Global stability in discrete models of nonautonomous Lotka-Volterra type

Yoshiaki MUROYA

(Received February 14, 2002; Revised October 17, 2002)

Abstract. In this paper, we establish sufficient conditions for the global asymptotic stability of the following discrete models of nonautonomous Lotka-Volterra type:

\[
N_i(p + 1) = N_i(p) \exp \left\{ c_i(p) - a_i(p)N_i(p) - \sum_{j=1}^{n} \sum_{l=0}^{m} a_{ij}^l(p)N_j(p - k_l) \right\},
\]

\[1 \leq i \leq n, \text{ for } p = 0, 1, 2, \ldots,\]

\[N_i(p) = N_{ip} \geq 0, \text{ for } p \leq 0, \text{ and } N_{i0} > 0, 1 \leq i \leq n,\]

where each \(c_i(p), a_i(p)\) and \(a_{ij}^l(p)\) are bounded for \(p \geq 0\) and

\[
\inf_{p \geq 0} a_i(p) > 0, \quad a_{ii}^0(p) \equiv 0, \quad a_{ij}^l(p) \geq 0, 1 \leq i \leq j \leq n, 0 \leq l \leq m,
\]

\[k_0 = 0, \text{ integers } k_l \geq 0, 1 \leq l \leq m.\]

We establish a condition for the permanence of system and applying the former work (2002, J. Math. Anal. Appl. 273 492–511) to this system, we improve a known condition for the global asymptotic stability of system.

Key words: permanence, global asymptotic stability, discrete model of nonautonomous Lotka-Volterra type.

1. Introduction

Consider the following discrete system of nonautonomous Lotka-Volterra type:

\[
N_i(p + 1) = N_i(p) \exp \left\{ c_i(p) - a_i(p)N_i(p) - \sum_{j=1}^{n} \sum_{l=0}^{m} a_{ij}^l(p)N_j(p - k_l) \right\},
\]

\[1 \leq i \leq n, \text{ for } p = 0, 1, 2, \ldots,\]

\[N_i(p) = N_{ip} \geq 0, \text{ for } p \leq 0, \text{ and } N_{i0} > 0, 1 \leq i \leq n,\]

(1.1)

2000 Mathematics Subject Classification : 34D23.

1Research partially supported by Waseda University Grant for Special Research Projects 2001A-571.
where we assume that each \( c_i(p), a_i(p) \) and \( a_{ij}^l(p) \) are bounded for \( p \geq 0 \) and
\[
\begin{cases}
\inf_{p \geq 0} a_i(p) > 0, & a_{ii}^0(p) \equiv 0, & a_{ij}^l(p) \geq 0, \\
1 \leq i \leq j \leq n, & 0 \leq l \leq m, \\
k_0 = 0, & \text{integers } k_l \geq 0, & 1 \leq l \leq m.
\end{cases}
\tag{1.2}
\]

For the system (1.1)–(1.2), there are several literatures in autonomous cases. In the case of a prey-predator system for \( n = 2 \) and \( m = 0 \), or the two species are competitive, then Hofbauer, Hutson and Jansen [7] offered a theorem that the existence of positive equilibrium in the system guarantees its permanence. But Lu and Wang [8] show that if the system is cooperative, it can not be permanent in any case. For no delay case \( m = 0 \), Lu and Wang [8] also give sufficient conditions for permanence. In the case \( n = 2 \) and any \( m \geq 0 \), Saito, Ma and Hara [16] and Saito, Hara and Ma [15] generalized them and established the necessary and sufficient conditions for permanence (see also Muroya [11]).

On the other hand, Wang and Lu [19] and Wang et al. [20] fined further conditions in the case of prey-predator and competitive system for \( n = 2 \) and \( m \geq 0 \), to ensure that the discrete system is globally asymptotically stable.

Recently, for the cases \( n \geq 2 \) and \( m \geq 0 \), applying the techniques offered by Ahmad and Lazer [2] and Muroya [12], Muroya [13] established sufficient conditions for the persistence and global asymptotic stability of the system (1.1)–(1.2).

In this paper, using results in Muroya [13] to the discrete system (1.1)–(1.2) of nonautonomous Lotka-Volterra type, we establish a condition for the permanence of system (cf. Wang et al. [20]). For the global asymptotic stability of the system (1.1)–(1.2), we apply also Muroya’s results in [13]. In particular, for the autonomous case of \( n = 1 \), we improve the result offered by Muroya [9] for conditions of the global asymptotic stability. This is other type condition than those established by So and Yu [17] and Muroya [10] (see Remark 2.4).

For a given sequence \( \{g(p)\}_{p=0}^\infty \), we set
\[
\begin{cases}
g_{M} = \sup \{ g(p) \mid p = 0, 1, 2, \ldots \}, \\
g_{L} = \inf \{ g(p) \mid p = 0, 1, 2, \ldots \},
\end{cases}
\tag{1.3}
\]
and put
\[
\begin{align*}
& a_{ijL}^l = a_{ijL}^{-} + a_{ijL}^{+}, \quad a_{ijL}^{-} \leq a_{ijL}^{+}, \\
& a_{ijM}^l = a_{ijM}^{-} + a_{ijM}^{+}, \quad a_{ijM}^{-} \leq a_{ijM}^{+}, \\
& b_{ijL}^{-} = \sum_{l=0}^{m} a_{ijL}^{-}, \quad b_{ijM}^{-} = \sum_{l=0}^{m} a_{ijM}^{-}, \quad b_{ijM}^{+} = \sum_{l=0}^{m} a_{ijM}^{+}, \\
& 1 \leq i, j \leq n, \\
& A_L = \text{diag}(a_{1L}, a_{2L}, \ldots, a_{nL}), \quad B_L^{-} = [b_{ijL}^{-}], \quad B_M^{+} = [b_{ijM}^{+}] \quad \text{and} \\
& D_M^{+} = \text{diag}(b_{11M}^{+}, b_{22M}^{+}, \ldots, b_{nnM}^{+}) \quad \text{are } n \times n \text{ matrices, and} \\
& c_L = [c_{iL}] \quad \text{and} \quad c_M = [c_{iM}] \quad \text{are } n\text{-dimensional vectors.}
\end{align*}
\]

For the system (1.1)–(1.2), assume the following condition:
\[
(A_L + B_L^{-})^{-1} c_M > 0. \tag{1.5}
\]

For (1.4), put
\[
\begin{align*}
& \tilde{c}_{iM} = c_{iM} - \sum_{j=1}^{i-1} b_{ijL}^{-} \tilde{N}_j, \\
& \tilde{N}_i = \begin{cases} \\
\tilde{c}_{iM}/a_{iL}, & \tilde{c}_{iM} \leq 1, \\
\tilde{c}_{iM}^{-1}/a_{iL}, & \tilde{c}_{iM} > 1,
\end{cases} \\
& \bar{a}_{iL} = \min\left(a_{iL}, \frac{2}{\tilde{N}_i} - a_{iM}\right) > 0, \quad 1 \leq i \leq n, \\
& \bar{A}_L = \text{diag}(\bar{a}_{1L}, \bar{a}_{2L}, \ldots, \bar{a}_{nL}), \\
& \text{and} \quad \bar{N} = [\tilde{N}_i].
\end{align*}
\]  

We refer that the system (1.1)–(1.2) is permanent, if there are positive constants $\delta$ and $\gamma$ such that
\[
0 < \delta \leq \liminf_{p \to \infty} N_i(p) \leq \limsup_{p \to \infty} N_i(p) \leq \gamma < +\infty, \\
1 \leq i \leq n. \tag{1.7}
\]

We shall establish the following results to the system (1.1)–(1.2).

**Theorem 1.1** Assume the conditions (1.5) and
\[
c_{iL} - \sum_{j=1}^{i-1} b_{ijM}^{-} \tilde{N}_j - \sum_{j \neq i} b_{ijM}^{+} \tilde{N}_j > 0, \quad 1 \leq i \leq n, \tag{1.8}
\]
where
\[
\begin{align*}
\bar{a}_{iM} &= a_{iM} + \sum_{l=1}^{m} a_{ilM}^{l} \\
&\quad \times \exp\left\{-k_{l}\left(c_{iL} - a_{iM} \bar{N}_{i} - \sum_{j=1}^{i-1} b_{ijM}^{j} \bar{N}_{j} - \sum_{j=1}^{n} b_{ijM}^{+} \bar{N}_{j}\right)\right\}, \\
\bar{N}_{i} &= \left(c_{iL} - \sum_{j=1}^{i-1} b_{ijM}^{j} \bar{N}_{j} - \sum_{j \neq i}^{n} b_{ijM}^{+} \bar{N}_{j}\right)/\bar{a}_{iM}, \\
N_{i} &= \min\left(\bar{N}_{i}, \bar{N}_{i} \exp\left\{\left(c_{iL} - \sum_{j=1}^{i-1} b_{ijM}^{j} \bar{N}_{j} - \sum_{j \neq i}^{n} b_{ijM}^{+} \bar{N}_{j}\right) - \bar{a}_{iM} \bar{N}_{i}\right\}\right) > 0, \ 1 \leq i \leq n.
\end{align*}
\] (1.9)

Then, the system (1.1)–(1.2) is permanent.

Moreover, if
\[
\bar{A}_{L} - (B_{M}^{+} - B_{M}^{-}) \text{ is an } M\text{-matrix},
\] (1.10)
then for any two solutions \(\{M_{i}(p)\}_{p=0}^{\infty}\) and \(\{N_{i}(p)\}_{p=0}^{\infty}\), 1 ≤ i ≤ n of the system (1.1)–(1.2), it holds
\[
\lim_{p \to \infty} (M_{i}(p) - N_{i}(p)) = 0, \ 1 \leq i \leq n.
\] (1.11)

The organization of this paper is as follows. In Section 2, for the permanence of the system (1.1)–(1.2), in addition to upper bounds \(\bar{N}_{i}\) of \(\limsup_{p \to \infty} N_{i}(p)\), 1 ≤ i ≤ n of Lemma 2.2 in Muroya [13], we obtain new lower bounds \(\underline{N}_{i} > 0\) of \(\liminf_{p \to \infty} N_{i}(p)\), 1 ≤ i ≤ n (see Theorem 2.1). For the global asymptotic stability of the system (1.1)–(1.2), applying the recent results in Muroya [13], we establish Theorem 1.1. In particular, for the special case of n = 1, we find an improved result than those of Muroya [9] and [10] (cf. So and Yu [17]).

2. Conditions of permanence and global asymptotic stability

Using the techniques of Muroya [13], we consider the permanence and the global asymptotic stability of the discrete system (1.1)–(1.2) of nonautonomous Lotka-Volterra type.

Lemma 2.1 For the system (1.1)–(1.2),
\[ N_i(p) = N_i(0) \exp \left( \sum_{q=0}^{p-1} \left\{ c_i(q) - a_i(q)N_i(q) - \sum_{j=1}^{n} \sum_{l=0}^{m} a_{ij}^l(q)N_j(q - k_l) \right\} \right), \]
\[ p \geq 1, \quad 1 \leq i \leq n, \quad (2.1) \]

and every solutions \( \{N_i(p)\}_{p=0}^{\infty}, 1 \leq i \leq n \) exist and remain positive for all \( p = 0, 1, 2, \ldots \).

**Proof.** By (1.1), we obtain (2.1), from which we get the conclusion. \( \square \)

**Remark 2.1** Consider the following differential equations with piecewise constant delays:

\[
\begin{aligned}
\frac{dx_i(t)}{dt} &= x_i(t) \left\{ c_i([t]) - a_i([t])x_i([t]) - \sum_{j=1}^{n} \sum_{l=0}^{m} a_{ij}^l([t])x_j([t - k_l]) \right\}, \\
&t \geq 0, \quad 1 \leq i \leq n, \\
x_i(p) &= \phi_i(p) \geq 0, \quad \text{for } p \leq 0, \quad \text{and } \phi_i(0) > 0, \quad 1 \leq i \leq n,
\end{aligned}
\]

where \([t]\) denotes the maximal integer less than or equal to \( t \) and \( \phi_i(p) = N_{ip}, \quad \text{for } p \leq 0. \)

Then, we easily see that for any \( p < t < p + 1, \) for \( p \geq 0, \)

\[
\frac{d}{dt} \left\{ \frac{1}{x_i(t)} \exp \left( \int_{p}^{t} \left\{ c_i(s) - a_i(s)x_i(s) - \sum_{j=1}^{n} \sum_{l=0}^{m} a_{ij}^l(s)x_j(s - k_l) \right\} ds \right) \right\} = 0.
\]

Thus, integrating both sides with respect to \( t \) on \([p, p + 1], \) we obtain (1.1) and \( N_i(p) = x_i(p), \) for \( p = 0, 1, \ldots. \)

For the permanence of the system (1.1)–(1.2), in addition to upper bounds \( \bar{N}_i \) of \( \limsup_{p \to \infty} N_i(p), 1 \leq i \leq n \) of Lemma 2.2 in Muroya [13], we obtain new lower bounds \( \underline{N}_i \) \( \geq 0 \) of \( \liminf_{p \to \infty} N_i(p), 1 \leq i \leq n. \)

**Theorem 2.1** Under the condition

\[ (A_L + B_L)^{-1}c_M > 0, \]

for any solutions \( N_i(p), 1 \leq i \leq n \) of the system (1.1)–(1.2), it holds that

\[ \limsup_{p \to \infty} N_i(p) \leq \bar{N}_i, \quad 1 \leq i \leq n, \quad (2.5) \]
where $\tilde{N}_i$, $1 \leq i \leq n$ are defined by (1.6).

Moreover, if (1.8) holds, then

$$\liminf_{p \to \infty} N_i(p) \geq \tilde{N}_i, \quad 1 \leq i \leq n,$$

(2.6)

where $\tilde{N}_i$, $1 \leq i \leq n$ are defined by (1.9).

Proof. Since $A_L + B_L^-$ is an $M$-matrix, it is well known that there is a diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ such that $d_i > 0$, $1 \leq i \leq n$ and $(A_L + B_L^-)D$ is a diagonally dominant matrix.

Thus, we may assume, without loss of generality, that $A_L + B_L^-$ is diagonally dominant, that is, $a_{iL} > 0$ and $a_{iL} + \sum_{j=1}^{i-1} b_{ijL} > 0$, $1 \leq i \leq n$.

Then, (2.5) follows from the proof of Lemma 2.2 in Muroya [13].

Hence, for any $\epsilon > 0$, there exists a constant $\tilde{p}_n \geq \max_{0 \leq i \leq m} k_i$ such that

$$N_i(p) \leq \tilde{N}_i + \epsilon, \quad p \geq \tilde{p}_n, \quad 1 \leq i \leq n.$$

Moreover, assume (1.8) and for some $1 \leq i \leq n$, suppose that for any sufficiently small fixed constant $\epsilon > 0$, there exists a constant $\tilde{p}_i \geq \tilde{p}_n$ such that

$$\liminf_{p \to \infty} N_j(p) \geq N_j - \epsilon > 0, \quad 1 \leq j \leq i - 1.$$

If $N_i(p) < \tilde{N}_i - \left\{(-\sum_{j=1}^{i-1} b_{ijM}^- + \sum_{j \neq i} b_{ijM}^+)/\tilde{a}_{iM}\right\} \epsilon$ for some $p \geq \tilde{p}_n$, then by (1.1) and the definition of $\tilde{N}_i$,

$$N_i(p + 1) \geq N_i(p) \exp\left\{c_{iL} - \sum_{j=1}^{i-1} b_{ijM}^- (N_j - \epsilon) - \sum_{j \neq i} b_{ijM}^+ (\tilde{N}_j + \epsilon) - \tilde{a}_{iM} N_i(p)\right\} > N_i(p).$$

For the case that $N_i(p)$ is eventually increasing and bounded upper by $\tilde{N}_i - \left\{(-\sum_{j=1}^{i-1} b_{ijM}^- + \sum_{j \neq i} b_{ijM}^+)/\tilde{a}_{iM}\right\} \epsilon$, we apply the similar discussions to the case $i = 1$ in the proof of Lemma 2.2 in Muroya [13], and we get

$$\liminf_{p \to \infty} N_i(p) = \tilde{N}_i - \left\{(-\sum_{j=1}^{i-1} b_{ijM}^- + \sum_{j \neq i} b_{ijM}^+)/\tilde{a}_{iM}\right\} \epsilon.$$
On the other hand, if \( N_i(p) \geq \bar{N}_i - \left\{ \left( - \sum_{j=1}^{i-1} b_{ijM}^- + \sum_{j \neq i} b_{ijM}^+ \right) / \bar{a}_{iM} \right\} \epsilon \) for some \( p \geq \bar{p}_n \) then by (1.1)

\[
N_i(p + 1) \geq N_i(p) \exp \left\{ \left( c_{iL} - \sum_{j=1}^{i-1} b_{ijM}^- N_j - \sum_{j \neq i} b_{ijM}^+ \bar{N}_j \right) - \bar{a}_{iM} N_i(p) \right\} \times \exp \left\{ - \left( - \sum_{j=1}^{i-1} b_{ijM}^- + \sum_{j \neq i} b_{ijM}^+ \right) \epsilon \right\}.
\]

Since for \( a > 0 \), \( \min_{\underline{x} \leq x \leq \bar{x}} xe^{-ax} = \min(\underline{x}e^{-a\underline{x}}, \bar{x}e^{-a\bar{x}}) \), we get

\[
\liminf_{p \to \infty} N_i(p) \geq \min \left( \bar{N}_i \exp \left\{ - \left( - \sum_{j=1}^{i-1} b_{ijM}^- + \sum_{j \neq i} b_{ijM}^+ \right) \epsilon \right\} - M_i^\epsilon, \right. \]

\[
\left. (\bar{N}_i + \epsilon) \exp \left\{ c_{iL} - \bar{a}_{iM}(\bar{N}_i + \epsilon) \right\} \right),
\]

where

\[
M_i^\epsilon = \left\{ \left( - \sum_{j=1}^{i-1} b_{ijM}^- + \sum_{j \neq i} b_{ijM}^+ \right) / \bar{a}_{iM} \right\} \epsilon \times \exp \left\{ - \left( - \sum_{j=1}^{i-1} b_{ijM}^- + \sum_{j \neq i} b_{ijM}^+ \right) \epsilon \right\},
\]

\[
c_{iL} = c_{iL} - \sum_{j=1}^{i-1} b_{ijM}^- N_j - \sum_{j \neq i} b_{ijM}^+ \bar{N}_j, \quad 1 \leq i \leq n.
\]

Since \( \epsilon > 0 \) is any sufficiently small positive constant, we have that

\[
\liminf_{p \to \infty} N_i(p) \geq N_i, \quad 1 \leq i \leq n.
\]

Hence, by inductions, we complete the proof. \( \square \)

**Remark 2.2**  Note that the condition (1.8) is a sufficient condition for the permanence of the system (1.1)–(1.2) (cf. (1.22) and (1.24) in Muroya [13]).

Recently, Muroya [13] has established a result which is an improved version of known conditions for the global asymptotic stability in discrete system (1.1)–(1.2) (cf. Gopalsamy [4]–[6], Tineo and Alvarez [18], Redheffer [14], Ahmad and Lazer [1]–[2] and Wang et al. [20]).
Lemma 2.2  For the system (1.1)–(1.2), assume the conditions (1.5) and (1.8), and suppose that there exist positive constants $\alpha_1, \alpha_2, \ldots, \alpha_n, \eta > 0$ and a positive integer $p_0$ such that for $p \geq p_0$,

$$\alpha_i \tilde{a}_i(p) - \sum_{j \neq i} \alpha_j |a_{ji}^0(p)| - \sum_{j=1}^{n} \alpha_j \sum_{l=1}^{m} |a_{ji}^l(p + k_l)| \geq \eta, \quad 1 \leq i \leq n,$$

(2.7)

where

$$\tilde{a}_i(p) = \min \left( a_i(p), \frac{2}{N_i} - a_i(p) \right), \quad 1 \leq i \leq n.$$  (2.8)

Then any two solutions $\{M_i(p)\}_{p=0}^{\infty}$, $\{N_i(p)\}_{p=0}^{\infty}$, $1 \leq i \leq n$ of the system (1.1)–(1.2), satisfy the condition

$$\lim_{p \to \infty} (M_i(p) - N_i(p)) = 0, \quad 1 \leq i \leq n.$$  (2.9)

Proof. The proof of this theorem follows from Theorem 2.1 and the proof of Lemma 2.8 in Muroya [13]. □

By Lemma 2.2, in order to prove Theorem 1.1, it is sufficient to show that the condition (1.10) implies the existence of $\alpha_i > 0$, $1 \leq i \leq n$ such that (2.7) holds. We can see that (1.10) implies stronger inequalities in (2.10) (cf. Ahmad and Lazer [2, Lemma 3.2]).

Lemma 2.3  (See Berman and Plemmons [3, p. 137])  (1.10) holds, if and only if there exist constants $\alpha_i > 0$, $1 \leq i \leq n$ such that for $1 \leq i \leq n$,

$$\alpha_i \tilde{a}_{iL} - \sum_{j \neq i} \alpha_j (a_{ji}^{0+} - a_{ji}^{0-}) - \sum_{j=1}^{n} \alpha_j \sum_{l=1}^{m} (a_{ji}^l - a_{ji}^{l-}) > 0,$$

$$1 \leq i \leq n.$$  (2.10)

From Theorem 1.1, we easily obtain the following corollary.

Corollary 2.1  (Cf. Theorem 1 in Wang et al. [20])  For the system (1.1)–(1.2) and (1.6), if

$$\left\{ \begin{array}{l}
\tilde{c}_{iM} \leq a_{iL}/a_{iM} \leq 1 \quad \text{and} \quad \tilde{N}_i = \tilde{c}_{iM}/a_{iL}, \quad 1 \leq i \leq n, \\
(A_L + B_L^-)^{-1} c_M > 0, \quad c_L > (B_M^+ - D_M^+)(A_L + B_L^-)^{-1} c_M, \quad (2.11) \\
\text{and} \quad A_L - (B_M^+ - B_L^-) \quad \text{is an} \ M\text{-matrix,} 
\end{array} \right.$$
then for any two solutions \( \{M_i(p)\}_{p=1}^{\infty} \) and \( \{N_i(p)\}_{p=1}^{\infty} \), 1 \( \leq i \leq n \) of the system (1.1)–(1.2), (1.11) holds.

Proof. By (1.6) and (2.11), we have \( \tilde{N}_i = c_{iM} / a_{iL} \leq 1/a_{iM} \), 2/\( \tilde{N}_i - a_{iM} \geq 1/\tilde{N}_i = a_{iL} / c_{iM} \geq a_{iL} \), 1 \( \leq i \leq n \) and \( \bar{N} = (A_L + B_L^{-1})^{-1}c_M \). Then, in this case, (2.11) implies (1.5), (1.8) and (1.10). Therefore, by Theorem 1.1, we obtain (1.7) and (1.11).

\[\square\]

**Remark 2.3** In Theorem 1 in Wang et al. [20], it is assumed the condition that the system (1.1)–(1.2) is strongly persistent, that is, \( \lim_{p \to \infty} \tilde{N}_i(p) > 0 \), 1 \( \leq i \leq n \). For the competitive system with \( n = 1 \) and 2, Wang et al. [20] has given sufficient conditions similar to (2.11), which are improved and extended by Theorem 1.1 to \( n \geq 1 \) (see also Corollary 2.2 and Remark 2.4).

Consider the following differential equation with piecewise constant arguments:

\[
\begin{aligned}
\frac{dN(t)}{dt} &= N(t)r(t) \left\{ 1 - b_0N(p) - \sum_{i=1}^{m} b_iN(p - i) \right\}, \\
& \quad p \leq t < p + 1, \ p = 0, 1, 2, \ldots, \\
N(0) &= N_0 > 0, \quad \text{and} \quad N(-j) = N_{-j} \geq 0, \ 1 \leq j \leq m,
\end{aligned}
\]  
(2.12)

where each \( r(t) \) is a nonnegative continuous function on \([0, \infty)\), \( r(t) \neq 0 \),

\[
b_0 > 0, \quad \text{and} \quad b_i \geq 0, \ 1 \leq i \leq m.
\]  
(2.13)

Then, similar to the proof in Remark 2.1, we have that

\[
\begin{aligned}
N(p + 1) &= N(p) \exp \left\{ r_p(1 - b_0N(p) - \sum_{i=1}^{m} b_iN(p - i)) \right\}, \\
& \quad p = 0, 1, 2, \ldots, \\
N(0) &= N_0 > 0, \quad \text{and} \quad N(-j) = N_{-j} \geq 0, \ 1 \leq j \leq m,
\end{aligned}
\]  
(2.14)

where

\[
r_p = \int_{p}^{p+1} r(t)dt, \quad p \geq 0.
\]  
(2.15)

To this system, we apply Theorem 1.1 and get the following corollary.
Corollary 2.2  For (2.12)–(2.13), assume that

\[
\begin{cases}
    b_0 > \sum_{i=1}^{m} b_i, \quad \liminf_{p \to \infty} r_p > 0 \quad \text{and} \\
    \bar{r} = \sup_{p \geq 0} r_p < 1 + \ln \left\{ \frac{2b_0}{\left( \sum_{i=0}^{m} b_i \right)} \right\} < 1 + \ln 2.
\end{cases}
\]  

Then, the positive equilibrium \( N^* = 1 / \left( \sum_{i=0}^{m} b_i \right) \) of (2.12)–(2.13) is globally asymptotically stable.

Proof.  For (2.14), the condition \( \liminf_{p \to \infty} r_p > 0 \) implies (1.8). Now, put

\[
\bar{N} = \begin{cases}
    \frac{1}{b_0}, & \bar{r} \leq 1, \\
    \frac{e^{\bar{r} - 1}}{\bar{r} b_0}, & \bar{r} > 1.
\end{cases}
\]

If \( \bar{r} \leq 1 \), then from the conditions (2.13) and \( b_0 > \sum_{i=1}^{m} b_i \), we have

\[
\min \left( \bar{r} b_0, \frac{2}{\bar{N}} - \bar{r} b_0 \right) = \bar{r} b_0 > \bar{r} \sum_{i=1}^{m} b_i,
\]

which implies (1.10) for (2.14).

If \( \bar{r} > 1 \), then by (2.16), \( e^{\bar{r} - 1} < \frac{2b_0}{\left( \sum_{i=0}^{m} b_i \right)} \) and from \( \bar{r} \sum_{i=0}^{m} b_i < \frac{2eb_0}{e^{\bar{r} - 1}} = \frac{2}{\bar{N}} \) we have that

\[
\min \left( \bar{r} b_0, \frac{2}{\bar{N}} - \bar{r} b_0 \right) > \bar{r} \sum_{i=1}^{m} b_i,
\]

from which we get (1.10) for (2.14). Thus, by Theorem 1.1, we obtain the conclusion of this corollary. \( \square \)

Remark 2.4  The condition (2.16) in Corollary 2.2 improve the sufficient conditions of (2.9) that \( b_0 > \sum_{i=1}^{m} b_i \) and \( r_p \leq 1 \) in Muroya [9] which has the "contractivity condition". Similarly, Theorem 3 in Wang et al. [20] is also improved by Theorem 1.1.

So and Yu [17] have established sufficient conditions that \( \limsup_{p \to \infty} r_p > 0 \) and \( r_p \leq \frac{3}{2(m+1)} \) for (2.12)–(2.13), and Muroya [10] offer the sufficient condition \( r_p \leq \hat{r} (\hat{a}) = f(0; \hat{r} (\hat{a})) = \sup_{t<1} f(t; \hat{r} (\hat{a})) = 2b_0 / (\sum_{i=0}^{m} b_i) \), \( \hat{a} = 1 - (\sum_{i=0}^{m} b_i) / b_0 < 0 \), where \( \hat{r} (\hat{a}) \), \( f(t; r) \) are defined in [10]. Thus, the condition (2.16) is other type condition than those of So and Yu [17] and Muroya [10]. From these results, we have a conjecture that there is a larger region of sufficient conditions which contains all the above conditions. To solve this conjecture is our future work.
Acknowledgement  The author grateful to the referee for his careful reading and many constructive comments.

References


Department of Mathematical Sciences
Waseda University
3-4-1 Ohkubo Shinjuku-ku
Tokyo 169-8555, Japan
E-mail: ymuoya@waseda.jp