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Non-closed minimal hypersurfaces of $\mathbb{S}^4(1)$ with identically zero Gauß-Kronecker curvature

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Abstract. We give a partial local description of minimal hypersurfaces M^3 with identically zero Gauß-Kronecker curvature function in the unit 4-sphere $\mathbb{S}^4(1)$, without assumption on the compactness of M^3.

Key words: minimal hypersurfaces in spheres, isoparametric hypersurfaces, identically zero Gauß-Kronecker curvature, nowhere zero second fundamental form.

1. Introduction

Let $x\colon M^3 \longrightarrow \mathbb{S}^4(1) \subset \mathbb{R}^5$ be a hypersurface immersion of a connected and orientable 3-dimensional manifold M^3 of class C^∞ into $\mathbb{S}^4(1) \subset \mathbb{R}^5$. Let λ_1, λ_2 and λ_3 be the three principal curvature functions. The normalized elementary symmetric curvature functions of the immersion x are given by:

$$\begin{aligned} H &:= \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3), \\ H_2 &:= \frac{1}{3}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3), \\ K &:= \lambda_1\lambda_2\lambda_3. \end{aligned}$$

S. Almeida and F. Brito [1] suggested to classify closed hypersurface immersions for which two of the three functions H, H_2, K are constant. The paper [3] gives a survey of results on closed hypersurfaces in $\mathbb{S}^4(1)$ with two constant curvature functions.

Particularly, the paper [2] investigated closed minimal hypersurfaces with constant Gauß-Kronecker curvature function, corresponding to $H \equiv 0$ and $K \equiv$ const. There it is proved that closed minimal hypersurfaces with constant Gauß-Kronecker curvature $K \neq 0$ are isoparametric, therefore closed minimal hypersurfaces with constant Gauß-Kronecker curvature

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$K \neq 0$ are classified. Brito conjectured that all hypersurfaces in $\mathbb{S}^4(1)$ with $K \equiv \text{const} \neq 0$ and $H \equiv \text{const}$ (or $H_2 \equiv \text{const}$) must be isoparametric (personal communication). If $K \equiv 0$ on M^3, the following is well known: a closed minimal hypersurface immersion in $\mathbb{S}^4(1)$ with nowhere zero second fundamental form is a boundary of a tube which is built over a non-degenerate minimal 2-dimensional surface immersion in $\mathbb{S}^4(1)$ with geodesic radius $\frac{\pi}{2}$. This nice result proves the existence of non-isoparametric closed minimal hypersurfaces with $K \equiv 0$ in $\mathbb{S}^4(1)$. But so far no explicit non-isoparametric example has been given. In this paper we investigate local descriptions of minimal hypersurfaces (not necessarily closed) in $\mathbb{S}^4(1)$ with identically zero Gauß-Kronecker curvature K, but with nowhere zero second fundamental form. In particular we present the following two explicit non-isoparametric examples:

Example 1.1 The mapping

$$\begin{aligned} &x_1 \colon \mathbb{R}^3 \longrightarrow \mathbb{S}^4(1) \subset \mathbb{R}^5 \\ &x_1(u,v,z) = \frac{1}{\sqrt{1+z^2}}\big(\cos(\sqrt{2}u)C_1 + \sin(\sqrt{2}u)C_2 \\ &\qquad\qquad + \cos(\sqrt{2}v)C_3 + \sin(\sqrt{2}v)C_4 + zC_5\big), \end{aligned}$$

where $C_1, C_2, C_3, C_4, C_5 \in \mathbb{R}^5$ are constant orthogonal vectors in $\mathbb{R}^5$ such that

$$\frac{1}{2} = \langle C_1, C_1\rangle = \langle C_2, C_2\rangle = \langle C_3, C_3\rangle = \langle C_4, C_4\rangle \quad \text{and} \quad \langle C_5, C_5\rangle = 1,$$

defines a minimal hypersurface immersion with zero Gauß-Kroneker curvature. The principal curvature functions take the values $\lambda_1(u,v,z) = \sqrt{z^2+1}$, $\lambda_2(z) = -\sqrt{z^2+1}$ and $\lambda_3(z) = 0$; they depend only on z.

Example 1.2 Let $I \subset \mathbb{R}$ be an open interval, $0 < c_1$, $c_2 \in \mathbb{R}$ such that $c_2e^{2v} - 1 - c_1^2e^{4v} > 0$ for all $v \in I$, and $g, h \colon I \longrightarrow \mathbb{R}$ two differentiable functions on I which are linearly independent solutions of the second order differential equation

$$(c_2e^{2v} - 1 - c_1^2e^{4v})A_5''(v) + (1 - c_1^2e^{4v})A_5'(v) + 2A_5(v) = 0,$$

and such that $g^2(v) + h^2(v) = 1 - \frac{e^{-2v}}{c_2}$, for all $v \in I$. The existence of such functions will be proved below, see Lemma 3.4 and Remark 3.5. Then the mapping

$$x_2\colon \mathbb{R}\times I\times\mathbb{R}\longrightarrow \mathbb{S}^4(1)\subset\mathbb{R}^5,$$

$$x_2(u,v,z)=\frac{e^{-v}}{\sqrt{c_2(z^2+1)}}\big(\cos(u)C_1+\sin(u)C_2\big) + \frac{1}{\sqrt{z^2+1}}\big(zC_3+g(v)C_4+h(v)C_5\big),$$

where $C_1, C_2, C_3, C_4, C_5 \in \mathbb{R}^5$ are constant orthonormal vectors, defines a minimal hypersurface immersion in $\mathbb{S}^4(1)$ with identically zero Gauß-Kronecker curvature. The principal curvatures take the values

$$\lambda_1(v,z)=c_1e^{2v}\sqrt{z^2+1},\quad \lambda_2(v,z)=-c_1e^{2v}\sqrt{z^2+1} \quad\text{and}\quad \lambda_3(v,z)=0,$$

thus they depend only on v and z.

The principal curvatures of both examples do depend on at most two parameters. We prove the following local classification of such hypersurfaces.

Main result Let $x\colon M^3 \longrightarrow \mathbb{S}^4(1)\subset\mathbb{R}^4$ be a minimal hypersuface immersion (with nowhere zero second fundamental form) of a connected and orientable C^∞-manifold M^3 in $\mathbb{S}^4(1)$ with identically zero Gauß-Kronecker curvature. If one of the two nowhere zero principal curvature functions is constant along its associated principal curvature line, then there exist local coordinates so that the immersion x locally can be described by one of the two non-isoparametric hypersurfaces x_1 and x_2 (see Example 1.1 and Example 1.2) above, or locally by Cartan's minimal isoparametric hypersurface with principal curvatures $\sqrt{3}$, $-\sqrt{3}$ and 0.

2. Notations and integrability conditions

Let $x : M^3 \longrightarrow \mathbb{S}^4(1)\subset\mathbb{R}^5$ be an immersion of a connected, orientable 3-dimensional C^∞-manifold M^3 into the unit 4-sphere $\mathbb{S}^4(1)$. Denote by y a unit normal vector field on $\mathbb{S}^4(1)$ along the immersion x, by $\langle\ ,\ \rangle$ the canonical inner product of the Euclidean structure, and by $\bar\nabla$ the flat connection of $\mathbb{R}^5$. Referring to [4] for details on geometry of submanifolds, recall that as immersion of codimension 2 in $\mathbb{R}^5$ the structure equations (Gauß and

Weingarten equations) for x state:

$$\begin{cases} \bar{\nabla}_u dx(v) := dx(\nabla_u v) + \mathrm{II}(u,v)y - \mathrm{I}(u,v)x, \\ \quad dy(u) := -dx(Su), \end{cases} \quad \text{for all } u, v \in TM^3 \quad (2.1)$$

where I denotes the first fundamental form (induced metric) with Levi-Civita connection ∇, II defines the second fundamental form and S denotes the shape operator.

The structure equations imply the following integrability conditions (Gauß formula and Codazzi equation) for any $u, v, w \in TM^3$:

$$R(u,v)w = \mathrm{I}(w,v)u - \mathrm{I}(w,u)v + \mathrm{II}(w,v)Su - \mathrm{II}(w,u)Sv, \quad (2.2)$$

$$(\nabla_u S)v = (\nabla_v S)u, \quad (2.3)$$

where R denotes the Riemannian curvature tensor for the induced metric I.

Let (e_1, e_2, e_3) be a I-orthonormal local differentiable frame of principal curvature vector fields on M^3:

$$Se_1 = \lambda_1 e_1, \quad Se_2 = \lambda_2 e_2 \quad \text{and} \quad Se_3 = \lambda_3 e_3.$$

There are 9 functions $\alpha_1, \ldots, \alpha_9$ such that

$$\begin{cases} \nabla_{e_1} e_1 = \alpha_1 e_2 + \alpha_2 e_3, & \nabla_{e_1} e_2 = -\alpha_1 e_1 + \alpha_3 e_3, \\ \nabla_{e_1} e_3 = -\alpha_2 e_1 - \alpha_3 e_2; & \\ \nabla_{e_2} e_1 = -\alpha_4 e_2 + \alpha_6 e_3, & \nabla_{e_2} e_2 = \alpha_4 e_1 + \alpha_5 e_3, \\ \nabla_{e_2} e_3 = -\alpha_6 e_1 - \alpha_5 e_2; & \\ \nabla_{e_3} e_1 = \alpha_9 e_2 - \alpha_7 e_3, & \nabla_{e_3} e_2 = -\alpha_9 e_1 - \alpha_8 e_3, \\ \nabla_{e_3} e_3 = \alpha_7 e_1 + \alpha_8 e_2. & \end{cases} \quad (2.4)$$

Remark 2.1 Consider the situation that the three principal curvature functions λ_1, λ_2, λ_3 are everywhere distinct; the fact that the frame (e_1, e_2, e_3) is orthonormal implies that the functions α_i, $i = 1, \ldots, 9$ in (2.4) are defined on M^3 uniquely up to sign.

Applying the Codazzi equation (2.3) to the vector fields e_1, e_2, e_3 and using (2.4), one gets the following equations:

$$\left\{\begin{aligned} e_1(\lambda_2) &= \alpha_4(\lambda_2-\lambda_1),\\ e_1(\lambda_3) &= \alpha_7(\lambda_3-\lambda_1),\\ e_2(\lambda_1) &= \alpha_1(\lambda_1-\lambda_2),\\ e_2(\lambda_3) &= \alpha_8(\lambda_3-\lambda_2),\\ e_3(\lambda_1) &= \alpha_2(\lambda_1-\lambda_3),\\ e_3(\lambda_2) &= \alpha_5(\lambda_2-\lambda_3),\\ \alpha_9(\lambda_1-\lambda_2) &= \alpha_3(\lambda_2-\lambda_3) = \alpha_6(\lambda_1-\lambda_3). \end{aligned}\right. \tag{2.5}$$

From now we assume that the immersion x (with nowhere zero second fundamental form) is minimal and has identically zero Gauß-Kronecker curvature ($K \equiv 0$). There exists a positive non-zero function λ such that the principal curvature functions associated to the immersion x are $\lambda_1 = \lambda$, $\lambda_2 = -\lambda$ and $\lambda_3 = 0$. From the equations (2.5), one gets

$$e_1(\lambda) = 2\alpha_4\lambda, \quad e_2(\lambda) = 2\alpha_1\lambda, \qquad e_3(\lambda) = \alpha_2\lambda; \tag{2.6}$$

$$\alpha_5 = \alpha_2, \qquad 2\alpha_9 = -\alpha_3 = \alpha_6, \qquad \alpha_7 = 0 = \alpha_8. \tag{2.7}$$

Applying the Gauß formula (2.2) to the vector fields e_1, e_2, e_3 and using the equations (2.4) and (2.7), one gets

$$\left\{\begin{aligned} e_1(\alpha_4)+e_2(\alpha_1) &= 1-\lambda^2+\alpha_1^2+\alpha_2^2+2\alpha_3^2+\alpha_4^2\\ e_3(\alpha_1)+\frac{1}{2}e_1(\alpha_3) &= \alpha_1\alpha_2-\frac{1}{2}\alpha_3\alpha_4\\ e_3(\alpha_4)-\frac{1}{2}e_2(\alpha_3) &= \alpha_2\alpha_4+\frac{1}{2}\alpha_1\alpha_3,\\ e_3(\alpha_2) &= 1+\alpha_2^2-\alpha_3^2,\\ e_3(\alpha_3) &= 2\alpha_2\alpha_3,\\ e_1(\alpha_2) &= e_2(\alpha_3),\\ e_1(\alpha_3) &= -e_2(\alpha_2). \end{aligned}\right. \tag{2.8}$$

Note that the Lie brackets with respect to the vector fields e_1, e_2 and e_3 are given by:

$$[e_1,e_2] = -\alpha_1e_1+\alpha_4e_2+2\alpha_3e_3, \quad [e_1,e_3] = -\alpha_2e_1-\frac{1}{2}\alpha_3e_2,$$
$$[e_2,e_3] = \frac{1}{2}\alpha_3e_1-\alpha_2e_2.$$

The fundamental equations (2.1) applied to the vector fields e_1, e_2, e_3

give rise to the following (partial) differential equations:

$$\begin{cases} \bar{\nabla}_{e_1} dx(e_1) = \alpha_1 dx(e_2) + \alpha_2 dx(e_3) + \lambda y - x \\ \bar{\nabla}_{e_1} dx(e_2) = -\alpha_1 dx(e_1) + \alpha_3 dx(e_3) \\ \bar{\nabla}_{e_1} dx(e_3) = -\alpha_2 dx(e_1) - \alpha_3 dx(e_2) \\ \bar{\nabla}_{e_2} dx(e_1) = -\alpha_4 dx(e_2) - \alpha_3 dx(e_3) \\ \bar{\nabla}_{e_2} dx(e_2) = \alpha_4 dx(e_1) + \alpha_2 dx(e_3) - \lambda y - x \\ \bar{\nabla}_{e_2} dx(e_3) = \alpha_3 dx(e_2) - \alpha_2 dx(e_2) \\ \bar{\nabla}_{e_3} dx(e_1) = -\frac{1}{2}\alpha_3 dx(e_2) \\ \bar{\nabla}_{e_3} dx(e_2) = \frac{1}{2}\alpha_3 dx(e_1) \\ \bar{\nabla}_{e_3} dx(e_3) = -x \\ dy(e_1) = -\lambda dx(e_1), \\ dy(e_2) = \lambda dx(e_2), \\ dy(e_3) = 0. \end{cases} \tag{2.9}$$

3. Proof of the main result

To describe locally hypersurface immersions in $\mathbb{S}^4(1)$ with $K \equiv 0$, one has to find local coordinates to solve the structure equations (2.9) using the integrability conditions (2.6) and (2.8). It seems to be very difficult to solve this problem in full generality.

In this section, we consider natural additional assumptions on the functions α_1, α_2, α_3 and α_4 to solve the structure equations (fundamental equations) for minimal hypersurface immersions in $\mathbb{S}^4(1)$ with $K \equiv 0$; namely we assume that the function λ is constant along the e_1-direction. This additional assumption is suggested by the Examples 1.1 and 1.2.

Proposition 3.1 *Let $x\colon M^3 \longrightarrow \mathbb{S}^4(1)$ be a minimal hypersurface immersion with identically zero Gauß-Kronecker curvature function and nowhere zero second fundamental form. Let $\alpha_1, \ldots, \alpha_9$ be the the functions as defined in (2.4). If the function α_4 vanishes identically on an open subset U of M^3, i.e., the function λ is constant along the curvature line of the vector field e_1, then also the function α_3 vanishes identically on U, or the immersion x is a minimal Cartan isoparametric hypersurface on U.*

Proof. Using the equations (2.6) and

$$e_1e_2(\lambda) - e_2e_1(\lambda) - [e_1, e_2](\lambda) = 0 = e_1e_3(\lambda) - e_3e_1(\lambda) - [e_1, e_3](\lambda),$$

one gets:

$$e_1(\alpha_1) = \alpha_2\alpha_3 \quad \text{and} \quad e_1(\alpha_2) = -\alpha_1\alpha_3.$$

Similarly, using (2.8) and

$$e_2e_3(\alpha_3) - e_3e_2(\alpha_3) - [e_2, e_3](\alpha_3) = 0,$$

one has:

$$\alpha_3 e_2(\alpha_2) = 0.$$

Assume now that $\alpha_3 \neq 0$ everywhere. This implies that $e_2(\alpha_2)$ vanishes identically. Inserting again the equations (2.6) into

$$e_1e_2(\alpha_3) - e_2e_1(\alpha_3) - [e_1, e_2](\alpha_3) = 0,$$

one gets:

$$\alpha_3^2\alpha_2 = 0.$$

Therefore $\alpha_2 = 0 = \alpha_1$ and $\alpha_3 = \pm 1$. Consequently, $\lambda^2 = 3$. Thus the immersion is isoparametric with principal curvatures $\lambda_1 = \sqrt{3}$, $\lambda_2 = -\sqrt{3}$ and $\lambda_3 = 0$, i.e. the immersion is a Cartan's minimal isoparametric hypersurface in $\mathbb{S}^4(1)$. □

Corollary 3.2 *Let $x \colon M^3 \longrightarrow \mathbb{S}^4(1) \subset \mathbb{R}^5$ be a closed minimal hypersurface immersion of a connected and orientable manifold M^3 into $\mathbb{S}^4(1) \subset \mathbb{R}^5$ with nowhere zero second fundamental form and $K \equiv 0$. Assume that one of the functions α_1 and α_4 vanish identically on M^3. Then $x(M)$ is a Cartan's minimal isoparametric hypersurface of $\mathbb{S}^4(1)$, i.e., the boundary of the tube* $\mathrm{Tube}(V^2, \frac{\pi}{2})$ *with radius $\frac{\pi}{2}$ around the Veronese surface $V^2 \subset \mathbb{S}^4(1)$.*

Proof. From the proposition above we have two possibilities:
(i) α_3 vanishes identically on M^3;
(ii) or $\alpha_1 = \alpha_2 = \alpha_4 \equiv 0$ and $\alpha_3^2 = 1$.

Assuming that the hypersurface M^3 is closed, the case (i) above cannot happen because the function $e_3(\alpha_2) = 1+\alpha_2^2-\alpha_3^2$ should be zero at the minimum and maximum points of the function α_2; but with $\alpha_3 \equiv 0$, $e_3(\alpha_2) = 1 + \alpha_2^2$ has no zeros. □

Proposition 3.3 *Let $x\colon M^3 \longrightarrow \mathbb{S}^4(1) \subset \mathbb{R}^5$ be a minimal hypersurface immersion of a connected and orientable manifold M^3 into $\mathbb{S}^4(1) \subset \mathbb{R}^5$ with nowhere zero second fundamental form and $K \equiv 0$. Assume that the immersion is non-isoparametric and the functions α_1 and α_4 vanish identically on M^3. Then there are local coordinates so that the immersion x can be locally described by the parametrization of the hypersurface given in Example* 1.1.

Proof. From Proposition 3.1 we may assume that the function α_3 vanishes identically on M^3. Then the following equations hold:

$$\begin{aligned} e_2(\alpha_2) &= 0 = e_1(\alpha_2) = e_1(\lambda) = e_2(\lambda), \\ e_3(\alpha_2) &= \alpha_2^2 + 1, \\ e_3(\lambda) &= \lambda\alpha_2, \\ \lambda^2 &= \alpha_2^2 + 1. \end{aligned}$$

The vector fields $\frac{1}{\lambda}e_1$, $\frac{1}{\lambda}e_2$ and $\frac{1}{\alpha_2^2+1}e_3$ satisfy:

$$0 = \left[\frac{1}{\lambda}e_1, \frac{1}{\lambda}e_2\right] = \left[\frac{1}{\lambda}e_1, \frac{1}{\alpha_2^2+1}e_3\right] = \left[\frac{1}{\lambda}e_2, \frac{1}{\alpha_2^2+1}e_3\right].$$

Therefore there are local coordinates (u, v, z) on M^3 such that

$$\frac{1}{\lambda}e_1 = \frac{\partial}{\partial u}, \quad \frac{1}{\lambda}e_2 = \frac{\partial}{\partial v}, \quad \frac{1}{\alpha_2^2+1}e_3 = \frac{\partial}{\partial z}.$$

The foregoing equations give

$$\alpha_2 = z \quad \text{and} \quad \lambda = \sqrt{z^2+1}.$$

With respect to the frame $\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial z}\right)$ the structure equations (Gauß and Weingarten equations) are given by the following system of second order partial differential equations:

$$\lambda^2 x_{uu} = z(z^2+1)x_z + \lambda y - x, \tag{3.10}$$

$$x_{vu} = 0 = x_{uv}, \tag{3.11}$$

$$x_{zu} = \frac{-z}{z^2+1}x_u = x_{uz}, \tag{3.12}$$

$$\lambda^2 x_{vv} = z(z^2+1)x_z - \lambda y - x, \tag{3.13}$$

$$x_{vz} = \frac{-z}{z^2+1}x_v = x_{vz}, \tag{3.14}$$

$$(z^2+1)^2 x_{zz} = -2z(z^2+1)x_z - x, \tag{3.15}$$
$$y_u = -\lambda x_u, \tag{3.16}$$
$$y_v = \lambda x_v, \tag{3.17}$$
$$y_z = 0. \tag{3.18}$$

Differentiating (3.10) with respect to u and using (3.12) and (3.16), one gets

$$x_{uuu} = -2x_u.$$

There are two vector valued functions $A_1 \equiv A_1(v,z)$ and $A_2 \equiv A_2(v,z)$ in $\mathbb{R}^5$ depending only on v and z such that

$$x_u = \sqrt{2}\big(-\sin(\sqrt{2}u)A_1 + \cos(\sqrt{2}u)A_2\big). \tag{3.19}$$

One has

$$\begin{aligned}\frac{1}{\lambda^2} &= \mathrm{I}(x_u, x_u)\\ &= 2\big(\langle A_1, A_1\rangle \sin^2(\sqrt{2}u) - \langle A_1, A_2\rangle \sin(2\sqrt{2}u) + \langle A_2, A_2\rangle \cos^2(\sqrt{2}u)\big)\\ &= \langle A_1, A_1\rangle + \langle A_2, A_2\rangle + (\langle A_2, A_2\rangle - \langle A_1, A_1\rangle)\cos(2\sqrt{2}u)\\ &\quad - 2\langle A_1, A_2\rangle \sin(2\sqrt{2}u).\end{aligned}$$

The linear independence of the functions 1, $\sin(2\sqrt{2}u)$ and $\cos(2\sqrt{2}u)$ implies

$$\langle A_1, A_1\rangle = \frac{1}{2\lambda^2} = \langle A_2, A_2\rangle \quad \text{and} \quad \langle A_1, A_2\rangle = 0. \tag{3.20}$$

Furthermore there is a vector valued function $A_3 \equiv A_3(v,z)$ depending only on v and z such that

$$x(u,v,z) = \cos(\sqrt{2}u)A_1(v,z) + \sin(\sqrt{2}u)A_2(v,z) + A_3(v,z).$$

One has

$$\begin{aligned}1 &= \langle x, x\rangle\\ &= \frac{1}{2\lambda^2} + 2\langle A_1, A_3\rangle \cos(\sqrt{2}u) + 2\langle A_2, A_3\rangle \sin(\sqrt{2}u) + \langle A_3, A_3\rangle.\end{aligned}$$

From the linear independence of the functions 1, $\sin(\sqrt{2}u)$ and $\cos(\sqrt{2}u)$, one gets

$$1 = \frac{1}{2\lambda^2} + \langle A_3, A_3 \rangle \quad \text{and} \quad \langle A_1, A_3 \rangle = 0 = \langle A_2, A_3 \rangle. \tag{3.21}$$

Differentiating (3.19) with respect to z (and with respect to v, resp.) and using the equation (3.12) (the equation (3.11), resp.) and the linear independence of the functions $1, \sin(\sqrt{2}u)$ and $\cos(\sqrt{2}u)$, one gets the following first order partial differential equations for the vector valued functions $A_1(v,z)$ and $A_2(v,z)$:

$$(z^2+1)\frac{\partial A_1}{\partial z} = -zA_1, \quad (z^2+1)\frac{\partial A_2}{\partial z} = -zA_2$$
$$\text{and} \quad \frac{\partial A_1}{\partial v} = 0 = \frac{\partial A_2}{\partial v}.$$

There are constant vectors C_1 and C_2 in $\mathbb{R}^5$ such that

$$A_1 = \frac{C_1}{\sqrt{z^2+1}} \quad \text{and} \quad A_2 = \frac{C_2}{\sqrt{z^2+1}}.$$

Because of (3.20), one has

$$\langle C_1, C_1 \rangle = \frac{1}{2} = \langle C_2, C_2 \rangle \quad \text{and} \quad \langle C_1, C_2 \rangle = 0.$$

The immersion x takes the form

$$x(u,v,z) = \frac{1}{\sqrt{z^2+1}}\big(\cos(\sqrt{2}u)C_1 + \sin(\sqrt{2}u)C_2\big) + A_3(v,z).$$

After inserting the above expression for x into the equation (3.15), one gets the following second order partial differential equation for the vector valued function $A_3(v,z)$:

$$(z^2+1)^2\frac{\partial^2 A_3}{\partial z^2} + 2z(z^2+1)\frac{\partial A_3}{\partial z} + A_3 = 0.$$

There are two vector valued functions $A_4 \equiv A_4(v)$ and $A_5 \equiv A_5(v)$ in $\mathbb{R}^5$ such that

$$A_3(v,z) = \frac{A_4(v)}{\sqrt{z^2+1}} + \frac{zA_5(v)}{\sqrt{z^2+1}}.$$

The equation (3.14) implies that the vector valued function $A_5(v)$ is constant:

$$A_5(v) = C_5 \equiv \text{const}\,.$$

Because of (3.21), one has

$$0 = \langle C_5, A_4(v)\rangle = \langle C_1, A_4(v)\rangle = \langle C_2, A_4(v)\rangle = \langle C_1, C_5\rangle = \langle C_2, C_5\rangle,$$

$$\langle A_4(v), A_4(v)\rangle = \frac{1}{2} \quad \text{and} \quad \langle C_5, C_5\rangle = 1.$$

Now eliminating y from the equations (3.10) and (3.13), one has that the vector valued function $A_4(v)$ is a solution of the following second order differential equation:

$$A_4''(v) = -2A_4(v).$$

Therefore there are constant vectors $C_3, C_4 \in \mathbb{R}^3$ such that

$$A_4(v) = \cos(\sqrt{2}v)C_3 + \sin(\sqrt{2}v)C_4.$$

The constant vectors C_3, C_4 are orthogonal to C_1, C_2, C_5 and satisfy

$$\langle C_3, C_3\rangle = \frac{1}{2} = \langle C_4, C_4\rangle \quad \text{and} \quad \langle C_3, C_4\rangle = 0.$$

Finally, the local description of the immersion x is given by

$$x(u,v,z) = \frac{1}{\sqrt{z^2+1}}\big(\cos(\sqrt{2}u)C_1 + \sin(\sqrt{2}u)C_2 + \cos(\sqrt{2}v)C_3 + \sin(\sqrt{2}v)C_4 + zC_5\big). \quad (3.22)$$

Note that the vector field y defined by

$$y(u,v,z) = -\cos(\sqrt{2}u)C_1 - \sin(\sqrt{2}u)C_2 + \cos(\sqrt{2}v)C_3 + \sin(\sqrt{2}v)C_4$$

is unit and normal to x. It is then easy to check that the mapping (3.22) defines a minimal hypersurface immersion with $K \equiv 0$ in $\mathbb{S}^4(1)$. Clearly this hypersurface immersion is non-isoparametric. □

Now we want to characterize Example 1.2. In order to succeed, we need to prove the following lemma:

Lemma 3.4 *Let $I \subset \mathbb{R}$ be an open interval and $0 < c_1, c_2 \in \mathbb{R}$ be two constant positive real numbers such that $c_2e^{2v} - 1 - c_1^2e^{4v} > 0$ for every $v \in I$. Consider the following second order differential equation for some* *function A on I :*

$$(c_2e^{2v} - 1 - c_1^2e^{4v})A''(v) + (1 - c_1^2e^{4v})A'(v) + 2A(v) = 0. \tag{3.23}$$

Then there exists a function $\phi\colon I \longrightarrow \mathbb{R}$ such that the general solution $A(v)$ for the equation (3.23) takes the form

$$A(v) = a\sqrt{c_2 - e^{-2v}}\cos\big(\phi(v)\big) + b\sqrt{c_2 - e^{-2v}}\sin\big(\phi(v)\big),$$

where $a, b \in \mathbb{R}$ are constants.

Proof. Let $B\colon I \longrightarrow \mathbb{R}$ be the function defined by

$$A(v) =: \sqrt{c_2 - e^{-2v}}B(v).$$

Inserting this expression of $A(v)$ into the equation (3.23), one gets the following second order differential equation in $B(v)$:

$$\begin{aligned} B''(v) - \frac{(3 + c_1^2e^{4v} + c_2c_1^2e^{6v} - 3c_2e^{2v})B'(v)}{(c_2e^{2v} - 1)(c_2e^{2v} - 1 - c_1^2e^{4v})} & \\ + \frac{c_2c_1^2e^{6v}B(v)}{(c_2e^{2v} - 1)^2(c_2e^{2v} - 1 - c_1^2e^{4v})} &= 0. \end{aligned} \tag{3.24}$$

Now take ϕ to be the function on I such that

$$\phi'(v) = \sqrt{\frac{c_2c_1^2e^{6v}}{(c_2e^{2v} - 1)^2(c_2e^{2v} - 1 - c_1^2e^{4v})}}.$$

It follows that

$$\frac{\phi''(v)}{\phi'(v)} = \frac{(3 + c_1^2e^{4v} + c_2c_1^2e^{6v} - 3c_2e^{2v})}{(c_2e^{2v} - 1)(c_2e^{2v} - 1 - c_1^2e^{4v})}.$$

Thus the equation (3.24) becomes

$$B''(v) - \frac{\phi''(v)}{\phi'(v)}B'(v) + \phi'^2(v)B(v) = 0.$$

Therefore, there are constants $a, b \in \mathbb{R}$ such that

$$B(v) = a\cos\big(\phi(v)\big) + b\sin\big(\phi(v)\big).$$

□

Remark 3.5 The particular solutions $g(v)$ and $h(v)$ of the equation (3.23) given by

$$g(v) = \frac{1}{\sqrt{c_2}}\sqrt{c_2 - e^{-2v}}\cos\big(\phi(v)\big) \quad \text{and}$$
$$h(v) = \frac{1}{\sqrt{c_2}}\sqrt{c_2 - e^{-2v}}\sin\big(\phi(v)\big)$$

are linearly independent and satisfy $g^2(v) + h^2(v) = 1 - \frac{e^{-2v}}{c_2}$.

Proposition 3.6 *Let $x\colon M^3 \longrightarrow \mathbb{S}^4(1) \subset \mathbb{R}^5$ be a minimal hypersurface immersion of a connected and orientable manifold M^3 into $\mathbb{S}^4(1) \subset \mathbb{R}^5$ with identically zero Gauß-Kronecker curvature, but with nowhere zero second fundamental form. Assume that the function α_4 vanishes identically and α_1 is nowhere zero on M^3. Then there are local coordinates (u, v, z) such that the immersion x can be locally described by the parametrization of the hypersurface given in Example* 1.2.

Proof. The vector fields $\frac{1}{\alpha_1}e_2$ and $\frac{1}{\alpha_2^2+1}e_3$ satisfy

$$\left[\frac{1}{\alpha_1}e_2, \frac{1}{\alpha_2^2+1}e_3\right] = 0.$$

Define the function f on M^3 by

$$f := \frac{1}{\sqrt{1+\lambda^2+\alpha_1^2+\alpha_2^2}}. \tag{3.25}$$

The function f satisfies the following equations:

$$e_1(f) = 0, \quad e_2(f) = -\alpha_1 f \quad \text{and} \quad e_3(f) = -\alpha_2 f.$$

Consequently,

$$\left[fe_1, \frac{1}{\alpha_1}e_2\right] = 0 = \left[fe_1, \frac{1}{\alpha_2^2+1}e_3\right].$$

Therefore there are local coordinates (u, v, z) on M^3 such that

$$\frac{\partial}{\partial z} = \frac{1}{\alpha_2^2+1}e_3, \quad \frac{\partial}{\partial v} = \frac{1}{\alpha_1}e_2, \quad \frac{\partial}{\partial u} = fe_1.$$

From above we get the following equations for α_2, λ and α_1.

Equations for α_2:

$$\frac{\partial \alpha_2}{\partial u} = f e_1(\alpha_2) = 0, \quad \frac{\partial \alpha_2}{\partial v} = \frac{1}{\alpha_1} e_2(\alpha_2) = 0, \quad \frac{\partial \alpha_2}{\partial z} = \frac{1}{1+\alpha_2^2} e_3(\alpha_2) = 1.$$

So

$$\alpha_2 = z. \tag{3.26}$$

Equations for λ:

$$\frac{\partial \lambda}{\partial u} = 0, \quad \frac{\partial \lambda}{\partial v} = 2\lambda, \quad \text{and} \quad \frac{\partial \lambda}{\partial z} = \frac{z\lambda}{z^2+1}.$$

Therefore

$$\lambda \equiv \lambda(v,z) = c_1 e^{2v}\sqrt{z^2+1}, \tag{3.27}$$

where $0 < c_1 \in \mathbb{R}$.

Equations for α_1:

$$\frac{\partial \alpha_1}{\partial z} = \frac{z\alpha_1}{z^2+1}, \quad \frac{\partial \alpha_1}{\partial v} = \frac{\alpha_1^2 + z^2 + 1 - \lambda^2}{\alpha_1}, \quad \text{and} \quad \frac{\partial \alpha_1}{\partial u} = 0;$$

thus

$$\alpha_1^2 \equiv \alpha_1^2(v,z) = (z^2+1)(c_2 e^{2v} - 1 - c_1^2 e^{4v}), \tag{3.28}$$

where $v \in I \subset \mathbb{R}$ an open interval and $0 < c_2 \in \mathbb{R}$ is a constant such that $c_2 e^{2v} - 1 - c_1^2 e^{4v} > 0$ for all $v \in I$. Inserting the expressions (3.26), (3.27) and (3.28) (of α_2, λ and α_1, respectively) into (3.25), we see that the function f satisfies

$$(1+z^2) f^2(v) = \frac{e^{-2v}}{c_2}.$$

With respect to the frame $\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial z}\right)$ the structure equations are given by the following system of second order partial differential equations:

$$x_{vz} = -\frac{\alpha_2}{\alpha_2^2+1} x_v = x_{zv}, \tag{3.29}$$

$$x_{zu} = -\frac{\alpha_2}{\alpha_2^2+1} x_u = x_{uz}, \tag{3.30}$$

$$x_{vu} = -x_u = x_{uv}, \tag{3.31}$$

$$x_{uu} = f^2\left(\alpha_1^2 x_v + \alpha_2(\alpha_2^2+1)x_z + \lambda y - x\right), \tag{3.32}$$

$$x_{zz} = -\frac{2\alpha_2}{\alpha_2^2+1}x_z - \frac{1}{(\alpha_2^2+1)^2}x, \tag{3.33}$$

$$x_{vv} = \frac{1}{\alpha_1^2}\left(-(\alpha_1^2+\alpha_2^2+1-\lambda^2)x_v + \alpha_2(\alpha_2^2+1)x_z - \lambda y - x\right), \tag{3.34}$$

$$y_u = -\lambda x_u, \tag{3.35}$$

$$y_v = \lambda x_v, \tag{3.36}$$

$$y_z = 0. \tag{3.37}$$

Differentiating the equation (3.32) with respect to u, one has:

$$x_{uuu} = f^2(-\alpha_1^2 - \alpha_2^2 - \lambda^2 - 1)x_u = -x_u. \tag{3.38}$$

There are vector valued functions $A_1 \equiv A_1(v,z)$ and $A_2 \equiv A_2(v,z)$ in $\mathbb{R}^5$ depending only on v and z such that

$$x_u = -\sin(u)A_1 + \cos(u)A_2. \tag{3.39}$$

One has:

$$\begin{aligned} f^2 &= \mathrm{I}(x_u, x_u) \\ &= \langle A_1, A_1\rangle \sin^2(u) - \langle A_1, A_2\rangle \sin(2u) + \langle A_2, A_2\rangle \cos^2(u) \\ &= \frac{\langle A_2, A_2\rangle + \langle A_1, A_1\rangle}{2} + \frac{\langle A_2, A_2\rangle - \langle A_1, A_1\rangle}{2}\cos(2u) \\ &\quad - \langle A_1, A_2\rangle \sin(2u). \end{aligned}$$

Using the linear independence of the functions 1, $\cos(2u)$ and $\sin(2u)$, from the equation above we deduce

$$\langle A_1, A_1\rangle = f^2 = \langle A_2, A_2\rangle \quad \text{and} \quad \langle A_1, A_2\rangle = 0. \tag{3.40}$$

Furthermore, there is a vector valued function $A_3 \equiv A_3(v,z)$ depending only on v and z such that

$$x(u,v,z) = \cos(u)A_1(v,z) + \sin(u)A_2(v,z) + A_3(v,z). \tag{3.41}$$

One has

$$\begin{aligned} 1 &= \langle x, x\rangle \\ &= f^2 + \langle A_3, A_3\rangle + 2\langle A_1, A_3\rangle \cos(u) + 2\langle A_2, A_3\rangle \sin(u). \end{aligned}$$

Using the linear independence of the functions 1, $\cos(u)$ and $\sin(u)$, we get

$$1 = f^2 + \langle A_3, A_3\rangle \quad \text{and} \quad \langle A_1, A_3\rangle = 0 = \langle A_2, A_3\rangle. \tag{3.42}$$

We differentiate the equation (3.39) with respect to z and v and use the equations (3.30) and (3.31); then the linear independence of the functions $\cos(u)$ and $\sin(u)$ implies the following first order partial differential equations for the vector valued functions $A_1(v,z)$ and $A_2(v,z)$:

$$\frac{\partial A_1}{\partial z} = -\frac{z}{z^2+1}A_1, \quad \frac{\partial A_1}{\partial v} = -A_1, \tag{3.43}$$

$$\frac{\partial A_2}{\partial z} = -\frac{z}{z^2+1}A_2, \quad \frac{\partial A_2}{\partial v} = -A_2. \tag{3.44}$$

Therefore,

$$A_1(v,z) = f(v,z)C_1 \quad \text{and} \quad A_2(v,z) = f(v,z)C_2, \tag{3.45}$$

where $C_1, C_2 \in \mathbb{R}^5$ are constant vectors; they are orthonormal because of (3.40), and from (3.42) they are orthogonal to A_3.

Differentiating (3.41) with respect to z we have:

$$x_z = \frac{\partial f}{\partial z}(\cos(u)C_1 + \sin(u)C_2) + \frac{\partial A_3}{\partial z},$$

$$x_{zz} = \frac{\partial^2 f}{\partial z^2}(\cos(u)C_1 + \sin(u)C_2) + \frac{\partial^2 A_3}{\partial z^2}.$$

Using the equation (3.33), we get the following partial differential equation for A_3, depending only on z:

$$\frac{\partial^2 A_3}{\partial z^2} = -\frac{2z}{z^2+1}\frac{\partial A_3}{\partial z} - \frac{1}{(z^2+1)^2}A_3.$$

Therefore there are vector valued functions $A_4 \equiv A_4(v)$ and $A_5 \equiv A_5(v)$ depending only on v such that:

$$A_3(v,z) = \frac{z}{\sqrt{z^2+1}}A_4(v) + \frac{1}{\sqrt{z^2+1}}A_5(v). \tag{3.46}$$

The equation (3.41) becomes

$$\begin{aligned} x(u,v,z) &= f(v,z)\cdot(\cos(u)C_1 + \sin(u)C_2) \\ &\quad + \frac{z}{\sqrt{z^2+1}}A_4(v) + \frac{1}{\sqrt{z^2+1}}A_5(v). \end{aligned}$$

Differentiating the equation above and using (3.29), we have:

$$0 = x_{vz} + \frac{z}{z^2+1}x_v = \frac{1}{\sqrt{z^2+1}}A_4'(v).$$

Therefore the vector valued function $A_4(v)$ is constant: $A_4(v) \equiv C_3 \in \mathbb{R}^5$.

From the equation (3.42), one has

$$1 = f^2 + \frac{1}{z^2+1}\left(z^2\langle C_3, C_3\rangle + 2z\langle C_3, A_5(v)\rangle + \langle A_5(v), A_5(v)\rangle\right)$$

and thus

$$\langle C_3, C_3\rangle = 1, \quad \langle C_3, A_5(v)\rangle = 0,$$

and

$$\langle A_5(v), A_5(v)\rangle = 1 - (1+z^2)f^2 = 1 - \frac{e^{-2v}}{c_2}. \tag{3.47}$$

Eliminating y from the equations (3.32) and (3.34), we get

$$\begin{aligned} 0 &= \alpha_1^2 x_{vv} + f^{-2}x_{uu} + (z^2+1-\lambda^2)x_v - 2z(z^2+1)x_z + 2x \\ &= \sqrt{z^2+1}\left((c_2e^{2v} - 1 - c_1^2e^{4v})A_5''(v) + (1 - c_1^2e^{4v})A_5'(v) + 2A_5(v)\right). \end{aligned}$$

Therefore the vector valued function $A_5(v)$ satisfies the following linear ordinary differential equation of second order:

$$(c_2e^{2v} - 1 - c_1^2e^{4v})A_5''(v) + (1 - c_1^2e^{4v})A_5'(v) + 2A_5(v) = 0. \tag{3.48}$$

By the Lemma 3.4, one can conclude that the general solution of the equation (3.48) is

$$A_5(v) = g(v)C_4 + h(v)C_5,$$

where $C_4, C_5 \in \mathbb{R}^5$ are constant vectors, and g and h are the functions given in Remark 3.5.

But since from (3.47) the vector valued function A_5 satisfies

$$\langle A_5(v), A_5(v)\rangle = 1 - \frac{e^{-2v}}{c_2},$$

we have that C_4 and C_5 are orthonormal. They constitute together with C_1, C_2, C_3 an orthonormal basis of $\mathbb{R}^5$. This proves Proposition 3.6. □

Our classification theorem summarizes the results from Propositions 3.1–3.6.

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