Hokkaido Mathematical Journal Vol. 33 (2004) p. 47–64

$\mathcal{A}$-unimodal map-germs into the plane

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(Received October 19, 2001)

Abstract. Singularities of map-germs of the plane of $\mathcal{K}$-modality 1 were classified by Dimca and Gibson [3]. Map-germs from $\mathbb{R}^n$ ($n \geq 2$) to $\mathbb{R}^2$ of $\mathcal{A}$-modality 0 were classified in [15], here we list those with $\mathcal{A}$-modality 1 and describe their adjacencies. It turns out that any such $\mathcal{A}$-orbit of modality 1 is contained in one of the $\mathcal{K}$-orbits of type A_3, A_5 or D_4.

Key words: singularities, modality, $\mathcal{A}$-classification.

1. Introduction

The modality of a point $p \in X$ under the action of a Lie group G on X is the smallest m such that a sufficiently small neighborhood of p can be covered by a finite number of m-parameter families of orbits. The $\mathcal{A}$-modality of a map-germ f at x is the modality of an $\mathcal{A}$-sufficient jet $j^k f$ in $J^k(n,p)_{x,f(x)}$ under the action of the Lie group $\mathcal{A}^k$ of k-jets of elements of $\mathcal{A}$. Map-germs of modality 0 are said to be simple. The $\mathcal{A}$-simple corank-1 germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ were classified in [14], and the $\mathcal{A}$-simple germs of maps from $\mathbb{R}^n$ ($n \geq 2$) to $\mathbb{R}^2$ of any corank were classified in [15].

In the present paper we classify map-germs from $\mathbb{R}^n$ ($n \geq 2$) to $\mathbb{R}^2$ of $\mathcal{A}$-modality 1 (Theorem 1.1). The $\mathcal{K}$-unimodal germs from the plane to the plane were classified by Dimca and Gibson [3] and all have corank 2. It turns out that the $\mathcal{A}$-unimodal germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^2, 0$ all have rank 1, in fact they are all contained in one of the $\mathcal{K}$-orbits of type A_3, A_5 or D_4. We also list all the $\mathcal{A}$-orbits within the $\mathcal{K}$-orbit A_3 (Proposition 1.2).

We summarize our main result in the following statement. (The notation for the types of singularities in Table 1 is consistent with the one used for the simple germs in [14] and [15] and with the notation in Table 3 below. The types **I**, **II**, **III**, **IV**, $\mathbf{V}_3$, $\mathbf{V}_4$, $\mathbf{VI}_5$ and $\mathbf{VI}_7$ in Table 2 correspond to N_1, N_2, N_4, N_6, N_3, N_7, N_5 and N_{11} in the classification of germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of $\mathcal{A}_e$-codimension ≤ 4 by Nabarro [11], see also Chapter 5 of [12].

2000 Mathematics Subject Classification : 58K40, 58K50, 58K60.

We use boldface symbols for these types to distinguish them from Mather's notation for certain corank 2 $\mathcal{K}$-classes, see Section 3.)

Theorem 1.1 *Any $\mathcal{A}$-unimodal map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^2, 0$, $n \geq 2$, is $\mathcal{A}$-equivalent to one of the germs in Tables* 1 *or* 2 *(if necessary, after adding a sum of squares in some extra variables to the second component function of the map-germs in these tables). The tables show the $\mathcal{A}_e$-codimension (and, in brackets, the $\mathcal{A}_e$-codimension of the modular stratum); $c(f)$ and $d(f)$ denote the cusp and double-fold numbers, respectively.*

Table 1. $\mathcal{A}$-unimodal germs.

Type	$f(x,y) =$	cod$(\mathcal{A}_e, f)$	$c(f)$	$d(f)$
19	$(x, y^4 + x^3y + \alpha x^2y^2 + x^3y^2)$, $\alpha \neq -3/2$	5 [4]	6	3
19[−3/2]	$(x, y^4 + x^3y - 3/2 \cdot x^2y^2 + x^3y^2)$	5	7	3
22	$(x, y^4 + x^3y + \alpha x^2y^2 + x^4y^2)$, $\alpha \neq -3/2$	6 [5]	6	3
22[−3/2]	$(x, y^4 + x^3y - 3/2 \cdot x^2y^2 + x^4y^2)$	6	8	3
23	$(x, y^4 + x^3y + \alpha x^2y^2)$, $\alpha \neq -3/2$	7 [6]	6	3
24_k	$(x, y^4 + x^3y - 3/2 \cdot x^2y^2 + x^ky)$, $k \geq 6$	$k+1$	$k+3$	3
25_k	$(x, y^4 \pm x^2y^2 + x^ky)$, $k \geq 4$	$k+1$	6	k
26_k	$(x, y^4 + x^ky \pm x^{k-1}y^2)$, $k = 4, 5$ $\pm$ agree for even k	$2k-2$	$2k$	k
$27_{k,l}$	$(x, y^4 + x^ky \pm x^ly^2)$, $k = 4, 5$ $k \leq l \leq 2k-2$, $\pm$ agree for odd l	$k+l-1$	$2k$	k
28_k	$(x, y^4 + x^ky)$, $k = 4, 5$	$3k-2$	$2k$	k
$29_{3,l}$	$(x, y^4 + x^3y^2 + x^ly)$, $l \geq 5$	$l+2$	9	l
8	$(x, xy + y^6 \pm y^8 + \alpha y^9)$	4 [3]	4	6
9	$(x, xy + y^6 + y^9)$	4	4	6
20	$(x, xy + y^6 \pm y^{14})$	5	4	6
21	$(x, xy + y^6)$	6	4	6
15	$(x, xy^2 + y^6 + y^7 + \alpha y^9)$	5 [4]	5	8

Proposition 1.2 *Any $\mathcal{A}$-finite map-germ in $\mathcal{K}(x, y^4)$ is $\mathcal{A}$-equivalent to one of the germs in Table* 3. *The notation is the same as in Table* 1, *and $M(f)$ indicates the modality.*

Table 2. more $\mathcal{A}$-unimodal germs.

Type	$f(x,y,z) =$	$\text{cod}(\mathcal{A}_e, f)$	$c(f)$
I	$(x, xy + y^3 + \alpha y^2 z + z^3 \pm z^5)$, $\alpha \neq 0, \pm(27/4)^{1/3}$	3 [2]	4
I′	$(x, xy + y^3 + (27/4)^{1/3} y^2 z + z^3 \pm y^5)$	3	4
II	$(x, xy + y^3 + \alpha y^2 z + z^3)$, $\alpha \neq 0, -(27/4)^{1/3}$	4 [3]	4
III	$(x, xy + \epsilon_1 y^2 z + z^3 + \epsilon_2 z^5)$, $\epsilon_i = \pm 1$	3	4
IV	$(x, xy \pm y^2 z + z^3)$	4	4
$\mathbf{V}_k$	$(x, xy + y^3 + z^3 \pm y^k z)$, $k \geq 3$	k	$k+2$
$\mathbf{VI}_{2k+1}$	$(x, xy \pm y^3 + yz^2 + z^{2k+1})$, $k \geq 2$	$k+1$	4

Table 3. $\mathcal{A}$-orbits in $\mathcal{K}(x, y^4)$.

Type	$f(x,y) =$	$\text{cod}(\mathcal{A}_e, f)$	$M(f)$	$c(f)$	$d(f)$
5	$(x, y^4 + xy)$	1	0	2	1
11_{2k+1}	$(x, y^4 + xy^2 + y^{2k+1})$, $k \geq 2$	k	0	3	k
16	$(x, y^4 + x^2 y \pm y^5)$	3	0	4	2
17	$(x, y^4 + x^2 y)$	4	0	4	2
19	$(x, y^4 + x^3 y + \alpha x^2 y^2 + x^3 y^2)$, $\alpha \neq -3/2$	5 [4]	1	6	3
$19[-3/2]$	$(x, y^4 + x^3 y - 3/2 \cdot x^2 y^2 + x^3 y^2)$	5	1	7	3
22	$(x, y^4 + x^3 y + \alpha x^2 y^2 + x^4 y^2)$, $\alpha \neq -3/2$	6 [5]	1	6	3
$22[-3/2]$	$(x, y^4 + x^3 y - 3/2 \cdot x^2 y^2 + x^4 y^2)$	6	1	8	3
23	$(x, y^4 + x^3 y + \alpha x^2 y^2)$, $\alpha \neq -3/2$	7 [6]	1	6	3
24_k	$(x, y^4 + x^3 y - 3/2 \cdot x^2 y^2 + x^k y)$, $k \geq 6$	$k+1$	1	$k+3$	3
25_k	$(x, y^4 \pm x^2 y^2 + x^k y)$, $k \geq 4$	$k+1$	1	6	k
26_k	$(x, y^4 + x^k y \pm x^{k-1} y^2)$, $k \geq 4$	$2k-2$	1 ($k{=}4,5$)	$2k$	k
	$\pm$ agree for even k		2 ($k \geq 6$)		
$27_{k,l}$	$(x, y^4 + x^k y \pm x^l y^2)$, $k \geq 4$	$k{+}l{-}1$	1 ($k{=}4,5$)	$2k$	k
	$k \leq l \leq 2k-2$, $\pm$ agree for odd l		2 ($k \geq 6$)		
28_k	$(x, y^4 + x^k y)$, $k \geq 4$	$3k-2$	1 ($k{=}4,5$)	$2k$	k
			2 ($k \geq 6$)		
$29_{k,l}$	$(x, y^4 \pm x^k y^2 + x^l y)$	$k{+}l{-}1$	1 ($k{=}3$)	$\min(3k, 2l)$	l
	$k \geq 3$, $l \geq k+2$, $2l \neq 3k$		2 ($k \geq 4$)		
	$\pm$ agree for odd k				
$30_{k,l}$	$(x, y^4 \pm x^{2k} y^2 + \alpha x^{3k} y + x^l y)$	$2k{+}l{-}1$	2	$6k$	$3k$
	$k \geq 2$, $3k+1 \leq l \leq 6k$	$[2k{+}l{-}2]$			
	$\alpha \neq 0$; for $30^-_{k,l}$: $\alpha \neq \pm(2/3)^{3/2}$				
$30^-_{k,l}[\pm(2/3)^{3/2}]$	$(x, y^4 - x^{2k} y^2 \pm (2/3)^{3/2} x^{3k} y + x^l y)$	$2k{+}l{-}1$	2	$3k+l$	$3k$
	$k \geq 2$, $l \geq 3k+1$				
31_k	$(x, y^4 \pm x^{2k} y^2 + \alpha x^{3k} y)$, $k \geq 2$	$8k$ $[8k{-}1]$	2	$6k$	$3k$
	$\alpha \neq 0$; for 31^-_k: $\alpha \neq \pm(2/3)^{3/2}$				

Du Plessis has given a much more 'compact' classification of $\mathcal{A}$-orbits in A_3 (in which some orbits may not be distinct) by first reducing to the prenormal form $(x, y^4+P(x)y+Q(x)y^2)$ (see Prop. 4.10 in [13]). From the – rather less 'compact' – classification above, which is based on an $\mathcal{A}$-invariant stratification of the jet-space, the adjacencies between and the modalities of orbits can be determined more easily. It is also interesting to compare the above classification with the classification of C^0-$\mathcal{A}$-orbits in A_3 in [6], the latter orbits all have weighted homogeneous representatives.

2. Notation and techniques

As a starting point of the present classification we take the $\mathcal{A}^k$-orbits of positive modality at the "boundary" of the simple orbits classified in [14, 15], and determine the $\mathcal{A}^l$-orbits ($l > k$) over these, for increasing l, until an $\mathcal{A}$-sufficient orbit or an orbit of modality > 1 appears. To find the $\mathcal{A}^k$-orbits over a given $(k-1)$-jet we use a combination of coordinate changes, Mather's Lemma (Lemma 3.1 in [9]) and complete transversals (Theorem 2.9 in [2]), to determine the order of $\mathcal{A}$-determinacy we use a combination of Theorem 2.1 in [1], Corollary 3.9 in [13] and Mather's Lemma. A very brief summary of notation and concepts from determinacy theory is given below (for details we refer to the survey in [16]).

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a C^∞-germ, the group $\mathcal{A} = \mathrm{Diff}(\mathbb{R}^n, 0) \times \mathrm{Diff}(\mathbb{R}^p, 0)$ acts on the space of smooth germs f as follows: $(h, k) \cdot f = h \circ f \circ k^{-1}$, $(k, h) \in \mathcal{A}$. Let C_n and C_p denote rings of function-germs at the origin in source and target, and let m_n and m_p denote the corresponding maximal ideals. We write $J^k(n,p)$ for the space of kth-order Taylor polynomials at the origin, and $j^k f$ for the k-jet of the map f. Similarly $\mathcal{A}^k = j^k(\mathcal{A})$ denotes k-jets of elements of $\mathcal{A}$. The Lie group $\mathcal{A}^k$ acts smoothly on $J^k(n,p)$, and when we speak of equivalence of k-jets we shall always mean $\mathcal{A}^k$-equivalence. Instead of writing $T_{j^k f(0)}\mathcal{A}^k \cdot j^k f(0)$ we shall write $T\mathcal{A}^k \cdot f$. A map-germ f is said to be k-determined (for some given group of equivalences) if every map g with the same k-jet as f is equivalent to f, in that case any jet $j^l f$ with $l \geq k$ is said to be sufficient.

Let θ_f denote the C_n-module of vector fields over f (i.e. sections of $f^*T\mathbb{R}^p$). Set $\theta_n = \theta(1_{\mathbb{R}^n})$ and $\theta_p = \theta(1_{\mathbb{R}^p})$; then the homomorphisms tf and wf are defined as follows:

$$tf : \theta_n \to \theta_f, \quad tf(\psi) = df \cdot \psi,$$

(where *df* is the differential of f), and

$$wf : \theta_p \to \theta_f, \quad wf(\phi) = \phi \circ f.$$

Apart from $\mathcal{A}$, we need the groups $\mathcal{A}_1$, $\mathcal{A}_e$ and $\mathcal{K}_e$: $\mathcal{A}_1$ is the subgroup of $\mathcal{A}$ of elements whose 1-jet is the identity, $\mathcal{A}_e$ is the extended pseudo-group of non-origin-preserving diffeomorphisms, and $\mathcal{K}_e$, resp. $\mathcal{K}$, is the (pseudo-) group obtained by allowing invertible $p \times p$ matrices with entries in C_n to act on the left, the right action is the same as for $\mathcal{A}_e$, resp. $\mathcal{A}$. The following tangent spaces are associated with these latter groups: $T\mathcal{A}_e \cdot f = tf(\theta_n) + wf(\theta_p)$ and $T\mathcal{K}_e \cdot f = tf(\theta_n) + f^* m_p \cdot \theta_f$, for $\mathcal{A}$ and $\mathcal{K}$ one multiplies by the first and for $\mathcal{A}_1$ by the second powers of the relevant maximal ideals, respectively.

The modality of an orbit depends on the orbits it is adjacent to. Recall that a class of germs X is adjacent to another class Y, denoted by $X \to Y$, if any representative f of X can be embedded in an unfolding $F(u, f_u(x))$, where $f = f_0$, such that the set $\{(u,x)\}$ for which $f_u(x) \in Y$ contains $(0,0)$ in its closure. In order to rule out certain adjacencies the following $\mathcal{A}$-invariants, which are upper-semicontinuous under deformations, are useful: apart from standard invariants, like the $\mathcal{A}_e$-codimension or the Milnor number of the critical set, the cusp and double-fold numbers, denoted by $c(f)$ and $d(f)$, are such invariants associated with map-germs into the plane. For germs of rank 1 (there are no $\mathcal{A}$-unimodal germs into the plane of rank 0, see Proposition 3.1) these can be calculated as follows.

For $n=2$ and $f(x,y) = (x, g(x,y))$, we have that $c(f) = \dim C_2/\langle g_y, g_{yy}\rangle$ and $d(f) = 1/2 \cdot \dim C_3/I$, where

$$I = \langle g_y(x,y), h := t^{-2}(g(x,y+t) - g(x,y) - t \cdot g_y(x,y)), \partial h/\partial t\rangle.$$

(For germs of rank 0 there is a corresponding formula for $c(f)$, see [4], but for $d(f)$ no such formula seems to be available.)

For $n=3$ and $f(x,y,z) = (x, g(x,y,z))$, we have

$$c(f) = \dim C_3/\langle g_y, g_z, g_{yy}g_{zz} - g_{yz}^2\rangle.$$

(In the rank 1 case above the cusps are defined as complete intersections, Fukui *et al.* [5] have shown that the corresponding local ring for rank 0 germs fails to be Cohen-Macaulay. I do not have a formula for $d(f)$, not even for rank 1 germs.)

Finally, a remark on notation: X, Y denote target coordinates, $x, y, \ldots$

source coordinates, and greek letters α, β denote moduli (for "general" coefficients we use $a, b, c, \ldots$). The singularity types 1 to 19 refer to the $\mathcal{A}$-simple germs or to germs of $\mathcal{A}_e$-codimension ≤ 4 (of corank 1, from the plane to the plane) in Table 1 of [14], new additional singularities (of modality ≥ 1 and $\mathcal{A}_e$-codimension ≥ 5) are of type ≥ 20.

3. The classification

The first result shows that there are no $\mathcal{A}$-unimodal germs from n-space, $n \geq 2$, into the plane of rank 0.

Proposition 3.1 *A map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^2, 0$ of rank 0 is either $\mathcal{A}$-equivalent to some member of one of the $\mathcal{A}$-simple series of germs of type $\mathrm{I}_{2,2}^{l,m}$ or $\mathrm{II}_{2,2}^{l}$ from* [15] *or it has $\mathcal{A}$-modality ≥ 2.*

Proof. For $n \geq 5$ the $\mathcal{K}$-modality is ≥ 2 (see p. 629 of [17]). Amongst the remaining cases, we first consider $n = 2$. The $\mathcal{K}$-simple orbits were classified by Mather [10] and Lander [7] has described the adjacencies between the $\mathcal{K}$-orbits of type $\Sigma^{2,0}$, i.e. between the series $\mathrm{I}_{k,l}$, $\mathrm{II}_{k,l}$ ($l \geq k \geq 2$) and IV_k ($k \geq 3$) (note: this is Mather's notation for real $\mathcal{K}$-orbits and should not be confused with the $\mathcal{A}$-classes **I**, **II** and so on in Table 2). It has been shown in [15] that all the $\mathcal{A}$-orbits in $\mathrm{I}_{2,2}$ belong to the doubly indexed series of simple germs $\mathrm{I}_{2,2}^{l,m}$, and those in $\mathrm{II}_{2,2}$ belong to the series of simple germs $\mathrm{II}_{2,2}^{l}$. The remaining $\mathcal{K}$-orbits are either adjacent to $\mathrm{I}_{2,3}$ or to IV_3, and Lemma 2.3.3 of [15] states that all $\mathcal{A}$-orbits in $\mathrm{I}_{2,3}$ are non-simple, but the proof of this lemma actually implies that their modality is ≥ 2. Hence we can conclude the case $n = 2$ by showing that all $\mathcal{A}$-orbits in $\mathrm{IV}_3 = \mathcal{K}(x^2 + y^2, x^3)$ are at least bi-modal. A general 3-jet in IV_3 is given by

$$\sigma = (x^2 + y^2, x^3 + ax^2y + bxy^2 + cy^3),$$

and for the subspace $\mathbb{R}\{x^2y, xy^2, y^3\} \cdot \partial/\partial Y$ there is only 1 generator, namely $t\sigma(y, 0) - t\sigma(0, x) + a \cdot w\sigma(0, Y)$.

For $n = 3$ and $n = 4$ there are the following complex $\mathcal{K}$-orbits of rank 0 to which all others are adjacent to, namely $\mathcal{K}(x^2 + y^2, x^2 + z^2)$ and $\mathcal{K}(x^2 + y^2 + z^2, y^2 + \alpha \cdot z^2 + w^2)$, where $\alpha \neq 0, 1$ (the latter is usually denoted by $T_{2,2,2,2}$). The proof of Lemma 2.3.5 in [15] shows that the $\mathcal{A}$-orbits in the former have modality ≥ 2, and somewhat more lengthy calculations show that the $\mathcal{A}$-orbits in the latter have modality ≥ 4. Over the reals, the above two $\mathcal{K}$-orbits split into various real orbits, and the other real $\mathcal{K}$-orbits of rank 0 are adjacent to at least one of these. Now we argue as follows: let S be the $\mathcal{A}$-modular stratum in $\mathcal{K}(x^2+y^2, x^2+z^2)$ of minimal codimension (over $\mathbb{C}$ there is only one such connected S). Then the modality of any $\mathcal{A}$-orbit in one of the real forms of $\mathcal{K}(x^2+y^2, x^2+z^2)$ is bounded from below by the modality of S, and hence ≥ 2. The same argument applied to $\mathcal{K}(x^2+y^2+z^2, y^2+\alpha\cdot z^2+w^2)$ shows that any $\mathcal{A}$-orbit in some real form of this $\mathcal{K}$-orbit has modality ≥ 4. □

Next consider germs of rank 1: any such germ is $\mathcal{A}$-equivalent to some

$$h(x,y,z) = \left(x, g(x,y_1,\ldots,y_m) + \sum_{i=1}^{n-m-1} \epsilon_i z_i^2\right), \qquad (*)$$

where $g(0,y_1,\ldots,y_m)$ is in the third power of the maximal ideal and $\epsilon_i = \pm 1$ (see Lemma 1.1 of [15]). With m as above we have the following.

Lemma 3.2 *Any map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^2, 0$ of rank 1, which is $\mathcal{A}$-equivalent to some h as above with $m \geq 3$, has $\mathcal{A}$-modality ≥ 2.*

Proof. Take $m = 3$ and $n = 4$: there are two $\mathcal{A}^2$-orbits satisfying the conditions on h, namely (x, xy_1) and $(x, 0)$, where the latter is adjacent to the former. One checks that any $\mathcal{A}$-orbit over the first $\mathcal{A}^2$-orbit (and hence also over the second) has modality ≥ 2: note that a general 3-jet over (x, xy_1) has the form:

$$(x, xy + h(y_1, y_2, y_3)), \quad h \in H^3,$$

where H^3 is the space of cubic forms in y_1, y_2, y_3, which has dimension 10. But the subspace $H^3 \cdot \partial/\partial Y \subset T\mathcal{A}^3 \cdot f$ has only 8 generators.

Finally, increasing n, for m fixed, doesn't affect the above argument, and increasing m increases the difference between $\dim H^3$ and the number of generators. □

Note that a deformation of h, for given m and n, does not contain germs that are $\mathcal{A}$-equivalent to some h' with $m' > m$ (where h' and m' refer to a representative of the form $(*)$ above). In order to classify the $\mathcal{A}$-unimodal germs of rank 1 (and hence all $\mathcal{A}$-unimodal germs) it is therefore sufficient to consider the two cases $m = 1$, $n = 2$ and $m = 2$, $n = 3$.

3.1. Case $m = 1$ and $n = 2$

In this case, we have to determine the $\mathcal{A}$-orbits in $A_k = \mathcal{K}(x, y^{k+1})$ of modality 1. The modality of the $\mathcal{A}$-orbits in $A_{\geq 6}$ is ≥ 2, and all the $\mathcal{A}$-orbits in $A_{\leq 2}$ are simple, see [14]. We will see that the modality of the $\mathcal{A}$-orbits in A_3 is 0, 1 or 2, in A_4 it is 0 or ≥ 2 and in A_5 it is ≥ 1.

To find the $\mathcal{A}$-unimodal orbits we have to expand the following subtrees of the classification tree for corank 1 germs of the plane in [14] (see **A** to **D** below) until the modality becomes two or greater.

A. $j^2 f = (x, xy)$ (see classification tree Fig. 1 of [14]): all germs f in A_k, $k = 3$ or 4, with this 2-jet are simple, and $f = (x, xy + y^6)$ is 14-determined (Section 3.1 of [14]). Using complete transversals or Mather's lemma one easily determines the following $\mathcal{A}$-orbits over f:

$$\begin{array}{rl} (x, xy + y^6 \pm y^8 + \alpha y^9) & \text{type } 8 \\ (x, xy + y^6 + y^9) & \text{type } 9 \\ (x, xy + y^6 \pm y^{14}) & \text{type } 20 \\ (x, xy + y^6) & \text{type } 21 \end{array}$$

The determinacy degrees of these are 9, 9, 14 and 14, and the $\mathcal{A}_e$-codimensions are 4 (and 3 for the modular stratum), 4, 5 and 6.

The $\mathcal{A}$-orbit of minimal codimension within the $\mathcal{K}$-orbit A_6 is the bimodal germ $(x, xy + y^7 \pm y^9 + \alpha y^{10} + \beta y^{11})$, type 10 in [14], which has $\mathcal{A}_e$-codimension 6 – the codimension of the modular stratum being 4. The closure of this modular stratum contains all $\mathcal{A}$-orbits in $A_{\geq 6}$, the modality of these orbits is therefore ≥ 2.

B. $j^3 f = (x, xy^2)$ (see classification tree Fig. 3 of [14]): the $\mathcal{A}$-orbits above this 3-jet in A_3 and A_4 are all simple (and denoted by 11_{2k+1}, 12, 13 and 14 in [14]).

We claim that there is only one unimodal germ in A_5 (with $j^3 f = (x, xy^2)$):

$$(x, xy^2 + y^6 + y^7 + \alpha y^9) \qquad \text{type } 15,$$

having $\mathcal{A}_e$-codimension 5 (the codimension of the modular stratum is 4). Note that a complete k-transversal, $k > 6$, for $j^{k-1} f = (x, xy^2 + y^6)$ is either given by $(x, xy^2 + y^6 + cy^k)$ (for odd k) or else by $(x, xy^2 + y^6)$, and that there are two $\mathcal{A}^{2k+1}$-orbits, $k \geq 3$: $(x, xy^2 + y^6 + y^{2k+1})$ and $(x, xy^2 + y^6)$. For $k = 3$, one obtains type 15 above as the only case. Some more substantial calculations then show that the germs

$$g_k := (x, xy^2 + y^6 + y^{4k+1} + \alpha y^{4k+2} + \beta y^{4k+3}), \quad k \geq 2$$

are $(4k+3)$-determined and have modality ≥ 2. The $\mathcal{A}$-orbits over $j^{2k+1}f = (x, xy^2 + y^6 + y^{2k+1})$, where $k \geq 4$ is not a multiple of 2, lie in the closure of $\mathcal{A} \cdot g_{(k-1)/2}$ and hence have modality ≥ 2. Type 15 is therefore the only unimodal $\mathcal{A}$-orbit over $(x, xy^2 + y^6)$.

Finally, one checks that all $\mathcal{A}$-orbits in $A_{\geq 6}$ over the 3-jet (x, xy^2) belong to the closure of

$$(x, xy^2 + y^7 + y^8 + \alpha y^{10} + \beta y^{11}).$$

This is 11-determined for generic choices of (α, β), has $\mathcal{A}_e$-codimension 7 (codimension of stratum being 5) and modality ≥ 2.

C. $j^3 f = (x, x^2 y)$ (see classification tree Fig. 4 of [14]): the $\mathcal{A}$-orbits in A_3 over this 3-jet (types 16 and 17) are all simple, and those in $A_{\geq 4}$ lie in the closure of

$$(x, x^2 y + xy^3 + \alpha y^5 + \beta y^7) \qquad \text{type 18,}$$

which has $\mathcal{A}_e$-codimension 6 (the codimension of the modular stratum being 4) and modality ≥ 2.

D. $j^3 f = (x, 0)$ (see classification tree Fig. 5 of [14]): one checks that the $\mathcal{A}$-orbits in $A_{\geq 4}$ over this 3-jet belong to the closure of type 18 above and hence have modality ≥ 2.

We will now determine all the $\mathcal{A}$-orbits in A_3 over $j^3 f = (x, 0)$. These have modality 1 and 2 and, together with the simple $\mathcal{A}$-orbits in A_3 (types 5, 11_{2k+1}, 16 and 17 in [14]), yield a complete classification of $\mathcal{A}$-orbits in A_3. A general 4-jet over such a 3-jet is given by $\sigma = (x, ax^3 y + bx^2 y^2 + y^4)$, and the $\mathcal{A}^4$-orbits can be determined by integrating the vector field

$$t\sigma(x \cdot \partial/\partial x) - w\sigma(X \cdot \partial/\partial X) = 3ax^3 y \cdot \partial/\partial Y + 2bx^2 y^2 \cdot \partial/\partial Y,$$

which yields the following orbits (see 3.2.3 of [14]):

$$f_\alpha = (x, y^4 + x^3 y + \alpha x^2 y^2) \qquad \text{(a)}$$
$$(x, y^4 \pm x^2 y^2) \qquad \text{(b)}$$
$$(x, y^4) \qquad \text{(c)}$$

In case of (a) the modular stratum has one special orbit corresponding to $\alpha = -3/2$.

For future reference we record the following four cases, namely (c), (b), (a) with $\alpha = -3/2$, and (a) with $\alpha \neq -3/2$, which correspond to a stratification of the $(x^3y, x^2y^2) \cdot \partial/\partial Y$-plane with coordinates u, v into the origin, the line $u = 0$ minus the origin, the special orbit mentioned above (an open half-parabola, cutting the line $u = 1$ in $v = -3/2$ and tending to the orign) and the rest of the plane. Notice that $\mathcal{A}$-orbits lying over different 1-dimensional strata, given by the second and third case, cannot be adjacent to each other (this will be used below).

Next, one checks that f_α is 6-determined for all $\alpha \neq -3/2$ and that, in this case, there are the following $\mathcal{A}$-orbits over this 4-jet:

$$\begin{array}{rl} (x, y^4 + x^3y + \alpha x^2y^2 + x^3y^2) & \text{type } 19 \\ (x, y^4 + x^3y + \alpha x^2y^2 + x^4y^2) & \text{type } 22 \\ (x, y^4 + x^3y + \alpha x^2y^2) & \text{type } 23 \end{array}$$

The $\mathcal{A}_e$-codimensions of these are 5, 6 and 7 (the codimensions of the modular strata being 4, 5 and 6) and the determinacy degrees are 5, 6 and 6, respectively. In the first two cases the special orbits 19[−3/2] and 22[−3/2] are also 5- respectively 6-determined. Type 23 is not finitely determined for $\alpha = -3/2$, in that case it is the stem of the series

$$(x, y^4 + x^3y - 3/2 \cdot x^2y^2 + x^ky), \quad k \geq 6, \qquad \text{type } 24_k,$$

which is $(k+1)$-determined and has $\mathcal{A}_e$-codimension $k+1$.

In the case of (b) there are two $\mathcal{A}^{k+1}$-orbits, $k \geq 4$, over $j^kf = (x, y^4 \pm x^2y^2)$, namely $(x, y^4 \pm x^2y^2)$ and

$$(x, y^4 \pm x^2y^2 + x^ky) \qquad \text{type } 25_k.$$

The latter is $(k+1)$-determined and has $\mathcal{A}_e$-codimension $k+1$.

Finally, in case (c) there are the following $\mathcal{A}^k$-orbits, $k \geq 5$, over $j^4f = (x, y^4)$:

$$\begin{array}{r} (x, y^4 + x^{k-1}y \pm x^{k-2}y^2) \\ (x, y^4 + x^{k-1}y) \\ (x, y^4 \pm x^{k-2}y^2) \\ (x, y^4), \end{array}$$

where $\pm$ coincide for odd k. One checks that the first k-jet is sufficient.

Hence we have the series

$$(x, y^4 + x^k y \pm x^{k-1} y^2), \quad k \geq 4, \qquad \text{type } 26_k$$

having $\mathcal{A}_e$-codimension $2k-2$.

The second k-jet $\sigma := (x, y^4 + x^{k-1}y)$ is $(2k+1)$-determined, and there are the following $\mathcal{A}$-orbits over σ:

$$\begin{array}{lll} (x, y^4 + x^k y \pm x^l y^2), \; k \geq 4, & k \leq l \leq 2k-2 & \text{type } 27_{k,l} \\ (x, y^4 + x^k y), & k \geq 4, & \text{type } 28_k \end{array}$$

having $\mathcal{A}_e$-codimension $k+l-1$ and $3k-2$, respectively. The orbits $27^{\pm}_{k,l}$ agree for odd l and are $(l+2)$-determined.

Taking $(x, y^4 \pm x^k y^2)$, $k \geq 3$, as a representative of the third orbit above we find the following $\mathcal{A}^l$-orbits, $l \geq k+2$:

$$\begin{array}{c} (x, y^4 \pm x^k y^2 + x^l y), \quad 2l \neq 3k \\ (x, y^4 \pm x^k y^2 + \alpha x^{3k/2} y) \\ (x, y^4 \pm x^k y^2). \end{array}$$

The first jet is sufficient, giving the doubly indexed series $29_{k,l}$, where $k \geq 3$, $l \geq k+2$, $2l \neq 3k$ and where $\pm$ agree for odd k. Rewriting the second jet as

$$f_k^{\pm} = (x, y^4 \pm x^{2k} y^2 + \alpha x^{3k} y), \quad k \geq 2,$$

and using the weighted homogeneity of $f_k^{\pm}$, one shows that it is $(6k+1)$-determined for $\alpha \neq 0$ (in case of f_k^+) and for $\alpha \neq 0, \pm(2/3)^{3/2}$ (in case of f_k^-). For such generic choices of α we then find the following $\mathcal{A}$-orbits over the $(3k+1)$-jet $f_k^{\pm}$:

$$\begin{array}{lll} (x, y^4 \pm x^{2k} y^2 + \alpha x^{3k} y + x^l), \; k \geq 2, & 3k < l \leq 6k, & \text{type } 30_{k,l} \\ (x, y^4 \pm x^{2k} y^2 + \alpha x^{3k} y), & k \geq 2, & \text{type } 31_k \end{array}$$

These are $(l+1)$- and $(6k+1)$-determined, respectively.

It remains to investigate the special values of the modulus α: for $\alpha = 0$ we are back to one of the cases already considered, and for $\alpha = \pm(2/3)^{3/2}$ we find the following doubly indexed series, type $30^-_{k,l}[\pm(2/3)^{3/2}]$, over the $(3k+1)$-jet f_k^-:

$$(x, y^4 - x^{2k} y^2 \pm (2/3)^{3/2} x^{3k} y + x^l y), \quad k \geq 2, \; l \geq 3k+1.$$

This is $(l+1)$-determined, and completes the classification of $\mathcal{A}$-orbits in A_3 (Proposition 1.2). It also completes the expansion of the classification subtrees **A** to **D**, all $\mathcal{A}$-orbits further down these subtrees have modality ≥ 2.

Amongst the orbits in **A** to **D** of modality ≥ 1 we now have to find the ones of modality 1, we also determine their adjacencies. In order to rule out certain adjacencies we calculate the cusp and double-fold numbers $c(f)$ and $d(f)$ (using the formulas in Section 2), the Milnor numbers of the critical sets and the local multiplicities of the germs f. All these invariants are upper-semicontinuous (for $c(f)$ and $d(f)$ the results of these calculations are shown in Table 1, the other to invariants are very easy to calculate). Notice that the $\mathcal{A}$-orbits in A_k, for $k \neq 3, 5$, are either simple or have modality ≥ 2, hence we only have to consider A_3 and A_5 further.

The only $\mathcal{A}$-orbits in A_5, whose modality could be less than 2, are those with 2-jet (x, xy) (types 8, 9, 20 and 21) and type 15. The Milnor numbers of the critical sets of all these germs is ≤ 1 and therefore smaller than that of any non-simple germ in $A_{\leq 4}$. These orbits are therefore not adjacent to any non-simple orbit in $A_{\leq 4}$, and the adjacencies between these orbits is shown in Table 4. For brevity we use the following conventions in the adjacency diagrams: (i) when two classes of germs X and Y have several real forms (differing by some $\pm$ signs) then $X \leftarrow Y$ means that each real form of Y is adjacent to all real forms of X unless the contrary is stated, (ii) we don't show the simple orbits to which a given unimodal orbit is adjacent to. In the diagrams the $\mathcal{A}_e$-codimensions of the modular strata are increasing from left to right.

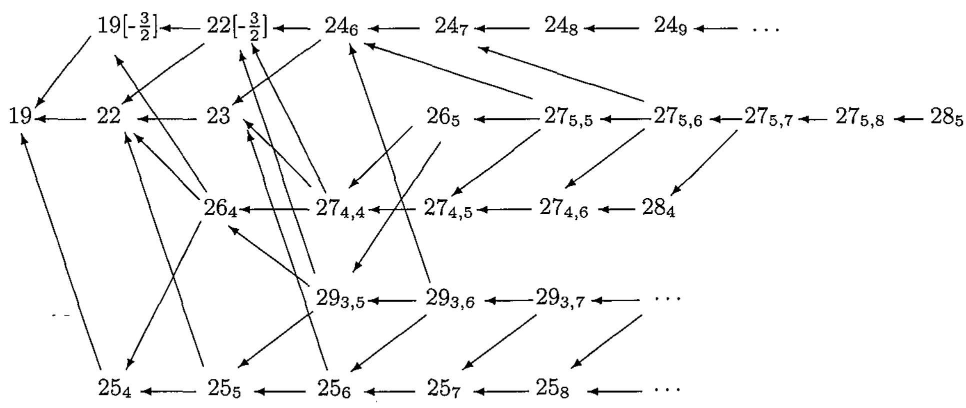
Table 4. Adjacencies between $\mathcal{A}$-unimodal orbits in A_5.

$$\begin{array}{ccccccc} 8 & \leftarrow & 9 & \leftarrow & 20 & \leftarrow & 21 \\ & \nwarrow & & & & & \\ & & 15 & & & & \end{array}$$

The adjacencies and the normal forms in Table 1 imply that all these orbits are unimodal. The adjacencies between the germs having 6-jet $(x, xy+y^6)$ follow trivially from the conditions for the membership in the corresponding $\mathcal{A}^k$-orbits, $7 \leq k \leq 14$, over this 6-jet. The adjacency $8 \leftarrow 15$ can be checked by deforming the germ of type 15 by a term $(0, t \cdot xy)$ and by verifying that, for $t \neq 0$, this is equivalent to $8^{\pm}$ by coordinate changes. For the more extensive adjacency diagrams below, we will suppress such routine arguments.

Now consider the $\mathcal{A}$-orbits in A_3 (see Table 3), these can only be adjacent to $\mathcal{A}$-orbits in $A_{\leq 3}$ and all $\mathcal{A}$-orbits in $A_{\leq 2}$ are simple. The $\mathcal{A}$-orbits of modality 2 all belong to the closure of type $30_{2,7}$. Those $\mathcal{A}$-orbits in the closure of type 19, which do not also belong to the closure of type $30_{2,7}$, are all unimodal and their adjacencies are shown in Table 5. The following rules out most of the *a priori* possible adjacencies: the upper semi-continuity of the cusp and double-fold numbers shown in Table 1, the non-adjacency of $\mathcal{A}$-orbits arising in the subcases (a), with $\alpha = -3/2$, and (b) of **D** (recall our remark above) and the non-adjacency between the members of the series $29_{3,l}$ and any of the orbits $27_{4,m}$ ($m = 4, 5, 6$) and 28_4 (which is due to the structure of the $\mathcal{A}^5$-orbits over $j^4 f = (x, y^4)$). The possible adjacencies that remain can be checked by tedious calculations (using $\mathcal{A}$-versal unfoldings and coordinate changes), which show that all but three actually do occur. (The three adjacencies that do not occur are: $26_5 \to 24_6$, $27_{5,5} \to 24_7$ and $29_{3,5} \to 23$.) In an appendix we shall list bases for the normal spaces of the series of germs found in the present paper (Table 7), these determine the $\mathcal{A}$-versal unfoldings used in the adjacency calculations.

Table 5. Adjacencies between $\mathcal{A}$-unimodal orbits in A_3.



3.2. Case $m = 2$ and $n = 3$

We take Nabarro's classification [11] of germs of the form $f = (x, g(x, y, z))$, where $g(0, y, z) \in m_n^3$, of $\mathcal{A}_e$-codimension ≤ 4 as our starting point. The proof of Theorem 2.3 in [11] implies that there are two $\mathcal{A}^2$-orbits, namely $(x, 0)$ and (x, xy), and that any $\mathcal{A}$-orbit over the former is at least trimodal (because it lies in the closure of the orbit of N_{12}, which has 3 moduli). Amongst the nine $\mathcal{A}^3$-orbits over the 2-jet (x, xy) listed in [11] the following four lie in the $\mathcal{K}$-orbit D_4 and lead to $\mathcal{A}$-unimodal orbits:

$$(x, xy + y^3 + \alpha y^2 z + z^3),\ \alpha \neq 0, \qquad \text{(a)}$$
$$(x, xy + y^3 + z^3) \qquad \text{(b)}$$
$$(x, xy \pm y^2 z + z^3) \qquad \text{(c)}$$
$$(x, xy \pm y^3 + yz^2) \qquad \text{(d)}$$

One checks that the other five $\mathcal{A}^3$-orbits lead to $\mathcal{A}$-orbits of modality ≥ 2 which lie in the closure of D_5.

Over the 3-jet in (a) we find the following $\mathcal{A}^4$-orbits:

$$f_\alpha = (x, xy + y^3 + \alpha y^2 z + z^3), \quad \alpha \neq -(27/4)^{1/3}$$
$$(x, xy + y^3 - (27/4)^{1/3} y^2 z + z^3 + z^4)$$
$$(x, xy + y^3 - (27/4)^{1/3} y^2 z + z^3).$$

According to [11] the first of these is 5-determined. General 5-jets over f_α are given by $(x, xy + y^3 + \alpha y^2 z + z^3 + cz^5)$ (for $\alpha \neq (27/4)^{1/3}$) and $(x, xy + y^3 + (27/4)^{1/3} y^2 z + z^3 + cy^5)$ (for $\alpha = (27/4)^{1/3}$). This yields the following $\mathcal{A}$-orbits:

$$(x, xy + y^3 + \alpha y^2 z + z^3 \pm z^5),\ \alpha \neq 0, \pm(27/4)^{1/3}, \qquad \text{type } \mathbf{I}$$
$$(x, xy + y^3 + (27/4)^{1/3} y^2 z + z^3 \pm y^5) \qquad \text{type } \mathbf{I'}$$
$$(x, xy + y^3 + \alpha y^2 z + z^3),\ \alpha \neq 0, -(27/4)^{1/3}, \qquad \text{type } \mathbf{II}$$

Note that type **I** corresponds to N_1 in [11] and that the union of types **I** and **I′** form a unimodal stratum for which there is no global normal form (the orbit **I′** is "special", because the z^5-term has to be replaced by y^5). When the coefficients c of both z^5 and y^5 vanish, we can combine both cases again to a single normal form **II**, which corresponds to N_2 in [11]. The $\mathcal{A}_e$-codimensions of **I**, **I′** and **II** are 3 (2 for modular stratum), 3 and 4 (3 for modular stratum).

The third $\mathcal{A}^4$-orbit above lies in the closure of the second, and the second is equivalent to:

$$(x, 27xy + 27y^3 - 27y^2z + 4z^3 + z^4).$$

Above this there is a single $\mathcal{A}^6$-orbit, namely $h_\beta = (x, 27xy+27y^3-27y^2z+4z^3 + z^4 + \beta y^6)$, which is 6-determined and has $\mathcal{A}_e$-codimension 4 (the codimension of the modular stratum being 3). The orbit of h_β, which is adjacent to the unimodal germ **I**, is at least bimodal and has cusp number $c(h_\beta) = 5$.

Over the 3-jet in case (b) we find the $\mathcal{A}^{k+1}$-orbits, $k \geq 3$, given by $(x, xy + y^3 + z^3)$ and $(x, xy + y^3 + z^3 \pm y^kz)$. The latter is sufficient, hence we obtain the series:

$$(x, xy + y^3 + z^3 \pm y^kz), \quad k \geq 3, \qquad \text{type } \mathbf{V}_k,$$

having $\mathcal{A}_e$-codimension k ($\mathbf{V}_3$ and $\mathbf{V}_4$ correspond to N_3 and N_7 in [11]).

The $\mathcal{A}$-orbits over the 3-jet in case (c) have been classified completely in [11]: they are of type $N_4 = \mathbf{III}$ and $N_6 = \mathbf{IV}$, which have $\mathcal{A}_e$-codimension 3 and 4, respectively, and are both 5-determined.

Finally, one checks that the $\mathcal{A}$-orbits over the 3-jet in case (d) all belong to the series:

$$(x, xy \pm y^3 + yz^2 + z^{2k+1}), \quad k \geq 2, \qquad \text{type } \mathbf{VI}_{2k+1}.$$

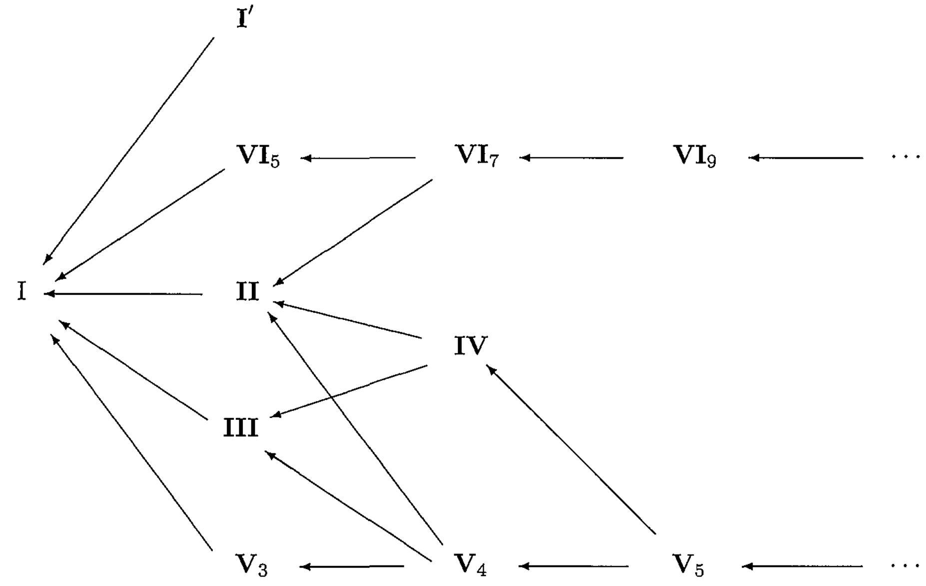
This is $(2k + 1)$-determined and has $\mathcal{A}_e$-codimension $k + 1$. The first two members of this series correspond to types N_5 and N_{11} in [11].

All the germs in Table 2 lie in the closure of **I**, their $\mathcal{A}$-modality is therefore at least 1, and their $\mathcal{K}$-type is D_4. Hence they can only be adjacent to $\mathcal{A}$-orbits within the $\mathcal{K}$-orbits of type $A_{\leq 3}$ and D_4. All $\mathcal{A}$-orbits in $A_{\leq 2}$ are simple, and the $\mathcal{K}$-types of the critical sets of all non-simple $\mathcal{A}$-orbits in A_3 are not of type A_k, for any k. On the other hand, the critical sets of all the germs in Table 2 are of type A_k (in the case of $\mathbf{V}_k$ of type A_{k-1}, in all other cases of type A_1) – hence these germs can only be adjacent to $\mathcal{A}$-simple orbits in $A_{\leq 3}$. One calculates that the cusp numbers of the germs in Table 2 are 4, except for type $\mathbf{V}_k$, where $c(\mathbf{V}_k) = k + 2$. Recall that the $\mathcal{A}$-orbits in D_4 in the closure of

$$h_\beta = (x, 27xy + 27y^3 - 27y^2z + 4z^3 + z^4 + \beta y^6)$$

above are at least bimodal and have at least 5 cusps. These orbits lie over the special orbit $\alpha = -(27/4)^{1/3}$ of the closure of the **I** stratum, whereas the $\mathbf{V}_k$ orbits lie over the special orbit $\alpha = 0$. Hence none of the $\mathbf{V}_k$ is adjacent to the closure of $\mathcal{A} \cdot h_\beta$, and none of the other germs in Table 2 is adjacent to $\mathcal{A} \cdot h_\beta$ because of the upper semicontinuity of the cusp number. It now follows that all germs in Table 2 are $\mathcal{A}$-unimodal. Table 6 below shows the adjacencies between these unimodal germs.

Table 6. Adjacencies between $\mathcal{A}$-unimodal orbits in D_4.



Appendix: $\mathcal{A}$-normal spaces for series of germs

Here we list the normal spaces $N\mathcal{A} \cdot f := m_n\theta_f/T\mathcal{A} \cdot f$ for the series of non-simple germs f (the normal spaces for the exceptional germs, not belonging to a series, can easily be calculated and are not listed to economize on space). Note that, in the adjacency calculations, it is more convenient to work with (origin preserving) $\mathcal{A}$-versal unfoldings.

Table 7. $\mathcal{A}$-normal spaces of non-simple series.

Series	basis for normal space
$24_k,\ k \geq 6$	$\mathbb{R}\{y, y^2, y^3, xy^2, xy, \ldots, x^{k-1}y\} \cdot \partial/\partial Y$
$25_k,\ k \geq 4$	$\mathbb{R}\{y, y^2, y^3, xy^2, xy, \ldots, x^{k-1}y\} \cdot \partial/\partial Y$
$26_k,\ k \geq 4$	$\mathbb{R}\{y, y^2, y^3, xy, \ldots, x^{k-1}y, xy^2, \ldots, x^{k-2}y^2\} \cdot \partial/\partial Y$
$27_{k,l}$	$\mathbb{R}\{y, y^2, y^3, xy, \ldots, x^{k-1}y, xy^2, \ldots, x^{l-1}y^2\} \cdot \partial/\partial Y$
	$k \geq 4,\ k \leq l \leq 2k-2$
$28_k,\ k \geq 4$	$\mathbb{R}\{y, y^2, y^3, xy, \ldots, x^{k-1}y, xy^2, \ldots, x^{2k-2}y^2\} \cdot \partial/\partial Y$
$29_{k,l}$	$\mathbb{R}\{y, y^2, y^3, xy, \ldots, x^{l-1}y, xy^2, \ldots, x^{k-1}y^2\} \cdot \partial/\partial Y$
	$k \geq 3,\ l \geq k+2,\ 2l \neq 3k$
$30_{k,l}$	$\mathbb{R}\{y, y^2, y^3, xy, \ldots, x^{l-1}y, xy^2, \ldots, x^{k-1}y^2\} \cdot \partial/\partial Y$
	$k \geq 2,\ 3k+1 \leq l \leq 6k$
$30^-_{k,l}[\pm(2/3)^{3/2}]$	$\mathbb{R}\{y, y^2, y^3, xy, \ldots, x^{l-1}y, xy^2, \ldots, x^{k-1}y^2\} \cdot \partial/\partial Y$
	$k \geq 2,\ l \geq 3k+1$
$31_k,\ k \geq 2$	$\mathbb{R}\{y, y^2, y^3, xy, \ldots, x^{6k}y, xy^2, \ldots, x^{k-1}y^2\} \cdot \partial/\partial Y$
$\mathbf{V}_k,\ k \geq 3$	$\mathbb{R}\{y, y^2, z, z^2, yz, \ldots, y^{k-1}z\} \cdot \partial/\partial Y$
$\mathbf{VI}_{2k+1},\ k \geq 2$	$\mathbb{R}\{y, z, y^2, yz, z^2; z^3, z^5, z^7, \ldots, z^{2k-1}\} \cdot \partial/\partial Y$

Acknowledgement I am grateful to Ana Nabarro for valuable correspondence about the classification of map-germs from 3-space into the plane.

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