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Geodesic spheres in a nonflat complex space form and their integral curves of characteristic vector fields

(Dedicated to Professor Yusuke Sakane on his sixtieth birthday)

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Abstract. For Hopf hypersurfaces in a nonflat complex space form $M^n(c;\mathbb{C})$, integral curves of their characteristic vector fields are "nice" curves in the sense that their extrinsic shapes in $M^n(c;\mathbb{C})$ are Kähler circles. In this paper we mainly study geodesic spheres in a nonflat complex space form $M^n(c;\mathbb{C})$. On these geodesic spheres we classify smooth curves whose extrinsic shapes are Kähler circles in $M^n(c;\mathbb{C})$, $c \neq 0$. We also give a characterization of complex space forms among Kähler manifolds by extrinsic shapes of integral curves of characteristic vector fields on their geodesic spheres.

Key words: complex space forms, geodesic spheres, integral curves of characteristic vector fields, Kähler Frenet curves, Kähler circles, structure torsion, Hopf hypersurfaces.

1. Introduction

Let $(\widetilde{M}, \langle\ ,\ \rangle)$ be a Kähler manifold with complex structure J. A smooth curve γ on $\widetilde{M}$ parameterized by its arclength is called a *Kähler circle* if it satisfies either $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = kJ\dot{\gamma}$ or $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = -kJ\dot{\gamma}$ with some positive constant k. Here $\widetilde{\nabla}_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Riemannian connection $\widetilde{\nabla}$ of $\widetilde{M}$. This constant k is called the *curvature* of this Kähler circle. Since we can interpret Kähler circles as trajectories for some uniform magnetic fields, we regard geodesics as Kähler circles of null curvature (c.f. [A1]). The notion of Kähler circles is closely related to the complex structure J of $\widetilde{M}$.

For a smooth curve γ on a Riemannian submanifold M in $\widetilde{M}$, regarding this curve as a curve in $\widetilde{M}$, we call it the extrinsic shape of γ in $\widetilde{M}$. In this paper we study extrinsic shapes of smooth curves on geodesic spheres in a

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Kähler manifold. For a geodesic sphere of small radius in a Kähler manifold $\widetilde{M}$, we have the canonical almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$. The characteristic vector field ξ is given by $\xi = -J\mathcal{N}$ with outward unit normal vector field $\mathcal{N}$ of a geodesic sphere. When $\widetilde{M}$ is a complex space form $M^n(c;\mathbb{C})$, which is locally congruent to one of a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c (>0), a complex Euclidean space $\mathbb{C}^n$ and a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c (<0), every integral curve of the characteristic vector field of a geodesic sphere is a geodesic, and moreover its extrinsic shape is a Kähler circle in $M^n(c;\mathbb{C})$.

In view of the above features of integral curves for characteristic vector fields, we naturally come to pose the following problems.

(1) On geodesic spheres in $M^n(c;\mathbb{C})$, $c \neq 0$, are there smooth curves whose extrinsic shapes are Kähler circles except integral curves of characteristic vector fields?

(2) Among real hypersurfaces in $M^n(c;\mathbb{C})$, $c \neq 0$, is it possible to characterize geodesic spheres by these smooth curves?

In this paper, we classify smooth curves on geodesic spheres in $M^n(c;\mathbb{C})$, $c \neq 0$ whose extrinsic shapes are Kähler circles, and show that there are infinitely many such smooth curves which are not congruent each other. This result is a complete answer to the first problem. We also give a characterization of complex space forms among Kähler manifolds from this point of view.

For about the second problem, we give a partial answer. We say a real hypersurface M of real dimension $2n-1$ in a nonflat $M^n(c;\mathbb{C})$ to be a *Hopf hypersurface* if ξ is a principal curvature vector of M in the ambient space $M^n(c;\mathbb{C})$. Geodesic spheres $G_x(r)$ are known as the simplest examples of Hopf hypersurfaces. We give a sufficient condition for a real hypersurface M to be a Hopf hypersurface by the condition that the extrinsic shape of each integral curve of the characteristic vector field is a Kähler circle in the ambient space $M^n(c;\mathbb{C})$.

2. Extrinsic shapes of smooth curves on geodesic spheres

Let M be a real hypersurface of a Kähler manifold $\widetilde{M}$ of complex dimension n $(\geqq 2)$ with Riemannian metric $\langle\ ,\ \rangle$ and Kähler structure J. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX,\, Y\rangle \mathcal{N}, \tag{2.1}$$
$$\widetilde{\nabla}_X \mathcal{N} = -AX, \tag{2.2}$$

for vector fields X and Y tangent to M, where A is the shape operator of M in $\widetilde{M}$ and $\mathcal{N}$ is a unit normal vector field. The canonical almost contact metric structure $(\phi,\, \xi,\, \eta,\, \langle\ ,\ \rangle)$ induced from J is a quartet of the characteristic vector field ξ, a tensor field ϕ of type (1,1), the induced metric $\langle\ ,\ \rangle$ and a 1-form η on M defined by $\langle \phi X,\, Y\rangle = \langle JX,\, Y\rangle$, $\langle \xi,\, X\rangle = \eta(X) = \langle JX,\, \mathcal{N}\rangle$. They satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \langle \xi,\, \xi\rangle = 1, \quad \phi\xi = 0. \tag{2.3}$$

Since $\widetilde{\nabla} J = 0$, those equalities (2.1), (2.2) and (2.3) show $\nabla_X \xi = \phi AX$. Indeed,

$$\begin{aligned}\nabla_X \xi &= \widetilde{\nabla}_X \xi - \langle AX,\, \xi\rangle \mathcal{N} = J\widetilde{\nabla}_X(-\mathcal{N}) + \langle AX,\, J\mathcal{N}\rangle \mathcal{N}\\ &= JAX - \langle JAX,\, \mathcal{N}\rangle \mathcal{N} = \phi AX.\end{aligned}$$

We now study smooth curves on a geodesic sphere $G(r)$ of radius r in $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$. Here, the radius r satisfies $0 < r < \pi/\sqrt{c}$ when $c > 0$, and $0 < r < \infty$ when $c < 0$. We should note that on these geodesic spheres their shape operators satisfy $\phi A = A\phi$ and $\langle (\nabla_u A)u,\, u\rangle = 0$ for every tangent vector u (see [NR]). In the following, we shall use these equalities repeatedly. For a smooth curve γ on $G(r)$, we define its *structure torsion* ρ_γ by $\rho_\gamma = \langle \dot\gamma,\, \xi\rangle$. In general, ρ_γ is not constant along γ. However, when γ is a geodesic, we can see that its structure torsion is constant in the following way. As $\nabla_{\dot\gamma}\xi = \phi A\dot\gamma$, we have

$$\nabla_{\dot\gamma}\langle \dot\gamma,\, \xi\rangle = \langle \dot\gamma,\, \phi A\dot\gamma\rangle = \langle \dot\gamma,\, A\phi\dot\gamma\rangle = \langle A\dot\gamma,\, \phi\dot\gamma\rangle = -\langle \phi A\dot\gamma,\, \dot\gamma\rangle,$$

so the structure torsion ρ_γ of a geodesic γ is constant. For geodesics on $G(r)$, their structure torsions are quite important. They are classified by their structure torsions: Two geodesics on $G(r)$ in $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$ are congruent each other if and only if the absolute values of their structure torsions coincide (see [AMY]).

The extrinsic shapes of geodesics on a geodesic sphere in $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$ were studied in [AMY] (see Propositions 2.1 and 3.1 in [AMY]).

Proposition 1 *The extrinsic shape of a geodesic γ on a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c})$ in $\mathbb{C}P^n(c)$ is as follows*:

(1) *If $\rho_\gamma = \pm 1$, which is the case that γ is an integral curve of ξ, then the extrinsic shape in $\mathbb{C}P^n(c)$ is a Kähler circle of curvature $\sqrt{c}\cot(\sqrt{c}r)$,*

(2) *if $\rho_\gamma = \pm\cot(\sqrt{c}r/2)$ for the case $\pi/(2\sqrt{c}) < r < \pi/\sqrt{c}$, then the extrinsic shape in $\mathbb{C}P^n(c)$ is a geodesic,*

(3) *otherwise, the extrinsic shape in $\mathbb{C}P^n(c)$ is not a Kähler circle.*

Proposition 2 *The extrinsic shape of a geodesic γ on a geodesic sphere $G(r)$ of radius r $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$ is as follows*:

(1) *If $\rho_\gamma = \pm 1$, then the extrinsic shape in $\mathbb{C}H^n(c)$ is a Kähler circle of curvature $\sqrt{|c|}\coth(\sqrt{|c|}r)$,*

(2) *otherwize, the extrinsic shape in $\mathbb{C}H^n(c)$ is not a Kähler circle.*

The features of Kähler circles on $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$ are known (see [A1, C]). On $\mathbb{C}P^n(c)$ a Kähler circle of curvature k is simple and closed with length $2\pi/\sqrt{k^2+c}$ and lies on a totally geodesic $\mathbb{C}P^1(c)$. On $\mathbb{C}H^n(c)$ a Kähler circle of curvature k is simple and lies on a totally geodesic $\mathbb{C}H^1(c)$. When $k > \sqrt{|c|}$, it is closed with length $2\pi/\sqrt{k^2+c}$ but when $k \leqq \sqrt{|c|}$, it is an unbounded curve.

In view of these Propositions we are hence interested in other smooth curves on geodesic spheres in $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$ whose extrinsic shapes are Kähler circles.

Theorem 1 *Let γ be a smooth curve on a geodesic sphere $G(r)$ of radius $r \leqq \pi/(2\sqrt{c})$ in $\mathbb{C}P^n(c)$. If the extrinsic shape of γ in $\mathbb{C}P^n(c)$ is a Kähler circle of curvature k $(\geqq 0)$, then the curve γ is one of the following curves*;

(1) *a geodesic with structure torsion $\rho_\gamma = \pm 1$, where $k = \sqrt{c}\cot(\sqrt{c}r)$,*

(2) *a non-geodesic curve satisfying $\nabla_{\dot\gamma}\dot\gamma = \pm k\phi\dot\gamma$ for $k > \sqrt{c}\cot(\sqrt{c}r)$ with structure torsion $\rho_\gamma = \pm c^{-1/2}(\sqrt{k^2+c}-k)\cot(\sqrt{c}r/2)$, where double signs take the same signatures.*

Theorem 2 *Let γ be a smooth curve on a geodesic sphere $G(r)$ of radius r with $\pi/(2\sqrt{c}) < r < \pi/\sqrt{c}$ in $\mathbb{C}P^n(c)$. If the extrinsic shape of γ in $\mathbb{C}P^n(c)$ is a Kähler circle of curvature k $(\geqq 0)$, then the curve γ is one of the following curves*;

(1) *a geodesic with structure torsion $\rho_\gamma = \pm 1$, where $k = -\sqrt{c}\cot(\sqrt{c}r)$,*

(2) *a geodesic with structure torsion $\rho_\gamma = \pm\cot(\sqrt{c}r/2)$, where $k = 0$,*

(3) *a non-geodesic curve satisfying $\nabla_{\dot\gamma}\dot\gamma = \pm k\phi\dot\gamma$ for arbitrary positive k*

with structure torsion $\rho_\gamma = \pm c^{-1/2}(\sqrt{k^2+c}-k)\cot(\sqrt{c}r/2)$*, where double signs take the same signatures,*

(4) *a non-geodesic curve satisfying* $\nabla_{\dot\gamma}\dot\gamma = \pm k\phi\dot\gamma$ *for* $0<k<-\sqrt{c}\cot(\sqrt{c}r)$ *with structure torsion* $\rho_\gamma = \mp c^{-1/2}(\sqrt{k^2+c}+k)\cot(\sqrt{c}r/2)$*, where double signs take the opposite signatures.*

Proof of Theorems 1 and 2. If the extrinsic shape of a smooth curve γ on $G(r)$ parameterized by its arclength is a Kähler circle of curvature k, its equation $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \pm kJ\dot\gamma$ is equivalent to the equations

$$\nabla_{\dot\gamma}\dot\gamma = \pm k\phi\dot\gamma, \tag{2.4}$$

$$\langle A\dot\gamma,\, \dot\gamma\rangle = \pm k\rho_\gamma \tag{2.5}$$

by the Gauss equation (2.1). Smooth curves satisfying (2.4) were also treated in [A2] from the viewpoint of magnetic fields. Our interest here is in curves satisfying both (2.4) and (2.5).

For a smooth curve γ satisfying (2.4) on $G(r)$, we see that it has constant structure torsion, because we have

$$\begin{aligned}\nabla_{\dot\gamma}\langle\dot\gamma,\, \xi\rangle &= \langle \pm k\phi\dot\gamma,\, \xi\rangle + \langle\dot\gamma,\, \phi A\dot\gamma\rangle \\ &= \langle\dot\gamma,\, \phi A\dot\gamma\rangle = \langle\dot\gamma,\, A\phi\dot\gamma\rangle = -\langle\phi A\dot\gamma,\, \dot\gamma\rangle.\end{aligned}$$

So we are enough to study the condition (2.5) at an initial point. In consideration of Proposition 1, it suffices to study the case of $\rho_\gamma \neq \pm 1$.

Since the shape operator A of $G(r)$ satisfies $A\xi = \sqrt{c}\cot(\sqrt{c}r)\xi$ and $Au = (\sqrt{c}/2)\cot(\sqrt{c}r/2)u$ for every tangent vector u orthogonal to ξ, the equality (2.5) turns to

$$\rho_\gamma^2\sqrt{c}\cot(\sqrt{c}r) + (1-\rho_\gamma^2)\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}r}{2}\Big) = \pm k\rho_\gamma.$$

As $2\cot 2\theta = \cot\theta - \tan\theta$, we see that ρ_γ satisfies

$$\sqrt{c}\tan\Big(\frac{\sqrt{c}r}{2}\Big)\cdot\rho_\gamma^2 \pm 2k\rho_\gamma - \sqrt{c}\cot\Big(\frac{\sqrt{c}r}{2}\Big) = 0.$$

Hence ρ_γ is a solution of the quadratic equation

$$\rho^2 \pm \Big(\frac{2k}{\sqrt{c}}\Big)\cot\Big(\frac{\sqrt{c}r}{2}\Big)\cdot\rho - \cot^2\Big(\frac{\sqrt{c}r}{2}\Big) = 0. \tag{2.6}$$

We denote by $f_\pm(\rho)$ the left side quadratic function in this equation. Since $f_\pm(0) < 0$, the equation (2.6) has a solution in the open interval $(0,\, 1)$ if and only if $f_{\pm}(1) > 0$ and has a solution in the open interval $(-1, 0)$ if and only if $f_{\pm}(-1) > 0$. One can easily see that $f_{+}(1) > 0$ if and only if $k > \sqrt{c}\cot(\sqrt{c}r)$ and that $f_{+}(-1) > 0$ if and only if $-k > \sqrt{c}\cot(\sqrt{c}r)$. Therefore, when $0 < r \leqq \pi/(2\sqrt{c})$, as $\cot(\sqrt{c}r) \geqq 0$, we know that the equation $f_{+}(\rho) = 0$ has a unique solution $c^{-1/2}\big(\sqrt{k^2+c}-k\big)\cot(\sqrt{c}r/2)$ in the interval $(-1, 1)$ if and only if $k > \sqrt{c}\cot(\sqrt{c}r)$. When $\pi/(2\sqrt{c}) < r < \pi/\sqrt{c}$, as $\cot(\sqrt{c}r) < 0$, we see that it has a solution $c^{-1/2}\big(\sqrt{k^2+c} - k\big)\cot(\sqrt{c}r/2)$ in the interval $(-1, 1)$ for every positive k, and has another solution $-c^{-1/2}\big(k+\sqrt{k^2+c}\big)\cot(\sqrt{c}r/2)$ in the interval $(-1, 1)$ if and only if $k < -\sqrt{c}\cot(\sqrt{c}r)$. By the same argument as in the equation $f_{-}(\rho) = 0$ we get our conclusions. □

For a geodesic sphere $G(r)$ in $\mathbb{C}H^n(c)$ of radius r, its shape operator satisfies $A\xi = \sqrt{|c|}\coth(\sqrt{|c|}r)\xi$ and $Au = (\sqrt{|c|}/2)\coth(\sqrt{|c|}r/2)u$ for every tangent vector u orthogonal to ξ. Therefore the equality (2.5) turns to

$$\rho_\gamma^2\sqrt{|c|}\coth(\sqrt{|c|}r) + (1-\rho_\gamma^2)\frac{\sqrt{|c|}}{2}\coth\Big(\frac{\sqrt{|c|}r}{2}\Big) = \pm k\rho_\gamma.$$

Since $2\coth 2\theta = \coth\theta + \tanh\theta$, we find that ρ_γ satisfies

$$\sqrt{|c|}\tanh\Big(\frac{\sqrt{|c|}r}{2}\Big)\cdot\rho_\gamma^2 \mp 2k\rho_\gamma + \sqrt{|c|}\coth\Big(\frac{\sqrt{|c|}r}{2}\Big) = 0.$$

Hence ρ_γ is a solution of the quadratic equation

$$\rho^2 \mp \Big(\frac{2k}{\sqrt{|c|}}\Big)\coth\Big(\frac{\sqrt{|c|}r}{2}\Big)\cdot\rho + \coth^2\Big(\frac{\sqrt{|c|}r}{2}\Big) = 0, \tag{2.7}$$

which corresponds to (2.6). We only need to consider the case $k^2 \geqq |c|$, which is equivalent to that this equation has solutions. Under this condition in the interval $(-1, 1)$ this equation has at most one solution. We denote by $g_{\pm}(\rho)$ the left side quadratic function in (2.7). Here double signs take the opposite signatures. As $g_{\pm}(0) > 0$, we see the equation $g_{+}(\rho) = 0$ has a solution in the interval $(0, 1)$ if and only if $g_{+}(1) < 0$, that is $k > \sqrt{|c|}\coth(\sqrt{|c|}r)$, and the equation $g_{-}(\rho) = 0$ has a solution in the interval $(-1, 0)$ if and only if $g_{-}(-1) < 0$. We hence obtain the following.

Theorem 3 *Let γ be a smooth curve on a geodesic sphere $G(r)$ of radius r $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$. If the extrinsic shape of γ in $\mathbb{C}H^n(c)$ is a Kähler circle of curvature k $(\geqq 0)$, then the curve γ is one of the following curves;*

(1) *a geodesic with structure torsion* $\rho_\gamma = \pm 1$, *where* $k = \sqrt{|c|}\coth(\sqrt{|c|}\, r)$,
(2) *a non-geodesic curve satisfying* $\nabla_{\dot\gamma}\dot\gamma = \pm k\phi\dot\gamma$ *for* $k > \sqrt{|c|}\coth(\sqrt{|c|}r)$ *with structure torsion* $\rho_\gamma = \pm |c|^{-1/2}\big(k - \sqrt{k^2+c}\big)\coth(\sqrt{|c|}r/2)$, *where double signs take the same signatures.*

We say two curves γ_1, γ_2 to be congruent if there exist an isometry φ and a constant s_0 with $\gamma_1(s) = (\varphi\circ\gamma_1)(s+s_0)$ for every s.

By these theorems we find that there are infinitely many smooth curves on geodesic spheres in $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$ whose extrinsic shapes are Kähler circles. But if we restrict ourselves on the smooth curves whose extrinsic shapes are Kähler circles of *given* curvature, we can say the following.

Corollary 1 *Consider a geodesic sphere* $G(r)$ *in* $\mathbb{C}P^n(c)$ *of radius* r *with* $0 < r < \pi/(2\sqrt{c})$.
(1) *For* $k \geqq \sqrt{c}\cot(\sqrt{c}\, r)$, *there is only one congruence class of smooth curves whose extrinsic shapes are Kähler circles of curvature* k.
(2) *For* $0 \leqq k < \sqrt{c}\cot(\sqrt{c}\, r)$, *there does not exist smooth curves whose extrinsic shapes are Kähler circles of curvature* k.

Corollary 2 *On a geodesic sphere* $G(r)$ *in* $\mathbb{C}H^n(c)$ *we find the following.*
(1) *For* $k \geqq \sqrt{|c|}\coth(\sqrt{|c|}\, r)$, *there is only one congruence class of smooth curves whose extrinsic shapes are Kähler circles of curvature* k.
(2) *For* $0 \leqq k < \sqrt{|c|}\cot(\sqrt{|c|}\, r)$, *there does not exist smooth curves whose extrinsic shapes are Kähler circles of curvature* k.

On the contrary, on a geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$ with $\pi/(2\sqrt{c}) < r < \pi/\sqrt{c}$, there exist *non-congruent* curves with respect to the full isometry group $\mathrm{Iso}(G(r))$ of $G(r)$ whose extrinsic shapes are Kähler circles of the same curvature k in the ambient space $\mathbb{C}P^n(c)$ for some positive constant k (c.f. [A2]).

Corollary 3 *Consider a geodesic sphere* $G(r)$ *in* $\mathbb{C}P^n(c)$ *of radius* r *with* $\pi/(2\sqrt{c}) < r < \pi/\sqrt{c}$.
(1) *For* $0 < k \leqq -\sqrt{c}\cot(\sqrt{c}r)$, *there are two congruence classes of smooth curves with respect to the full isometry group of* $G(r)$ *whose extrinsic shapes are Kähler circles of curvature* k.
(2) *For* $k > -\sqrt{c}\cot(\sqrt{c}r)$ *or* $k = 0$, *there is only one congruence class of smooth curves whose extrinsic shapes are Kähler circles of curvature* k

Remark Though two smooth curves corresponding to congruence classes in Corollary 3 (1) are not congruent with respect to the full isometry group of $G(r)$, their extrinsic shapes are congruent each other with respect to the full isometry group of $\mathbb{C}P^n(c)$.

3. A characterization of Hopf hypersurfaces

In this section we give a characterization of Hopf hypersurfaces in a nonflat complex space form by extrinsic shapes of integral curves of their characteristic vector fields.

As a generalization of Kähler circles we say a smooth curve $\gamma = \gamma(s)$ on a Kähler manifold M parameterized by its arclength s to be a *Kähler Frenet curve* if it satisfies either $\nabla_{\dot\gamma}\dot\gamma(s) = \kappa(s)J\dot\gamma(s)$ or $\nabla_{\dot\gamma}\dot\gamma(s) = -\kappa(s)J\dot\gamma(s)$ with some positive smooth function $\kappa(s)$ (c.f. [MT]). Needless to say we regard a geodesic as a Kähler Frenet curve in a trivial sense.

Theorem 4 *Let M be a real hypersurface in either $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$. Then the following are equivalent.*

(1) *M is a Hopf hypersurface in $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$.*
(2) *The extrinsic shape of each integral curve of the characteristic vector field ξ on M is a Kähler Frenet curve in $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$.*
(3) *The extrinsic shape of each integral curve of the characteristic vector field ξ on M is a Kähler circle in $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$.*

Proof. (1) $\Longrightarrow$ (3). Suppose that our real hypersurface M satisfies $A\xi = \alpha\xi$. Then the principal curvature function α is locally constant on M (see [NR]). Hence, from (2.1) the vector ξ satisfies $\widetilde\nabla_\xi\xi = \alpha\mathcal{N} = \alpha J\xi$. Thus the extrinsic shape of every integral curve of ξ on M is a Kähler circle of curvature $|\alpha|$ in the ambient manifold $M^n(c, \mathbb{C})$.

(2) $\Longrightarrow$ (1). For each integral curve $\gamma = \gamma(s)$ of ξ, we have $\dot\gamma = \xi_\gamma$, hence we have $\widetilde\nabla_{\xi_\gamma}\xi_\gamma = \pm\kappa_\gamma(s)\mathcal{N}_\gamma$ with a function $\kappa_\gamma(s)$ which is either a positive smooth function or $\kappa_\gamma \equiv 0$. This, together with (2.1) and $\nabla_{\xi_\gamma}\xi_\gamma = \phi A\xi_\gamma$, implies that

$$\phi A\xi_\gamma + \langle A\xi_\gamma, \xi_\gamma\rangle\mathcal{N}_\gamma = \pm\kappa_\gamma(s)\mathcal{N}_\gamma.$$

In view of the tangential component for M of this equality, we find that $\phi A\xi_\gamma = 0$, hence ξ_γ is principal. Since γ is an arbitrary integral curve of the characteristic vector field ξ, we find that our real hypersurface M is a Hopf hypersurface. □

4. A characterization of complex space forms

In this section we give a characterization of complex space forms among Kähler manifolds by extrinsic shapes of integral curves of characteristic vector fields of geodesic spheres. To do this we use an expansion for the second fundamental form of geodesic spheres due to Chen and Vanhecke ([CV]). For a Riemannian manifold M of dimension greater than 2, we denote by $A_{x,r}$ the shape operator of a geodesic sphere $G_x(r)$ in M of sufficiently small radius r centered at $x \in M$ with respect to the outward unit normal vector field $\mathcal{N}$. We adopt the following signature of the Riemannian curvature tensor $\widetilde{R}$ of M; $\widetilde{R}(X, Y)Z = \widetilde{\nabla}_{[X,Y]}Z - [\widetilde{\nabla}_X, \widetilde{\nabla}_Y]Z$.

Lemma ([CV, Theorem 3.1]) *For nonzero tangent vectors v, $w \in T_xM$ at a point $x \in M$, we choose a unit tangent vector $u \in T_xM$ orthogonal to both v and w. We denote by v_r, $w_r \in T_{\exp_x(ru)}M$ the parallel displacements of v, w along the geodesic segment $\exp_x(su)$, $0 \leqq s \leqq r$. Then for sufficiently small r we have*

$$\langle A_{x,r}v_r,\, w_r\rangle = \frac{1}{r}\langle v,\, w\rangle - \frac{r}{3}\langle \widetilde{R}(u,\, v)w,\, u\rangle + O(r^2). \tag{4.1}$$

We shall prove the following:

Theorem 5 *For a Kähler manifold M of complex dimension greater than 1, the following are equivalent.*

(1) *M is a complex space form.*

(2) *At an arbitrary point $x \in M$, for each geodesic sphere $G_x(r)$ of sufficiently small radius r, the extrinsic shape of each integral curve of the characteristic vector field ξ on $G_x(r)$ is a Kähler Frenet curve in M.*

(3) *At an arbitrary point $x \in M$, for each geodesic sphere $G_x(r)$ of sufficiently small radius r, the extrinsic shape of each integral curve of the characteristic vector field ξ on $G_x(r)$ is a Kähler circle in M.*

Proof. It suffices to prove that the condition (2) implies the condition (1). By the same argument as in the proof of Theorem 4, for every integral curve γ of the characteristic vector field we know that $\phi A_{x,r}\xi_\gamma = 0$ and hence our geodesic sphere $G_x(r)$ is a Hopf hypersurface.

Given a unit tangent vector $v \in T_xM$ we take a unit tangent vector $w \in T_xM$ which is orthogonal to both v and Jv and use Lemma by putting $u = Jv$. Since u_r is a normal vector of $G_x(r)$ in M at $y = \exp_x(ru)$, the vector $v_r = -Ju_r$ is the characteristic vector of $G_x(r)$ at y, so that v_r is a principal curvature vector of $G_x(r)$. This, combined with equation (4.1), shows that the curvature tensor $\widetilde{R}$ of M satisfies $\langle \widetilde{R}(u, Ju)w, u\rangle = 0$. This means that $\widetilde{R}(u, Ju)u$ is proportional to Ju for every unit vector u at each point x of M, so that our Kähler manifold M is a complex space form $M^n(c;\mathbb{C})$ (see [T]). $\square$

References

[A1] Adachi T., *Kähler magnetic flows on a manifold of constant holomorphic sectional curvature.* Tokyo J. Math. **18** (1995), 473–483.

[A2] Adachi T., *Trajectories on geodesic spheres in a non-flat complex space form.* preprint.

[AMY] Adachi T., Maeda S. and Yamagishi M., *Length spectrum of geodesic spheres in a non-flat complex space form.* J. Math. Soc. Japan. **54** (2002), 373–408.

[C] Comtet A., *On the Landau levels on the hyperbolic plane.* Ann. Phys. **173** (1987), 185–209.

[CV] Chen B.Y. and Vanhecke L., *Differential geometry of geodesic spheres.* J. Reine Angew. **325** (1981), 28–67.

[MT] Maeda S. and Tanabe H., *Totally geodesic immersions of Kähler manifolds and Kähler Frenet curves* Math. Z. **252** (2006), 787–795.

[NR] Niebergall R. and Ryan P.J., *Real hypersurfaces in complex space forms.* Tight and Taut Submanifolds Cecil T.E. and Chern S.S. eds., Cambridge University Press, 1998, pp. 233–305.

[T] Tanno S., *Constancy of holomorphic sectional curvature in almost Hermitian manifolds* Kodai Math. Sem. Rep. **25** (1973), 190–201.

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