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The abstract Fatou theorem and the signal transmission on Thomson cables

(Dedicated to the late Professor Kôsaku Yosida on his centennial anniversary)

Hikosaburo Komatsu

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Abstract. The Fatou theorem on the Poisson representation of bounded harmonic functions on a half space is generalized to the bounded solutions $u(t)$ of the second order equation

$$u''(t) = Au(t), \quad 0 < t < \infty,$$

in a dual Banach space $X = X_*{}'$, when A is the dual of a non-negative operator A_* with dense domain in X_*. Any bounded weak* solution is represented as $u(t) = \exp(-t\sqrt{A})f$ with the weak* initial value f. Its prototype is in A. V. Balakrishnan's paper in 1960 on fractional powers of non-negative operators.

This is applied to prove the uniqueness of solutions in the theory of signal transmission on submarine cables by W. Thomson in 1855.

Key words: Fatou theorem, Thomson cable, uniqueness.

1. Introduction

The classical Fatou theorem [5] says that every bounded harmonic function $u(t, x)$ on the half space $\{(t, x) \in \mathbf{R}^{n+1}; t > 0\}$ has a non-tangential limit $f(x) \in L^\infty(\mathbf{R}^n)$ almost everywhere as $t \to 0$ and the original $u(t, x)$ is represented as the Poisson integral of $f(x)$.

Let $X = L^p(\mathbf{R}^n)$ for $1 < p < \infty$. Then, Hardy−Littlewood [6] have characterized those harmonic functions $u(t, x)$ on the half space $\{(t, x) \in \mathbf{R}^{n+1}; t > 0\}$ that have uniformly bounded X-norms $\|u(t, \cdot)\|_X$ as the Poisson integral of an $f(x) \in X$, which is the strong limit in X of $u(t, x)$ as $t \to 0$. If $p = 1$, then the corresponding result holds with an $f(x)$ in the space $\mathcal{M}^1(\mathbf{R}^n)$ of bounded measures on $\mathbf{R}^n$. The convergence $u(t, x) \to f(x)$

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is in measure, that is, in the weak* topology on $\mathcal{M}^1(\mathbf{R}^n)$ as the dual of the space $C_\infty(\mathbf{R}^n)$ of continuous functions vanishing at ∞.

Let A be the negative $-\Delta$ of the Laplacian in X. Then, the results show that the evolution equation of the second order

$$\frac{d^2}{dt^2}u(t) = A\,u(t), \quad t > 0, \tag{1}$$

with the uniformly bounded condition

$$\sup_{t>0} \|u(t)\|_X < \infty, \tag{2}$$

is equivalent to the equation of the first order

$$\frac{d}{dt}u(t) = -\sqrt{A}\,u(t), \quad t > 0, \tag{3}$$

for the square root of A, so that every solution $u(t)$ is represented as the Poisson integral

$$u(t) = \exp\big(-t\sqrt{A}\big)\, f \tag{4}$$

of a single initial value $f \in X$.

We extend this results to the case where A is a non-negative operator in the dual Banach space X, and apply it to reconstruct the theory of W. Thomson [20] on the signal transmission on submarine cables.

2. The Abstract Fatou Theorem

We consider the situation in which X is the strong dual of a complex Banach space X_*, and A is the dual of a non-negative operator A_* with dense domain in X_*, where a closed linear operator A in a Banach space X is said to be *non-negative* if the negative real line $(-\infty, 0)$ is included in the resolvent set of A and if the resolvent $(\lambda + A)^{-1}$ has the uniform estimate

$$M_0 = \sup_{0<\lambda<\infty} \|\lambda(\lambda + A)^{-1}\| < \infty. \tag{5}$$

Then, it follows from (5) that there is a $\theta > 0$ such that

$$M_\theta = \sup_{|\arg\lambda|\le\theta} \|\lambda(\lambda+A)^{-1}\| < \infty. \tag{6}$$

The *type* (or the spectral angle) ω of A is defined to be the infimum of $\pi-\theta$ for all θ for which (6) holds. A non-negative operator A_* with dense domain is of type $\omega < \pi/2$ if and only if $-A_*$ is the infinitesimal generator of a bounded holomorphic semigroup $\exp(-tA_*)$ defined for $|\arg t| < \pi/2-\omega$. The Hille–Yosida theorem [21] asserts that if $-A_*$ is the infinitesimal generator of a bounded continuous semigroup of operators, then A_* is a non-negative operator of type $\pi/2$ with dense domain, but the converse does not hold. Cf. Komatsu [10, Sections 10–12].

Under our assumptions A is a non-negative operator but its domain $D(A)$ is not necessarily dense as in the case of $-\Delta$ in $L^\infty(\mathbf{R}^n)$ or in $\mathcal{M}^1(\mathbf{R}^n)$. On the other hand, every non-negative operator A in a reflexive space has a dense domain (see [10, Section 2]), so that we can choose the strong dual X' as its predual X_* and the dual operator A' as the predual A_* of A.

Balakrishnan [2] and Komatsu [10] have developed the theory of fractional powers A_*^α of non-negative operators A_* with dense domain. For any $\operatorname{Re}\alpha > 0$ we can define a closed linear operator A_*^α equipped with the properties the powers of operators should have. In particular, the square root $A_*^{1/2}$ of A_* is a non-negative operator with dense domain of type $\omega/2$ satisfying

$$A_*^{1/2}A_*^{1/2} = A_* \tag{7}$$

as the product of operators. Thus, its negative $-A_*^{1/2}$ generates the bounded holomorphic semigroup $\exp(-tA_*^{1/2})$ defined for $|\arg t| < (\pi-\omega)/2$, strongly continuous and uniformly bounded on each subsector

$$\Sigma_\phi = \{t \in \mathbf{C}; |\arg t| \le \phi\} \cup \{0\}, \quad \phi < (\pi-\omega)/2.$$

We denote by $\sqrt{A}$ the dual of $A_*^{1/2}$, and by $\exp(-t\sqrt{A})$ the dual of $\exp(-tA_*^{1/2})$. We have

$$\sqrt{A}\sqrt{A} = A, \tag{8}$$

as the dual of (7). Hence the domain $D(A)$ is always included in $D(\sqrt{A})$ but, if X is not reflexive, $D(\sqrt{A})$ may not be dense. The semigroup $\exp(-t\sqrt{A})$

is holomorphic for $|\arg t| < (\pi - \omega)/2$ and uniformly bounded on each Σ_ϕ, $\phi < (\pi - \omega)/2$, but it may not be strongly continuous in t at the origin. It is, however, continuous in the weak* topology on each Σ_ϕ.

For any $f \in X$, the function

$$u(t) = \exp\big(-t\sqrt{A}\big)f, \quad t \geq 0, \tag{9}$$

is an analytic and uniformly bounded solution of equation (3). If $t > 0$, it is included in $D(\sqrt{A}^k)$ for any $k = 1, 2, \dots$, and because of (8) it also satisfies (1).

The following theorem gives the converse, which is an improvement of Theorem 6.1 in Balakrishnan [2] and Theorem 6.3.2 in Martínez – Sanz [14].

Theorem 1 (Abstract Fatou) *Let a non-negative operator $A = {A_*}'$ be as above in the dual Banach space $X = {X_*}'$.*

Suppose that $u(t)$ is a function on $(0, \infty)$ with values in X and satisfies the following conditions:

(i) *$u(t)$ is twice continuously differentiable in the weak* topology on $(0, \infty)$;*
(ii) *$u(t)$ belongs to the domain $D(A)$ for any $t \in (0, \infty)$;*
(iii) $\dfrac{d^2}{dt^2}u(t) = Au(t), \quad t \in (0, \infty)$;
(iv) $\|u(t)\| \leq C < \infty, \quad t \in (0, \infty)$.

Then, there is a unique $f \in X$ such that

$$u(t) = \exp\big(-t\sqrt{A}\big)f, \quad t \in (0, \infty).$$

In particular, $u(t)$ converges to f in the weak topology* (*strongly if $D(A)$ is dense*) *as t tends to* 0.

Proof. First we add two more assumptions

(v) $\|u'(t)\| \leq C < \infty, \quad t \in (0, \infty)$;
(vi) $\big\|\sqrt{A}u(t)\big\| \leq C < \infty, \quad t \in (0, \infty)$,

and give a proof.

We have, in the weak* topology,

$$\frac{d}{dt}\big(\exp(-t\sqrt{A})u'(t)\big) = -\sqrt{A}\exp\big(-t\sqrt{A}\big)u'(t) + \exp\big(-t\sqrt{A}\big)u''(t)$$
$$= -\sqrt{A}\exp\big(-t\sqrt{A}\big)\big(u'(t) - \sqrt{A}u(t)\big) = -\sqrt{A}\frac{d}{dt}\big(\exp(-t\sqrt{A})u(t)\big)$$

because $\exp(-t\sqrt{A})$ is differentiable in the operator norm.

Integrating both sides on the interval (ϵ, t), we have

$$\exp\big(-t\sqrt{A}\big)u'(t) - \exp\big(-\epsilon\sqrt{A}\big)u'(\epsilon)$$
$$= -\sqrt{A}\big(\exp(-t\sqrt{A})u(t) - \exp(-\epsilon\sqrt{A})u(\epsilon)\big)$$

as $\sqrt{A}$ is closed in the weak* topology.

Since the closed ball $\{x \in X, \|x\| \le C\}$ is compact in the weak* topology, it follows from assumptions (iv) and (v) that there is a net $\epsilon_\nu \to 0$ such that $u(\epsilon_\nu) \rightharpoonup u_0$ and $u'(\epsilon_\nu) \rightharpoonup u_1$ in the weak* topology.

Hence we have

$$\exp\big(-t\sqrt{A}\big)\big(u'(t) + \sqrt{A}u(t)\big) = u_1 + \sqrt{A}u_0, \quad t > 0, \tag{10}$$

since the predual $\exp(-\epsilon_\nu A_*^{1/2})$ tends to the identity strongly.

If we choose a positive $\phi < \omega/2 - \pi/2$, we have for any $s \in \Sigma_\phi$

$$\exp\big(-(s+t)\sqrt{A}\big)\big(u'(t) + \sqrt{A}u(t)\big) = \exp\big(-s\sqrt{A}\big)\big(u_1 + \sqrt{A}u_0\big). \tag{11}$$

The right hand side is a bounded holomorphic function in s on the interior of Σ_ϕ and the left hand side shows that it can be continued analytically to the interior of $\Sigma_\phi - t$ for any $t > 0$. Thus it can be continued to an entire function. It is, moreover, bounded because of assumptions (v) and (vi). Hence it is a constant by the Liouville theorem. Letting $s \in \Sigma_\phi$ tend to 0, we find that its value is equal to $u_1 + \sqrt{A}u_0$. Since $s + t$ in the left hand side can also be arbitrary, we have

$$u'(t) + \sqrt{A}u(t) = u_1 + \sqrt{A}u_0, \quad t > 0. \tag{12}$$

In order to prove that this vanishes, we first note that

$$\sqrt{A}\big(u'(t) + \sqrt{A}u(t)\big) = \sqrt{A}\big(u_1 + \sqrt{A}u_0\big) = 0, \quad t > 0, \tag{13}$$

which is equivalent to the invariance of (12) under the action of $\exp(-s\sqrt{A})$.

We set

$$v(t) = \exp\big(-t\sqrt{A}\big)u_0 + t\big(u_1 + \sqrt{A}u_0\big).$$

Then, it follows from (12) and (13) that

$$v'(t) + \sqrt{A}v(t) = u'(t) + \sqrt{A}u(t), \quad 0 < t < \infty. \tag{14}$$

We assume for the moment that $u(t)$ is continuous in the weak* topology up to the origin, so that $u_0 = u(0)$. Then,

$$w(t) = u(t) - v(t)$$

is a function on $[0, \infty)$ continuous in the weak* topology and satisfying

$$\begin{cases} w'(t) + \sqrt{A}w(t) = 0, \quad 0 < t < \infty, \\ w(0) = 0. \end{cases} \tag{15}$$

In order to prove that $w(t) = 0$ for $t > 0$, we consider the function $\exp((t-s)\sqrt{A})w(t)$ defined for $0 < t < s$. It follows from (14) that its derivative in t vanishes on $(0, s)$. Hence we have

$$\exp\big((t-s)\sqrt{A}\big)w(t) = 0, \quad 0 \le t < s.$$

We let $s > t$ tend to t. Then, we have $w(t) = 0$ as the weak* limit.

Thus we have shown that

$$u(t) = \exp\big(-t\sqrt{A}\big)u_0 + t\big(u_1 + \sqrt{A}u_0\big).$$

In order that this be bounded, the second term must vanish, so that we have

$$u(t) = \exp\big(-t\sqrt{A}\big)u_0. \tag{16}$$

To be exact, we have proved the theorem under the extra conditions (v), (vi) and the weak* continuity of $u(t)$ at $t = 0$. To prove it in general, let $u(t)$ be a solution satisfying only the conditions of the theorem.

Then, for any η and $\epsilon > 0$,

$$v(t) = \exp\big(-\eta\sqrt{A}\big)u(t+\epsilon) \tag{17}$$

is a solution fulfilling all conditions (i)–(vi) and the weak* continuity at 0.

In fact, since $\exp(-\eta\sqrt{A})$, $\sqrt{A}\exp(-\eta\sqrt{A})$ and $A\exp(-\eta\sqrt{A})$ are bounded linear operators, it is easy to see that $\|v\|$, $\|\sqrt{A}v\|$ and $\|Av\| = \|v''\|$ are uniformly bounded. The inequality

$$\sup\|v'\| \le 2\sqrt{\sup\|v\|\sup\|v''\|} \tag{18}$$

implies condition (v). (See Carleman [3] or Cartan [4] for a proof of the inequality. The referee has kindly informed us of a long history of inequality (18) starting with a paper of Hardy and Littlewood in 1913 and ending up with Kolmogoroff [9] in 1939. We can trace the development by the review of [9] by R. P. Boas, Jr. in Mathematical Reviews.)

Thus, we have

$$\exp\big(-\eta\sqrt{A}\big)u(t+\epsilon) = \exp\big(-(t+\eta)\sqrt{A}\big)u(\epsilon)$$

for any t, η and $\epsilon > 0$. Let $\eta \searrow 0$, and we have

$$u(t+\epsilon) = \exp\big(-t\sqrt{A}\big)u(\epsilon). \tag{19}$$

This shows that $u(t)$ is analytic in $t > 0$. Then, substituting the net ϵ_ν such that $u(\epsilon_\nu) \rightharpoonup u_0$ for ϵ in (19), we obtain (16) as the weak* limits. Since the right hand side of (16) converges to u_0 in the weak* topology as $t \searrow 0$, there are no other choice of the initial values u_0.

The last statement in the parentheses follows from Phillips [16]. This completes the proof of Theorem 1. □

The ordinary Fatou and the Hardy−Littlewood theorems are reduced to this theorem. Let

$$X = \mathcal{M}^1(\mathbf{R}^n) \text{ for } p = 1, \quad \text{and} \quad L^p(\mathbf{R}^n) \text{ for } 1 < p \le \infty \tag{20}$$

with the predual

$$X_* = C_\infty(\mathbf{R}^n) \text{ for } p = 1, \quad \text{and} \quad L^{p'}(\mathbf{R}^n) \text{ for } 1 < p \le \infty, \tag{21}$$

where $p' = p/(p-1)$ is the exponent conjugate to p.

Since the Laplacian

$$-A_* = \Delta \quad \text{with domain} \quad D(A_*) = \{f_* \in X_*; \Delta f_* \in X_*\} \tag{22}$$

is the generator of the Gauss semigroup

$$G(t)f_*(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{-\frac{|x-y|^2}{4t}} f_*(y)dy, \tag{23}$$

which is bounded and strongly continuous in X_* [21], A_* and its dual $A = A_*'$ are nonnegative operators satisfying the conditions of Theorem 1.

Moreover, since the space $\mathcal{D}(\mathbf{R}^n)$ of the Schwartz test functions is dense in the domain $D(A_*)$ with the graph norm, it follows that $f \in X$ is in $D(A)$ if and only if Δf in the sense of distribution belongs to X.

Now suppose that $u(t,x)$ is a harmonic function on the half space $\{(t,x) \in \mathbf{R}^{n+1}, t > 0\}$ such that

$$\sup_{t>0} \|u(t,\cdot)\|_X < \infty. \tag{24}$$

In order to prove that

$$u(t) = u(t,\cdot) \in X \tag{25}$$

satisfies the conditions of Theorem 1, we first note that for any $\varepsilon > 0$ we can find an infinitely differentiable function $\psi \in \mathcal{D}(\mathbf{R}^{n+1})$ with support in the ε ball with center at 0 and such that

$$u(t,x) = \int_{\mathbf{R}} \int_{\mathbf{R}^n} \psi(s,y)u(t-s,x-y)dsdy \tag{26}$$

for $t > \varepsilon$. A choice of ψ is any radial function $\psi(t,x) \in \mathcal{D}(\mathbf{R}^{n+1})$ with $\int\int \psi dtdx = 1$ and with support in the ε ball as proved by the mean value theorem for harmonic functions.

Or, we may construct such a function ψ by

$$\psi(t,x) = \begin{cases} 0 & \text{if } (t,x) \text{ is near the origin,} \\ -\Delta_{t,x}(\chi(t,x)N(t,x)) & \text{otherwise,} \end{cases} \tag{27}$$

with the use of a fundamental solution N of Δ and a cutoff function $\chi \in \mathcal{D}(\mathbf{R}^{n+1})$ which is equal to 1 near the origin and with support in the ε ball as in Schwartz' proof of the hypo-ellipticity of the Laplacian [19].

Then, we have

$$\partial_t^\alpha \partial_x^\beta u(t,x) = \int_{\mathbf{R}} \int_{\mathbf{R}^n} \partial_s^\alpha \partial_y^\beta \psi(s,y) u(t-s, x-y) ds dy \tag{28}$$

for any differential operator $\partial_t^\alpha \partial_x^\beta$. In particular, we have for any fixed $t > \varepsilon$,

$$\left\| u^{''}(t) \right\|_X \leq \int_{-\epsilon}^{\epsilon} \left\| \partial_s^2 \psi(s,\cdot) \right\|_{L^1} \| u(t-s,\cdot) \|_X ds < \infty, \tag{29}$$

proving condition (i) of Theorem 1. Conditions (ii) and (iii) are proved in the same way.

3. Thomson's theory of submarine cables

In 1855 W. Thomson published the first paper [20] on the signal transmission of submarine cables for telegraphy. The then submarine cables are similar to the present coaxial cables for connecting television receivers with antennas. The outer sheath was more or less directly submerged into the sea water. Thomson assumed, with some reservation, that its electric potential was maintained at rigorously zero everywhere at each instant, or that the sheath was a perfect conductor with no electric resistance. In an earlier paper he had proved that the electro-static capacity C per unit length of a cable was $\frac{I}{2 \log b/a}$, where I is the specific inductive capacitance of the insulator between the wire and the sheath, and a and b are the radii of the wire and the inner surface of the sheath, respectively. Then, he took into consideration only the electric resistance R per unit length of the wire except for the capacitance and claimed that the voltage $v(t,x)$ and the current $j(t,x)$ of the wire obey the equations

$$-\frac{\partial v}{\partial x} = Rj, \quad -\frac{\partial j}{\partial x} = C\frac{\partial v}{\partial t}. \tag{30}$$

For the sake of simplicity we consider an infinitely long cable $x \geq 0$. Let the cable be quiet for $t < 0$. Then, we put an electro-motive force $\phi(t)$ at $x = 0$ and observe the electric current $j(t,x)$ at $x > 0$. Eliminating j, we have the problem:

$$\begin{cases} \dfrac{\partial^2 v(t,x)}{\partial x^2} = CR\dfrac{\partial v(t,x)}{\partial t}, & x \geq 0, \\ v(t,x) = 0, & t < 0,\ x \geq 0, \\ v(t,0) = \phi(t), & t \geq 0. \end{cases} \tag{31}$$

The partial differential equation is the same as Fourier's on heat conduction but we have to solve it as an evolution equation in the unusual direction of the space $x \geq 0$.

In order to discuss the problem in our framework, we consider it in one of the Banach spaces

$$X = \mathcal{M}^1([0,\infty)) \text{ for } p = 1, \quad \text{and} \quad L^p([0,\infty)) \text{ for } 1 < p \leq \infty, \tag{32}$$

regarded as the strong dual of the quotient Banach spaces

$$X_* = C_\infty(\mathbf{R})/C_\infty((-\infty,0)) \quad \text{and} \quad L^{p'}(\mathbf{R})/L^{p'}((-\infty,0)), \tag{33}$$

respectively, where the denominator spaces are imbedded in the numerator by extension by 0.

The operators A in X and A_* in X_* are the negative of generators of semigroups

$$\exp(-xA)f(t) = \begin{cases} f(t-CRx) & \text{if } t \geq CRx, \\ 0 & \text{otherwise,} \end{cases} \tag{34}$$

$$\exp(-xA_*)f_*(t) = f_*(t+CRx). \tag{35}$$

Balakrishnan [1] and Yosida [21] have shown that

$$\exp\big(-xA_*^{1/2}\big)f_* = \int_0^\infty \frac{x}{\sqrt{4\pi s^3}} e^{-\frac{x^2}{4s}} \exp(-sA_*)f_* ds \tag{36}$$

for any bounded continuous semigroup $\exp(-xA_*)$. Hence it follows that

$$\exp\big(-x\sqrt{A}\big)f = \int_0^\infty \frac{x}{\sqrt{4\pi s^3}} e^{-\frac{x^2}{4s}} \exp(-sA)f ds \tag{37}$$

as an absolutely convergent integral in the weak* topology.

Thus, given an arbitrary $\phi(t) \in X$, we obtain

$$v(t,x) = \frac{\sqrt{CR}\,x}{\sqrt{4\pi}} \int_0^t \frac{1}{\sqrt{(t-s)^3}} e^{-\frac{CRx^2}{4(t-s)}} \phi(s)ds \tag{38}$$

as a unique solution of Theorem 1.

This is exactly the solution given in [20] of problem (31). Referring to Fourier, Thomson derived the integral kernel as the integral

$$\frac{2}{\pi}\int_0^\infty e^{-zn^{1/2}} \cos(2nt - zn^{1/2})dn = \begin{cases} \dfrac{z}{\sqrt{4\pi}t^{3/2}} e^{-\frac{z^2}{4t}}, & \text{if } t > 0, \\ 0, & \text{if } t < 0, \end{cases} \tag{39}$$

but his arguments are more physical than mathematical and difficult to follow. However, the paper [20] is composed of three excerpts from the correspondence he exchanged with G. G. Stokes and it contains a proof by Stokes that (38) is a solution to (31) with the use of Fourier's sine integral formula. (The referee has brought our attention to Picard's course at the Sorbonne [17] in which he gives a proof of (39) by complex contour integrals like ours in [13].)

Heaviside [7] invented his operational calculus first to make Thomson's theory more accessible to engineers, and then with this new tool he succeeded in developing the theory of submarine cables much further to make the telephony over the transatlantic cables possible by taking into consideration not only the capacitance and the resistance but also the self-inductance and the leakage conductance of cables. But he was also accused for the lack of rigor in his theory.

In the later justification of operational calculus, formula (39) is interpreted as the Bromwich integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\tau t} e^{-z\sqrt{\tau}} d\tau = \begin{cases} \dfrac{z}{\sqrt{4\pi t^3}} e^{-\frac{z^2}{4t}}, & t > 0, \\ 0, & t \le 0, \end{cases} \tag{40}$$

for $z > 0$. Its rigorous proofs are given in textbooks like van der Pol–Bremmer [18], Jeffreys–Jeffreys [8] and Mikusiński [15] but they are not so straightforward as Heaviside. For easier proofs, see Komatsu [12, p. 250] and [13], which are not very far from Heaviside's original.

If we look at (31) as a mathematical problem, there remains a serious defect, however. The uniqueness of solutions does not hold.

In fact, as shown by Fourier (1822), the initial value problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t}, & t, x \in \mathbf{R}, \\ v(t,0) = \phi(t), & t \in \mathbf{R}, \\ \dfrac{\partial}{\partial x} v(t,0) = \psi(t), & t \in \mathbf{R}, \end{cases} \tag{41}$$

in the direction x of space as time, has a solution in the formal power series

$$\begin{aligned} v(t,x) = \phi(t) &+ \frac{x^2}{2!}\phi'(t) + \frac{x^4}{4!}\phi^{(2)}(t) + \frac{x^6}{6!}\phi^{(3)}(t) + \cdots \\ &+ x\psi(t) + \frac{x^3}{3!}\psi'(t) + \frac{x^5}{5!}\psi^{(2)}(t) + \frac{x^7}{7!}\psi^{(3)}(t) + \cdots, \end{aligned} \tag{42}$$

which converges if the data ϕ and ψ are in Gevrey class of index (2), i.e. if their derivatives have the estimates

$$\sup_{t \in K} |f^{(m)}(t)| \le C_{\varepsilon,K} \varepsilon^m (m!)^2 \tag{43}$$

for any $\varepsilon > 0$ and compact set $K \subset \mathbf{R}$ with a constant $C_{\varepsilon,K}$.

Therefore, if we choose $\phi = 0$ and a nontrivial function ψ in Gevrey class of index (2) with compact support in $\{t \in \mathbf{R}; t > 0\}$ [3], then formula (42) gives an infinitely differentiable solution $v(t,x)$ of (41) which vanishes for $t \le 0$, and for $x = 0$ but does not on a neighborhood of $\{(t,0); t \in \operatorname{supp} \psi\}$.

Our Theorem 1 excludes such a solution under the uniform boundedness condition of the Fatou type.

Theorem 2 *Let $v(t,x)$ be a distribution solution on $\{(t,x) \in \mathbf{R}^2; x > 0\}$ of*

$$\frac{\partial^2 v(t,x)}{\partial x^2} = CR \frac{\partial v(t,x)}{\partial t}, \tag{44}$$

which vanishes for $t < 0$ and satisfies

$$\sup_{x>0} \|v(\cdot, x)\|_{L^p(\mathbf{R})} < \infty \tag{45}$$

for a $1 \leq p \leq \infty$.

Then, there is a unique $\phi \in X$ such that (38) *holds, where X is the Banach space defined by* (32)*.*

PROOF is almost the same as ours of the ordinary Fatou theorem. Since the heat kernel

$$W(t, x) = \begin{cases} -\sqrt{\dfrac{CR}{4\pi t}} e^{-\frac{CRx^2}{4t}}, & t > 0, \\ 0, & t \leq 0 \end{cases} \tag{46}$$

is a fundamental solution to the differential operator

$$H_{x,t} = \frac{\partial^2}{\partial x^2} - CR\frac{\partial}{\partial t}, \tag{47}$$

it follows that given an $\varepsilon > 0$ we have

$$v(t, x) = \int_{\mathbf{R^2}} \psi(s, y) v(t - s, x - y) ds dy \tag{48}$$

for $x > \varepsilon$, with

$$\psi(t, x) = \begin{cases} 0 & \text{if } (t, x) \text{ is near the origin,} \\ -H_{x,t}(\chi(t, x) W(t, x)) & \text{otherwise,} \end{cases} \tag{49}$$

where χ is a cut-off function with support in the ε ball with center at the origin.

Hence we have the infinite differentiability

$$\partial_t^\alpha \partial_x^\beta u(t, x) = \int_{\mathbf{R^2}} \partial_s^\alpha \partial_y^\beta \psi(s, y) u(t - s, x - y) ds dy \tag{50}$$

and also the fact that $v(x) = v(\cdot, x)$ satisfies the conditions of Theorem 1 as a function with values in X.

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Graduate School of Mathematical Sciences
University of Tokyo
Komaba 3-8-1, Tokyo, 153-8914 Japan
E-mail: komatsu@ms.u-tokyo.ac.jp