Y_1, Y_2) = -\frac{1}{(\mu, \mu)^2}\boldsymbol{\Psi}(\mu, [[\mu, Y_1], Y_2]), \quad \forall Y_1 \in \mathfrak{m}_1, \forall Y_2 \in \mathfrak{m}_2. \tag{3.10}$$

Proof. First we prove

$$\boldsymbol{\Psi}(Y_0, Y_0') = 0, \quad \forall Y_0, Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \text{ satisfying } (Y_0, Y_0') = 0. \tag{3.11}$$

We may assume that $Y_0, Y_0' \neq 0$. Then, by Proposition 7 (1), we know that there is an element $k_0 \in K$ satisfying $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$. Since $(Y_0, Y_0') = 0$, we have $Y_0' \in \mathrm{Ad}(k_0)\mathfrak{m}_2$. Then, by Proposition 5, we know $Y_0' \in \mathbf{Ker}(\boldsymbol{\Psi}_{Y_0})$. Hence $\boldsymbol{\Psi}(Y_0, Y_0') = 0$, completing the proof of (3.11).

Now (3.8) can be proved by (3.11) as follows: Let Y_0 and Y_0' be two elements of $\mathfrak{a} + \mathfrak{m}_2$ of the same length. Since $(Y_0 + Y_0', Y_0 - Y_0') = 0$, we obtain $\boldsymbol{\Psi}(Y_0 + Y_0', Y_0 - Y_0') = 0$. Hence, we have $\boldsymbol{\Psi}(Y_0, Y_0) = \boldsymbol{\Psi}(Y_0', Y_0')$. This implies that $\boldsymbol{\Psi}(Y_0, Y_0)/(Y_0, Y_0)$ $(Y_0 \in \mathfrak{a} + \mathfrak{m}_2, Y_0 \neq 0)$ takes a constant value $\mathbf{A}$ $(\in \boldsymbol{N})$. Therefore, we have $\boldsymbol{\Psi}(Y_0, Y_0) = (Y_0, Y_0)\mathbf{A}$ for any $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Now (3.8) follows immediately from this equality.

In a similar manner, by applying Proposition 7 (2) we can prove (3.9).

Finally, we prove (3.10). Without loss of generality, we may assume that $Y_1 \neq 0$. Apply Proposition 8 to this Y_1 ($\in \mathfrak{m}_1$). Then there is an element $k_1' \in K$ satisfying (3.6) and (3.7). By (3.1) we have

$$\begin{aligned} 0 &= \mathbf{\Psi}\big(\mathrm{Ad}(k_1')\mu,\ \mathrm{Ad}(k_1')Y_2\big) \\ &= \frac{1}{2}\mathbf{\Psi}\Big(\mu + \frac{|\mu|}{|Y_1|}Y_1,\ Y_2 + \frac{1}{|\mu|^3|Y_1|}[[\mu,\ Y_1],\ Y_2]\Big). \end{aligned}$$

Note that $\big[[\mu,\ Y_1],\ Y_2\big] \in \mathfrak{m}_1$ (see Proposition 4 (1)) and $\big[[\mu,\ Y_2],\ Y_1\big] = 2\big[[\mu,\ Y_1],\ Y_2\big]$ (see Lemma 5.3 of [5]). Then, we have

$$\big(Y_1,\ [[\mu,\ Y_1],\ Y_2]\big) = \frac{1}{2}\big(Y_1,\ [[\mu,\ Y_2],\ Y_1]\big) = -\frac{1}{2}\big([Y_1,\ Y_1],\ [\mu,\ Y_2]\big) = 0.$$

Hence by (3.9) we have $\mathbf{\Psi}\big(Y_1,\ [[\mu,\ Y_1],\ Y_2]\big) = 0$. This together with $\mathbf{\Psi}(\mu,\ Y_2) = 0$ proves (3.10). □

To calculate the left hand side of the Gauss equation (2.2), we prepare one more proposition, which will be proved in the last section of this paper.

Proposition 10 (1) *Let $Y_0,\ Y_0' \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Then:*

$$\big[Y_0,\ [Y_0,\ Y_1]\big] = -(\mu,\ \mu)(Y_0,\ Y_0)Y_1, \tag{3.12}$$

$$\big[Y_0,\ [Y_0,\ Y_0']\big] = \begin{cases} -4(\mu,\ \mu)(Y_0,\ Y_0)Y_0', & \text{if } (Y_0,\ Y_0') = 0, \\ 0, & \text{if } Y_0' \in \boldsymbol{R}Y_0. \end{cases} \tag{3.13}$$

(2) *Let $Y_1,\ Y_1' \in \mathfrak{m}_1$ and $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Then:*

$$\big[Y_1,\ [Y_1,\ Y_1']\big] = \begin{cases} -4(\mu,\ \mu)(Y_1,\ Y_1)Y_1', & \text{if } (Y_1,\ Y_1') = 0, \\ 0, & \text{if } Y_1' \in \boldsymbol{R}Y_1, \end{cases} \tag{3.14}$$

$$\big[Y_1,\ [Y_1,\ Y_0]\big] = -(\mu,\ \mu)(Y_1,\ Y_1)Y_0. \tag{3.15}$$

With these preparations, we start the proof of Theorem 3. We first show a series of lemmas by using the Gauss equation (2.2) and Proposition 9.

Lemma 11 $\langle \mathbf{A},\ \mathbf{A}\rangle = \langle \mathbf{B},\ \mathbf{B}\rangle = 4(\mu,\ \mu)$.

Proof. Take an element $Y_2 \in \mathfrak{m}_2$ satisfying $(Y_2,\ Y_2) = 1$. Put $X = Z = \mu$ and $Y = W = Y_2$ into the Gauss equation (2.2). Then, since $\mathbf{\Psi}(\mu,\ Y_2) = 0$, we have

$$\big([[\mu,\ Y_2],\ \mu],\ Y_2\big) = \big\langle \mathbf{\Psi}(\mu,\ \mu),\ \mathbf{\Psi}(Y_2,\ Y_2)\big\rangle.$$

Since $\mathbf{\Psi}(\mu,\ \mu)/(\mu,\ \mu) = \mathbf{\Psi}(Y_2,\ Y_2) = \mathbf{A}$ and $\big([[\mu,\ Y_2],\ \mu],\ Y_2\big) = 4(\mu,\ \mu)^2$, we have $\langle \mathbf{A}, \mathbf{A} \rangle = 4(\mu, \mu)$.

Next, we prove $\langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$. Take elements Y_1, Y_1' of $\mathfrak{m}_1$ satisfying $(Y_1, Y_1) = (Y_1', Y_1') = 1$ and $(Y_1, Y_1') = 0$. Put $X = Z = Y_1$ and $Y = W = Y_1'$ into (2.2). Then, since $\boldsymbol{\Psi}(Y_1, Y_1') = 0$, we have

$$\big([[Y_1, Y_1'], Y_1], Y_1'\big) = \big\langle \boldsymbol{\Psi}(Y_1, Y_1), \boldsymbol{\Psi}(Y_1', Y_1')\big\rangle.$$

Since $\boldsymbol{\Psi}(Y_1, Y_1) = \boldsymbol{\Psi}(Y_1', Y_1') = \mathbf{B}$ and $\big[[Y_1, Y_1'], Y_1\big] = 4(\mu, \mu)Y_1'$ (see (3.14)), we have $\langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$. □

Lemma 12 $\big\langle \mathbf{A}, \boldsymbol{\Psi}_\mu(\mathfrak{m}_1)\big\rangle = \big\langle \mathbf{B}, \boldsymbol{\Psi}_\mu(\mathfrak{m}_1)\big\rangle = 0$.

Proof. Let Y_1 be an arbitrary element of $\mathfrak{m}_1$. Take an element $Y_2 \in \mathfrak{m}_2$ satisfying $(Y_2, Y_2) = 1$. Put $X = Z = Y_2$, $Y = \mu$ and $W = Y_1$ into (2.2). Then, since $\boldsymbol{\Psi}(\mu, Y_2) = 0$, we have

$$\big([[Y_2, \mu], Y_2], Y_1\big) = \big\langle \boldsymbol{\Psi}(Y_2, Y_2), \boldsymbol{\Psi}(\mu, Y_1)\big\rangle.$$

Since $\boldsymbol{\Psi}(Y_2, Y_2) = \mathbf{A}$ and $\big[[Y_2, \mu], Y_2\big] = 4(\mu, \mu)\mu$ (see (3.13)), we have $\big\langle \mathbf{A}, \boldsymbol{\Psi}(\mu, Y_1)\big\rangle = 4(\mu, \mu)(\mu, Y_1) = 0$. Since Y_1 is an arbitrary element of $\mathfrak{m}_1$, we have $\big\langle \mathbf{A}, \boldsymbol{\Psi}_\mu(\mathfrak{m}_1)\big\rangle = 0$.

Next, let Y_1 be an arbitrary element of $\mathfrak{m}_1$. Take an element $Y_1' \in \mathfrak{m}_1$ satisfying $(Y_1', Y_1) = 0$ and $(Y_1', Y_1') = 1$. Put $X = Z = Y_1'$, $Y = \mu$ and $W = Y_1$ into (2.2). Then, since $\boldsymbol{\Psi}(Y_1, Y_1') = 0$, we have

$$\big([[Y_1', \mu], Y_1'], Y_1\big) = \big\langle \boldsymbol{\Psi}(Y_1', Y_1'), \boldsymbol{\Psi}(\mu, Y_1)\big\rangle.$$

& Since $\boldsymbol{\Psi}(Y_1', Y_1') = \mathbf{B}$ and $\big[[Y_1', \mu], Y_1'\big] = (\mu, \mu)\mu$ (see (3.15)), we have $\big\langle \mathbf{B}, \boldsymbol{\Psi}(\mu, Y_1)\big\rangle = (\mu, \mu)(\mu, Y_1) = 0$. Since Y_1 is an arbitrary element of $\mathfrak{m}_1$, we have $\big\langle \mathbf{B}, \boldsymbol{\Psi}_\mu(\mathfrak{m}_1)\big\rangle = 0$. □

Viewing Proposition 4 (3), we have $\dim \boldsymbol{N} = \dim \mathfrak{m}_1 + 1$. Since $\mathbf{Ker}(\boldsymbol{\Psi}_\mu) \cap \mathfrak{m}_1 = \mathfrak{m}_2 \cap \mathfrak{m}_1 = 0$, we have $\dim \boldsymbol{\Psi}_\mu(\mathfrak{m}_1) = \dim \mathfrak{m}_1 = \dim \boldsymbol{N} - 1$. Consequently, by Lemma 12 and Lemma 11, we easily have $\mathbf{B} = \pm\mathbf{A}$. More strongly, we can show

Lemma 13 $\mathbf{A} = \mathbf{B}$.

Proof. By the above discussion, it suffices to prove $\langle \mathbf{A}, \mathbf{B} \rangle > 0$. Let $Y_1 \in \mathfrak{m}_1$ satisfy $(Y_1, Y_1) = 1$. In (2.2), we put $X = Z = \mu$ and $Y = W = Y_1$. Then, we have

$$\big([[\mu, Y_1], \mu], Y_1\big) = \big\langle \boldsymbol{\Psi}(\mu, \mu), \boldsymbol{\Psi}(Y_1, Y_1)\big\rangle - \big\langle \boldsymbol{\Psi}(\mu, Y_1), \boldsymbol{\Psi}(Y_1, \mu)\big\rangle.$$

Since $\mathbf{\Psi}(\mu,\,\mu) = (\mu,\,\mu)\mathbf{A}$, $\mathbf{\Psi}(Y_1,\,Y_1) = \mathbf{B}$ and $\big[[\mu,\,Y_1],\,\mu\big] = (\mu,\,\mu)^2 Y_1$, we have

$$(\mu,\,\mu)\langle \mathbf{A},\,\mathbf{B}\rangle = (\mu,\,\mu)^2(Y_1,\,Y_1) + \big\langle \mathbf{\Psi}(\mu,\,Y_1),\,\mathbf{\Psi}(\mu,\,Y_1)\big\rangle \geq (\mu,\,\mu)^2.$$

This proves $\langle \mathbf{A},\,\mathbf{B}\rangle > 0$. □

Utilizing Lemma 13, we have

Lemma 14 *Let $Y_1,\,Y_1' \in \mathfrak{m}_1$. Then*

$$\big\langle \mathbf{\Psi}(\mu,\,Y_1),\,\mathbf{\Psi}(\mu,\,Y_1')\big\rangle = 3(\mu,\,\mu)^2(Y_1,\,Y_1'). \tag{3.16}$$

Proof. Put $X = Z = \mu$, $Y = Y_1$ and $W = Y_1'$ into (2.2). Then we have

$$\big([[\mu,\,Y_1],\,\mu],\,Y_1'\big) = \big\langle \mathbf{\Psi}(\mu,\,\mu),\,\mathbf{\Psi}(Y_1,\,Y_1')\big\rangle - \big\langle \mathbf{\Psi}(\mu,\,Y_1'),\,\mathbf{\Psi}(Y_1,\,\mu)\big\rangle.$$

Since $\mathbf{\Psi}(\mu,\,\mu) = (\mu,\,\mu)\mathbf{A}$, $\mathbf{\Psi}(Y_1,\,Y_1') = (Y_1,\,Y_1')\mathbf{B}$ and $\mathbf{A} = \mathbf{B}$, the first term of the right hand side becomes $\big\langle \mathbf{\Psi}(\mu,\,\mu),\,\mathbf{\Psi}(Y_1,\,Y_1')\big\rangle = 4(\mu,\,\mu)^2(Y_1,\,Y_1')$ (see Lemma 11). Therefore, by $\big[[\mu,\,Y_1],\,\mu\big] = (\mu,\,\mu)^2 Y_1$, we have

$$\begin{aligned}\big\langle \mathbf{\Psi}(\mu,\,Y_1),\,\mathbf{\Psi}(\mu,\,Y_1')\big\rangle &= 4(\mu,\,\mu)^2(Y_1,\,Y_1') - (\mu,\,\mu)^2(Y_1,\,Y_1')\\ &= 3(\mu,\,\mu)^2(Y_1,\,Y_1').\end{aligned}$$

□

We are now in a position to complete the proof of Theorem 3. Let $Y_1 \in \mathfrak{m}_1$ $(Y_1 \neq 0)$ and $Y_2 \in \mathfrak{m}_2$ $(Y_2 \neq 0)$. Note that $[Y_1,\,Y_2] \in \mathfrak{k}_1$ (see Proposition 4 (1)). We also note that $[Y_1,\,Y_2] \neq 0$. In fact, if $[Y_1,\,Y_2] = 0$, then the 2-dimensional subspace generated by Y_1 and Y_2 forms an abelian subspace of $\mathfrak{m}$, which contradicts $\mathrm{rank}(G/K) = 1$. Now, set $Y_1' = \big[[Y_1,\,Y_2],\,\mu\big]$. Then it is clear that $Y_1' \in \mathfrak{m}_1$ (see Proposition 4 (1)). Moreover, we have $Y_1' \neq 0$, because $[\mu,\,Y_1'] = (\mu,\,\mu)^2[Y_1,\,Y_2] \neq 0$.

Now, put $X = Y_1$, $Y = Y_2$, $Z = \mu$ and $W = Y_1'$ into (2.2). Since $\mathbf{\Psi}(Y_2,\,\mu) = 0$, we have

$$\big([[Y_1,\,Y_2],\,\mu],\,Y_1'\big) = \big\langle \mathbf{\Psi}(Y_1,\,\mu),\,\mathbf{\Psi}(Y_2,\,Y_1')\big\rangle. \tag{3.17}$$

By (3.10) and (3.16), the right hand side of (3.17) becomes

$$\begin{aligned}\big\langle \mathbf{\Psi}(Y_1,\,\mu),\,\mathbf{\Psi}(Y_2,\,Y_1')\big\rangle &= -\big\langle \mathbf{\Psi}(\mu,\,Y_1),\,\mathbf{\Psi}(\mu,\,[[\mu,\,Y_1'],\,Y_2])\big\rangle/(\mu,\,\mu)^2\\ &= -3\big(Y_1,\,[[\mu,\,Y_1'],\,Y_2]\big)\\ &= 3\big([Y_1,\,Y_2],\,[\mu,\,Y_1']\big)\end{aligned}$$

$$=3\big([[Y_1, Y_2], \mu], Y_1'\big).$$

Putting this equality into (3.17), we have $\big([[Y_1, Y_2], \mu], Y_1'\big) = 0$, which contradicts our assumption $\big([[Y_1, Y_2], \mu], Y_1'\big) = (Y_1', Y_1') \neq 0$.

As we have shown above, starting from the assumption that the Gauss equation (2.2) admits a solution $\boldsymbol{\Psi}$, we finally arrive at a contradiction. Accordingly, we can conclude that if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, then the Gauss equation (2.2) does not admit any solution in case $\dim \boldsymbol{N} = q(G/K) - \dim G/K$. This completes the proof of Theorem 3. □

4. The action of the isotropy group $\mathrm{Ad}(K)$

In this section we prove Propositions 7, 8 and 10, which are needed in the proof of Theorem 3.

Lemma 15 *Let $X_i \in \mathfrak{k}_i$ $(i = 1, 2)$. Then*

$$\big[X_i, [X_i, \mu]\big] = -i^2(\mu, \mu)(X_i, X_i)\mu. \tag{4.1}$$

Proof. By (2.5) we have $\big[X_i, [X_i, \mu]\big] \in \mathfrak{a} + \mathfrak{m}_{2i}$. By the Jacobi identity we have

$$\Big[\mu, \big[X_i, [X_i, \mu]\big]\Big] = [[\mu, X_i], [X_i, \mu]] + \Big[X_i, \big[[\mu, X_i], \mu\big]\Big] = 0,$$

because $\big[[\mu, X_i], \mu\big] \in \boldsymbol{R}X_i$. Therefore, we have $\big[X_i, [X_i, \mu]\big] \in \mathfrak{a}$. Since $\mathfrak{a} = \boldsymbol{R}\mu$, there is a scalar $c \in \boldsymbol{R}$ satisfying $\big[X_i, [X_i, \mu]\big] = c\mu$. Then we have $c = -i^2(\mu, \mu)(X_i, X_i)$, because

$$\begin{aligned} c(\mu, \mu) &= \big(\big[X_i, [X_i, \mu]\big], \mu\big) = \big(X_i, \big[[X_i, \mu], \mu\big]\big) \\ &= -(i\mu, \mu)^2(X_i, X_i). \end{aligned}$$

□

By the above lemma, we obtain

Lemma 16 *Let $X_i \in \mathfrak{k}_i$ $(i = 1, 2)$ satisfy $X_i \neq 0$. Then*

$$\begin{aligned} \mathrm{Ad}\big(\exp(tX_i)\big)\mu = {} & \cos(i|\mu||X_i|t)\mu \\ & + \frac{\sin(i|\mu||X_i|t)}{i|\mu||X_i|}[X_i, \mu], \quad \forall t \in \boldsymbol{R}. \end{aligned} \tag{4.2}$$

Proof. Let n be a non-negative integer. By induction of n, we can easily show

$$(\operatorname{ad} X_i)^{2n}\mu = (-1)^n(i|\mu||X_i|)^{2n}\mu,$$
$$(\operatorname{ad} X_i)^{2n+1}\mu = (-1)^n(i|\mu||X_i|)^{2n}[X_i,\, \mu].$$

Consequently, for all $t \in \boldsymbol{R}$ we have

$$\begin{aligned}\operatorname{Ad}(\exp(tX_i))\mu &= \sum_{n=0}^{\infty}\left\{\frac{t^{2n}}{(2n)!}(\operatorname{ad} X_i)^{2n}\mu + \frac{t^{2n+1}}{(2n+1)!}(\operatorname{ad} X_i)^{2n+1}\mu\right\}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(i|\mu||X_i|t)^{2n}\mu\\ &\quad + \frac{1}{i|\mu||X_i|}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(i|\mu||X_i|t)^{2n+1}[X_i,\, \mu]\\ &= \cos(i|\mu||X_i|t)\mu + \frac{\sin(i|\mu||X_i|t)}{i|\mu||X_i|}[X_i,\, \mu].\end{aligned}$$

□

With these preparations, we proceed to the proof of Proposition 7. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. If $Y_0 \in \mathfrak{a}$, then we have only to set $k_0 = e$, where e is the identity element of K.

Now we assume that $Y_0 \notin \mathfrak{a}$ and write $Y_0 = c\mu + Y_2$ ($c \in \boldsymbol{R}$, $Y_2 \in \mathfrak{m}_2$, $Y_2 \neq 0$). Set $X_2 = [Y_0,\, \mu]$. Then we easily have $X_2 = [Y_2,\, \mu] \in \mathfrak{k}_2$ and $[X_2,\, \mu] = -4(\mu,\, \mu)^2 Y_2$. Moreover, we have $|X_2| = 2|\mu|^2|Y_2|$, because

$$(X_2,\, X_2) = \big([Y_2,\, \mu],\, [Y_2,\, \mu]\big) = -\big(\big[[Y_2,\, \mu],\, \mu\big],\, Y_2\big) = 4(\mu,\, \mu)^2(Y_2,\, Y_2).$$

Putting this X_2 into Lemma 16, we have

$$\operatorname{Ad}(\exp(tX_2))\mu = \cos(4|\mu|^3|Y_2|t)\mu - \frac{|\mu|}{|Y_2|}\sin(4|\mu|^3|Y_2|t)Y_2, \quad \forall t \in \boldsymbol{R}.$$

Take $t_0 \in \boldsymbol{R}$ satisfying $\cos(4|\mu|^3|Y_2|t_0) = c(|\mu|/|Y_0|)$ and $\sin(4|\mu|^3|Y_2|t_0) = -|Y_2|/|Y_0|$. Let us set $k_0 = \exp(t_0 X_2)$. Then we have $k_0 \in K$ and

$$\operatorname{Ad}(k_0)\mu = \operatorname{Ad}(\exp(t_0 X_2))\mu = \frac{|\mu|}{|Y_0|}(c\mu + Y_2) = \frac{|\mu|}{|Y_0|}Y_0.$$

Thus we get $\operatorname{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$. By (2.5) we immediately have $[X_2,\, \mathfrak{a} + \mathfrak{m}_2] \subset$

$\mathfrak{a}+\mathfrak{m}_2$. Hence, we have $\mathrm{Ad}(k_0)(\mathfrak{a}+\mathfrak{m}_2)=\mathfrak{a}+\mathfrak{m}_2$. Since $\mathrm{Ad}(k_0)$ is an orthogonal transformation of $\mathfrak{m}$, we know that $\mathrm{Ad}(k_0)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_0$ in $\mathfrak{a}+\mathfrak{m}_2$. This finishes the proof of Proposition 7 (1).

To prove Proposition 7 (2), we first show

Lemma 17 *Let* $X_1 \in \mathfrak{k}_1$. *Then*

$$\big[X_1,\,[X_1,\,Y_2]\big]=-(\mu,\,\mu)(X_1,\,X_1)Y_2, \quad \forall Y_2 \in \mathfrak{m}_2.$$

Proof. By (4.1), we have

$$\big[X_1,\,[X_1,\,\mu]\big]=-(\mu,\,\mu)(X_1,\,X_1)\mu. \tag{4.3}$$

Let Y_2 be a non-zero element of $\mathfrak{m}_2$. Then, as in the proof of Proposition 7 (1), we know that there is a scalar $t_0 \in \boldsymbol{R}$ such that the element $k_0=\exp(t_0X_2) \in K$ satisfies $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_2$, where we set $X_2=[Y_2,\,\mu] \in \mathfrak{k}_2$. Then, we have $\mathrm{Ad}(k_0)\mathfrak{k}_1=\mathfrak{k}_1$, because $[X_2,\,\mathfrak{k}_1] \subset \mathfrak{k}_1$ (see Proposition 4 (1)).

Now, applying $\mathrm{Ad}(k_0)$ to the both sides of (4.3), we have

$$\begin{aligned}\big[\mathrm{Ad}(k_0)X_1,\,[\mathrm{Ad}(k_0)X_1,\,Y_2]\big]&=-(\mu,\,\mu)(X_1,\,X_1)Y_2\\&=-(\mu,\,\mu)(\mathrm{Ad}(k_0)X_1,\,\mathrm{Ad}(k_0)X_1)Y_2.\end{aligned}$$

Writing X_1 instead of $\mathrm{Ad}(k_0)X_1 \in \mathfrak{k}_1$, we get the lemma. □

Now we return to the proof of Proposition 7 (2). Set $X_1=[Y_1,\,\mu]$. In the same way as in the proof of (1), we can easily prove $X_1 \in \mathfrak{k}_1$, $[X_1,\,\mu]=-(\mu,\,\mu)^2Y_1$ and $|X_1|=|\mu|^2|Y_1|$. Applying Lemma 16 to this X_1, we have

$$\begin{aligned}\mathrm{Ad}(\exp(tX_1))\mu&=\cos(|\mu|^3|Y_1|t)\mu\\&\quad-\frac{|\mu|}{|Y_1|}\sin(|\mu|^3|Y_1|t)Y_1, \quad \forall t \in \boldsymbol{R}.\end{aligned} \tag{4.4}$$

Let $Y_2 \in \mathfrak{m}_2$. By Lemma 17, we have

$$\begin{aligned}(\mathrm{ad}\,X_1)^{2n}Y_2&=(-1)^n(|\mu||X_1|)^{2n}Y_2,\\(\mathrm{ad}\,X_1)^{2n+1}Y_2&=(-1)^n(|\mu||X_1|)^{2n}[X_1,\,Y_2].\end{aligned}$$

From these equalities, it follows

$$\begin{aligned}\mathrm{Ad}\big(\exp(tX_1)\big)Y_2&=\cos(|\mu|^3|Y_1|t)Y_2\\&\quad+\frac{\sin(|\mu|^3|Y_1|t)}{|\mu|^3|Y_1|}\big[[Y_1,\,\mu],\,Y_2\big], \quad \forall t \in \boldsymbol{R}.\end{aligned} \tag{4.5}$$

Let us take $t_1 \in \boldsymbol{R}$ satisfying $|\mu|^3|Y_1|t_1 = -\pi/2$ and set $k_1 = \exp(t_1 X_1)$. Then we can easily show that $k_1 \in K$, $\mathrm{Ad}(k_1)\mu = (|\mu|/|Y_1|)Y_1 \in \mathfrak{m}_1$ and

$$\mathrm{Ad}(k_1)Y_2 = -\frac{1}{|\mu|^3|Y_1|}\big[[Y_1, \mu], Y_2\big]. \tag{4.6}$$

Hence, we have $\mathrm{Ad}(k_1)\mu \in \boldsymbol{R}Y_1$ and $\mathrm{Ad}(k_1)\mathfrak{m}_2 \subset \big[[Y_1, \mu], \mathfrak{m}_2\big]$. Since $\big[[Y_1, \mu], \mathfrak{m}_2\big] \subset \mathfrak{m}_1$ (see Proposition 4 (1)), we have $\mathrm{Ad}(k_1)(\mathfrak{a} + \mathfrak{m}_2) \subset \mathfrak{m}_1$. Therefore, we have $\mathrm{Ad}(k_1)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{m}_1$, because $\dim(\mathfrak{a} + \mathfrak{m}_2) = \dim \mathfrak{m}_1$ (see Proposition 4 (3)). Since $\mathrm{Ad}(k_1)$ is an orthogonal transformation of $\mathfrak{m}$, we know that $\mathrm{Ad}(k_1)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_1$ in $\mathfrak{m}_1$. This completes the proof of Proposition 7 (2). □

Next we prove Proposition 8. Under the same situation as in the proof of Proposition 7 (2), let us set $k_1' = \exp(t_1 X_1/2)$. Then by the equalities (4.4) and (4.5) we easily obtain (3.6) and (3.7). □

Finally, we prove Proposition 10. First we show Proposition 10 (1). If $Y_0 \in \mathfrak{a}$, then there is nothing to prove. Hence we may assume that $Y_0 \notin \mathfrak{a}$. Applying Proposition 7 (1), we have an element $k_0 \in K$ satisfying $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$ and $\mathrm{Ad}(k_0)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{a} + \mathfrak{m}_2$. Then, it is easily seen that $\mathrm{Ad}(k_0)\mathfrak{m}_1 = \mathfrak{m}_1$. If we write $\mathrm{Ad}(k_0)\mu = cY_0$ ($c \in \boldsymbol{R}$), then we have $c^2 = (\mu, \mu)/(Y_0, Y_0)$. Let Y_i be an element of $\mathfrak{m}_i$ ($i = 1, 2$). Apply $\mathrm{Ad}(k_0)$ to the both sides of the equality $\big[\mu, [\mu, Y_i]\big] = -i^2(\mu, \mu)^2 Y_i$ ($i = 1, 2$). Then, since $c^2 = (\mu, \mu)/(Y_0, Y_0)$, we have

$$\big[Y_0, [Y_0, \mathrm{Ad}(k_0)Y_i]\big] = -i^2(\mu, \mu)(Y_0, Y_0)\,\mathrm{Ad}(k_0)Y_i, \quad i = 1, 2.$$

Now, (3.12) and (3.13) follow immediately from the above equality. (Note the equality (3.4) and the fact $\mathrm{Ad}(k_0)\mathfrak{m}_1 = \mathfrak{m}_1$.)

By applying Proposition 7 (2), Proposition 10 (2) can be also shown in a similar manner. Details are left to the readers. □

Thus, we have completed the proofs of Propositions 7, 8 and 10.

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