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Some estimates for strong uniform approximation on sphere

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Abstract. Let $d \geq 3$ and Σ_{d-1} be the unit sphere in $\mathbb{R}^d$. For $f \in L(\Sigma_{d-1})$, denote by σ_n^δ the Cesàro means of its Fourier-Laplace series of order δ. In this paper, we study the strong uniform approximation by σ_n^δ with the critical index in certain continuous function spaces, which shows that the results in our previous paper are best possible in a suitable sense.

Key words: spherical harmonics, Fourier-Laplace series, Cesáro means, strong uniform approximation, modulus of continuity.

1. Introduction and Main Results

Let Σ_{d-1} be the unit sphere in $\mathbb{R}^d$ centered at the origin. We will assume that $d \geq 3$ and write $\lambda = (d-2)/2$ throughout this paper. Denote by $x \cdot y$ the usual inner product of x and y in $\mathbb{R}^d$. The Lebesgue measure on $\mathbb{R}^d$ induces a measure on Σ_{d-1} invariant for rotations around the origin. Let $\mathcal{H}_k^d$ be the space of spherical harmonics of degree k. If $f \in L(\Sigma_{d-1})$ then f has its expansion in spherical harmonics, so-called the Fourier-Laplace series,

$$\sigma(f)(x) = \sum_{k=0}^{\infty} Y_k(f, x), \tag{1}$$

where $Y_k(f, \cdot)$ is the projection of f on $\mathcal{H}_k^d$ and can be expressed as a spherical convolution,

$$Y_k(f, x) = \frac{(k+\lambda)\Gamma(\lambda)}{2\pi^{\lambda+1}} \int_{\Sigma_{d-1}} f(y) P_k^\lambda(x \cdot y)\, dy, \quad k = 0, 1, 2, \ldots .$$

Here, $P_k^\lambda(t)$ is the ultraspherical (or Gegenbauer) polynomial of degree k associated with λ, which can be defined in terms of a generating equation

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as follows,

$$(1-2rt+r^2)^{-\lambda} = \sum_{k=0}^{\infty} P_k^{\lambda}(t)r^k, \quad 0 \le r < 1, \ |t| \le 1. \tag{2}$$

It is well-known that if $f \in L^2(\Sigma_{d-1})$ then there exists a unique representation, $f(x) = \sum_{k=0}^{\infty} Y_k(f,x)$, where the series on the right-hand side converges to f in the L^2 norm.

For more information on the spherical harmonic analysis and approximation theory, we refer the readers to [7], [11] and [17].

Denote by $S_N(f)$ the partial sum operator of the series in (1), says, $S_N(f)(x) = \sum_{k=0}^{N} Y_k(f,x)$. It is known that the Lebesgue constant of the operator S_N has the order N^{λ}, see [17] p. 48 or [5] for details. Thus, in general, the convergence properties of the partial sum operator is not good. Indeed, in 1973, Bonami and Clerc [2] pointed out that, for any $p \neq 2$ and $1 \le p \le \infty$, there is a function $f \in L^p(\Sigma_{d-1})$ such that S_N does not converge to f in L^p norm (also see [17] p. 49). So, it is natural and important to consider the linear summation of the Fourier-Laplace series. However, among others, Cesàro means is the most important one. For $f \in L(\Sigma_{d-1})$ and $\alpha > -1$, the nth Cesàro means of order α of the series in (1) are defined by

$$\sigma_n^{\alpha}(f)(x) = \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha} Y_k(f,x), \quad n = 0, 1, 2, \ldots,$$

where

$$A_k^{\alpha} = \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+1)}, \quad k = 0, 1, 2, \ldots .$$

In the same paper mentioned above, Bonami and Clerc proved that there exists a function $f \in L^p(\Sigma_{d-1})$ such that $\sigma_n^{\alpha}(f)$ does not converge to f in L^p norm when $0 < \alpha \le \lambda = \frac{d-2}{2}$ and $1 \le p \le \frac{2(d-1)}{d+2\alpha}$ or $p \ge \frac{2(d-1)}{d-2-2\alpha}$ (also see [17] p. 49). For more information on Cesàro means on sphere, we refer the readers to [1], [2], [4] and [14] ~ [17].

It is known that the special value of α, $(d-2)/2$, is critical, and the Cesàro means of critical order, σ_n^{λ}, plays a role as like as the partial sum operator of the classical Fourier series. So, it is natural and significant to study the strong summability of $\sigma_n^{\lambda}(f)$ with index $q > 0$, which is defined by

$$H_{q,n}(f)(x) = \frac{1}{n+1}\sum_{k=0}^{n} |f(x) - \sigma_k^{\lambda}(f)(x)|^q, \quad \text{for} \quad q > 0.$$

In 1994, Wang and the author [18] studied the uniform approximation of $H_{q,n}$. In 1995, the author considered the pointwise convergence of $H_{q,n}$ in [19].

To state our results, let us first recall some definitions and notations. Denote by $C(\Sigma_{d-1})$ the continuous function space on Σ_{d-1} with the norm $\|f\|_C = \max\{|f(x)| : x \in \Sigma_{d-1}\}$, by $E_k(f)$ the best approximation of $f \in C(\Sigma_{d-1})$ by spherical polynomials of order k, see [17] p. 161 for its definition.

Let $\xi \in \Sigma_{d-1}$ and $0 < \gamma \le \pi$, denote by $\ell_{\xi,\gamma} = \{\eta \in \Sigma_{d-1} : \xi \cdot \eta = \cos\gamma\}$ and by $\ell(\gamma)$ its length, obviously, $\ell(\gamma) = |\Sigma_{d-2}| \sin^{d-2}\gamma$. For $f \in L(\Sigma_{d-1})$, we define the spherical translation operator by

$$\mathcal{S}_\gamma(f)(\xi) = \frac{1}{\ell(\gamma)} \int_{\ell_{\xi,\gamma}} f(\eta) d\ell_{\xi,\gamma}(\eta).$$

The idea of the spherical translation operator may be found in [6] and [8], cf. [1] and [17] p. 57.

For $f \in C(\Sigma_{d-1})$, the modulus of continuity of f is defined by

$$\omega(f,t) = \sup\{|\mathcal{S}_{\gamma'}(f)(x) - \mathcal{S}_{\gamma''}(f)(x)| : |\gamma' - \gamma''| \le t, \ x \in \Sigma_{d-1}\}.$$

In 1994, we established the following uniform approximation theorem for the strong summation $H_{q,n}$ in [18].

Theorem 1 ([18]) *Let $f \in C(\Sigma_{d-1})$ and $q > 1$, then there is a positive constant $C_{d,q}$ depending on d and q, such that*

$$\left\| \frac{1}{n+1}\sum_{k=0}^{n} |f - \sigma_k^{\lambda}(f)|^q \right\|_C \le \frac{C_{d,q}}{n+1}\sum_{k=0}^{n} \left[\omega\Big(f, \frac{1}{k+1}\Big)\right]^q.$$

In this paper, we continue the study of [18]. We will prove that Theorem 1 is best possible in certain suitable sense.

Let $\{F_k\}_{k=0}^{\infty}$ be a monotone sequence of positive numbers with $\lim_{k\to\infty} F_k = 0$ and $\omega(t)$ be a modulus of continuity. Write

$$C(F) = \{f \in C(\Sigma_{d-1}) : E_k(f) \le F_k, \quad k = 0, 1, 2, \ldots\}$$

and

$$H^{\omega} = \{f \in C(\Sigma_{d-1}) : \omega(f,t) \le \omega(t), \ \ 0 \le t \le \pi\}.$$

In the sequel, we reserve the notation $\phi \sim \psi$, which means that there exist absolute constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\psi \le \phi \le C_2\psi$. Now, our main results can be stated as follows.

Theorem 2 *Let $\{F_k\}_{k=0}^{\infty}$ be a monotone sequence of positive numbers with $\lim_{k\to\infty} F_k = 0$, then for $q > 1$ there has*

$$\sup_{f\in C(F)} \left\| \frac{1}{n+1} \sum_{k=0}^{n} |f - \sigma_k^{\lambda}(f)|^q \right\|_C \sim \frac{1}{n+1} \sum_{k=0}^{n} F_k^q.$$

Theorem 3 *If there is an absolute constant $B > 0$ such that*

$$\sum_{\nu=0}^{k} \omega\Big(\frac{1}{\nu+1}\Big) \le B(k+1)\omega\Big(\frac{1}{k+1}\Big), \tag{3}$$

then for any $q > 1$, the following holds

$$\sup_{f\in H^{\omega}} \left\| \frac{1}{n+1} \sum_{k=0}^{n} |f - \sigma_k^{\lambda}(f)|^q \right\|_C \sim \frac{1}{n+1} \sum_{k=0}^{n} \Big[\omega\Big(\frac{1}{k+1}\Big)\Big]^q.$$

Our theorems are the spherical analogues of the corresponding results for the classical Fourier series (see [9]). The basic idea of the proof is from [9], [10] and [12].

2. Auxiliary Lemmas

In this section, we recall and prove some elementary inequalities to be used.

Lemma 1 ([13]) *Let $P_k^{\lambda}(t)$ be the ultraspherical polynomials defined by (2), then*

$$P_k^{\lambda}(1) = A_k^{2\lambda-1} \quad \textit{and} \quad |P_k^{\lambda}(t)| \le A_k^{2\lambda-1}.$$

Lemma 2 (Chebysev's inequality, [3] p. 9) *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are real numbers such that $a_1 \le \cdots \le a_n$ and $b_1 \le \cdots \le b_n$ or $a_1 \ge \cdots \ge$*

a_n *and* $b_1 \geq \cdots \geq b_n$ *then*

$$\sum_{k=1}^{n} a_k b_k \geq \frac{1}{n} \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} b_k.$$

Lemma 3 *Let* $\{F_k\}_{k=0}^{\infty}$ *be a sequence of real numbers and satisfy the following condition*

$$F_k \geq F_{k+1} > 0, \quad k = 0, 1, 2, \ldots .$$

Then for any $\alpha > 0$ *there has*

$$\frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} F_k \geq \frac{C_\alpha}{n+1} \sum_{k=0}^{n} F_k, \tag{4}$$

where $C_\alpha = \alpha$ *when* $0 < \alpha < 1$ *and* $n \geq 1$, *and otherwise* $C_\alpha = 1$.

Proof. When $n = 0$ or $\alpha = 1$, it is easy to see that (4) becomes an identity with $C_\alpha = 1$. Now, we assume that $n \geq 1$ and $\alpha \neq 1$.

(i) If $0 < \alpha < 1$ then $A_{n-k}^{\alpha-1} \geq A_n^{\alpha-1}$, $k = 0, 1, \ldots, n$, and then

$$\frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} F_k \geq \frac{A_n^{\alpha-1}}{A_n^\alpha} \sum_{k=0}^{n} F_k = \frac{\alpha}{n+\alpha} \sum_{k=0}^{n} F_k \geq \frac{\alpha}{n+1} \sum_{k=0}^{n} F_k. \tag{5}$$

(ii) If $\alpha > 1$ then $A_{n-k}^{\alpha-1} > A_{n-k-1}^{\alpha-1}$, $k = 0, 1, \ldots, n$. Applying the Chebysev's inequality (Lemma 2) for $a_k = A_{n-k}^{\alpha-1}$ and $b_k = F_k$, we obtain the following inequality

$$\sum_{k=0}^{n} A_{n-k}^{\alpha-1} F_k \geq \frac{1}{n+1} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} \sum_{k=0}^{n} F_k. \tag{6}$$

From (6) and noting that $\sum_{k=0}^{n} A_{n-k}^{\alpha-1} = A_n^\alpha$, we have

$$\frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} F_k \geq \frac{1}{n+1} \sum_{k=0}^{n} F_k. \tag{7}$$

It follows from (5) and (7) that (4) is true when $\alpha \neq 1$ and $n \geq 1$. Summing up the discussion above, we finish the proof of Lemma 3.

□

3. Proof of the Theorems

In this section, we prove Theorems 2 and 3. To this end, we first establish a inequality, in which we control the strong uniform approximation in terms of the best approximation.

Proposition 1 *If $f \in C(\Sigma_{d-1})$ and $q > 1$ then*

$$\left\| \frac{1}{n+1} \sum_{k=0}^{n} |f - \sigma_k^\lambda(f)|^q \right\|_C \le \frac{C_{d,q}}{n+1} \sum_{k=0}^{n} [E_k(f)]^q.$$

Proof. By Theorems 6.2.3 and 6.2.6 in [17] (also see Theorem 4.3.1 and Lemma 4.3.2 in [4]), we have

$$\omega\Big(f, \frac{1}{k+1}\Big) \le \frac{C_d}{k+1} \sum_{i=0}^{k} E_i(f). \tag{8}$$

Applying Theorem 1 and the Hardy-Landau's inequality (see [3] p. 144), we have

$$\begin{aligned} \left\| \frac{1}{n+1} \sum_{k=0}^{n} |f - \sigma_k^\lambda(f)|^q \right\|_C &\le \frac{C_{d,q}}{n+1} \sum_{k=0}^{n} \left[\frac{C_d}{k+1} \sum_{i=0}^{k} E_i(f) \right]^q \\ &\le \frac{C_{d,q}}{n+1} \sum_{k=0}^{n} [E_k(f)]^q. \end{aligned}$$

This is what we want. So complete the proof. □

Proposition 1 can be regarded as a spherical analogue of the corresponding result of the classical Fourier series, cf. [9].

Proof of Theorem 2. Recall the definition of $C(F)$, it follows from Proposition 1 that

$$\sup_{f \in C(F)} \left\| \frac{1}{n+1} \sum_{k=0}^{n} |f - \sigma_k^\lambda(f)|^q \right\|_C \le \frac{C_{d,q}}{n+1} \sum_{k=0}^{n} F_k^q. \tag{9}$$

Next, we shall deduce the reserve inequality of (9) by using the idea from Sun [12]. Fix a point $e \in \Sigma_{d-1}$ as the north pole of Σ_{d-1}. Set

$$f_0(x) = \sum_{\nu=1}^{\infty} (F_{\nu-1} - F_\nu) \big(A_\nu^{2\lambda-1}\big)^{-1} P_\nu^\lambda(x \cdot e), \quad \text{for all } x \in \Sigma_{d-1}.$$

This function was constructed by Ugulava in [14]. It is not difficult to see that $f_0 \in C(\Sigma_{d-1})$. Denote by $S_k(f_0)$ the kth partial sums of f_0, says,

$$S_k(f_0)(x) = \sum_{i=0}^{k} Y_i(f_0, x).$$

From the second inequality in Lemma 1, we have

$$\begin{aligned} E_k(f_0) &\le \left\| f_0 - S_k(f_0) \right\|_C \\ &= \left\| \sum_{\nu=k+1}^{\infty} (F_{\nu-1} - F_\nu)(A_\nu^{2\lambda-1})^{-1} P_\nu^\lambda(x \cdot e) \right\|_C \\ &\le \sum_{\nu=k+1}^{\infty} (F_{\nu-1} - F_\nu) = F_k. \end{aligned}$$

This shows that $f_0 \in C(F)$. Since

$$\sigma_n^\alpha(f_0)(x) = \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^\alpha (F_{k-1} - F_k)(A_k^{2\lambda-1})^{-1} P_k^\lambda(x \cdot e), \quad \alpha > 0,$$

and noting that $A_0^\alpha = A_0^{\alpha-1} = 1$ and $A_{n-k-1}^\alpha - A_{n-k}^\alpha = -A_{n-k}^{\alpha-1}$ $(k = 0, 1, \ldots, n-1)$. It follows from the first inequality of Lemma 1 that

$$\begin{aligned} \sigma_n^\alpha(f_0)(e) &= \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^\alpha (F_{k-1} - F_k) \\ &= \frac{1}{A_n^\alpha} \sum_{k=0}^{n-1} (A_{n-k-1}^\alpha - A_{n-k}^\alpha) F_k + F_0 - \frac{A_0^\alpha}{A_n^\alpha} F_n \\ &= F_0 - \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} F_k. \end{aligned}$$

Since

$$f_0(e) = \sum_{\nu=1}^{\infty} (F_{\nu-1} - F_\nu)(A_\nu^{2\lambda-1})^{-1} P_\nu^\lambda(1) = F_0,$$

then

$$f_0(e) - \sigma_n^\alpha(f_0)(e) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} F_k, \quad \alpha > 0. \tag{10}$$

Applying Lemma 3 for $\alpha = \lambda = (d-2)/2$, we have

$$|f_0(e) - \sigma_n^\lambda(f_0)(e)| \geq \frac{C_d}{n+1}\sum_{k=0}^{n} F_k.$$

Noting that $F_k \geq F_{k+1}$ for $k = 0, 1, \dots$, we have, for $q > 1$,

$$\begin{aligned}\frac{1}{n+1}\sum_{k=0}^{n}|f_0(e) - \sigma_k^\lambda(f_0)(e)|^q &\geq \frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{C_d}{k+1}\sum_{i=0}^{k} F_i\right|^q \\ &\geq \frac{C_{d,q}}{n+1}\sum_{k=0}^{n} F_k^q.\end{aligned}$$

This shows that

$$\sup_{f\in C(F)}\left\|\frac{1}{n+1}\sum_{k=0}^{n}|f - \sigma_k^\lambda(f)|^q\right\|_C \geq \frac{C_{d,q}}{n+1}\sum_{k=0}^{n} F_k^q. \tag{11}$$

Theorem 2 follows from (9) and (11). So finish the proof. □

Proof of Theorem 3. Fix a point $e \in \Sigma_{d-1}$ as the north pole of Σ_{d-1}. Set

$$f_1(x) = \varepsilon\sum_{k=1}^{\infty}\left[\omega\Big(\frac{1}{k}\Big) - \omega\Big(\frac{1}{k+1}\Big)\right](A_k^{2\lambda-1})^{-1}P_k^\lambda(x\cdot e), \quad \forall x \in \Sigma_{d-1},$$

where ε is an undetermined constant.

We first check that $f \in H^\omega$ for certain suitable ε. Denote by $S_k(f_1)(x)$ the kth partial sum of $f_1(x)$. Obviously, we have

$$\begin{aligned}E_k(f_1) &\leq \|f_1 - S_k(f_1)\|_C \\ &= \varepsilon\left\|\sum_{\nu=k+1}^{\infty}\left[\omega\Big(\frac{1}{\nu}\Big) - \omega\Big(\frac{1}{\nu+1}\Big)\right](A_\nu^{2\lambda-1})^{-1}P_\nu^\lambda(x\cdot e)\right\|_C \\ &\leq \varepsilon\sum_{\nu=k+1}^{\infty}\left[\omega\Big(\frac{1}{\nu}\Big) - \omega\Big(\frac{1}{\nu+1}\Big)\right] \\ &= \varepsilon\omega\Big(\frac{1}{k+1}\Big).\end{aligned} \tag{12}$$

From (8), (12) and (3), we have

$$\omega\Big(f_1, \frac{1}{k+1}\Big) \le \frac{C_d}{k+1}\sum_{i=0}^{k} E_i(f_1)$$
$$\le \frac{\varepsilon C_d}{k+1}\sum_{i=0}^{k}\omega\Big(\frac{1}{i+1}\Big)$$
$$\le \varepsilon C_d B\omega\Big(\frac{1}{k+1}\Big),$$

where B is the constant in (3).

Let $\varepsilon = \big(2C_dB\big)^{-1}$, then the above inequality gives that $\omega\big(f_1, \frac{1}{k+1}\big) \le \omega\big(\frac{1}{k+1}\big)$, which implies $f_1 \in H^\omega$. Set $F_k = \omega\big(\frac{1}{k+1}\big)$, $k = 0, 1, \dots$. Reasoning as (10), we have

$$f_1(e) - \sigma_n^\alpha(f_1)(e) = \frac{\varepsilon}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}\omega\Big(\frac{1}{k+1}\Big), \quad \alpha > 0. \tag{13}$$

By Lemma 3 and (13), we have, for any $\alpha = \lambda$

$$|f_1(e) - \sigma_n^\lambda(f_1)(e)| \ge \frac{\varepsilon C_d}{n+1}\sum_{k=0}^{n}\omega\Big(\frac{1}{k+1}\Big) \ge C_{d,q,B}\,\omega\Big(\frac{1}{n+1}\Big).$$

And then

$$\sup_{f\in H^\omega}\Big\|\frac{1}{n+1}\sum_{k=0}^{n}|f - \sigma_k^\lambda(f)|^q\Big\|_C \ge \frac{C_{d,q,B}}{n+1}\sum_{k=0}^{n}\Big[\omega\Big(\frac{1}{k+1}\Big)\Big]^q. \tag{14}$$

On the other hand, the reverse inequality of (14) follows from Theorem 1. So complete the proof. □

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