Hokkaido Mathematical Journal Vol. 33 (2004) p. 83–113

Decay estimates for hyperbolic systems

Vladimir GEORGIEV, Sandra LUCENTE and Guido ZILIOTTI

(Received January 28, 2002; Revised October 21, 2002)

Abstract. In this work we study the Sobolev spaces generated by pseudo-differential operators associated with the group of symmetry of general first order hyperbolic systems. In these spaces we establish pointwise estimates of the solutions of a class of first order systems having convex eigenvalues. Various physical models belong to this class. For example, we consider crystal optics systems and anisotropic elasticity equations.

Key words: first order hyperbolic systems, generalized Sobolev spaces, a priori estimates for wave type equations.

1. Introduction

The aim of this work is to study the Sobolev spaces generated by pseudo-differential operators associated with the group of symmetry of general first order hyperbolic systems. More precisely we consider the following first order system

$$\partial_t u - \sum_{j=1}^{n} A_j \partial_j u = F, \tag{1.1}$$

where $u, F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^N$, and A_j are constant $(N \times N)$ matrices. For the principal symbol

$$A(\xi) = \sum_j A_j \xi_j,$$

we assume that

$$\begin{cases} A_j^* = A_j \quad \text{for } \ j = 1, \dots, n; \\ A(\xi) \ \text{ has real eigenvalues of constant multiplicity} \end{cases} \tag{H_1}$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$.

Here and below A^* denotes the complex conjugate to the matrix A.

2000 Mathematics Subject Classification : 35L40, 35B45.

It is easy to see that the above assumption implies that the number of different non-zero eigenvalues is even; in particular these eigenvalues can be ordered as follows

$$\lambda_{-K}(\xi) < \cdots < \lambda_{-1}(\xi) < \lambda_1(\xi) < \cdots < \lambda_K(\xi).$$

These eigenvalues are homogeneous functions of degree 1 in ξ. Moreover, we require that

$$\text{the surface } \Sigma_{\lambda_k} := \{\lambda_k(\xi) = 1\}, \ \text{ is strictly convex} \qquad (\mathrm{H}_2)$$

for each non-zero eigenvalue λ_k, with $k = 1, \ldots, K$.

This class of systems includes, for example

- Dirac-Pauli system;
- Maxwell system for uniaxial crystals;
- Wave equations with different speeds of propagation;
- Equations of elasticity in some anisotropic media.

The natural group of translations in the space and time leaves the solutions of (1.1) invariant for $F = 0$. This group generates the classical Sobolev spaces $H^m(\mathbb{R}^n)$ and the corresponding Banach spaces $\bigcap_{j=0}^m C^{m-j}(I; H^j(\mathbb{R}^n))$ for any integer $m \geq 0$ and for any time interval $I \subseteq \mathbb{R}$. The classical L^2 estimate for (1.1) gives the boundedness of the map

$$F \in \bigcap_{j=0}^{m} C^{m-j}([0,T]; H^j(\mathbb{R}^n)) \to u \in \bigcap_{j=0}^{m} C^{m-j}([0,T]; H^j(\mathbb{R}^n))$$

provided $T > 0$ and u satisfies the zero initial condition $u(x,0) = 0$. A Lie algebra of pseudo-differential operators Γ_i satisfying

$$\Big[\Gamma_i, \partial_t - \sum_j A_j \partial_{x_j}\Big] = c_i\Big(\partial_t - \sum_j A_j \partial_{x_j}\Big),$$

has been introduced in [2]. Here $[A, B]$ denotes commutator of the operators A, B.

The main goal of this work is to obtain a decay estimate of the L^∞ norm of the solution of the form

$$|u(x,t)| \leq C(t + |x|)^{-\frac{n-1}{2}} \sup_{0 \leq s \leq t} (1+s)^a \sum_{|\alpha| \leq M} \|\Gamma^\alpha F(\cdot, s)\|_2. \qquad (1.2)$$

It is well-known (see [6]) that such estimate in combination with energy type estimates can be used to obtain global existence results for small data solutions of nonlinear systems. In fact, using the estimate of Corollary 5.2 (see Section 5 below) one obtains the above estimate with $a = (n-1)/2$, if $n \geq 4$ (similar result for the wave equation is obtained in [5]). If the nonlinearity is $F(u)$ is quadratic in u and the space dimension is $n \geq 4$, then by means of the above estimate one can look for a global solution provided the initial data are small enough. If the space dimension is $n = 3$, then $a = 1 + \varepsilon$, where ε is arbitrary positive number (a similar estimate for the wave equation is true, see [5]).

In order to establish L^2-L^∞ estimate, we reduce the system to n pseudo-differential equations. For each of these equations we apply the stationary phase method and get a suitable dispersive inequality. The proof is contained in Section 3. Section 4 is the main part of this paper: here we show how to associate generalized Sobolev spaces to first order systems. In Section 5 we obtain estimates of type (1.2) for the solution of the system. In the last section we treat the examples mentioned above. A particular attention is devoted to some elliptic conditions which arise in the case of vanishing eigenvalues.

Finally, we shall mention some previous results. In [11] we have established L^1-L^∞ estimate for the solutions of homogeneous systems using a reduction to pseudo-differential equations. In this case the initial data appear in classical Sobolev norm. Instead here we treat the inhomogeneous estimates and we use modified Sobolev spaces generate by $\{\Gamma_i\}$.

The relevance of the convexity of Σ_{λ_k} is discussed by Liess in [9] where an important counter-example of a non-convex surface Σ_{λ_k} is studied and a loss of decay rate appears. In this direction Sugimoto, in [14], studied some generalizations of L^p-$L^{p'}$ estimates for higher order hyperbolic equations such that Σ_{λ_k} is convex, but not strictly convex. The order of the contact between the surface and its tangent hyperplane plays a crucial role in that paper.

For the case of non-elliptic $\sum_j A_j \partial_j$, *i.e.* when $\ker A(\xi) \neq \{0\}$, we refer to [12], where the ellipticity assumption is relaxed to suitable elliptic complex assumptions.

Notations The inner product in $\mathbb{R}^n$ will be denoted by $\langle \cdot, \cdot \rangle$.
Our choice for the coefficients of the Fourier transform is

$$\hat{f}(\xi) = \mathcal{F}(f) = \int \mathrm{e}^{-\mathrm{i}\langle \xi, y \rangle} f(y)\, \mathrm{d}y.$$

If f depends also on $t \in \mathbb{R}$, then we use Fourier transform with respect to the space variable only.

Let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. The support of the function $x \to F(x,t)$, for fixed $t \in \mathbb{R}$, is denoted by $\operatorname{supp}_x F(x,t)$.

We omit to write $\mathbb{R}^n$ if it is a domain of a function space; we also denote by $\|\cdot\|_p$ the $L^p(\mathbb{R}^n)$-norm.

We indicate with $p(x,D)$ a pseudo-differential operator with symbol $p(x,\xi)$:

$$P(x,D)(f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} p(x,\xi)\hat{f}(\xi)d\xi$$

for any function $f \in S(\mathbb{R}^n)$. In particular $p(D)$ is called convolution type operator.

Finally $A(x) \sim B(x)$ means that there exists a constant $C > 0$, independent of x, such that $C^{-1}B \leq A \leq CB$.

2. Generalized Sobolev spaces on Σ_λ

Consider the following Cauchy problem

$$\begin{cases} u_t - \mathrm{i}\lambda(D)u = F & x \in \mathbb{R}^n \\ u(x,0) = 0 \end{cases} \tag{2.1}$$

where $\lambda(D)$ is a pseudo-differential operator with symbol $\lambda(\xi)$ homogeneous of order one. We suppose that $\lambda : \mathbb{R}^n \to \mathbb{R}$ is positive, homogeneous of degree one. Moreover let

$$\Sigma_\lambda := \{\lambda(\xi) = 1\}$$

be the unit surface associated with this symbol.

Our first step is to find a family of operators Y that satisfy the commutator relations

$$[Y, P] = cP, \tag{2.2}$$

where $P = \partial_t - \mathrm{i}\lambda(D)$ and $c = c(Y)$ is a real constant. It is clear that $\partial_t, \partial_{x_j}$, $j = 1, \ldots, n$ are $n+1$ vector fields that commute with P.

For the case of $\lambda(\xi) = |\xi|$ one can use the generators

$$x_j\partial_{x_k} - x_k\partial_{x_j}$$

of $SO(n)$, so that these generators commute with $|D|$. In order to consider the general case of $\lambda(\xi)$ homogeneous of degree 1, following [2], we consider a pseudo-differential operators $\Omega_{j,k}(\lambda)$ with symbol

$$-\mathrm{i}/2(x_k\partial_j\lambda^2(\xi) - x_j\partial_k\lambda^2(\xi)). \tag{2.3}$$

Thus, we have

$$\begin{aligned} &\Omega_{j,k}(\lambda)f(x) \\ &= -\mathrm{i}/2(2\pi)^{-n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\left(x_k\partial_{\xi_j}\lambda^2(\xi) - x_j\partial_{\xi_k}\lambda^2(\xi)\right)\hat{f}(\xi)\,\mathrm{d}\xi \end{aligned} \tag{2.4}$$

and also

$$\Omega_{j,k}(\lambda) = -\mathrm{i}/2\left(x_k\partial_j\lambda^2(D) - x_j\partial_k\lambda^2(D)\right).$$

We see that these operators commute with $\lambda(D)$.

Lemma 2.1 *With the above notations, one has:*

$$[\lambda(D), \Omega_{j,k}(\lambda)] = 0.$$

Proof. Combining (2.4), the algebraic rule $[A, BC] = [A, B]C + B[A, C]$ and the fact that two operators of convolution type commute, we check

$$[\lambda(D), \Omega_{j,k}(\lambda)] = -\frac{\mathrm{i}}{2}[\lambda(D), x_k]\partial_j\lambda^2(D) + \frac{\mathrm{i}}{2}\partial_k\lambda^2(D)[\lambda(D), x_j].$$

Since $(x_jf)^\wedge = \mathrm{i}\partial_j\hat{f}$, integrating by parts, we get

$$[a(D), x_j] = -\mathrm{i}(\partial_j a)(D) \tag{2.5}$$

for any convolution type operator $a(D)$. Whence,

$$[\lambda(D), \Omega_{j,k}(\lambda)] = -\frac{1}{2}\partial_j\lambda^2(D)(\partial_k\lambda)(D) + \frac{1}{2}\partial_k\lambda^2(D)(\partial_j\lambda)(D) = 0.$$

This gives the conclusion. □

Let us denote by S the scaling operator

$$S = t\partial_t + \sum_j x_j\partial_{x_j} = t\partial_t + \langle x, \nabla_x\rangle. \tag{2.6}$$

Note that $[\lambda(D), \partial_t] = [\lambda(D), \partial_{x_i}] = 0$; on the contrary, from (2.5) it follows

$$[\lambda(D), S] = -\mathrm{i}\langle\nabla\lambda(D), \nabla_x\rangle = \lambda(D);$$

then $[\partial_t - \mathrm{i}\lambda(D), S] = \partial_t - \mathrm{i}\lambda(D)$. This relation is a natural analogue of the well-known property for wave equation: $[\partial_{tt} - \Delta, S] = 2(\partial_{tt} - \Delta)$.

Next step is to introduce the Sobolev spaces associated with the generators

$$\{\partial_{x_j}, \partial_t, S, \Omega_{j,k}(\lambda)\}_{j,k=1,\dots,n,\ j<k}. \tag{2.7}$$

For brevity these generators shall be denoted by $Y_1, \dots, Y_N$. Here $N = (n^2+n+4)/2$. Thus, Y_j are pseudo-differential operators and they generate a Lie algebra if $[Y_l, Y_m] = \sum_{r=1}^N c_{l,m}^r Y_r$. It is possible to define a Sobolev space associated with these generators whenever the structural coefficients c_{jk}^m satisfy suitable a-priori estimates. In our specific case we shall see that these coefficients are zero order homogeneous pseudo-differential operators of convolution type.

Lemma 2.2 *Let Y_l, Y_m be two elements of* (2.7). *Then*

$$[Y_l, Y_m] = \sum_{r=1}^N c_{l,m}^r(D) Y_r,$$

where $c_{l,m}^r(D)$ is a pseudo-differential operator of convolution type with symbol homogeneous of degree 0.

Proof. It is evident that $[\partial_{x_i}, \partial_t] = [\partial_{x_i}, \partial_{x_k}] = [\Omega_{j,k}(\lambda), \partial_t] = 0$, while $[S, \partial_t] = -\partial_t$, $[S, \partial_{x_j}] = -\partial_{x_j}$. We recall that $\mathrm{i}a(D) = \langle \nabla a(D), \nabla \rangle$ for all a homogeneous of degree 1; we gain

$$\begin{aligned}[\Omega_{j,k}(\lambda), \partial_{x_r}] &= \mathrm{i}(\delta_{k,r}\partial_j\lambda(D) - \delta_{j,r}\partial_k\lambda(D))\lambda(D) \\ &= \mathrm{i}(\delta_{k,r}\partial_j\lambda(D) - \delta_{j,r}\partial_k\lambda(D))\Big(\sum_m \partial_m\lambda(D)\partial_m\Big).\end{aligned} \tag{2.8}$$

By using (2.5), we see that

$$[\Omega_{j,k}(\lambda), x_r] = -\frac{\mathrm{i}}{2}\left(\partial_{j,r}^2\lambda^2(D)x_k - \partial_{k,r}^2\lambda^2(D)x_j\right),$$

hence $[\Omega_{j,k}(\lambda), S] = 0$. It remains to consider the commutators of $\Omega_{j,k}(\lambda)$:

$$\begin{aligned}[\Omega_{j,k}(\lambda), \Omega_{l,m}(\lambda)] &= \partial_{k,l}^2\lambda^2(D)\Omega_{j,m}(\lambda) - \partial_{m,k}^2\lambda^2(D)\Omega_{j,l}(\lambda) \\ &\quad - \partial_{j,l}^2\lambda^2(D)\Omega_{k,m}(\lambda) + \partial_{j,m}^2\lambda^2(D)\Omega_{k,l}(\lambda).\end{aligned} \tag{2.9}$$

The last relation is obtained applying once more (2.5). □

This lemma assures that the operators (2.7) generate a Lie algebra $\mathcal{A}_\lambda$; one can consider the Sobolev norm associated to $\mathcal{A}_\lambda$:

$$\|f\|_{k,\mathcal{A}_\lambda} := \sum_{|\alpha|\le k} \|Y^\alpha f\|_{L^2(\mathbb{R}^n)}.$$

On the other hand usual Sobolev spaces on Σ_λ are defined by means of the norm

$$\|u\|^2_{H^s(\Sigma_\lambda)} \simeq \sum_{|\alpha|\le s} \|\tilde{\Omega}^\alpha(\lambda)u\|^2_{L^2(\Sigma_\lambda)} \quad s\in\mathbb{N}. \tag{2.10}$$

Here $\tilde{\Omega}_{jk}$ are the operators

$$\tilde{\Omega}_{jk} = (\partial_j\lambda^2)(D)\partial_{\xi_k} - (\partial_k\lambda^2)(D)\partial_{\xi_j}. \tag{2.11}$$

They generate the tangent space to Σ_λ. In term of symbols we see that

$$(\Omega_{j,k}(\lambda)f)^\wedge(\xi) = \frac{1}{2}\tilde{\Omega}_{j,k}(\lambda)\hat{f}(\xi). \tag{2.12}$$

In [11] we studied fractional version of these spaces and gave a proof of the following result based on stationary phase method.

Theorem 2.3 *Consider a smooth function $\lambda : \mathbb{R}^n\setminus\{0\} \to \mathbb{R}$ homogeneous of degree* 1*. Suppose $\Sigma_\lambda := \{\lambda(\xi) = 1\}$ is strictly convex. Let $s > (n-1)/2$, then*

$$\left|\int_{\Sigma_\lambda} e^{i\langle x,\omega\rangle} g(\omega)\,d\sigma_{n-1}(\omega)\right| \le C|x|^{-\frac{n-1}{2}}\|g\|_{H^s(\Sigma_\lambda)} \quad x\ne 0.$$

We remark that the assumption of strict convexity for Σ_λ means that $\{\lambda(x)\le 1\}$ is a convex and compact set whose boundary Σ_λ has strictly positive Gaussian curvature. If λ^k is smooth, for some $k\in\mathbb{N}$, then these surfaces are called *ovaloids* (see [4]). A basic example is the ellipsoid

$$\lambda(\xi) = \sum_{i=1}^n a_i^2\xi_i^2 \quad a_i\in\mathbb{R}.$$

In this case we have symmetry in ξ. This assumption appears in [11] Theorem 2.10, but a slight different proof shows that the result holds in non-symmetric case.

A relation between $H^s(\Sigma_\lambda)$ and the Sobolev spaces generated by the algebra $\mathcal{A}$ involves real power of $\lambda(D)$. Let $a(D)$ a convolution type operator; it is clear that for any $s \in \mathbb{N}$, $(a(D))^s$ has symbol $a^s(\xi)$; in particular

$$\|(a(D))^s f\|_2 \simeq \|a^s(\xi)\hat{f}(\xi)\|_2. \tag{2.13}$$

On the right side we can use complex interpolation and define $(a(D))^s$ for any real $s \geq 0$ in such a way that (2.13) holds. After this preparation one has the following.

Lemma 2.4 *For any integer $s \geq 0$, and any $\delta \geq 0$, it holds*

$$\int_0^{+\infty} \rho^{n-1+2\delta} \|\hat{g}(\rho\cdot)\|^2_{H^s(\Sigma_\lambda)} \, \mathrm{d}\rho \leq C \sum_{|\alpha| \leq s} \|\lambda(D)^\delta \Omega^\alpha(\lambda) g\|_2^2.$$

Proof. Using the explicit norm (2.10) it follows that

$$\begin{aligned}
&\int_0^{+\infty} \rho^{n-1+2\delta} \|\hat{g}(\rho\cdot)\|^2_{H^s(\Sigma_\lambda)} \, \mathrm{d}\rho \\
&\leq C \sum_{|\alpha| \leq s} \int_0^{+\infty} \rho^{n-1+2\delta} \int_{\Sigma_\lambda} |\tilde{\Omega}^\alpha(\lambda)\hat{g}(\rho\omega)|^2 \, \mathrm{d}\sigma_{n-1}(\omega) \, \mathrm{d}\rho \\
&\leq C \sum_{|\alpha| \leq s} \|\lambda(\xi)^\delta (\tilde{\Omega}^\alpha(\lambda)\hat{g})\|_2^2 \\
&\leq C \sum_{|\alpha| \leq s} \|\lambda(D)^\delta (\tilde{\Omega}^\alpha(\lambda)\hat{g})^\vee\|_2^2 \simeq \sum_{|\alpha| \leq s} \|\lambda(D)^\delta \Omega^\alpha(\lambda) g\|_2^2.
\end{aligned}$$

Here we use the relation (2.12). □

From this lemma we deduce the next result.

Corollary 2.5 *For any integer $s \geq 0$. For any $d, \delta \in \mathbb{R}$ which satisfies $0 < d+1 < \delta$, one has*

$$\int_0^{+\infty} \rho^{\frac{n}{2}+d} \|\hat{g}(\rho\cdot)\|_{H^s(\Sigma_\lambda)} \, \mathrm{d}\rho \leq C_{d,\delta,s} \sum_{|\alpha| \leq s} \|\Omega^\alpha(\lambda)\langle\lambda(D)\rangle^\delta g\|_{L^2(\mathbb{R}^n)}.$$

Proof. As usual, by Schwartz inequality we have

$$\int_0^{+\infty} \rho^{\frac{n}{2}+d} \|\hat{g}(\rho\cdot)\|_{H^s(\Sigma_\lambda)} \,\mathrm{d}\rho$$
$$\leq \left(\int_0^{\infty} \rho^{1+2d} \langle\rho\rangle^{-2\delta} \,\mathrm{d}\rho\right)^{1/2} \left(\int_0^{\infty} \rho^{n-1} \langle\rho\rangle^{2\delta} \|\hat{g}(\rho\cdot)\|^2_{H^s(\Sigma_\lambda)} \,\mathrm{d}\rho\right)^{1/2}.$$

The right side is finite if $0 < d+1 < \delta$. From the previous lemma we get the claim. □

3. L^2-L^∞ estimate for non homogeneous scalar equation

In this section we plan to evaluate the L^∞ norm of the solution of (2.1) by means of Sobolev norm of the data F.

Before stating the main result we shall justify a suitable localization of the inhomogeneous term F in the pseudo-differential equation $u_t - \mathrm{i}\lambda(D)u = F$.

Lemma 3.1 *Let $t \geq 1$ and λ homogeneous of degree 1. Set*

$$C_0 = \max_{|\omega|=1} |\nabla\lambda(\omega)|.$$

Assume that

$$\operatorname{supp}_y F(y,s) \subseteq \{|y| \geq 3C_0 t\}, \quad 0 \leq s \leq t. \tag{3.1}$$

Then, for $|x| \leq C_0 t$, the solution of $u_t - \mathrm{i}\lambda(D)u = F$ having zero initial data satisfies

$$|u(x,t)| \leq C(t+|x|)^{-\frac{n}{2}} \int_0^t \|F(\cdot,s)\|_2 \,\mathrm{d}s.$$

Proof. We start with the representation

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} K(t,s,x,y) F(s,y) dy\, ds, \tag{3.2}$$

where the kernel K is given by the following oscillatory integral

$$K(t,s,x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(\langle x-y,\xi\rangle + (t-s)\lambda(\xi))} \,\mathrm{d}\xi.$$

The assumptions (3.1) and $|x| \leq C_0 t$ imply that the gradient with respect to ξ of the phase function

$$\psi := \langle x-y,\xi\rangle + (t-s)\lambda(\xi)$$

has norm greater than $C_0 t$ so integrating by parts, we get

$$|K(t,s,x,y)| \leq C|y|^{-n}. \tag{3.3}$$

More precisely, we make the decomposition

$$\begin{aligned} K &= (2\pi)^{-n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\psi}\varphi(|\xi||y|)\,\mathrm{d}\xi + (2\pi)^{-n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\psi}(1-\varphi(|\xi||y|))\,\mathrm{d}\xi \\ &:= K_1 + K_2. \end{aligned}$$

φ is a smooth cut-off function identically one near the origin. For K_1 we have

$$|K_1| \leq C\int_{|\xi|\leq 1/|y|} \mathrm{d}\xi \leq C|y|^{-n}.$$

In order to estimate K_2 we employ the operator

$$L := \left\langle \frac{\nabla_\xi \psi}{|\nabla_\xi \psi|^2}, \nabla_\xi \right\rangle.$$

Being $L\mathrm{e}^{\mathrm{i}\psi} = \mathrm{e}^{\mathrm{i}\psi}$, integrating by parts $n+1$ times we obtain

$$K_2 = (2\pi)^{-n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\psi}(L^*)^{n+1}(1-\varphi(|\xi||y|))\,\mathrm{d}\xi.$$

We claim that

$$|(L^*)^{n+1}(1-\varphi(|\xi||y|))| \leq C|y|^{-n-1}|\xi|^{-n-1}. \tag{3.4}$$

From this the assertion follows, since we find

$$|K_2| \leq C|y|^{-n-1}\int_{|\xi|\geq 1/|y|} \frac{\mathrm{d}\xi}{|\xi|^{n+1}} = C_1|y|^{-n}.$$

In turn this implies (3.3). Coming back to the representation formula (3.2) we get the desired estimate. It remains to establish (3.4). This estimate is consequence of the relations

$$\begin{aligned} &|\nabla_\xi \psi| \leq C|y|; \\ &|D_\xi^\alpha \varphi(|\xi||y|)| \leq C|\xi|^{-|\alpha|}; \\ &|D_\xi^\alpha \nabla_\xi \psi| \leq Ct|\xi|^{-|\alpha|}; \\ &\left| D_\xi^\alpha \frac{\nabla_\xi \psi}{|\nabla_\xi \psi|^2} \right| \leq C|y|^{-1}|\xi|^{-|\alpha|}. \end{aligned}$$

This completes the proof. □

The main result of this section is the following.

Theorem 3.2 *Let $n \geq 3$. Consider a smooth function $\lambda : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ which verifies*

(i) *λ is homogeneous of degree* 1;

(ii) *$\lambda(\xi) > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$;*

(iii) *Σ_λ is strictly convex.*

Let $t \geq 1$. There exists a suitable constant $\sigma > 0$ such that for the solution of (2.1), *the following estimates hold:*

$$|u(x,t)| \leq (1+t)^{-\frac{n-1}{2}} \sum_{|\alpha| \leq [n/2]+1} \int_0^t \|Y^\alpha F(\cdot,s)\|_2 \, \mathrm{d}s + \sum_{1 \leq |\alpha| \leq [n/2]} \|Y^\alpha F(\cdot,t)\|_2 \qquad \text{if } |x| \leq \sigma t;$$

$$|u(x,t)| \leq (1+|x|)^{-\frac{n-1}{2}} \sum_{|\alpha| \leq [n/2]+k} \int_0^t \|Y^\alpha F(\cdot,s)\|_2 \, \mathrm{d}s \qquad \text{if } |x| \geq \sigma t;$$

$$|u(x,t)| \leq C(1+t+|x|)^{-\frac{n-1}{2}} \sum_{|\alpha| \leq [n/2]+k} \int_0^t \|Y^\alpha(\lambda) F(\cdot,s)\|_2 \, \mathrm{d}s + \sum_{1 \leq |\alpha| \leq [n/2]} \|Y^\alpha(\lambda) F(\cdot,t)\|_2.$$

Here $k = 1$ for even n, $k = 2$ for odd n. Moreover $Y_i(\lambda) \in \mathcal{A}_\lambda$ the Lie algebra generated by $\{\partial_t, \nabla, S, \Omega_{j,k}(\lambda)\}$.

Proof. The solution of (2.1) is given by

$$u(x,t) = (2\pi)^{-n/2} \int_0^t \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(\langle x,\xi\rangle + (t-s)\lambda(\xi))} \hat{F}(\xi,s) \, \mathrm{d}\xi \, \mathrm{d}s,$$

Using polar coordinates

$$\begin{cases} \rho = \lambda(\xi) \\ \omega = \xi/\lambda(\xi) \end{cases}$$

we find

$$u(x,t) = (2\pi)^{-n/2} \int_0^t \int_0^{+\infty} \rho^{n-1} \int_{\Sigma_\lambda} \mathrm{e}^{\mathrm{i}\rho(\langle x,\omega\rangle + (t-s))} \hat{F}(\rho\omega, s)\, \mathrm{đ}\omega\, \mathrm{d}\rho\, \mathrm{d}s,$$

where

$$\mathrm{đ}\omega = |\nabla\lambda(\omega)|^{-1}\, \mathrm{d}\sigma_{n-1}(\omega).$$

We can write

$$u(x,t) = \int_0^{+\infty} \rho^{n-1} \int_{\Sigma_\lambda} \mathrm{e}^{\mathrm{i}\rho t \phi_{x,t}(\omega)} \tilde{F}_t(\rho\omega, \rho)\, \mathrm{đ}\omega\, \mathrm{d}\rho$$

where $\tilde{F}_t$ is the truncated Fourier transform

$$\begin{aligned} \tilde{F}_t(\xi,\tau) &:= (2\pi)^{-n/2} \int_0^t \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}(\langle y,\xi\rangle + \tau s)} F(y,s)\, \mathrm{d}s \\ &= \int_0^t \mathrm{e}^{-\mathrm{i}\tau s} \hat{F}(\xi, s)\, \mathrm{d}s. \end{aligned}$$

Notice that this is a truncated space-time Fourier transform; indeed if we define

$$\tilde{F}(\tau,\xi) = \int_{-\infty}^{\infty} \mathrm{d}s \int_{\mathbb{R}^n_y} \mathrm{e}^{\mathrm{i}(s\tau + \langle y,\xi\rangle)} F(s,y)\, \mathrm{d}y,$$

we get $\tilde{F}_t(\tau,\xi) = (\chi_{[0,t]}F)\tilde{}(\tau,\xi)$. Moreover, we have

$$|\tilde{F}_t(\xi,\tau)| \leq \int_0^t |\hat{F}(\xi,s)|\, \mathrm{d}s. \tag{3.5}$$

The involved phase is

$$\phi_{x,t}(\omega) = \langle x,\omega\rangle/t + 1.$$

In order to integrate by parts with respect to ρ, it is necessary to distinguish the cases $\phi_{x,t}(\omega) \geq C$ and $\phi_{x,t}(\omega) \leq C$ for a fixed constant C. To this aim we put

$$\tilde{\sigma} = \frac{1}{2} \inf_{\xi\in S^{n-1}} \left[\sup_{\omega\in\Sigma_\lambda} |\langle \xi,\omega\rangle| \right]^{-1}.$$

This is well-posed due to our assumption on λ. Moreover for all $x \in \mathbb{R}^n$,

$t > 0$, $\omega \in \Sigma_\lambda$ we have

$$0 \leq |x|/t \leq \tilde{\sigma} \quad \Rightarrow \quad \phi_{x,t}(\omega) \geq 1/2.$$

Indeed

$$\begin{aligned}\frac{\langle x, \omega\rangle}{t} + 1 = 1 + \frac{|x|}{t}\frac{\langle x,\omega\rangle}{|x|} &\geq 1 - \frac{|x|}{t}\sup_{\omega\in\Sigma_\lambda}\left|\left\langle \frac{x}{|x|}, \omega\right\rangle\right| \\ &\geq 1 - \tilde{\sigma}\sup_{\omega\in\Sigma_\lambda}\left|\left\langle \frac{x}{|x|}, \omega\right\rangle\right| \geq \frac{1}{2}.\end{aligned}$$

In what follows, we put $\sigma = \min\{\tilde{\sigma}, C_0\}$, where $C_0 = \max|\nabla\lambda|$.

The case $|x| \leq \sigma t$. After integration by parts, we get

$$u(x,t) = \mathrm{i}t^{-1}\int_0^{+\infty}\rho^{n-2}\int_{\Sigma_\lambda}\phi_{x,t}^{-1}(\omega)\mathrm{e}^{\mathrm{i}\rho t\phi_{x,t}(\omega)}\rho\partial_\rho\tilde{F}_t(\rho\omega,\rho)\,\mathrm{đ}\omega\,\mathrm{d}\rho.$$

Let S be the scaling operator defined in (2.6); since

$$\rho\partial_\rho\tilde{F}_t(\rho\omega,\rho) = -(n+1)\tilde{F}_t(\rho\omega,\rho) - (SF)_t^{\sim}(\rho\omega,\rho) + t\mathrm{e}^{-\mathrm{i}\rho t}\hat{F}(\rho\omega,t),$$

we find

$$\begin{aligned}u(x,t) = t^{-1}\sum_{j=0}^{1}c_j\int_0^{+\infty}\rho^{n-2}\int_{\Sigma_\lambda}\phi_{x,t}^{-1}(\omega)\mathrm{e}^{\mathrm{i}\rho t\phi_{x,t}(\omega)}(S^jF)_t^{\sim}(\rho\omega,\rho)\,\mathrm{đ}\omega\,\mathrm{d}\rho \\ + \mathrm{i}\int_0^{+\infty}\rho^{n-2}\mathrm{e}^{-\mathrm{i}\rho t}\int_{\Sigma_\lambda}\phi_{x,t}^{-1}(\omega)\mathrm{e}^{\mathrm{i}\rho t\phi_{x,t}(\omega)}\hat{F}(\rho\omega,t)\,\mathrm{đ}\omega\,\mathrm{d}\rho.\end{aligned}$$

We apply further integration by parts and obtain by induction:

$$\begin{aligned}u(x,t) \;&=\; t^{-k}\sum_{j=0}^{k}\tilde{c}_j\int_0^{+\infty}\rho^{n-1-k}\int_{\Sigma_\lambda}\phi_{x,t}^{-k}(\omega)\mathrm{e}^{\mathrm{i}\rho t\phi_{x,t}(\omega)}(S^jF)_t^{\sim}(\rho\omega,\rho)\,\mathrm{đ}\omega\,\mathrm{d}\rho \\ &+ \sum_{h=1}^{k}d_ht^{-h+1}\int_0^{+\infty}\rho^{n-1-k}\mathrm{e}^{-\mathrm{i}\rho t}\int_{\Sigma_\lambda}\phi_{x,t}^{-k}(\omega)\mathrm{e}^{\mathrm{i}\rho t\phi_{x,t}(\omega)}(S^{h-1}F)^{\wedge}(\rho\omega,t)\,\mathrm{đ}\omega\,\mathrm{d}\rho \\ &=: t^{-k}\sum_{j=0}^{k}\bar{c}_jT_j + \sum_{h=1}^{k}d_ht^{-h+1}B_h.\end{aligned}$$

First we estimate T_j. Fix $a \in \mathbb{N}$, $b \in \mathbb{R}$; using the Cauchy-Schwartz inequality with respect to $\mathrm{d}\rho$, we have

$$T_j^2 \leq C\int_0^{+\infty} \frac{\rho^{n-1-2k+2b}}{\langle\rho\rangle^{2a}}\,\mathrm{d}\rho \int_0^{+\infty} \rho^{n-1-2b}\langle\rho\rangle^{2a}\int_{\Sigma_\lambda} |(S^jF)_t^{\sim}(\rho\omega,\rho)|^2\,\mathrm{d}\omega\,\mathrm{d}\rho.$$

The first integral converges whenever the following condition is fulfilled:

$$-b < n/2 - k < a - b. \tag{3.6}$$

Coming back to the original coordinates, since $\lambda(\xi) \simeq |\xi|$, we get

$$|T_j| \leq C\||\xi|^{-b}\langle\xi\rangle^a(S^jF)_t^{\sim}(\xi,\lambda(\xi))\|_{L^2(\mathbb{R}^n_\xi)}.$$

Combining (3.5) with Minkowski inequality, we find

$$T_j \leq \sum_{|\alpha|\leq a} C\int_0^t \||\xi|^{-b}(D^\alpha S^jF)^\wedge(\cdot,s)\|_{L^2(\mathbb{R}^n)}\,\mathrm{d}s.$$

In this contest, extended Hardy inequality implies (see [3]):

$$\||\xi|^{-b}\hat{f}(\xi)\|_2 \leq \||x|^bf\|_2 \quad b\in[0,n/2).$$

In our case, this gives

$$\||\xi|^{-b}(D^\alpha S^jF)^\wedge(\xi,s)\|_{L^2(\mathbb{R}^n_\xi)} \leq C\||y|^bD^\alpha S^jF(y,s)\|_{L^2(\mathbb{R}^n_y)}.$$

Being $|x|\leq C_0t$, Lemma 3.1 shows that we lose no generality assuming

$$\operatorname{supp}_y F(y,s) \subseteq \{|y|\leq 3C_0t\},$$

we find

$$|T_j| \leq Ct^b\sum_{|\alpha|\leq a}\int_0^t \|D^\alpha S^jF(\cdot,s)\|_2\,\mathrm{d}s \quad b\in[0,n/2).$$

A similar argument works for B_h:

$$\begin{aligned}
&|B_h|^2\\
&\leq \int_0^{+\infty}\frac{\rho^{n-1-2k+2b_h}}{\langle\rho\rangle^{2a_h}}\,\mathrm{d}\rho\int_0^{\infty}\rho^{n-1-2b_h}\langle\rho\rangle^{2a_h}\int_{\Sigma_\lambda}|(S^{h-1}F)^\wedge(\rho\omega,t)|^2\,\mathrm{d}\omega\,\mathrm{d}\rho\\
&\leq C\||\xi|^{-b_h}\langle\xi\rangle^{a_h}(S^{h-1}F)^\wedge(\xi,t)|\|^2_{L^2(\mathbb{R}^n_\xi)}\\
&\leq Ct^{2b_h}\|\langle D\rangle^{a_h}S^{h-1}F(\cdot,t)\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where the conditions

$$-b_h < n/2 - h < a_h - b_h, \quad b_h \in [0, n/2) \tag{3.7}$$

are fulfilled. In conclusion, for any $k \in \mathbb{N}$, if $|x| \leq \sigma t$ then

$$\begin{aligned} |u(x,t)| \leq t^{-k} \sum_{j=0}^{k} t^b \int_0^t \|\langle D\rangle^\alpha F(\cdot, s)\|_2 \,\mathrm{d}s \\ + \sum_{h=1}^{k} t^{-h+1} t^{b_h} \|\langle D\rangle^{a_h} S^{h-1} F(\cdot, t)\|_2, \end{aligned} \tag{3.8}$$

where $a, b, a_h, b_h \in \mathbb{R}$ satisfy (3.6) and (3.7).

The case $|x| \geq \sigma t$. We put $|u(x,t)| \leq I + II$, where

$$I = \int_0^{1/|x|} \rho^{n-1} \int_{\Sigma_\lambda} \mathrm{e}^{i\rho t \phi_{x,t}(\omega)} \tilde{F}_t(\rho\omega, \rho) \,\text{đ}\omega \,\mathrm{d}\rho,$$

$$II = \int_{1/|x|}^{\infty} \rho^{n-1} \mathrm{e}^{\mathrm{i}\rho t} \int_{\Sigma_\lambda} \mathrm{e}^{\mathrm{i}\rho\langle x,\omega\rangle} \tilde{F}_t(\rho\omega, \rho) \,\text{đ}\omega \,\mathrm{d}\rho.$$

Combining Cauchy-Schwartz inequality and Lemma 2.4, we get

$$\begin{aligned} |I| &\leq c|x|^{-\frac{n}{2}} \left(\int_0^{\frac{1}{|x|}} \|\tilde{F}_t(\rho\omega, \rho)\|^2_{L^2(\Sigma_\lambda)} \rho^{n-1} \,\mathrm{d}\rho \right)^{1/2} \\ &\leq |x|^{-\frac{n}{2}} \|\tilde{F}_t(\xi, \lambda(\xi))\|_{L^2(\mathbb{R}^n)} \leq c|x|^{-\frac{n}{2}} \int_0^t \|F(\cdot, s)\|_{L^2(\mathbb{R}^n)} \,\mathrm{d}s. \end{aligned}$$

In the last estimate we use (3.5) and Minkowski inequality. In order to deal with II we need Theorem 2.3, so that

$$|II| \leq C|x|^{-\frac{n-1}{2}} \int_0^{+\infty} \rho^{\frac{n-1}{2}} \|\tilde{F}_t(\rho\cdot, \rho)\|_{H^s(\Sigma_\lambda)} \,\mathrm{d}\rho.$$

On the other hand, we know

$$\|\tilde{F}_t(\rho\cdot, \rho)\|_{H^s(\Sigma_\lambda)} \leq \sum_{|\alpha| \leq s} \int_0^t \|\tilde{\Omega}^\alpha \hat{F}(\rho\cdot, s)\|_{L^2(\Sigma_\lambda)} \,\mathrm{d}s.$$

At this point we can apply Corollary 2.5 with $d = -1/2$, $a = 1$, obtaining

$$|II| \leq C|x|^{-\frac{n-1}{2}} \sum_{|\alpha| \leq s} \int_0^t \int_0^{+\infty} \rho^{\frac{n-1}{2}} \|\tilde{\Omega}^\alpha \hat{F}(\rho\cdot, s)\|_{L^2(\Sigma_\lambda)} \,\mathrm{d}\rho \,\mathrm{d}s$$

$$\leq C|x|^{-\frac{n-1}{2}} \sum_{|\alpha| \leq s} \int_0^t \|\langle \lambda(D) \rangle \Omega^\alpha F(\cdot, s)\|_2 \, \mathrm{d}s$$
$$\leq C|x|^{-\frac{n-1}{2}} \sum_{|\alpha| \leq s+1} \int_0^t \|Y^\alpha F(\cdot, s)\|_2 \, \mathrm{d}s.$$

Being $t \geq 1$ and $|x| \geq \sigma t$, we can conclude

$$|u(x,t)| \leq (1+|x|)^{-\frac{n-1}{2}} \sum_{|\alpha| \leq s+1} \int_0^t \|Y^\alpha F(\cdot, s)\|_2 \, \mathrm{d}s,$$
$$s > (n-1)/2, \ s \in \mathbb{N}. \tag{3.9}$$

It remains to combine this expression with (3.8). In particular we have to take $k \geq \frac{n-1}{2}$. Since k represents the loss of derivatives in the integration by parts, we can take $k = [n/2]$. This means that in (3.8) for odd n we can choose $b = b_h = 0$ and $a = a_h = 1$. The situation is slightly different for even n: we fix $b = 1/2$ and $a = 1$ while $b_1 = 0$ and $a_1 = n/2$, $b_h = 1/2$ and $a_h = n/2 + 1 - h$ for $h \geq 2$. This inequality and (3.9) give the conclusion. □

Remark 3.1 In the case $n = 2$, the proof of Theorem 3.2 is available except that the conditions (3.7) leads to

$$-b_1 < 0 < a_1 - b_1,$$

so that we can not take $b_1 = 0$. Than we choose $b_1 = \varepsilon > 0$ and $a_1 = 0$ obtaining

$$|u(x,t)| \leq C(t+|x|)^{-1/2} \sum_{|\alpha| \leq 2} \int_0^t \|Y^\alpha(\lambda) F(\cdot, s)\|_2 \, \mathrm{d}s + C_\varepsilon t^\varepsilon \|F(\cdot, t)\|_2. \tag{3.10}$$

Remark 3.2 Theorem 3.2 holds when λ is negative and $\Sigma_{-\lambda}$ is strictly convex.

4. First order systems

We consider a first order system

$$\partial_t u - \sum_j A_j \partial_j u = F$$

with $u, F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^N$ and A_j constant matrices of order N. We associate the matrix symbol $A(\xi) = \sum_j A_j \xi_j$ and assume that it satisfies (H_1). In correspondence to non-zero eigenvalues we can take the Lie algebra generated by

$$\{\nabla_x, \partial_t, S, \Omega_{j,k}(m) = \partial_j \lambda_m^2(D) x_k - \partial_k \lambda_m^2(D) x_j\}$$

On the other hand, we can consider the operator

$$\Omega_{j,k} = \sum_m \Omega_{j,k}(m) \pi_m(D) \tag{4.1}$$

where $\pi_m(\xi)$ is the projection on the eigenspace related to $\lambda_m(\xi)$. We recall that

$$\pi_j \pi_k = \delta_{j,k} \pi_k \tag{4.2}$$

$$I = \pi_{\ker A} + \sum_j \pi_j \tag{4.3}$$

$$A(\xi) = \sum_j \lambda_j(\xi) \pi_j(\xi). \tag{4.4}$$

First of all, we investigate the invariance properties of $\Omega_{j,k}$ with respect to the operator $\partial_t - \sum_j A_j \partial_j$.

Lemma 4.1 *Denoting by A the operator having symbol $A(\xi)$ we get*

$$[A, \Omega_{j,k}] = 0.$$

Proof. Since λ_m and π_m are convolution type operators, we can write

$$A = \sum_l \lambda_l(D) \pi_l(D).$$

Recalling that convolution type operators commute, we find

$$\begin{aligned}[A, \Omega_{j,k}] &= \sum_{m,l} [\lambda_l(D) \pi_l(D), \Omega_{j,k}(m) \pi_m(D)] \\ &= \sum_{m,l} \lambda_l(D) [\pi_l(D), \Omega_{j,k}(m)] \pi_m(D) \\ &\quad + \sum_{m,l} [\lambda_l(D), \Omega_{j,k}(m)] \pi_l(D) \pi_m(D).\end{aligned}$$

By using (4.2) and Lemma 2.1 we get $[\lambda_l(D), \Omega_{j,k}(m)] \pi_l(D) \pi_m(D) = 0$. It remains to prove that

$$[\pi_l(D), \Omega_{j,k}(m)]\pi_m(D) = 0 \qquad \forall m, l. \tag{4.5}$$

A consequence of (2.5) is the following:

$$[\pi_l(D), \Omega_{j,k}(m)]\pi_m(D) = 0 \qquad \forall m \neq l. \tag{4.6}$$

In fact we have

$$\begin{aligned}
&[\pi_l(D), \Omega_{j,k}(m)]\pi_m(D)\\
&\quad = [\pi_l(D), \partial_j \lambda_m^2(D)x_k - \partial_k \lambda_m^2(D)x_j]\pi_m(D)\\
&\quad = \partial_j \lambda_m^2(D)[\pi_l(D), x_k]\pi_m(D) - \partial_k \lambda_m^2(D)[\pi_l(D), x_j]\pi_m(D)\\
&\quad = -\mathrm{i}\partial_j \lambda_m^2(D)(\partial_k \pi_l)(D)\pi_m(D) + \mathrm{i}\partial_k \lambda_m^2(D)(\partial_j \pi_l)(D)\pi_m(D).
\end{aligned}$$

On the other hand (4.3) implies $(\partial_j \pi_l)(D)\pi_m(D) = -\pi_l(D)(\partial_j \pi_m)(D)$ so that

$$[\pi_l(D), \Omega_{j,k}(m)]\pi_m(D) = -[\pi_m(D), \Omega_{j,k}(m)]\pi_l(D) \quad \forall m \neq l.$$

Finally

$$\begin{aligned}
[\pi_l(D), \Omega_{j,k}(m)]\pi_m(D) &= [\pi_l(D), \Omega_{j,k}(m)]\pi_m(D)\pi_m(D)\\
&= -[\pi_m(D), \Omega_{j,k}(m)]\pi_l(D)\pi_m(D) = 0.
\end{aligned}$$

The same argument works for the identity

$$[\pi_{\ker A}(D), \Omega_{j,k}(m)]\pi_m(D) = 0.$$

Using (4.3) we see that last relation and (4.6) give

$$[\pi_m(D), \Omega_{j,k}(m)]\pi_m(D) = 0. \tag{4.7}$$

In detail

$$\begin{aligned}
&[\pi_m(D), \Omega_{j,k}(m)]\pi_m(D)\\
&\quad = \pi_m(D)\Omega_{j,k}(m)\pi_m(D) - \Omega_{j,k}(m)\pi_m(D)\\
&\quad = -\sum_{m \neq l} \pi_l(D)\Omega_{j,k}(m)\pi_m(D) - \pi_{\ker A}\Omega_{j,k}(m)\pi_m(D)\\
&\quad = -\sum_{m \neq l} \Omega_{j,k}(m)\pi_l(D)\pi_m(D) - \Omega_{j,k}(m)\pi_{\ker A}\pi_m(D) = 0.
\end{aligned}$$

From (4.6) and (4.7) we obtain the conclusion. □

Another consequence of (4.5) is the following:

$$[\Omega_{j,k}, \pi_r(D)] = 0. \tag{4.8}$$

In fact

$$\begin{aligned}[\Omega_{j,k}, \pi_r(D)] &= \sum_m [\Omega_{j,k}(m)\pi_m(D), \pi_r(D)] \\ &= \sum_m [\Omega_{j,k}(m), \pi_r(D)]\pi_m(D) = 0.\end{aligned}$$

The last lemma gives the possibility to consider the Lie algebra $\mathcal{A}$ generated by

$$\{\nabla_x, \partial_t, S, \Omega_{j,k}\}.$$

It is necessary to see that for $\Gamma_1, \Gamma_2 \in \mathcal{A}$ the operator $[\Gamma_1, \Gamma_2]$ is a linear combination of the operators belonging to $\mathcal{A}$ modulo coefficient in S^0. We have the usual relations $[\partial, \partial] = 0$ and $[S, \partial] = -\partial$. Moreover formulas (2.9), (4.5) imply

$$[\Omega_{j,k}, \Omega_{r,s}] = \sum_{p,q} c_{p,q}(D)\Omega_{p,q}.$$

It is also clear that $[\partial_t, \Omega_{j,k}] = 0$ and $[\partial_t, \pi_j] = 0$. An application of (2.8) leads to

$$\begin{aligned}[\Omega_{j,k}, \partial_k] &= \sum_m [\Omega_{j,k}(m), \partial_k]\pi_m(D) \\ &= -2\mathrm{i} \sum_m \partial_j \lambda_m(D) \Big(\sum_r \partial_r \lambda_m(D) \partial_r \Big) \pi_m(D).\end{aligned}$$

In order to obtain

$$[\Omega_{j,k}, \partial_r] = \sum_i c_i(D)\partial_i$$

with $c_i(D) \in OPS^0$, we have to suppose that the projections are zero order operators. It is easy to prove that if $\lambda_j(\xi)$ is a non-vanishing homogeneous function of degree one, then $\pi_j(\xi)$ is a zero order matrix symbol with homogeneous coefficient of degree zero. In this case we can use Euler formula and conclude $[S, \pi_m] = 0$. Combining this with (2.8) we get $[S, \Omega_{j,k}] = 0$. Finally we proved the following.

Lemma 4.2 *Let $\mathcal{A}$ be generated by $\{\nabla_x, \partial_t, S, \Omega_{j,k}\}$, with $\Omega_{j,k}$ associated with hyperbolic operator $\partial_t - \sum_j A_j \partial_j$ which satisfies* (H$_1$), (H$_2$) . *For any $\Gamma_l, \Gamma_m \in \mathcal{A}$ it holds*

$$[\Gamma_l, \Gamma_m] = \sum_{r=1}^{N} c^r_{l,m}(D)\Gamma_r,$$

where $c^r_{l,m}$ is a matrix-pseudo-differential operator of convolution type with symbol homogeneous of order 0 *and $N = (n^2+n+4)/2$.*

This lemma gives the possibility to define Sobolev spaces associate to $\mathcal{A}$. We want to relate the norm of this space with the norms of generalized Sobolev spaces associated to each eigenvalue. This means to prove the following equivalence

$$\sum_{|\alpha|\le s} \|\Gamma^\alpha u\|_2 \simeq \sum_l \sum_{|\alpha|\le s} \|\Gamma^\alpha(l)\pi_l u\|_2 \tag{4.9}$$

being $\Gamma \in \mathcal{A}$ and $\Gamma(l) \in \mathcal{A}_l$. It is obvious that

$$\sum_l \|\pi_l u\|_2 \simeq \|u\|_2.$$

Now, we recall that $\Omega_{j,k}(m)$ is skew-symmetric in L^2 inner product, so that

$$\begin{aligned}\|\Omega_{j,k}u\|_2^2 &= \sum_m \sum_r \left(\Omega_{j,k}(m)\pi_m u \mid \Omega_{j,k}(r)\pi_r u\right)\\ &= -\sum_m \sum_r \left(\pi_r\Omega_{j,k}(r)\Omega_{j,k}(m)\pi_m u \mid \pi_r u\right).\end{aligned}$$

Using (4.7) one can see that $\Omega_{j,k}(m)\pi_m = \pi_m\Omega_{j,k}(m)\pi_m$ hence

$$\pi_r\Omega_{j,k}(r)\Omega_{j,k}(m)\pi_m = \pi_r\Omega_{j,k}(r)\pi_r\pi_m\Omega_{j,k}(m)\pi_m.$$

This gives

$$\begin{aligned}\|\Omega_{j,k}u\|_2^2 &= -\sum_m \left(\pi_m\Omega_{j,k}(m)\pi_m\Omega_{j,k}(m)\pi_m u \mid \pi_m u\right)\\ &= \sum_m \|\Omega_{j,k}(m)\pi_m u\|_2^2.\end{aligned}$$

It follows that (4.9) holds when $|\alpha| = 1$.

For $|\alpha| \ge 2$ the equivalence (4.9) can be proved by induction applying different times the relations (4.5) and (4.8).

5. L^2-L^∞ estimate for first order system

Combining Theorem 3.2 and the results contained in the previous section we have the following

Theorem 5.1 *Let $n \geq 3$ and $t \geq 1$. Let $A(\xi) = \sum_j A_j \xi_j$ satisfy the assumption* (H_1), (H_2) *and* $\ker A(\xi) = \{0\}$ *for all ξ. Given $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^N$, the solution of*

$$\partial_t u(x,t) - \sum_j A_j \partial_{x_j} u(x,t) = F(x,t), \tag{5.1}$$

having zero initial data satisfies the following estimates:

$$\begin{aligned} |u(x,t)| \leq C(1+t)^{-\frac{n-1}{2}} & \sum_{|\alpha| \leq [n/2]+1} \int_0^t \|\Gamma^\alpha F(\cdot,s)\|_2 \, ds \\ & + C \sum_{1 \leq |\alpha| \leq [n/2]} \|\Gamma^\alpha F(\cdot,t)\|_2 \qquad \text{if } |x| \leq \sigma t \end{aligned} \tag{5.2}$$

$$\begin{aligned} |u(x,t)| \leq C(1+|x|)^{-\frac{n-1}{2}} & \sum_{|\alpha| \leq [n/2]+k} \int_0^t \|\Gamma^\alpha F(\cdot,s)\|_2 \, ds \\ & \qquad \text{if } |x| \geq \sigma t \end{aligned} \tag{5.3}$$

$$\begin{aligned} |u(x,t)| \leq C(1+t+|x|)^{-\frac{n-1}{2}} & \sum_{|\alpha| \leq [n/2]+k} \int_0^t \|\Gamma^\alpha F(\cdot,s)\|_2 \, ds \\ & + C \sum_{1 \leq |\alpha| \leq [n/2]} \|\Gamma^\alpha F(\cdot,t)\|_2 . \end{aligned} \tag{5.4}$$

Here $k = 1$ for even $n \geq 3$ and $k = 2$ for odd $n \geq 3$ and

$$\sigma = \max\left\{ \max |\nabla \lambda| ; \frac{1}{2} \min_j \inf_{\xi \in S^{n-1}} \left[\sup_{\omega \in \Sigma_{\lambda_j}} |\langle \xi, \omega \rangle| \right]^{-1} \right\}.$$

Moreover $\Gamma(\lambda) \in \mathcal{A}$ is the Lie algebra generated by $\{\partial_t, \nabla, S, \Omega_{j,k}\}$.

Proof. For simplicity we consider only the case $|x| \leq \sigma t$, since the case $|x| \geq \sigma t$ follows from (3.9) and the argument presented below. The relation (4.3) implies

$$u(x,t) = \sum_j \pi_j(D) u(x,t)$$

so that we can write (5.1) in the form

$$\sum_j \frac{\mathrm{d}}{\mathrm{d}t}(\pi_j(D)u) - \lambda_j(D)(\pi_j(D)u) = \sum_j \pi_j(D)F.$$

Applying $\pi_h(D)$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\pi_h(D)u) - \lambda_h(D)(\pi_h(D)u) = \pi_h(D)F. \tag{5.5}$$

By using Theorem 3.2 we get

$$\begin{aligned}|u(x,t)| &\leq \sum_j |\pi_j(D)u(x,t)| \\ &\leq C(t+1)^{-\frac{n-1}{2}} \sum_{|\alpha|\leq[n/2]+1} \sum_j \int_0^t \|\Gamma^\alpha(j)\pi_j(D)F(\cdot,s)\|_{L^2(\mathbb{R}^n)}\,\mathrm{d}s \\ &\quad + C \sum_{1\leq|\alpha|\leq[n/2]} \sum_j \|\Gamma^\alpha(j)\pi_j(D)F(\cdot,t)\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

On the right side we apply (4.9), obtaining the required L^2-L^∞ estimate.

□

Corollary 5.2 *Let $n \geq 3$ and $t \geq 1$. Let $A(\xi) = \sum_j A_j\xi_j$ satisfy the assumption* (H_1), (H_2) *and* $\ker A(\xi) = \{0\}$ *for all ξ. Given $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^N$, the solution of* (5.1) *having zero initial data satisfies the following estimate*:

$$|u(x,t)| \leq C(1+t+|x|)^{-\frac{n-1}{2}} \sum_{|\alpha|\leq[n/2]+k} \sup_{0\leq s\leq t} (1+s)^a \|\Gamma^\alpha F(\cdot,s)\|_2 \tag{5.6}$$

Here

$$k = \begin{cases} 1, & \text{if } n \geq 3 \text{ is even;} \\ 2, & \text{if } n \geq 3 \text{ is odd,} \end{cases}$$

$$a = \begin{cases} (n-1)/2, & \text{if } n \geq 4; \\ 1+\varepsilon, \varepsilon > 0, & \text{if } n = 3 \end{cases} \tag{5.7}$$

and $\Gamma(\lambda) \in \mathcal{A}$ is the Lie algebra generated by $\{\partial_t, \nabla, S, \Omega_{j,k}\}$.

Proof. First we consider the case, when $|x| \geq \sigma t$. Then (5.3) gives the desired inequality.

Let $|x| \le \sigma t$. The right side of the inequality (5.2) can be estimated from above by constant times

$$(1+t)^{-(n-1)/2} \int_0^t f(s)\,\mathrm{d}s + f(t),$$

where

$$f(s) = \sum_{|\alpha| \le [n/2]+k} \|\Gamma^\alpha F(\cdot, s)\|_2.$$

It is sufficient to note that

$$(1+t)^{-(n-1)/2} \int_0^t f(s)\,\mathrm{d}s + f(t)$$
$$\le C(1+t)^{-(n-1)/2} \sup_{0 \le s \le t} (1+s)^a f(s),$$

where the constant a is defined in (5.7). □

In the case $\ker A(\xi) \neq \{0\}$ the relation (4.3) gives

$$u(x,t) = \pi_{\ker A}(D)u(x,t) + \sum_j \pi_j(D)u(x,t),$$

$$F(x,t) = \pi_{\ker A}(D)F(x,t) + \sum_j \pi_j(D)F(x,t).$$

We can still deduce (5.5), but we have to consider the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\pi_{\ker A}(D)u) = \pi_{\ker A}(D)F.$$

If $\pi_{\ker A}(D)F = 0$, then we have the conservation law

$$\pi_{\ker A}(D)u(x,t) = \pi_{\ker A}(D)u(x,0).$$

In order to have L^2-L^∞ estimate we assume that the initial data satisfy

$$\pi_{\ker A}(D)u(x,0) = 0.$$

Theorem 5.3 *Let $n \ge 3$. Consider a Cauchy Problem associated to* (5.1). *Assume that all non-vanishing characteristic roots of $A(\xi)$ satisfies the assumptions of Theorem* 3.2. *Suppose in addition that*

$$\hat{F}(t) \perp \ker A(\xi), \quad \hat{u}(0) \perp \ker A(\xi). \tag{5.8}$$

Then (5.6) *holds.*

In the introduction we refer to the assumptions (5.8) like elliptic conditions which guarantee that all components of u decay.

Remark 5.1 By the aid of Remark 3.1, one can write a similar estimate to (5.6) for the case $n = 2$ with weights of type t^{ε} like in (3.10).

6. Examples

6.1. Dirac system

An important hyperbolic problem (see [16]) is the Dirac system

$$\mathrm{i}\gamma_0\partial_t + \mathrm{i}\sum_{j=1}^{3}\gamma_j\partial_j\psi(x,t) = 0. \tag{6.1}$$

The solution $\psi(t,x) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}^4$ is called *spinor* the coefficients γ_μ are the Dirac matrices defined as follows

$$\gamma_0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1,2,3.$$

The Pauli matrices σ_k are determined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We rewrite (6.1) in the form $(\partial_t - \sum_j A_j\partial_j)\psi(x,t)$ with

$$A(\xi) = \begin{pmatrix} 0 & \sigma_1\xi_1 + \sigma_2\xi_2 + \sigma_3\xi_3 \\ \sigma_1\xi_1 + \sigma_2\xi_2 + \sigma_3\xi_3 & 0 \end{pmatrix}.$$

This is hermitian matrix; our technique still works since we use the symmetry of A just to prove that the eigenvalues are real and the eigenspaces are orthogonal. The correspondent characteristic form $\det(A(\xi) - \lambda\mathbf{I}) = 0$ is

$$\lambda^4 - 2|\xi|\lambda^2 + |\xi|^4 = 0$$

We see that the eigenvalues are strictly convex:

$$\lambda_{1,2} = |\xi|; \quad \lambda_{3,4} = -|\xi|.$$

For this system the estimate (5.6) holds, taking only one field $\Omega_{j,k}(\lambda) = x_j\partial_k - x_k\partial_j$. In this example is evident that our methods works for not-strictly hyperbolic systems.

6.2. Maxwell system for uniaxial crystals

We consider Maxwell system in anisotropic media:

$$\begin{cases} \epsilon_0 \partial_t E = \operatorname{curl} H + F_1, \\ \partial_t H = -\operatorname{curl} E + F_2. \end{cases} \tag{6.2}$$

Here $E(x,t), H(x,t), F_1(x,t), F_2(x,t) \in \mathbb{R}^3$, and $x \in \mathbb{R}^3$, $t \in \mathbb{R}$. Moreover ϵ_0 is a diagonal 3 by 3 matrix:

$$\epsilon_0 = \operatorname{diag}(a^2, b^2, c^2).$$

Taking $\underline{u} = (\epsilon_0^{1/2} E, H)$, $F = (\epsilon_0^{-1/2} F_1, F_2)$ we can rewrite this system as

$$\partial_t \underline{u} - B\underline{u} = F,$$

with

$$B := \begin{bmatrix} 0 & \epsilon_0^{-1/2} \operatorname{curl} \\ -\operatorname{curl} \epsilon_0^{-1/2} & 0 \end{bmatrix}.$$

In particular

$$B(\xi) := \begin{bmatrix} 0 & -\epsilon_0^{-1/2} \xi \wedge \\ \xi \wedge \epsilon_0^{-1/2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & U \\ U^T & 0 \end{bmatrix},$$

$$U = \begin{bmatrix} 0 & -|a|^{-1}\xi_3 & |a|^{-1}\xi_2 \\ |b|^{-1}\xi_3 & 0 & -|b|^{-1}\xi_1 \\ -|c|^{-1}\xi_2 & |c|^{-1}\xi_1 & 0 \end{bmatrix}.$$

The characteristic form $\det(B(\xi) - \lambda \mathbf{I}) = 0$ is given by

$$\lambda^2(\lambda^4 - \psi(\xi)\lambda^2 + \phi(\xi)|\xi|^2) = 0, \tag{6.3}$$

where

$$\psi(\xi) = (|b|^{-2} + |c|^{-2})\xi_1^2 + (|a|^{-2} + |c|^{-2})\xi_2^2 + (|a|^{-2} + |b|^{-2})\xi_3^2,$$
$$\phi(\xi) = |b|^{-2}|c|^{-2}\xi_1^2 + |a|^{-2}|c|^{-2}\xi_2^2 + |a|^{-2}|b|^{-2}\xi_3^2.$$

In the case $b = c$, $B(\xi)$ has eigenvalues

$$\lambda_{1,2} = 0,$$

$$\lambda_{3,4}(\xi) = \pm\sqrt{b^{-2}\xi_1^2 + a^{-2}\xi_2^2 + a^{-2}\xi_3^2},$$
$$\lambda_{5,6}(\xi) = \pm|b|^{-1}|\xi|.$$

The most relevant point is that Σ_{λ_j} is strictly convex for $j \geq 3$. On the contrary for $b \neq c$, taking $\lambda = 1$ in (6.3), one finds Fresnel surface with four singular points. Hence Σ_λ is not strictly convex. This is consequence of the variable multiplicity of the characteristic roots. Our technique is not available.

This analytic difference correspond to different physical situations. Maxwell system describes the propagation of the light in crystals. In particular, in uniaxial crystal, there is a single axis along which light can propagate without exhibiting double refraction; along other axis a light beam splits into two different components which travel at different velocities. This corresponds to the choice $\epsilon_0 = \mathrm{diag}(a^2, b^2, b^2)$. On the contrary, in biaxial crystals conical refraction take place: a ray incident on a surface of the crystal in a certain direction splits into a family of rays which lie along a cone. To examine this case one takes $\epsilon_0 = \mathrm{diag}(a^2, b^2, c^2)$ with three different entries.

We state that our estimates works in uniaxial case. In this case

$$\Omega_{1,2}(\lambda_{3,4}) = 2(a^{-2}x_1\partial_2 - b^{-2}x_2\partial_1),$$
$$\Omega_{1,3}(\lambda_{3,4}) = 2(a^{-2}x_1\partial_3 - b^{-2}x_3\partial_1),$$
$$\Omega_{2,3}(\lambda_{3,4}) = 2a^{-2}(x_2\partial_3 - x_3\partial_2),$$
$$\Omega_{j,k}(\lambda_{5,6}) = 2b^{-2}(x_j\partial_k - x_k\partial_j) \quad j,k = 1,2,3.$$

We observe that the fields $\Omega_{1,2}(\lambda_{3,4})$, $\Omega_{1,3}(\lambda_{3,4})$, are different from those used in [6]. It remains to consider the conditions due to the presence of null eigenvalues. We notice that the following conditions are equivalent

$$(\hat{G}_1, \hat{G}_2) \perp \mathrm{Ker}\, B(\xi),$$
$$\mathrm{div}\, \epsilon_0^{1/2} G_1 + \mathrm{div}\, G_2 = 0.$$

In fact $w \in \mathrm{Ker}\, B(\xi)$ if and only if $w = (w_1, w_2)$ satisfies $\epsilon_0^{1/2}\xi \wedge w_2 = 0 = \xi \wedge \epsilon_0^{-1/2} w_1$ that is $w_2 = \beta\xi$, $w_1 = \alpha\epsilon_0^{1/2}\xi$, for some $\alpha, \beta \in \mathbb{R}$. Hence, the following conditions are equivalent

$$(\hat{G}_1, \hat{G}_2) \perp \mathrm{Ker}\, B(\xi),$$
$$\alpha\langle\epsilon_0^{1/2}\xi, \epsilon_0\hat{G}_1\rangle + \beta\langle\xi, \hat{G}_2\rangle = 0,$$

$$\alpha \operatorname{div} \epsilon_0^{1/2} G_1 + \beta \operatorname{div} G_2 = 0.$$

In conclusion, considering ϵ_0 having two different entries, if

$$\operatorname{div} \epsilon_0 E_0 = 0 = \operatorname{div} H_0$$
$$\operatorname{div} \epsilon_0 F_1 = 0 = \operatorname{div} F_2$$

then (5.6) holds for the solution of (6.2). In this case the elliptic conditions connected with null eigenvalue have a clear physical meaning.

We remark that L^1-L^∞ estimate in usual Sobolev space are known (see [10] for uniaxial crystals and [9] for biaxial case); the main point here is to associate to Maxwell system generalized Sobolev spaces.

6.3. Wave equations with different propagation speeds

Some recent papers (see [7]) concern system of semilinear wave equations with different speeds of propagation:

$$\begin{cases} (\partial_{tt} - c_1 \Delta) u_1 = f_1, \\ \dots \\ (\partial_{tt} - c_N \Delta) u_N = f_N. \end{cases} \tag{6.4}$$

Here $c_i \in \mathbb{R} \backslash \{0\}$ represents the propagation speed of u_i. Let us set

$$U_i = (\partial_t u_i, \sqrt{c_i} \partial_{x_1} u_i, \dots, \sqrt{c_i} \partial_{x_n} u_i), \quad U = (U_1, \dots, U_N),$$
$$F_i = (f_i, 0, \dots, 0), \quad F = (F_1, \dots, F_N).$$

The system (6.4) can be written in the form $\partial_t U - \sum_j A_j \partial_{x_j} U = F$ with

$$A(\xi) = \begin{bmatrix} B_1(\xi) & & & \\ & B_2(\xi) & & \\ & & \ddots & \\ & & & B_N(\xi) \end{bmatrix}.$$

The blocks on the diagonal have the following form:

$$B_i(\xi) = \begin{pmatrix} 0 & \sqrt{c_i}\xi_1, & \cdots & \sqrt{c_i}\xi_n \\ \sqrt{c_i}\xi_1 & 0 & \cdots & 0 \\ \cdots & 0 & \cdots & 0 \\ \sqrt{c_i}\xi_n & 0 & \cdots & 0 \end{pmatrix}.$$

We see that

$$\det(A(\xi)-\lambda\mathbf{I})=\prod_{i=1}^{N}\det(B_i(\xi)-\lambda\mathbf{I})=\lambda^{(n-1)N}\prod_{i=1}^{N}(\lambda^2-c_i|\xi|^2).$$

Non vanishing eigenvalues have the form $\lambda_{i,\pm}=\pm\sqrt{c_i}|\xi|$; the correspondent surfaces $\Sigma_{\lambda_{i,\pm}}$ are strictly convex and we can take eigenvectors $\underline{V}_{i,\pm}=(0,\dots,V_{i,\pm},0\dots0)$ with $V_{i\pm}$ orthogonal eigenvectors for B_i.

It remains to discuss the conditions due to the presence of the null eigenvalue. We observe that $\hat{V}=(\hat{V}_1,\dots,\hat{V}_N)\in\ker A(\xi)$ if and only if $\hat{V}_i\in\ker B_i(\xi)$. The last condition is equivalent to $\xi_k\hat{V}_{i0}=0$ and $\sum_k\xi_k\hat{V}_{ik}=0$. In turn this gives $V_{i0}=0$. In our case ξ is parallel to $\hat{U}_i'=\sqrt{c_i}(\partial_{x_1}u_i,\dots,\partial_{x_n}u_i)^\wedge$, so that $\hat{U}_i(0)\perp\ker B_i(\xi)$. The condition for $\hat{F}_i=(\hat{f}_i,0,\dots,0)$ is trivial, in fact $\hat{F}_i\perp\ker A(\xi)$ means $\hat{f}_i\hat{v}_0=0$ with $v_0=0$.

In conclusion without any assumptions on the initial data the solution of (6.4) satisfies (5.6). The involved Lie algebra contains the fields $\sum_i c_i(x_j\partial_k-x_k\partial_j)\pi_i(D)$.

6.4. Equations of elasticity in anisotropic media

The motion of homogeneous elastic material is described by the system

$$\partial_t^2u-(\nabla\cdot\sigma(\nabla u))^T=F. \tag{6.5}$$

Here

$$u(x,t)=\begin{pmatrix}u_1(t,x)\\u_2(t,x)\\u_3(t,x)\end{pmatrix}:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3,\quad \sigma=(\sigma_{jk})_{j,k=1,2,3}\in\mathcal{M}_3$$

and $\nabla\cdot\sigma=(v_1,v_2,v_3)$ is the vector with components

$$v_k=\sum_{j=1}^{3}\partial_{x_j}\sigma_{jk}\quad k=1,2,3.$$

Moreover,

$$(v_1,v_2,v_3)^T=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}.$$

For a physical discussion of this model see [1], [8], [13]. We consider the following ansatz for σ

$$\sigma(\nabla u) = c \cdot \nabla u + d \,\mathrm{tr}(\nabla u)\mathbf{I}$$

with c a constant 3×3 matrix, $A \cdot B$ is the matrix product, $d \in \mathbb{R}$ and ∇u represented by a matrix in which each column is the gradient of u_j, $j = 1, 2, 3$.

In isotropic case we have $c = \mu\mathbf{I}$, $d = \lambda + \mu$, $\lambda, \mu \in \mathbb{R}_+$ for the Lamé coefficients. In order to study some anisotropic media we assume

$$c = \{c_{j,k}\} \text{ positively defined, } d > 0. \tag{6.6}$$

Explicitly, we have

$$(\nabla \cdot \sigma(\nabla u))^T = \begin{pmatrix} \Delta_c u_1 \\ \Delta_c u_2 \\ \Delta_c u_3 \end{pmatrix} + d \begin{pmatrix} \partial_{x_1}(\operatorname{div} u) \\ \partial_{x_2}(\operatorname{div} u) \\ \partial_{x_3}(\operatorname{div} u) \end{pmatrix}$$

where $\Delta_c = \sum_{j,k} c_{j,k}\partial_j\partial_k$. In Fourier transform coordinates (6.5) can be rewritten as

$$\hat{u}'' - A(\xi)\hat{u} = \hat{F}; \tag{6.7}$$

$$A(\xi) = \begin{pmatrix} d\xi_1^2 + |\xi|_c^2 & d\xi_1\xi_2 & d\xi_1\xi_3 \\ d\xi_1\xi_2 & d\xi_2^2 + |\xi|_c^2 & d\xi_2\xi_3 \\ d\xi_1\xi_3 & d\xi_2\xi_3 & d\xi_3^3 + |\xi|_c^2 \end{pmatrix};$$

$$|\xi|_c^2 = \sum_{j,k} c_{j,k}\xi_j\xi_k.$$

The characteristic form $\det(A(\xi) - \lambda(\xi)\mathbf{I}) = 0$ has solutions

$$\lambda(\xi) = |\xi|_c^2,$$
$$\mu(\xi) = d|\xi|^2 + |\xi|_c^2.$$

Due to (6.6), the corresponding surfaces Σ_{λ_i} are strictly convex.

Next step is to reduce (6.7) in the form of first order system. We observe that $\xi = (\xi_1, \xi_2, \xi_3)$ is an eigenvector associated to $\mu(\xi)$, and $w_1 = (-\xi_3, 0, \xi_1)$, $w_2 = (-\xi_2, \xi_1, 0)$ are orthogonal eigenvectors associated to $\lambda(\xi)$. This means that we have two projectors:

$$\pi_1(\xi)v = \frac{\langle w_1 \cdot v\rangle w_1}{|w_1|^2} + \frac{\langle w_2 \cdot v\rangle w_2}{|w_2|^2}, \quad \pi_2(\xi)v = \frac{\langle \xi \cdot v\rangle \xi}{|\xi|^2}.$$

Taking $v_i = \pi_i(D)u$, $i = 1, 2$, from (6.7) we have

$$\begin{cases} (\partial_t^2 + \lambda(D))v_1 = \pi_1(D)F, \\ (\partial_t^2 + \mu(D))v_2 = \pi_2(D)F. \end{cases}$$

Considering $U = (\partial_t v_1, \sqrt{\lambda(D)}v_1, \partial_t v_2, \sqrt{\mu(D)}v_2)$ we arrive at

$$\hat{U}_t - \mathcal{A}(\xi)\hat{U} = \hat{\mathcal{F}}.$$

Here $\mathcal{F} = (\pi_1(D)F, 0, \pi_2(D)F, 0)$ and

$$\mathcal{A}(\xi) = \begin{pmatrix} 0 & -\sqrt{\lambda(\xi)} & 0 & 0 \\ -\sqrt{\lambda(\xi)} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\mu(\xi)} \\ 0 & 0 & -\sqrt{\mu(\xi)} & 0 \end{pmatrix}.$$

The corresponding eigenvalues are $\pm\lambda(\xi)$ and $\pm\mu(\xi)$. This enable us to associate to (6.5) a generalized Sobolev space and derive L^2-L^∞ estimates for these equations.

Acknowledgments The authors are partially supported by HYKE PROJECT HPRN-CT-2002-00282.

References

[1] Agemi R., *Global existence of nonlinear elastic waves*. Invent. Math. **142** (2000), 225–250.

[2] Covachev V.C. and Georgiev V., *A counterpart of the Poincaré group for pseudodifferential operators*. C. R. Acad. Bulgare Sci. **44** (1991), 17–20.

[3] D'Ancona P., Georgiev V. and Kubo H., *L^q Weighted Decay Estimate for Wave Equation*. J. Differential Equations **177** (2001), 146–208.

[4] Gallot S., Hulin D. and Lafontaine J., *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, 1987.

[5] Georgiev V., *Global solution of the system of wave and Klein – Gordon equations*. Math. Zeit. **203** (1990) 683–698.

[6] Klainerman S., *Uniform Decay Estimates and Lorentz Invariance of the Classical Wave Equation*. Comm. Pure Appl. Math. **38** (1985), 321–332.

[7] Kubo H. and Otha M., *Small Data Blowup for Systems of Semilinear Wave Equations with Different Propagation Speeds in Three Space Dimensions*. J. Differential Equations **163** (2000), 475–492.

[8] Landau L.D. and Lifshitz E.M., *Course of theoretical physics. Vol. 7. Theory of elasticity*. Translated from the Russian by J.B. Sykes and W.H. Reid. Pergamon Press, Oxford, 1986.

[9] Liess O., *Decay Estimates for the Solutions of the System of Crystal Optics*. Asymptotic Anal. **4** (1991), 61–95.

[10] Lucente S. and Ziliotti G., *Global Existence for a Quasilinear Maxwell System*. Rendiconti dell'Istituto Matematico Università di Trieste, Vol. XXXI Suppl. 2, (2000), 169–187.

[11] Lucente S. and Ziliotti G., *Pseudo-differential Equations applied to Systems of Crystal Optics*. Proceedings of the Workhop "Partial Differential Operators", Torino Maggio 2000, L. Rodino Ed. 83–100.

[12] Majda A.J., *Coercive inequalities for nonelliptic symmetric systems*. Comm. Pure Appl. Math. **28** (1975), 49–89.

[13] Noll W., *A mathematical theory of the mechanical behavior of continuous media*. Arch. Rational Mech. Anal. **2** (1958) 198–226.

[14] Sugimoto M., *Estimates for hyperbolic equations with non-convex characteristics*. Math. Z. **222** (1996), 521–531.

[15] Taylor M.E., *Pseudodifferential Operators*. Princepton Univ. Press, Princeton 1981.

[16] Tzvetkov N., *Global Existence for Semilinear Massless Dirac Equation with Small Data*. Tsukuba J. Math. **22** (1998), 193–211.

Vladimir Georgiev
Dipartimento di Matematica
Università degli Studi di Pisa
Via F. Buonarroti 2, I-56100 Pisa, Italy
E-mail: georgiev@dm.unipi.it.

Sandra Lucente
Dipartimento di Matematica
Università degli Studi di Bari
Via E. Orabona 4, I-70125 Bari, Italy
E-mail: lucente@dm.uniba.it.

Guido Ziliotti
Dipartimento di Matematica
Università degli Studi di Pisa
Via F. Buonarroti 2, I-56100 Pisa, Italy
E-mail: ziliotti@mail.dm.unipi.it.