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Two criteria of Wiener type for minimally thin sets and rarefied sets in a cylinder

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Abstract. We shall give two criteria of Wiener type which characterize minimally thin sets and rarefied sets in a cylinder. We shall also show that a positive superharmonic function on a cylinder behaves regularly outside a rarefied set in a cylinder.

Key words: superharmonic function, minimally thin set, rarefied set, cylinder.

1. Introduction

Lelong-Ferrand [14] investigates the regularity of value distribution of a positive superharmonic function on the half-space $\mathbf{T}_n$ through introducing the notion of a set "*effilé* at ∞" which is defind by a criterion of Wiener type.

Essén and Jackson [7] observed that a subset E of $\mathbf{T}_n$ is effilé at ∞ if and only if E is minimally thin at ∞, and led later developments to a different direction. Their investigation was motivated by Ahlfors and Heins [1], Hayman [11], Ušakova [18] and Azarin [4], who are concerned with regularity of value distribution of a subharmonic function defined on the half plane $\mathbf{T}_2$, the half-space $\mathbf{T}_n$ or cone, outside a exceptional set covered by a sequence of balls. By introducing a new type of exceptional set in $\mathbf{T}_n$ defined by another criterion of Wiener type, which is called a rarefied set, Essén and Jackson [8] gave a detailed covering theorem for it and sharpend their results by proving the reguralíty of value distribution outside the exceptional set, of a positive superharmonic function on T_n in place of a subharmonic function.

Essén and Jackson's concern is limited to a positive superharmonic function on $\mathbf{T}_n$ which is a special cone, while Azarin [4] treats subharmonic functions defined on general cones. Lelong-Ferrand [15] also refered to a set effilé at ∞ in a cone without giving explicitely a criterion of Wiener type and extended her results in [14] for a positive superharmonic function on

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a cone. In these senses, it seemed important to extend their results to a positive superharmonic functions on a cone and to try obtaining a result sharpening Azarin's result in a true sense. In the previous paper [16], we gave some results to this direction, including two criteria of Wiener types. In our recent paper [17], we obtained a result sharpening Azarin's result in a true sense by giving a covering theorem for a rarefied set in a cone.

On the other hand, Lelong-Ferrand [15] refered to a set effilé at ∞ in a cylinder without giving a criterion of Wiener type, and said that her results in [14] were also extended for a positive superharmonic function on a cylinder. Since a cylinder is a domain of completly different type from a cone in the sense that ∞ is a cusp of domain when it is changed into a bounded domain by a Kelvin transformation, it also seems valuable to observe how a series of results obtained with a cone follows when a cylinder is considered in place of a cone.

In this paper we shall first prove that a minimally thin set at ∞ in a cylinder is also defined by a criterion of Wiener type (Theorem 1). Next we shall define a rarefied set in a cylinder and show that it is also judged by another criterion of Wiener type (Theorem 2). We shall prove the regularity of boundary behavior of a positive superharmonic function on a cylinder outside a rarefied set (Theorem 3). Finally we shall give some connection between a minimally thin set and a rarefied set in a cylinder (Theorem 4).

2. Preliminaries

Let D be a bounded domain on $\mathbf{R}^{n-1} (n \geq 2)$ with smooth boundary. Consider the Dirichlet problem

$$\begin{aligned} (\Delta_n + \tau) f &= 0 \quad \text{on } D \\ f &= 0 \quad \text{on } \partial D. \end{aligned} \tag{2.1}$$

We denote the least positive eigenvalue of (2.1) by τ_D and the normalized positive eigenfunction corresponding to τ_D by $f_D(X)$;

$$\int_D f_D^2(X) dX = 1,$$

where dX is the $(n-1)$-dimensional volume element. By $\Gamma_n(D)$, we denote the set $\{P = (X, y) \in \mathbf{R}^n; X \in D, -\infty < y < +\infty\}$. We call it a cylinder. It is known that the Martin boundary of $\Gamma_n(D)$ is the set $\partial\Gamma_n(D) \cup \{\infty, -\infty\}$ (Yoshida [19, p. 285]). When we denote the Martin kernel by $K(P, Q)$ $(P \in$

$\Gamma_n(D)$, $Q \in \partial\Gamma_n(D) \cup \{\infty, -\infty\}$), we know

$$K(P, \infty) = e^{\sqrt{\tau_D}y} f_D(X),\ K(P, -\infty) = \kappa e^{-\sqrt{\tau_D}y} f_D(X) \quad (P = (X, y) \in \Gamma_n(D)),$$

where κ is a positive constant.

A subset E of $\Gamma_n(D)$ is called to be minimally thin at ∞ in $\Gamma_n(D)$ (Brelot [5, p. 122] and Doob [6, p. 208]), if there exists a point $P \in \Gamma_n(D)$ such that

$$\hat{R}^E_{K(\,\cdot\,,\infty)}(P) \neq K(P, \infty),$$

where $\hat{R}^E_{K(\,\cdot\,,\infty)}(P)$ is the regulalized reduced function of $K(\,\cdot\,, \infty)$ relative to E (Helms [12, p. 134]).

When we set

$$\Gamma_n(D; -\infty, b) = \{P = (X, y) \in \mathbf{R}^n; X \in D, y < b\} \quad (-\infty < b < +\infty)$$

and E is a subset of $\Gamma_n(D)$ such that there exists a real number b satisfying $E \subset \Gamma_n(D; -\infty, b)$, E is called to be bounded above. If $E \subset \Gamma_n(D)$ is bounded above, then $\hat{R}^E_{K(\,\cdot\,,\infty)}$ is bounded on $\Gamma_n(D)$ and hence the greatest harmonic minorant of $\hat{R}^E_{K(\,\cdot\,,\infty)}$ is zero. When we denote by $G(P, Q)$ ($P \in \Gamma_n(D)$, $Q \in \Gamma_n(D)$) the Green function of $\Gamma_n(D)$, we see from the Riesz decomposition theorem (Helms [12, p. 116]) that there exists a unique positive measure λ_E on $\Gamma_n(D)$ such that

$$\hat{R}^E_{K(\,\cdot\,,\infty)}(P) = G\lambda_E(P) \tag{2.2}$$

for any $P \in \Gamma_n(D)$ and λ_E is concentrated on B_E, where

$$B_E = \{P \in \Gamma_n(D); E \text{ is not thin at } P\}$$

(see Brelot [5, Theorem VIII, 11] and Doob [6, Theorem XI. 14(d)]).

The (Green) energy $\gamma(E)$ of λ_E is defined by

$$\gamma(E) = \int_{\Gamma_n(D)} (G\lambda_E) d\lambda_E$$

(see [12, p. 223]).

In the following, we put the strong assumption relative to D on $\mathbf{R}^{n-1}$: If $n \geq 3$, then D is a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) on $\mathbf{R}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see [9, pp. 88-89] for the definition of $C^{2,\alpha}$-domain). Then $f_D(X)$ is twice continuously differentiable on $\overline{D}$ ([9, Theorem 6.15]).

3. Statement of results

Let E be a subset of $\Gamma_n(D)$ and $E(k) = E \cap I_k$, where

$$I_k = \{(X, y) \in \Gamma_n(D) \colon k \leq y < k+1\}.$$

First, for a minimally thin set at ∞ with respect to $\Gamma_n(D)$ we shall give not only a criterion of Wiener type, but also another definition which is parallel to the difinition for a rarefied set at ∞ with respect to $\Gamma_n(D)$ (this definition can be state in more general form as in Armitage and Gardiner [3, Theorem 9.2.6]).

Theorem 1 *For a subset E of $\Gamma_n(D)$, the following statements are equivalent*:
(I) *E is minimally thin at ∞ with respect to $\Gamma_n(D)$.*
(II) $\sum_{k=0}^{\infty} \gamma\big(E(k)\big) e^{-2\sqrt{\tau_D}k} < +\infty$.
(III) *There exists a positive superharmonic function $v(P)$ on $\Gamma_n(D)$ such that*

$$\inf_{P \in \Gamma_n(D)} \frac{v(P)}{K(P, \infty)} = 0$$

and

$$E \subset M_v, \tag{3.1}$$

where

$$M_v = \{P = (X, y) \in \Gamma_n(D); v(P) \geq K(P, \infty)\}.$$

A subset E of $\Gamma_n(D)$ is said to be *rarefied* at ∞ with respect to $\Gamma_n(D)$, if there exists a positive superharmonic function $v(P)$ on $\Gamma_n(D)$ such that

$$\inf_{P \in \Gamma_n(D)} \frac{v(P)}{K(P, \infty)} = 0$$

and

$$E \subset H_v,$$

where

$$H_v = \{P = (X,\, y) \in \Gamma_n(D); v(P) \geq e^{\sqrt{\tau_D}y}\}$$

(for the definition of rarefied sets at ∞ with respect to the half-space, see Aikawa and Essén [2, DEFINITION 12.4 in p. 74] and Hayman [10, p. 474]).

Theorem 2 *A subset E of $\Gamma_n(D)$ is rarefied at ∞ with respect to $\Gamma_n(D)$ if and only if*

$$\sum_{k=0}^{\infty} e^{-\sqrt{\tau_D}k} \lambda_{(E(k))}\big(\Gamma_n(D)\big) < +\infty.$$

Theorem 3 *Let $v(P)$ be a positive superharmonic function on $\Gamma_n(D)$ and $c_\infty(v)$ be a constant defined by*

$$\inf_{P\in\Gamma_n(D)} \frac{v(P)}{K(P,\, \infty)} = c_\infty(v).$$

Then there exists a rarefied set E at ∞ with respect to $\Gamma_n(D)$ such that $v(P)e^{-\sqrt{\tau_D}y}$ uniformly converges to $c_\infty(v)f_D(X)$ on $\Gamma_n(D) - E$ as $y \to +\infty$ $(P = (X,\, y) \in \Gamma_n(D))$.

Remark We observe the following fact from the definition of a rarefied set. Given any rarefied set E at ∞ with respect to $\Gamma_n(D)$, there exists a positive superharmonic function $v(P)$ on $\Gamma_n(D)$ such that $v(P)e^{-\sqrt{\tau_D}y} \geq 1$ on E and

$$c_\infty(v) = \inf_{P=(X,\,y)\in\Gamma_n(D)} \frac{v(P)}{K(P,\, \infty)} = 0.$$

Hence $v(P)e^{-\sqrt{\tau_D}y}$ does not converge to $c_\infty(v)f_D(X) = 0$ at any point $P = (X,\, y)$ of $\Gamma_n(D) - E$ as $y \to +\infty$.

A cylinder $\Gamma_n(D')$ is called a subcylinder of $\Gamma_n(D)$, if $\overline{D'} \subset D$ ($\overline{D'}$ is the closure of D'). As in $\mathbf{T}_n$ (Essén and Jackson [8, Remark 3.2]), we have

Theorem 4 *Let E be a subset of $\Gamma_n(D)$. If E is rarefied at ∞ with respect to $\Gamma_n(D)$, then E is minimally thin at ∞ with respect to $\Gamma_n(D)$. If E is contained in a subcylinder of $\Gamma_n(D)$ and E is minimally thin at ∞ with respect to $\Gamma_n(D)$, then E is rarefied at ∞ with respect to $\Gamma_n(D)$.*

4. Lemmas

In the following we set

$$\Gamma_n(D; a, b) = \{P = (X, y) \in \mathbf{R}^n; X \in D,\, a \le y < b\}$$
$$(-\infty < a < b \le +\infty).$$

First of all, we remark that

$$C_1 e^{\sqrt{\tau_D} y} e^{-\sqrt{\tau_D} y'} f_D(X) f_D(X') \le G(P, Q)$$
$$\le C_2 e^{\sqrt{\tau_D} y} e^{-\sqrt{\tau_D} y'} f_D(X) f_D(X') \quad (4.1)$$

for any $P = (X, y) \in \Gamma_n(D)$ and any $Q = (X', y') \in \Gamma_n(D)$ satisfying $y < y' - 1$, where C_1 and C_2 are two positive constants (Yoshida [19]).

Lemma 1 *Let μ be a positive measure on $\Gamma_n(D)$ such that there is a sequence of points $P_i = (X_i, y_i) \in \Gamma_n(D)$, $y_i \to +\infty$ $(i \to +\infty)$ satisfying*

$$G\mu(P_i) = \int_{\Gamma_n(D)} G(P_i, Q) d\mu(Q) < +\infty$$
$$(i = 1, 2, 3, \ldots; Q \in \Gamma_n(D)).$$

Then for a real number l,

$$\int_{\Gamma_n(D; l, +\infty)} e^{-\sqrt{\tau_D} y'} f_D(X') d\mu(X', y') < +\infty \quad (4.2)$$

and

$$\lim_{L \to \infty} e^{-2\sqrt{\tau_D} L} \int_{\Gamma_n(D; -\infty, L)} e^{\sqrt{\tau_D} y'} f_D(X') d\mu(X', y') = 0. \quad (4.3)$$

Proof. Take a real number l satisfying $P_1 = (X_1, y_1) \in \Gamma_n(D)$, $y_1 + 1 \le l$. Then from (4.1), we have

$$C_1 e^{\sqrt{\tau_D} y_1} f_D(X_1) \int_{\Gamma_n(D; l, \infty)} e^{-\sqrt{\tau_D} y'} f_D(X') d\mu(X', y')$$
$$\le \int_{\Gamma_n(D)} G(P_1, Q) d\mu(Q) < +\infty,$$

which gives (4.2). For any positive number ε, from (4.2) we can take a large number A such that

$$\int_{\Gamma_n(D;A,\,\infty)} e^{-\sqrt{\tau_D}y'} f_D(X')d\mu(X',\,y') < \frac{\varepsilon}{2}.$$

If we take a point $P_i = (X_i,\,y_i) \in \Gamma_n(D)$, $y_i \geq A+1$, then we have from (4.1)

$$C_1 e^{-\sqrt{\tau_D}y_i} f_D(X_i) \int_{\Gamma_n(D;-\infty,\,A)} e^{\sqrt{\tau_D}y'} f_D(X')d\mu(X',\,y') \leq \int_{\Gamma_n(D)} G(P_i,\,Q)d\mu(Q) < +\infty.$$

If L ($L > A$) is sufficiently large, then

$$\begin{aligned}
&e^{-2\sqrt{\tau_D}L} \int_{\Gamma_n(D;-\infty,\,L)} e^{\sqrt{\tau_D}y'} f_D(X')d\mu(X',\,y') \\
&= e^{-2\sqrt{\tau_D}L} \int_{\Gamma_n(D;-\infty,\,A)} e^{\sqrt{\tau_D}y'} f_D(X')d\mu(X',\,y') \\
&\quad + e^{-2\sqrt{\tau_D}L} \int_{\Gamma_n(D;A,\,L)} e^{\sqrt{\tau_D}y'} f_D(X')d\mu(X',\,y') \\
&\leq e^{-2\sqrt{\tau_D}L} \int_{\Gamma_n(D;-\infty,\,A)} e^{\sqrt{\tau_D}y'} f_D(X')d\mu(X',\,y') \\
&\quad + \int_{\Gamma_n(D;A,\,\infty)} e^{-\sqrt{\tau_D}y'} f_D(X')d\mu(X',\,y') < \varepsilon,
\end{aligned}$$

which gives (4.3). □

Lemma 2 *Let $v(P)$ be a positive superharmonic function on $\Gamma_n(D)$ such that*

$$\inf_{P\in\Gamma_n(D)} \frac{v(P)}{K(P,\,\infty)} = 0.$$

Then for any positive number B the set

$$\{P = (X,\,y) \in \Gamma_n(D); v(P) \geq BK(P,\,\infty)\}$$

is minimally thin at ∞ with respect to $\Gamma_n(D)$.

Proof. Apply a result in Doob [6, p. 213] to the positive superharmonic function $v(P)$. Then

$$\operatorname*{mf\,lim}_{y\to\infty,\, P\in\Gamma_n(D)} \frac{v(P)}{K(P,\,\infty)} = \inf_{P\in\Gamma_n(D)} \frac{v(P)}{K(P,\,\infty)} = 0,$$

where "mf limit" means minimal-fine limit. This gives the conclusion. □

In the following we put

$$S_n(D; a,\, b) = \{P = (X,\, y) \in \mathbf{R}^n; X \in \partial D,\, a \le y < b\} \qquad (-\infty < a < b \le +\infty)$$

and

$$S_n(D; -\infty,\, b) = \{P = (X,\, y) \in \mathbf{R}^n; X \in \partial D,\, -\infty < y < b\} \qquad (-\infty < b \le +\infty).$$

Hence $S_n(D; -\infty,\, +\infty)$ denoted simply by $S_n(D)$ is $\partial\Gamma_n(D)$.

Lemma 3 *Let $v(P)$ be a positive superharmonic function on $\Gamma_n(D)$ and put*

$$c_\infty(v) = \inf_{P\in\Gamma_n(D)} \frac{v(P)}{K(P,\,\infty)}, \quad c_{-\infty}(v) = \inf_{P\in\Gamma_n(D)} \frac{v(P)}{K(P,\,-\infty)}. \tag{4.4}$$

Then there are a unique positive measure μ on $\Gamma_n(D)$ and a unique positive measure ν on $S_n(D)$ such that

$$\begin{aligned} v(P) = {} & c_\infty(v)K(P,\,\infty) + c_{-\infty}(v)K(P,\,-\infty) \\ & + \int_{\Gamma_n(D)} G(P,\,Q)d\mu(Q) + \int_{S_n(D)} \frac{\partial G(P,\,Q)}{\partial n_Q} d\nu(Q), \end{aligned}$$

where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $\Gamma_n(D)$.

Proof. By the Riesz decomposition theorem, we have a unique measure μ on $\Gamma_n(D)$ such that

$$v(P) = \int_{\Gamma_n(D)} G(P,\,Q)d\mu(Q) + h(P) \quad (P \in \Gamma_n(D)), \tag{4.5}$$

where h is the greatest harmonic minorant of v on $\Gamma_n(D)$. Further by the Martin representation theorem we have another positive measure ν' on

$\partial\Gamma_n(D)\cup\{\infty, -\infty\}$

$$\begin{aligned}h(P) &= \int_{\partial\Gamma_n(D)\cup\{\infty,-\infty\}} K(P, Q)d\nu'(Q)\\ &= K(P, \infty)\nu'(\{\infty\}) + K(P, -\infty)\nu'(\{-\infty\})\\ &\quad + \int_{S_n(D)} K(P, Q)d\nu'(Q) \quad (P\in\Gamma_n(D)).\end{aligned}$$

We see from (4.4) that $\nu'(\{\infty\}) = c_\infty(v)$ and $\nu'(\{-\infty\}) = c_{-\infty}(v)$ (see Yoshida [19, p. 292]). Since

$$K(P, Q) = \lim_{P_1\to Q,\, P_1\in\Gamma_n(D)} \frac{G(P, P_1)}{G(P^*, P_1)} = \frac{\partial G(P, Q)/\partial n_Q}{\partial G(P^*, Q)/\partial n_Q} \tag{4.6}$$

(P^* is a fixed reference point of the Martin kernel), we also obtain

$$\begin{aligned}h(P) &= c_\infty(v)K(P, \infty)\\ &\qquad + c_{-\infty}(v)K(P, -\infty) + \int_{S_n(D)} \frac{\partial G(P, Q)}{\partial n_Q} d\nu(Q)\end{aligned}$$

by taking

$$d\nu(Q) = \left\{\frac{\partial G(P^*, Q)}{\partial n_Q}\right\}^{-1} d\nu'(Q) \quad (Q\in S_n(D)).$$

Finally this and (4.5) give the conclusion of this lemma. □

We remark the following inequality which follows from (4.1).

$$\begin{aligned}&C_1 e^{\sqrt{\tau_D}y} e^{-\sqrt{\tau_D}y'} f_D(X)\frac{\partial}{\partial n_{X'}} f_D(X') \le \frac{\partial G(P, Q)}{\partial n_Q}\\ &\le C_2 e^{\sqrt{\tau_D}y} e^{-\sqrt{\tau_D}y'} f_D(X)\frac{\partial}{\partial n_{X'}} f_D(X')\end{aligned} \tag{4.7}$$

for any $P = (X, y) \in \Gamma_n(D)$ and any $Q = (X', y') \in S_n(D)$ satisfying $y < y' - 1$, where C_1 and C_2 are two positive constants.

Lemma 4 *Let ν be a positive measure on $S_n(D)$ such that there is a sequence of points $P_i = (X_i, y_i) \in \Gamma_n(D)$, $y_i \to +\infty$ $(i \to +\infty)$ satisfying*

$$\int_{S_n(D)} \frac{\partial G(P_i, Q)}{\partial n_Q} d\nu(Q) < +\infty \quad (i = 1, 2, 3, \ldots).$$

Then for a real number l

$$\int_{S_n(D;l,\infty)} e^{-\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu(X', y') < \infty.$$

and

$$\lim_{R\to\infty} e^{-2\sqrt{\tau_D}R} \int_{S_n(D;-\infty, R)} e^{\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu(X', y') = 0.$$

Proof. If we use (4.7) in place of (4.1), we obtain this lemma in the completely paralleled way to the proof of Lemma 1. □

Lemma 5 *Let $E \subset \Gamma_n(D)$ be bounded above and $u(P)$ be a positive superharmonic function on $\Gamma_n(D)$ such that $u(P)$ is represented as*

$$u(P) = \int_{\Gamma_n(D)} G(P, Q) d\mu_u(Q) + \int_{S_n(D)} \frac{\partial}{\partial n_Q} G(P, Q) d\nu_u(Q) \quad (P \in \Gamma_n(D)). \tag{4.8}$$

with two positive measures μ_u and ν_u on $\Gamma_n(D)$ and $S_n(D)$, respectively, and

$$u(P) \geq 1$$

for any $P \in E$. Then

$$\lambda_E\big(\Gamma_n(D)\big) \leq \int_{\Gamma_n(D)} e^{\sqrt{\tau_D}y'} f_D(X') d\mu_u(X', y') + \int_{S_n(D)} e^{\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu_u(X', y'). \tag{4.9}$$

When $u(P) = \hat{R}_1^E(P)$ $(P \in \Gamma_n(D))$, the equality holds in (4.9).

Proof. Since λ_E is concentrated on B_E and $u(P) \geq 1$ for any $P \in B_E$, we see from (4.8) that

$$\lambda_E\big(\Gamma_n(D)\big) = \int_{\Gamma_n(D)} d\lambda_E \leq \int_{\Gamma_n(D)} u(P) d\lambda_E(P)$$
$$= \int_{\Gamma_n(D)} \hat{R}^E_{K(\cdot,\infty)}(Q) d\mu_u(Q)$$

$$+\int_{S_n(D)}\left(\int_{\Gamma_n(D)}\frac{\partial}{\partial n_Q}G(P,Q)d\lambda_E(P)\right)d\nu_u(Q). \tag{4.10}$$

Now we have

$$\hat{R}^E_{K(\,\cdot\,,\infty)}(Q)\le K(Q,\infty)=e^{\sqrt{\tau_D}y'}f_D(X') \qquad (Q=(X',y')\in\Gamma_n(D)). \tag{4.11}$$

Since

$$\int_{\Gamma_n(D)}\frac{\partial}{\partial n_Q}G(P,Q)d\lambda_E(P)\le\varliminf_{\rho\to0}\frac{1}{\rho}\int_{\Gamma_n(D)}G(P,P_\rho)d\lambda_E(P)$$

for any $Q\in S_n(D)$ ($P_\rho=(X_\rho,y_\rho)=Q+\rho n_Q\in\Gamma_n(D)$, n_Q is the inward normal unit vector at Q) and

$$\int_{\Gamma_n(D)}G(P,P_\rho)d\lambda_E(P)=\hat{R}^E_{K(\,\cdot\,,\infty)}(P_\rho)\le K(P_\rho,\infty)=e^{\sqrt{\tau_D}y_\rho}f_D(X_\rho),$$

we have

$$\int_{\Gamma_n(D)}\frac{\partial}{\partial n_Q}G(P,Q)d\lambda_E(P)\le e^{\sqrt{\tau_D}y'}\frac{\partial}{\partial n_{X'}}f_D(X') \tag{4.12}$$

for any $Q=(X',y')\in S_n(D)$. Thus from (4.10), (4.11) and (4.12) we obtain (4.9).

When $u(P)=\hat{R}^E_1(P)$, $u(P)$ has the expression (4.8) by Lemma 3, because $\hat{R}^E_1(P)$ is bounded on $\Gamma_n(D)$. Then we easily have the equalities only in (4.10), because $\hat{R}^E_1(P)=1$ for any $P\in B_E$ (see Brelot [5, p. 61] and Doob [6, p. 169]). Hence if we can show that

$$\mu_u\big(\{P\in\Gamma_n(D);\hat{R}^E_{K(\,\cdot\,,\infty)}(P)<K(P,\infty)\}\big)=0 \tag{4.13}$$

and

$$\nu_u\Bigg(\bigg\{Q=(X',y')\in S_n(D);\int_{\Gamma_n(D)}\frac{\partial}{\partial n_Q}G(P,Q)d\lambda_E(P)<e^{\sqrt{\tau_D}y'}\frac{\partial}{\partial n_{X'}}f_D(X')\bigg\}\Bigg)=0, \tag{4.14}$$

then we can prove the equality in (4.9). To see (4.13), we remark that

$$\{P \in \Gamma_n(D); \hat{R}^E_{K(\cdot,\infty)}(P) < K(P, \infty)\} \subset \Gamma_n(D) - B_E$$

and

$$\mu_u(\Gamma_n(D) - B_E) = 0$$

(see Brelot [5, Theorem VIII,11] and Doob [6, Theorem XI.14(d)]). To prove (4.14), we set

$$B'_E = \{Q \in S_n(D); E \text{ is not minimally thin at } Q\} \tag{4.15}$$

and $e = \{P \in E; \hat{R}^E_{K(\cdot,\infty)}(P) < K(P, \infty)\}$. Then e is a polar set (see Doob [6, Theorem VI.3(b)]) and hence for any $Q \in S_n(D)$

$$\hat{R}^E_{K(\cdot,Q)} = \hat{R}^{E-e}_{K(\cdot,Q)}$$

(see Doob [6, Theorem VI.3(c)]). Thus at any $Q \in B'_E$, $E - e$ is not also minimally thin at Q and hence

$$\int_{\Gamma_n(D)} K(P, Q)d\eta(P) = \varliminf_{P' \to Q, P' \in E-e} \int_{\Gamma_n(D)} K(P, P')d\eta(P) \tag{4.16}$$

for any positive measure η on $\Gamma_n(D)$, where

$$K(P, P') = \frac{G(P, P')}{G(P^*, P')} \quad \big(P \in \Gamma_n(D),\ P' \in \Gamma_n(D)\big)$$

(see Brelot [5, Theorem XV,6]). Now, take $\eta = \lambda_E$ in (4.16). Since

$$\lim_{P \to Q,\, P \in \Gamma_n(D)} \frac{K(P, \infty)}{G(P^*, P)} = e^{\sqrt{\tau_D} y'} \frac{\partial}{\partial n_{X'}} f_D(X') \left\{\frac{\partial G(P^*, Q)}{\partial n_Q}\right\}^{-1} \quad \big(Q = (X', y') \in S_n(D)\big)$$

(for the existence of the limit in the left side, see Jerison and Kenig [13, (7.9) in p. 87]), we obtain from (4.6)

$$\int_{\Gamma_n(D)} \frac{\partial G(P, Q)}{\partial n_Q} d\lambda_E(P) = e^{\sqrt{\tau_D} y'} \frac{\partial}{\partial n_{X'}} f_D(X') \varliminf_{P' \to Q,\, P' \in E-e} \int_{\Gamma_n(D)} \frac{G(P, P')}{K(P', \infty)} d\lambda_E(P).$$

for any $Q = (X', y') \in B'_E$. Since

$$\int_{\Gamma_n(D)} \frac{G(P, P')}{K(P', \infty)} d\lambda_E(P) = \frac{1}{K(P', \infty)} \hat{R}^E_{K(\cdot, \infty)}(P') = 1$$

for any $P' \in E - e$, we have

$$\int_{\Gamma_n(D)} \frac{\partial G(P, Q)}{\partial n_Q} d\lambda_E(P) = e^{\sqrt{\tau_D} y'} \frac{\partial}{\partial n_{X'}} f_D(X')$$

for any $Q = (X', y') \in B'_E$, which shows

$$\Big\{ Q = (X', y') \in S_n(D); \int_{\Gamma_n(D)} \frac{\partial}{\partial n_Q} G(P, Q) d\lambda_E(P) < e^{\sqrt{\tau_D} y'} \frac{\partial}{\partial n_{X'}} f_D(X') \Big\} \subset S_n(D) - B'_E. \quad (4.17)$$

Let h be the greatest harmonic minorant of $u(P) = \hat{R}^E_1(P)$ and ν'_u be the Martin representing measure of h. If we can prove that

$$\hat{R}^E_h = h \quad (4.18)$$

on $\Gamma_n(D)$, then $\nu'_u(S_n(D) - B'_E) = 0$ (see Essén and Jackson [8, pp. 240–241], Brelot [5, Theorem XV,11] and, Aikawa and Essén [2, Part II, p. 188]). Since

$$d\nu'_u(Q) = \frac{\partial}{\partial n_Q} G(P_0, Q) d\nu_u(Q) \quad (Q \in S_n(D))$$

from (4.6), we also have $\nu_u(S_n(D) - B'_E) = 0$, which gives (4.14) from (4.17).

To prove (4.18), set $u^* = \hat{R}^E_1 - h$. Then

$$u^* + h = \hat{R}^E_1 = \hat{R}^E_{u^*+h} \le \hat{R}^E_{u^*} + \hat{R}^E_h$$

(see Brelot [5, VI, 10. d)] and Helms [12, THEOREM 7.12 (iv)]), and hence

$$\hat{R}^E_h - h \ge u^* - \hat{R}^E_{u^*} \ge 0,$$

from which (4.18) follows. □

5. Proof of Theorem 1

Proof of (I) ⇒ (II). Apply the Riesz decomposition theorem to the superharmonic function $\hat{R}^E_{K(\cdot, \infty)}$ on $\Gamma_n(D)$. Then we have a positive measure μ

on $\Gamma_n(D)$ satisfying

$$G\mu(P) < \infty$$

for any $P \in \Gamma_n(D)$ and a non-negative greatest harmonic minorant H of $\hat{R}^E_{K(\,\cdot\,,\infty)}$ such that

$$\hat{R}^E_{K(\,\cdot\,,\infty)} = G\mu + H. \tag{5.1}$$

We remark that $K(P, \infty)$ $(P \in \Gamma_n(D)$ is a minimal function at ∞.

Let E be a minimally thin set at ∞ with respect to $\Gamma_n(D)$. Then $\hat{R}^E_{K(\,\cdot\,,\infty)}$ is a potential (see Doob [6, p. 208]) and hence $H \equiv 0$ on $\Gamma_n(D)$. Since

$$\hat{R}^E_{K(\,\cdot\,,\infty)} = K(P, \infty) \tag{5.2}$$

for any $P \in B_E$ (Brelot [5, p. 61] and Doob [6, p. 169]), we see from (5.1)

$$G\mu(P) = K(P, \infty) \tag{5.3}$$

for any $P \in B_E$. Take a sufficiently large integer L from Lemma 1 such that

$$C_2 e^{-2\sqrt{\tau_D}L} \int_{\Gamma_n(D;-\infty,\,L)} e^{\sqrt{\tau_D}y'} f_D(X') d\mu(X', y') < \frac{1}{4},$$

where C_2 is the constant in (4.1). Then from (4.1)

$$\int_{\Gamma_n(D;-\infty,\,L)} G(P, Q) d\mu(Q) \leq \frac{1}{4} K(P, \infty)$$

for any $P = (X, y) \in \Gamma_n(D),\ y \geq L+1$, and hence from (5.3)

$$\int_{\Gamma_n(D;L,\,+\infty)} G(P, Q) d\mu(Q) \geq \frac{3}{4} K(P, \infty) \tag{5.4}$$

for any $P = (X, y) \in B_E,\ y \geq L+1$. Now, divide $G\mu$ into three parts:

$$G\mu(P) = A_1^{(k)}(P) + A_2^{(k)}(P) + A_3^{(k)}(P) \quad (P \in \Gamma_n(D)), \tag{5.5}$$

where

$$A_1^{(k)}(P) = \int_{\Gamma_n(D;k-1,\,k+2)} G(P, Q) d\mu(Q),$$
$$A_2^{(k)}(P) = \int_{\Gamma_n(D;-\infty,\,k-1)} G(P, Q) d\mu(Q),$$

$$A_3^{(k)}(P) = \int_{\Gamma_n(D;k+2,+\infty)} G(P, Q) d\mu(Q).$$

Then we shall show that there exists an integer N such that

$$B_E \cap \overline{I_k} \subset \left\{ P = (X, y) \in \Gamma_n(D);\ A_1^{(k)}(P) \geq \frac{1}{4} K(P, \infty) \right\} \quad (k \geq N). \tag{5.6}$$

Take any $P = (X, y) \in \overline{I_k} \cap \Gamma_n(D)$. When by Lemma 1 we choose a sufficiently large integer N_1 such that

$$e^{-2\sqrt{\tau_D}k} \int_{\Gamma_n(D;-\infty,k-1)} e^{\sqrt{\tau_D}y'} f_D(X') d\mu(X', y') \leq \frac{1}{4C_2} \quad (k \geq N_1)$$

and

$$\int_{\Gamma_n(D;k+2,\infty)} e^{-\sqrt{\tau_D}y'} f_D(X') d\mu(X', y') \leq \frac{1}{4C_2} \quad (k \geq N_1),$$

we have from (4.1) that

$$A_2^{(k)}(P) \leq \frac{1}{4} K(P, \infty) \quad (k \geq N_1) \tag{5.7}$$

and

$$A_3^{(k)}(P) \leq \frac{1}{4} K(P, \infty) \quad (k \geq N_1). \tag{5.8}$$

Put

$$N = \max\{N_1, L + 1\}.$$

If $P = (X, y) \in B_E \cap \overline{I_k}$ $(k \geq N)$, then we have from (5.4), (5.5), (5.7) and (5.8) that

$$\begin{aligned} A_1^{(k)}(P) &\geq \int_{\Gamma_n(D;L,+\infty)} G(P, Q) d\mu(Q) - A_2^{(k)}(P) - A_3^{(k)}(P) \\ &\geq \frac{1}{4} K(P, \infty), \end{aligned}$$

which gives (5.6).

Since the measure $\lambda_{E(k)}$ is concentrated on $B_{E(k)}$ and $B_{E_k} \subset \overline{E_k} \cap \Gamma_n(D)$, we finally obtain by (5.6) that

$$\begin{aligned}
\gamma\bigl(E(k)\bigr) &= \int_{\Gamma_n(D)} (G\lambda_{E(k)})d\lambda_{E(k)} \\
&\le \int_{B_{E(k)}} e^{\sqrt{\tau_D}y} f_D(X)d\lambda_{E(k)}(X,\, y) \\
&\le 4\int_{B_{E(k)}} A_1^{(k)}(P)d\lambda_{E(k)}(P) \\
&= 4\int_{\Gamma_n(D;k-1,\,k+2)} \left\{\int_{\Gamma_n(D)} G(P,\, Q)d\lambda_{E(k)}(P)\right\}d\mu(Q) \\
&\le 4\int_{\Gamma_n(D;k-1,\,k+2)} e^{\sqrt{\tau_D}y'} f_D(X')d\mu(X',\, y') \quad (k \ge N)
\end{aligned}$$

and hence

$$\begin{aligned}
&\sum_{k=N}^{\infty} \gamma\bigl(E(k)\bigr)e^{-2\sqrt{\tau_D}k} \\
&\le 4\sum_{k=N}^{\infty} \int_{\Gamma_n(D;k-1,\,k+2)} e^{\sqrt{\tau_D}y'} f_D(X')e^{-2\sqrt{\tau_D}k}d\mu(X',\, y') \\
&\le 12e^{4\sqrt{\tau_D}} \int_{\Gamma_n(D;N-1,\,\infty)} e^{-\sqrt{\tau_D}y'} f_D(X')d\mu(X',\, y') \\
&< \infty
\end{aligned}$$

from Lemma 1. This gives (II).

Proof of (II) $\Rightarrow$ (III). Since

$$\hat{R}^{E(k)}_{K(\,\cdot\,,\infty)}(Q) = K(Q,\, \infty)$$

for any $Q \in B_{E(k)}$ as in (5.2), we have

$$\begin{aligned}
\gamma\bigl(E(k)\bigr) &= \int_{B_{E(k)}} K(Q,\, \infty)d\lambda_{E(k)}(Q) \\
&\ge e^{\sqrt{\tau_D}k} \int_{B_{E(k)}} f_D(X')d\lambda_{E(k)}(X',\, y') \\
&\qquad\qquad (Q = (X',\, y') \in \Gamma_n(D))
\end{aligned}$$

and hence from (4.1)

$$\begin{aligned}&\hat{R}^{E(k)}_{K(\,\cdot\,,\infty)}(P)\\&\le C_2 e^{\sqrt{\tau_D}y} f_D(X)\int_{B_{E(k)}} e^{-\sqrt{\tau_D}y'} f_D(X')d\lambda_{E(k)}(X',\,y')\\&\le C_2 e^{\sqrt{\tau_D}y} f_D(X) e^{-2\sqrt{\tau_D}k}\gamma\bigl(E(k)\bigr)\end{aligned}\tag{5.9}$$

for any $P=(X,\,y)\in\Gamma_n(D)$ and any integer k satisfying $k-1\ge y$. If we define a measure μ on $\Gamma_n(D)$ by

$$\mu=\sum_{k=0}^{\infty}\lambda_{E(k)}$$

then from (I) and (5.9)

$$G\mu(P)=\int_{\Gamma_n(D)} G(P,\,Q)d\mu(Q)=\sum_{k=0}^{\infty}\hat{R}^{E(k)}_{K(\,\cdot\,,\infty)}(P)$$

is a finite-valued superharmonic function on $\Gamma_n(D)$ and

$$G\mu(P)\ge\int_{\Gamma_n(D)} G(P,\,Q)d\lambda_{E(k)}(Q)=\hat{R}^{E(k)}_{K(\,\cdot\,,\infty)}(P)=K(P,\,\infty)$$

for any $P=(X,\,y)\in B_{E(k)}$, and from (4.1)

$$G\mu(P)\ge C'K(P,\,\infty)$$

for any $P=(X,\,y)\in\Gamma_n(D;-\infty,\,0)$, where

$$C'=C_1\int_{\Gamma_n(D;1,\,+\infty)} e^{-\sqrt{\tau_D}y'} f_D(X')d\mu(X',\,y').$$

It is evident from (5.4) that C' is positive. If we set

$$E(-1)=E\cap\Gamma_n(D;-\infty,\,0),\quad E'=\bigcup_{k=-1}^{\infty}B_{E(k)}$$

and $B=\min(C',\,1)$, then

$$E'\subset\{P=(X,\,y)\in\Gamma_n(D);\ G\mu(P)\ge BK(P,\,\infty)\}.$$

Since E' is equal to E except a polar set S (see Brelot [5, p. 57] and Doob [6, p. 177]), we can take a positive measure η on $\Gamma_n(D)$ such that $G\eta$ is identically $+\infty$ on S (see Doob [6, p. 58]). If we define a measure ν on

$\Gamma_n(D)$ by

$$\nu = \frac{1}{B}(\mu + \eta),$$

then

$$E \subset \{P = (X,\, y) \in \Gamma_n(D);\ G\nu(P) \geq K(P,\, \infty)\}. \tag{5.10}$$

If we put $v(P) = G\nu(P)$, then (5.10) shows that $v(P)$ is the function required in (III).

Proof of (III) $\Rightarrow$ (I). Let $v(P)$ be the function in (III). By Lemma 3, we can find two positive measures μ on $\Gamma_n(D)$ and ν on $S_n(D)$ such that

$$\begin{aligned} v(P) = {} & c_{-\infty}(v)K(P,\, -\infty) + \int_{\Gamma_n(D)} G(P,\, Q)d\mu(Q) \\ & + \int_{S_n(D)} \frac{\partial G(P,\, Q)}{\partial n_Q} d\nu(Q) \quad (P \in \Gamma_n(D)). \end{aligned}$$

When we put

$$W(P) = \int_{\Gamma_n(D)} G(P,\, Q)d\mu(Q) + \int_{S_n(D)} \frac{\partial G(P,\, Q)}{\partial n_Q} d\nu(Q),$$

we have

$$\begin{aligned} W(P) & = v(P) - c_{-\infty}(v)K(P,\, -\infty) \\ & \geq \left\{e^{\sqrt{\tau_D}y} - c_{-\infty}(v)\kappa e^{-\sqrt{\tau_D}y}\right\} f_D(X) \geq \frac{1}{2}K(P,\, \infty) \end{aligned}$$

for any $P = (X,\, y) \in M_v,\ y \geq y_0$, with a sufficiently large y_0, which gives

$$\begin{aligned} & M_v \cap \Gamma_n(D; y_0,\, \infty) \\ & \qquad \subset \left\{P = (X,\, y) \in \Gamma_n(D); W(P) \geq \frac{1}{2}K(P,\, \infty)\right\}. \end{aligned} \tag{5.11}$$

We easily see that

$$\left\{P = (X,\, y) \in \Gamma_n(D); W(P) \geq \frac{1}{2}K(P,\, \infty)\right\} \subset U \cup V, \tag{5.12}$$

where

$$U = \left\{P = (X,\, y) \in \Gamma_n(D); \int_{\Gamma_n(D)} G(P,\, Q)d\mu(Q) \geq \frac{1}{4}K(P,\, \infty)\right\}$$

and

$$V = \left\{ P = (X,\, y) \in \Gamma_n(D);\, \int_{S_n(D)} \frac{\partial G(P,\, Q)}{\partial n_Q} d\nu(Q) \geq \frac{1}{4} K(P,\, \infty) \right\}.$$

By Lemma 2 applied to $\int_{\Gamma_n(D)} G(P,Q)d\mu(Q)$ and $\int_{S_n(D)} \partial G(P,Q)/\partial n_Q d\nu(Q)$, U and V are minimally thin sets at ∞ with respect to $\Gamma_n(D)$, respectively. When we observe

$$E \subset (M_v \cap \Gamma_n(D; y_0,\, \infty)) \cup \Gamma_n(D; -\infty,\, y_0),$$

we see from (3.1), (5.11) and (5.12) that E is a minimally thin set at ∞ with respect to $\Gamma_n(D)$. □

6. Proof of Theorem 2

Let E be a rarefied set at ∞ with respect to $\Gamma_n(D)$. Then there exists a positive superharmonic function $v(P)$ on $\Gamma_n(D)$ such that

$$\inf_{P=(X,\,y)\in\Gamma_n(D)} \frac{v(P)}{K(P,\, \infty)} = 0$$

and

$$E \subset H_v. \tag{6.1}$$

By Lemma 3, we can find two positive measures μ on $\Gamma_n(D)$ and ν on $S_n(D)$ such that

$$v(P) = c_{-\infty}(v) K(P,\, -\infty) + \int_{\Gamma_n(D)} G(P,\, Q) d\mu(Q) + \int_{S_n(D)} \frac{\partial G(P,\, Q)}{\partial n_Q} d\nu(Q) \quad (P \in \Gamma_n(D)).$$

Now we write

$$v(P) = c_{-\infty}(v) K(P,\, -\infty) + B_1^{(k)}(P) + B_2^{(k)}(P) + B_3^{(k)}(P), \tag{6.2}$$

where

$$B_1^{(k)}(P) = \int_{\Gamma_n(D; -\infty,\, k-1)} G(P,\, Q) d\mu(Q) + \int_{S_n(D; -\infty,\, k-1)} \frac{\partial G(P,\, Q)}{\partial n_Q} d\nu(Q),$$

$$B_2^{(k)}(P) = \int_{\Gamma_n(D;\, k-1,\, k+2)} G(P,\, Q) d\mu(Q) + \int_{S_n(D; k-1,\, k+2)} \frac{\partial G(P,\, Q)}{\partial n_Q} d\nu(Q)$$

and

$$B_3^{(k)}(P) = \int_{\Gamma_n(D;\, k+2,\, \infty)} G(P,\, Q) d\mu(Q) + \int_{S_n(D; k+2,\, \infty)} \frac{\partial G(P,\, Q)}{\partial n_Q} d\nu(Q) \quad (P \in \Gamma_n(D); k = 1,\, 2,\, 3,\, \ldots).$$

First we shall show the existence of an integer N such that

$$H_v \cap I_k \subset \left\{ P = (X,\, y) \in I_k; B_2^{(k)}(P) \geq \frac{1}{2} e^{\sqrt{\tau_D} y} \right\} \tag{6.3}$$

for any integer k, $k \geq N$. Since $v(P)$ is finite almost everywhere on $\Gamma_n(D)$, from Lemmas 1 and 4 applied to

$$\int_{\Gamma_n(D)} G(P,\, Q) d\mu(Q) \quad \text{and} \quad \int_{S_n(D)} \frac{\partial G(P,\, Q)}{\partial n_Q} d\nu(Q)$$

respectively, we can take an integer N such that for any k, $k \geq N$,

$$e^{-2\sqrt{\tau_D} k} \int_{\Gamma_n(D; -\infty,\, k-1)} e^{\sqrt{\tau_D} y'} f_D(X') d\mu(X',\, y') \leq \frac{1}{12 C_2 J_D}, \tag{6.4}$$

$$\int_{\Gamma_n(D; k+2,\, +\infty)} e^{-\sqrt{\tau_D} y'} f_D(X') d\mu(X',\, y') \leq \frac{1}{12 C_2 J_D}, \tag{6.5}$$

$$e^{-2\sqrt{\tau_D} k} \int_{S_n(D; -\infty,\, k-1)} e^{\sqrt{\tau_D} y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu(X',\, y') \leq \frac{1}{12 C_2 J_D} \tag{6.6}$$

and

$$\int_{S_n(D; k+2,\, +\infty)} e^{-\sqrt{\tau_D} y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu(X',\, y') \leq \frac{1}{12 C_2 J_D}, \tag{6.7}$$

where

$$J_D = \sup_{X \in D} f_D(X).$$

Then for any $P = (X,\, y) \in I_k$ $(k \geq N)$, we have

$$B_1^{(k)}(P) \leq C_2 e^{-\sqrt{\tau_D} y} f_D(X) \int_{\Gamma_n(D; -\infty,\, k-1)} e^{\sqrt{\tau_D} y'} f_D(X') d\mu(X',\, y')$$

$$+ C_2 e^{-\sqrt{\tau_D}y} f_D(X) \int_{S_n(D;-\infty,\, k-1)} e^{\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu(X',\, y')$$
$$\leq 2C_2 e^{-\sqrt{\tau_D}y} J_D e^{2\sqrt{\tau_D}k} \frac{1}{12 C_2 J_D} = \frac{1}{6} e^{\sqrt{\tau_D}y}$$

from (4.1), (4.7), (6.4) and (6.6), and

$$B_3^{(k)}(P) \leq C_2 e^{\sqrt{\tau_D}y} f_D(X) \int_{\Gamma_n(D;k+2,\, +\infty)} e^{-\sqrt{\tau_D}y'} f_D(X') d\mu(X',\, y')$$
$$+ C_2 e^{\sqrt{\tau_D}y} f_D(X) \int_{S_n(D;k+2,\, +\infty)} e^{-\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu(X',\, y')$$
$$\leq \frac{1}{6} e^{\sqrt{\tau_D}y}$$

from (4.1), (4.7), (6.5) and (6.7). Further we can assume that

$$6\kappa c_{-\infty}(v) J_D \leq e^{2\sqrt{\tau_D}y}$$

for any $P = (X,\, y) \in I_k$ $(k \geq N)$, hence if $P = (X,\, y) \in I_k \cap H_v$ $(k \geq N)$, then we obtain

$$B_2^{(k)}(P) \geq e^{\sqrt{\tau_D}y} - \frac{1}{6} e^{\sqrt{\tau_D}y} - \frac{1}{6} e^{\sqrt{\tau_D}y} - \frac{1}{6} e^{\sqrt{\tau_D}y} = \frac{1}{2} e^{\sqrt{\tau_D}y}$$

from (6.2), which gives (6.3).

Now we observe from (6.1) and (6.3) that

$$B_2^{(k)}(P) \geq \frac{1}{2} e^{\sqrt{\tau_D}k} \quad (k \geq N)$$

for any $P \in E(k)$. If we define a function $u_k(P)$ on $\Gamma_n(D)$ by

$$u_k(P) = 2e^{-\sqrt{\tau_D}k} B_2^{(k)}(P),$$

then

$$u_k(P) \geq 1 \quad (P \in E(k),\, k \geq N)$$

and

$$u_k(P) = \int_{\Gamma_n(D)} G(P,\, Q) d\mu_k(Q) + \int_{S_n(D)} \frac{\partial G(P,\, Q)}{\partial n_Q} d\nu_k(Q)$$

with two measures

$$d\mu_k(Q) = \begin{cases} 2e^{-\sqrt{\tau_D}k} d\mu(Q) & (Q \in \Gamma_n(D; k-1,\, k+2)) \\ 0 & (Q \in \Gamma_n(D; -\infty,\, k-1) \cup \Gamma_n(D; k+2,\, \infty)) \end{cases}$$

and

$$d\nu_k(Q) = \begin{cases} 2e^{-\sqrt{\tau_D}k} d\nu(Q) & (Q \in S_n(D; k-1,\, k+2)) \\ 0 & (Q \in S_n(D; -\infty,\, k-1) \cup S_n(D; k+2,\, \infty)). \end{cases}$$

Hence by applying Lemma 5 to $u_k(P)$, we obtain

$$\lambda_{E(k)}\big(\Gamma_n(D)\big) \le 2e^{-\sqrt{\tau_D}k} \bigg\{ \int_{\Gamma_n(D; k-1,\, k+2)} e^{\sqrt{\tau_D}y'} f_D(X') d\mu(X',\, y') + \int_{S_n(D; k-1,\, k+2)} e^{\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu(X',\, y') \bigg\}$$

$(k \ge N)$. Finally we have

$$\sum_{k=N}^{\infty} e^{-\sqrt{\tau_D}k} \lambda_{E(k)}\big(\Gamma_n(D)\big) \le 6e^{4\sqrt{\tau_D}} \bigg\{ \int_{\Gamma_n(D; N-1,\, \infty)} e^{-\sqrt{\tau_D}y'} f_D(X') d\mu(X',\, y') + \int_{S_n(D; N-1,\, \infty)} e^{-\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu(X',\, y') \bigg\}.$$

If we take a sufficiently large N, then the integrals of the right side are finite from Lemmas 1 and 4.

Suppose that a subset E of $\Gamma_n(D)$ satisfies

$$\sum_{k=0}^{\infty} e^{-\sqrt{\tau_D}k} \lambda_{E(k)}\big(\Gamma_n(D)\big) < +\infty.$$

Then from the second part of Lemma 5 applied to $E(k)$, we have

$$\sum_{k=0}^{\infty} e^{-\sqrt{\tau_D}k} \bigg(\int_{\Gamma_n(D)} e^{\sqrt{\tau_D}y'} f_D(X') d\mu_k^*(X',\, y') + \int_{S_n(D)} e^{\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu_k^*(X',\, y') \bigg) < \infty, \quad (6.8)$$

where μ_k^* and ν_k^* are two positive measures on $\Gamma_n(D)$ and $S_n(D)$, respectively, such that

$$\hat{R}_1^{E(k)}(P) = \int_{\Gamma_n(D)} G(P,\, Q) d\mu_k^*(Q)$$

$$+\int_{S_n(D)} \frac{\partial}{\partial n_Q} G(P,\, Q) d\nu_k^*(Q). \tag{6.9}$$

Consider the function $v_0(P)$ on $\Gamma_n(D)$ defined by

$$v_0(P) = \sum_{k=-1}^{\infty} e^{\sqrt{\tau_D}(k+1)} \hat{R}_1^{E(k)}(P) \quad (P \in \Gamma_n(D)).$$

Then $v_0(P)$ is a superharmonic function on $\Gamma_n(D)$ or identically ∞ on $\Gamma_n(D)$. Take any positive integer k_0 and write

$$v_0(P) = v_1(P) + v_2(P) \quad (P \in \Gamma_n(D)),$$

where

$$v_1(P) = \sum_{k=-1}^{k_0+1} e^{\sqrt{\tau_D}(k+1)} \hat{R}_1^{E(k)}(P),\ v_2(P) = \sum_{k_0+2}^{\infty} e^{\sqrt{\tau_D}(k+1)} \hat{R}_1^{E(k)}(P).$$

Since μ_k^* and ν_k^* are concentrated on $B_{E(k)} \subset \overline{E(k)} \cap \Gamma_n(D)$ and $B'_{E(k)} \subset \overline{E(k)} \cap S_n(D)$ (see (4.15) for the notation $B'_{E(k)}$), respectively (Brelot [5, Theorem XV,11]), we have from (4.1) and (4.7) that

$$\begin{aligned}
&e^{\sqrt{\tau_D}(k+1)} \int_{\Gamma_n(D)} G(P_0,\, Q) d\mu_k^*(Q) \\
&\le C_2 e^{\sqrt{\tau_D}k} e^{\sqrt{\tau_D}(y_0+1)} f_D(X_0) \int_{\Gamma_n(D)} e^{-\sqrt{\tau_D}y'} f_D(X') d\mu_k^*(X',\, y') \\
&\le C_2 e^{\sqrt{\tau_D}(y_0+1)} f_D(X_0) e^{-\sqrt{\tau_D}k} \int_{\Gamma_n(D)} e^{\sqrt{\tau_D}y'} f_D(X') d\mu_k^*(X',\, y')
\end{aligned}$$

and

$$\begin{aligned}
&e^{\sqrt{\tau_D}(k+1)} \int_{S_n(D)} \frac{\partial}{\partial n_Q} G(P_0,\, Q) d\nu_k^*(Q) \\
&\le C_2 e^{\sqrt{\tau_D}(y_0+1)} f_D(X_0) e^{-\sqrt{\tau_D}k} \int_{S_n(D)} e^{\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X') d\nu_k^*(X',\, y')
\end{aligned}$$

for a point $P_0 = (X_0,\, y_0) \in \Gamma_n(D),\ y_0 \le k_0 + 1$, and any integer $k \ge k_0 + 2$. Hence we know

$$v_2(P_0) \le C_2 e^{\sqrt{\tau_D}(y_0+1)} f_D(X_0)$$

$$\times \sum_{k_0+2}^{\infty} e^{-\sqrt{\tau_D}k}\bigg\{\int_{\Gamma_n(D)} e^{\sqrt{\tau_D}y'} f_D(X')d\mu_k^*(X',y') + \int_{S_n(D)} e^{\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X')d\nu_k^*(X',\, y')\bigg\}. \quad (6.10)$$

This and (6.8) show that $v_2(P_0)$ is finite and hence $v_0(P)$ is a positive superharmonic function on $\Gamma_n(D)$. To see

$$c_\infty(v_0) = \inf_{P\in\Gamma_n(D)} \frac{v_0(P)}{K(P,\,\infty)} = 0, \quad (6.11)$$

consider the representations of $v_0(P)$, $v_1(P)$ and $v_2(P)$

$$v_0(P) = c_\infty(v_0)K(P,\,\infty) + c_{-\infty}(v_0)K(P,\,-\infty) + \int_{\Gamma_n(D)} G(P,\,Q)d\mu_{(0)}(Q) + \int_{S_n(D)} \frac{\partial G(P,\,Q)}{\partial n_Q} d\nu_{(0)}(Q),$$
$$v_1(P) = c_\infty(v_1)K(P,\,\infty) + c_{-\infty}(v_1)K(P,\,-\infty) + \int_{\Gamma_n(D)} G(P,\,Q)d\mu_{(1)}(Q) + \int_{S_n(D)} \frac{\partial G(P,\,Q)}{\partial n_Q} d\nu_{(1)}(Q),$$

and

$$v_2(P) = c_\infty(v_2)K(P,\,\infty) + c_{-\infty}(v_2)K(P,\,-\infty) + \int_{\Gamma_n(D)} G(P,\,Q)d\mu_{(2)}(Q) + \int_{S_n(D)} \frac{\partial G(P,\,Q)}{\partial n_Q} d\nu_{(2)}(Q)$$

by Lemma 3. It is evident from (6.9) that $c_\infty(v_1) = 0$ for any k_0. Since

$$c_\infty(v_2) = \inf_{P\in\Gamma_n(D)} \frac{v_2(P)}{K(P,\,\infty)} \le \frac{v_2(P_0)}{K(P_0,\,\infty)}$$
$$\le C_2 e^{\sqrt{\tau_D}} \sum_{k_0+2}^{\infty} e^{-\sqrt{\tau_D}k}\bigg\{\int_{\Gamma_n(D)} e^{\sqrt{\tau_D}y'} f_D(X')d\mu_k^*(X',\, y') + \int_{S_n(D)} e^{\sqrt{\tau_D}y'} \frac{\partial}{\partial n_{X'}} f_D(X')d\nu_k^*(X',y')\bigg\} \to 0 \quad (k_0 \to +\infty)$$

from (6.8) and (6.10), we know $c_\infty(v_2) = 0$ and hence $c_\infty(v_0) = 0$, which is (6.11).

Since $\hat{R}_1^{E(k)} = 1$ on $B_{E(k)}$, $B_{E(k)} \subset \overline{E(k)} \cap \Gamma_n(D)$ (Brelot [5, p. 61] and

Doob [6, p. 169]), we see

$$v_0(P) \geq e^{\sqrt{\tau_D}(k+1)} \geq e^{\sqrt{\tau_D}y}$$

for any $P = (X, y) \in B_{E(k)}$ $(k = -1, 0, 1, 2, \ldots)$. If we set $E' = \cup_{k=-1}^{\infty} B_{E(k)}$, then

$$E' \subset H_{v_0}. \tag{6.12}$$

Since E' is equal to E except a polar set S, we can take another positive superharmonic function v_3 on $\Gamma_n(D)$ such that $v_3 = G\eta$ with a positive measure η on $\Gamma_n(D)$ and v_3 is identically $+\infty$ on S (see Doob [6, p. 58]). Finally, define a positive superharmonic function v on $\Gamma_n(D)$ by

$$v = v_0 + v_3.$$

Since $c_\infty(v_3) = 0$, it is easy to see from (6.11) that $c_\infty(v) = 0$. Also we see from (5.12) that $E \subset H_v$. Thus we complete to prove that E is a rarefied set at ∞ with respect to $\Gamma_n(D)$. □

7. Proofs of Theorems 3 and 4

Proof of Theorem 3. By Lemma 3 we have

$$\begin{aligned} v(P) = {} & c_\infty(v)K(P, \infty) + c_{-\infty}(v)K(P, -\infty) \\ & + \int_{\Gamma_n(D)} G(P, Q)d\mu(Q) + \int_{S_n(D)} \frac{\partial G(P, Q)}{\partial n_Q} d\nu(Q) \end{aligned}$$

for two positive measures μ and ν on $\Gamma_n(D)$ and $S_n(D)$, respectively. Then

$$v_1(P) = v(P) - c_\infty(v)K(P, \infty) - c_{-\infty}(v)K(P, -\infty) \qquad (P = (X, y) \in \Gamma_n(D))$$

also is a positive superharmonic function on $\Gamma_n(D)$ such that

$$\inf_{P=(X,y)\in\Gamma_n(D)} \frac{v_1(P)}{K(P, \infty)} = 0.$$

We shall prove the existence of a rarefied set E at ∞ with respect to $\Gamma_n(D)$ such that

$$v_1(P)e^{-\sqrt{\tau_D}y} \quad (P = (X, y) \in \Gamma_n(D))$$

uniformly converges to 0 on $\Gamma_n(D) - E$ as $y \to +\infty$. Let $\{\varepsilon_i\}$ be a sequence of positive numbers ε_i satisfying $\varepsilon_i \to 0$ $(i \to +\infty)$. Put

$$F_i = \{P = (X,\, y) \in \Gamma_n(D); v_1(P) \geq \varepsilon_i e^{\sqrt{\tau_D}y}\} \quad (i = 1,\, 2,\, 3,\, \ldots).$$

Then F_i $(i = 1, 2, 3, \ldots)$ is rarefied at ∞ with respect to $\Gamma_n(D)$ and hence

$$\sum_{k=0}^{\infty} e^{-\sqrt{\tau_D}k} \lambda_{F_i(k)}\big(\Gamma_n(D)\big) < \infty \quad (i = 1,\, 2,\, 3,\, \ldots)$$

by Theorem 2. Take a sequence $\{q_i\}$ such that

$$\sum_{k=q_i}^{\infty} e^{-\sqrt{\tau_D}k} \lambda_{F_i(k)}\big(\Gamma_n(D)\big) < \frac{1}{2^i} \quad (i = 1,\, 2,\, 3,\, \ldots)$$

and set

$$E = \bigcup_{i=1}^{\infty} \bigcup_{k=q_i}^{\infty} F_i(k).$$

Then

$$\lambda_{E(m)}\big(\Gamma_n(D)\big) \leq \sum_{i=1}^{\infty} \sum_{k=q_i}^{\infty} \lambda_{F_i \cap I_k \cap I_m}\big(\Gamma_n(D)\big) \quad (m = 1,\, 2,\, 3,\, \ldots),$$

because λ is a countably sub-additive set function as in Aikawa and Essén [2, Lemma 2.4 (iii)] and in Essén and Jakson [8, p. 241]. Since

$$\begin{aligned}
&\sum_{m=1}^{\infty} \lambda_{E(m)}\big(\Gamma_n(D)\big) e^{-\sqrt{\tau_D}m} \\
&\qquad \leq \sum_{i=1}^{\infty} \sum_{k=q_i}^{\infty} \sum_{m=1}^{\infty} \lambda_{F_i \cap I_k \cap I_m}\big(\Gamma_n(D)\big) e^{-\sqrt{\tau_D}m} \\
&\qquad = \sum_{i=1}^{\infty} \sum_{k=q_i}^{\infty} \lambda_{F_i(k)}\big(\Gamma_n(D)\big) e^{-\sqrt{\tau_D}k} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,
\end{aligned}$$

we know by Theorem 2 that E is a rarefied set at ∞ with respect to $\Gamma_n(D)$. It is easy to see that

$$v_1(P) e^{-\sqrt{\tau_D}y} \quad (P = (X,\, y) \in \Gamma_n(D))$$

uniformly converges to 0 on $\Gamma_n(D) - E$ as $y \to \infty$. □

Proof of Theorem 4. Since $\lambda_{E(k)}$ is concentrated on $B_{E(k)} \subset \overline{E(k)} \cap \Gamma_n(D)$, we see

$$\gamma\bigl(E(k)\bigr) = \int_{\Gamma_n(D)} \hat{R}^{E(k)}_{K(\cdot,\infty)}(P) d\lambda_{E(k)}(P)$$
$$\leq \int_{\Gamma_n(D)} K(P, \infty) d\lambda_{E(k)}(P) \leq J_D e^{\sqrt{\tau_D}(k+1)} \lambda_{E(k)}\bigl(\Gamma_n(D)\bigr)$$

and hence

$$\sum_{k=0}^{\infty} e^{-2\sqrt{\tau_D}k} \gamma\bigl(E(k)\bigr) \leq J_D e^{\sqrt{\tau_D}} \sum_{k=0}^{\infty} e^{-\sqrt{\tau_D}k} \lambda_{E(k)}\bigl(\Gamma_n(D)\bigr),$$

which gives the conclusion in the first part from Theorems 1 and 2.

To prove the second part, put $J'_D = \min_{X \in \overline{D}} f_D(X)$. Since

$$K(P, \infty) = e^{\sqrt{\tau_D}y} f_D(X) \geq J'_D e^{\sqrt{\tau_D}y} \geq J'_D e^{\sqrt{\tau_D}k}$$
$$(P = (X, y) \in E(k)),$$

and

$$\hat{R}^{E(k)}_{K(\cdot,\infty)}(P) = K(P, \infty)$$

for any $P \in B_{E(k)}$, we have

$$\gamma\bigl(E(k)\bigr) = \int_{\Gamma_n(D)} \hat{R}^{E(k)}_{K(\cdot,\infty)}(P) d\lambda_{E(k)}(P) \geq J'_D e^{\sqrt{\tau_D}k} \lambda_{E(k)}\bigl(\Gamma_n(D)\bigr).$$

Since

$$J'_D \sum_{k=0}^{\infty} e^{-\sqrt{\tau_D}k} \lambda_{E(k)}\bigl(\Gamma_n(D)\bigr) \leq \sum_{k=0}^{\infty} e^{-2\sqrt{\tau_D}k} \gamma\bigl(E(k)\bigr) < +\infty$$

from Theorem 1, it follows from Theorem 2 that E is rarefied at ∞ with respect to $\Gamma_n(D)$. □

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