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# Growth of solutions and oscillation of differential polynomials generated by some complex linear differential equations

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**Abstract.** This paper is devoted to studying the growth and the oscillation of solutions of the second order non-homogeneous linear differential equation

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)e^{Q(z)}f = F,$$

where $P(z)$, $Q(z)$ are nonconstant polynomials such that $\deg P = \deg Q = n$ and $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$, $F \not\equiv 0$ are entire functions with $\rho(A_j) < n$ $(j = 0, 1)$. We also investigate the relationship between small functions and differential polynomials $g_f(z) = d_2 f'' + d_1 f' + d_0 f$, where $d_0(z)$, $d_1(z)$, $d_2(z)$ are entire functions that are not all equal to zero with $\rho(d_j) < n$ $(j = 0, 1, 2)$ generated by solutions of the above equation.

*Key words*: linear differential equations, entire solutions, order of growth, exponent of convergence of zeros, exponent of convergence of distinct zeros.

## 1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [7], [10]). In addition, we will use $\lambda(f)$ and $\lambda(1/f)$ to denote respectively the exponents of convergence of the zero-sequence and the pole-sequence of a meromorphic function $f$, $\rho(f)$ to denote the order of growth of $f$, $\overline{\lambda}(f)$ and $\overline{\lambda}(1/f)$ to denote respectively the exponents of convergence of the sequence of distinct zeros and distinct poles of $f$. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r, \varphi) = o(T(r, f))$ as $r \to +\infty$, where $T(r, f)$ is the Nevanlinna characteristic function of $f$.

To give the precise estimate of fixed points, we define:

**Definition 1.1** ([3], [12], [13]) Let $f$ be a meromorphic function and let $z_1, z_2, \ldots$ $(|z_j| = r_j,\ 0 < r_1 \leq r_2 \leq \cdots)$ be the sequence of the fixed points of $f$, each point being repeated only once. The exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

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$$\overline{\tau}(f) = \inf\left\{\tau > 0 : \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty\right\}.$$

Clearly,

$$\overline{\tau}(f) = \overline{\lim}_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}, \tag{1.1}$$

where $\overline{N}\left(r, \frac{1}{f-z}\right)$ is the counting function of distinct fixed points of $f(z)$ in $\{|z| < r\}$.

Consider the second order linear differential equation

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)e^{Q(z)}f = 0, \tag{1.2}$$

where $P(z)$, $Q(z)$ are nonconstant polynomials, $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ are entire functions such that $\rho(A_1) < \deg P(z)$, $\rho(A_0) < \deg Q(z)$. Gundersen showed in [6, p. 419] that if $\deg P(z) \neq \deg Q(z)$, then every nonconstant solution of (1.2) is of infinite order. If $\deg P(z) = \deg Q(z)$, then (1.2) may have nonconstant solutions of finite order. For instance $f(z) = e^z + 1$ satisfies $f'' + e^z f' - e^z f = 0$.

In [9], Ki-Ho Kwon has investigated the hyper order of solutions of (1.2) when $\deg P(z) = \deg Q(z)$ and has proved the following:

**Theorem A** ([9]) *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials, where $a_i$, $b_i$ $(i = 0, 1, \ldots, n)$ are complex numbers, $a_n b_n \neq 0$, let $A_1(z)$ and $A_0(z)$ $(\not\equiv 0)$ be entire functions with $\rho(A_j) < n$ $(j = 0, 1)$. If either $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0 < c < 1)$, then every nonconstant solution $f$ of (1.2) has infinite order with $\rho_2(f) \geq n$.*

Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [14]). However, there are a few studies on the fixed points of solutions of differential equations. It was in year 2000 that Z. X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients (see [3]). In [2], Z. X. Chen and K. H. Shon have investigated the fixed points of solutions, their 1st and 2nd derivatives and the differential polynomials generated by solutions of second order linear differential equations with meromorphic coefficients. In [13], Wang and Yi investigated fixed points and hyper order of differential polynomials generated by solutions of some second order linear differential equations. In [11], Laine and Rieppo gave improvement of the results of [13] by considering fixed points and iterated order.

The first main purpose of this paper is to study the growth and the oscillation of solutions of the second order non-homogeneous linear differential equation

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)e^{Q(z)}f = F. \tag{1.3}$$

We will prove the following results:

**Theorem 1.1** *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *and* $Q(z) = \sum_{i=0}^{n} b_i z^i$ *be non-constant polynomials where* $a_i$, $b_i$ $(i = 0, 1, \ldots, n)$ *are complex numbers,* $a_n b_n \neq 0$ *such that* $\arg a_n \neq \arg b_n$ *or* $a_n = c b_n$ $(0 < c < 1)$. *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ *and* $F \not\equiv 0$ *be entire functions with* $\max\{\rho(A_j)(j = 0, 1), \rho(F)\} < n$. *Then every solution* $f$ *of equation* (1.3) *has infinite order and satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty. \tag{1.4}$$

**Remark 1.1** If $\rho(F) \geq n$, then equation (1.3) can posses solution of finite order. For instance equation $f'' + e^{-z}f' + e^z f = 1 + e^{2z}$ satisfies $\rho(F) = \rho(1 + e^{2z}) = 1$ and has finite order solution $f(z) = e^z - 1$.

**Theorem 1.2** *Let* $P(z)$, $Q(z)$, $A_1(z)$, $A_0(z)$ *satisfy the hypotheses of Theorem* 1.1, *and let* $F$ *be an entire function such that* $\rho(F) \geq n$. *Then every solution* $f$ *of equation* (1.3) *satisfies* (1.4) *with at most one finite order solution* $f_0$.

The second main purpose of this paper is to study the relation between small functions and some differential polynomials generated by solutions of second order linear differential equation (1.3). We obtain some estimates of their distinct fixed points. Let us denote by

$$\alpha_1 = d_1 - d_2 A_1 e^P, \quad \alpha_0 = d_0 - d_2 A_0 e^Q, \tag{1.5}$$

$$\beta_1 = d_2 A_1^2 e^{2P} - \big((d_2 A_1)' + P' d_2 A_1 + d_1 A_1\big)e^P - d_2 A_0 e^Q + d_0 + d_1', \tag{1.6}$$

$$\beta_0 = d_2 A_0 A_1 e^{P+Q} - \big((d_2 A_0)' + Q' d_2 A_0 + d_1 A_0\big) e^Q + d_0', \tag{1.7}$$

$$h = \alpha_1 \beta_0 - \alpha_0 \beta_1 \tag{1.8}$$

and

$$\psi = \frac{\alpha_1 \big(\varphi' - (d_2 F)' - \alpha_1 F\big) - \beta_1 (\varphi - d_2 F)}{h}. \tag{1.9}$$

**Theorem 1.3** *Let $P(z)$, $Q(z)$, $A_1(z)$, $A_0(z)$, $F$ satisfy the hypotheses of Theorem* 1.1. *Let $d_0(z)$, $d_1(z)$, $d_2(z)$ be entire functions that are not all equal to zero with $\rho(d_j) < n$ $(j = 0, 1, 2)$, $\varphi(z)$ is an entire function with finite order. If $f(z)$ is a solution of* (1.3), *then the differential polynomial $g_f(z) = d_2 f'' + d_1 f' + d_0 f$ satisfies $\overline{\lambda}(g_f - \varphi) = \infty$. In particularly the differential polynomial $g_f(z) = d_2 f'' + d_1 f' + d_0 f$ has infinitely many fixed points and satisfies $\overline{\lambda}(g_f - z) = \overline{\tau}(g_f) = \infty$.*

**Theorem 1.4** *Let $P(z)$, $Q(z)$, $A_1(z)$, $A_0(z)$, $F$ satisfy the hypotheses of Theorem* 1.2. *Let $d_0(z)$, $d_1(z)$, $d_2(z)$ be entire functions that are not all equal to zero with $\rho(d_j) < n$ $(j = 0, 1, 2)$, $\varphi(z)$ is an entire function with finite order such that $\psi(z)$ is not a solution of equation* (1.3). *If $f(z)$ is a solution of* (1.3), *then the differential polynomial $g_f(z) = d_2 f'' + d_1 f' + d_0 f$ satisfies $\overline{\lambda}(g_f - \varphi) = \infty$ with at most one finite order solution $f_0$.*

In the next, we investigate the relation between infinite order solutions of a pair non-homogeneous linear differential equations and we obtain the following result:

**Theorem 1.5** *Let $P(z)$, $Q(z)$, $A_1(z)$, $A_0(z)$, $d_j(z)$, $(j = 0, 1, 2)$ satisfy the hypotheses of Theorem* 1.3. *Let $F_1 \not\equiv 0$ and $F_2 \not\equiv 0$ be entire functions such that* $\max\{\rho(F_j) : j = 1, 2\} < n$ *and $F_1 - C F_2 \not\equiv 0$ for any constant $C$, $\varphi(z)$ is an entire function with finite order. If $f_1$ is a solution of equation*

$$f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f = F_1 \tag{1.10}$$

*and $f_2$ is a solution of equation*

$$f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f = F_2, \tag{1.11}$$

*then the differential polynomial $g_{f_1 - C f_2}(z) = d_2 (f_1'' - C f_2'') + d_1 (f_1' - C f_2') +$*

$d_0(f_1 - Cf_2)$ *satisfies* $\overline{\lambda}(g_{f_1 - Cf_2} - \varphi) = \infty$ *for any constant* $C$.

## 2. Preliminary Lemmas

We need the following lemmas in the proofs of our theorems.

**Lemma 2.1** ([5]) *Let* $f$ *be a transcendental meromorphic function of finite order* $\rho$, *let* $\Gamma = \{(k_1, j_1), (k_2, j_2), \ldots, (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers that satisfy* $k_i > j_i \geq 0$ *for* $i = 1, \ldots, m$ *and let* $\varepsilon > 0$ *be a given constant. Then, there exists a set* $E_1 \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\psi \in [0, 2\pi) - E_1$, *then there is a constant* $R_1 = R_1(\psi) > 1$ *such that for all* $z$ *satisfying* $\arg z = \psi$ *and* $|z| \geq R_1$ *and for all* $(k, j) \in \Gamma$, *we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho - 1 + \varepsilon)}. \tag{2.1}$$

**Lemma 2.2** ([1]) *Let* $P(z) = a_n z^n + \cdots + a_0$, $(a_n = \alpha + i\beta \neq 0)$ *be a polynomial with degree* $n \geq 1$ *and* $A(z)$ $(\not\equiv 0)$ *be a meromorphic function with* $\rho(A) < n$. *Set* $f(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_2 \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\theta \in [0, 2\pi) \backslash (E_2 \cup E_3)$, *where* $E_3 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ *is a finite set, then for sufficiently large* $|z| = r$, *we have*

(i) *If* $\delta(P, \theta) > 0$, *then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta) r^n\} \leq |f(z)| \leq \exp\{(1 + \varepsilon)\delta(P, \theta) r^n\}, \tag{2.2}$$

(ii) *If* $\delta(P, \theta) < 0$, *then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta) r^n\} \leq |f(z)| \leq \exp\{(1 - \varepsilon)\delta(P, \theta) r^n\}. \tag{2.3}$$

**Lemma 2.3** ([4]) *Let* $A_0$, $A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ *be finite order meromorphic functions. If* $f$ *is a meromorphic solution with* $\rho(f) = \infty$ *of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.4}$$

*then* $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$.

**Lemma 2.4** ([1]) *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where $a_i$, $b_i$ $(i = 0, 1, \ldots, n)$ are complex numbers, $a_n b_n \neq 0$ such that $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0 < c < 1)$. We denote index sets by*

$$\Lambda_1 = \{0, P\},$$
$$\Lambda_2 = \{0, P, Q, 2P, P + Q\}.$$

(i) *If $H_j (j \in \Lambda_1)$ and $H_Q \not\equiv 0$ are all meromorphic functions of orders that are less than $n$, setting $\Psi_1(z) = \sum_{j \in \Lambda_1} H_j(z) e^j$, then $\Psi_1(z) + H_Q e^Q \not\equiv 0$.*

(ii) *If $H_j (j \in \Lambda_2)$ and $H_{2Q} \not\equiv 0$ are all meromorphic functions of orders that are less than $n$, setting $\Psi_2(z) = \sum_{j \in \Lambda_2} H_j(z) e^j$, then $\Psi_2(z) + H_{2Q} e^{2Q} \not\equiv 0$.*

**Lemma 2.5** *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where $a_i$, $b_i$ $(i = 0, 1, \ldots, n)$ are complex numbers, $a_n b_n \neq 0$ such that $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0 < c < 1)$. Let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ be entire functions with $\rho(A_j) < n$ $(j = 0, 1)$. We denote*

$$L_f = f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f. \tag{2.5}$$

*If $f \not\equiv 0$ is a finite order entire function, then $\rho(L_f) \geq n$.*

*Proof.* We suppose that $\rho(L_f) < n$ and then we obtain a contradiction.

(i) If $\rho(f) < n$, then $f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f - L_f = 0$ has the form of $\Psi_1(z) + H_Q e^Q = f'' + A_1(z) e^{P(z)} f' - L_f + A_0(z) e^{Q(z)} f = 0$ and this is a contradiction by Lemma 2.4 (i).

(ii) If $\rho(f) \geq n$, we rewrite

$$\frac{L_f}{f} = \frac{f''}{f} + A_1(z) e^{P(z)} \frac{f'}{f} + A_0(z) e^{Q(z)}. \tag{2.6}$$

**Case 1** Suppose first that $\arg a_n \neq \arg b_n$. Then $\arg a_n$, $\arg b_n$, $\arg(a_n + b_n)$ are three distinct arguments. Set $\rho(L_f) = \beta < n$. Then, for any given $\varepsilon$ $(0 < \varepsilon < n - \beta)$, we have for sufficiently large $r$

$$|L_f| \leq \exp\left\{r^{\beta + \varepsilon}\right\}. \tag{2.7}$$

From Wiman-Valiron theory (see [8, p. 344]), we know that there exists a set $E$ with finite logarithmic measure such that for a point $z$ satisfying $|z| = r \notin E$ and $|f(z)| = M(r, f)$, we have

$$v_f(r) < [\log \mu_f(r)]^2, \tag{2.8}$$

where $\mu_f(r)$ is a maximum term of $f$. By Cauchy's inequality, we have $\mu_f(r) \le M(r, f)$. This and (2.8) yield

$$v_f(r) < \big[\log |f(z)|\big]^2, \quad (r \notin E). \tag{2.9}$$

By $f$ is transcendental function we know that $v_f(r) \to \infty$. Then for sufficiently large $|z| = r$ we have $|f(z)| = M(r, f) \ge 1$, then

$$\left|\frac{L_f}{f}\right| \le |L_f| \le \exp\{r^{\beta+\varepsilon}\}. \tag{2.10}$$

Also, by Lemma 2.1, for the above $\varepsilon$, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) - E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all $z$ satisfying $\arg z = \theta$ and $|z| \ge R_1$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le |z|^{k(\rho(f)-1+\varepsilon)}, \quad (k = 1, 2). \tag{2.11}$$

By Lemma 2.2, there exists a ray $\arg z = \theta \in [0, 2\pi)\backslash E_1 \cup E_2 \cup E_3$, $E_3 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0 \text{ or } \delta(Q, \theta) = 0\} \subset [0, 2\pi)$, $E_1 \cup E_2$ having linear measure zero, $E_3$ being a finite set, such that $\delta(P, \theta) < 0$, $\delta(Q, \theta) > 0$ and for any given $\varepsilon$ $(0 < \varepsilon < n - \beta)$, we have for sufficiently large $|z| = r$

$$\big|A_0 e^Q\big| \ge \exp\{(1-\varepsilon)\delta(Q, \theta) r^n\}, \tag{2.12}$$

$$\left|\frac{f'}{f} A_1 e^P\right| \le r^{\rho(f)-1+\varepsilon} \exp\{(1-\varepsilon)\delta(P, \theta) r^n\} < r^{\rho(f)-1+\varepsilon}. \tag{2.13}$$

By (2.6), (2.10)–(2.13), we have

$$\exp\{(1-\varepsilon)\delta(Q, \theta) r^n\} \le |A_0 e^Q| \le \exp\{r^{\beta+\varepsilon}\} + r^{\rho(f)-1+\varepsilon} + r^{2(\rho(f)-1+\varepsilon)}. \tag{2.14}$$

This is a contradiction by $\beta + \varepsilon < n$. Hence $\rho(L_f) \geq n$.

**Case 2** Suppose now $a_n = cb_n$ $(0 < c < 1)$. Then for any ray $\arg z = \theta$, we have

$$\delta(P, \theta) = c\delta(Q, \theta).$$

Then, by Lemma 2.1 and Lemma 2.2, for any given $\varepsilon$ $(0 < \varepsilon < \min(\frac{1-c}{2(1+c)}, n - \beta))$, there exist $E_j \subset [0, 2\pi)$ $(j = 1, 2, 3)$ such that $E_1, E_2$ having linear measure zero and $E_3$ being a finite set, where $E_1, E_2$ and $E_3$ are defined as in the Case 1 respectively. We take the ray $\arg z = \theta \in [0, 2\pi) \backslash E_1 \cup E_2 \cup E_3$ such that $\delta(Q, \theta) > 0$ and for sufficiently large $|z| = r$, we have (2.11), (2.12) and

$$\left| \frac{f'}{f} A_1 e^P \right| \leq r^{\rho(f)-1+\varepsilon} \exp\{(1+\varepsilon)c\delta(Q, \theta)r^n\}. \tag{2.15}$$

By (2.6), (2.10)–(2.12) and (2.15)

$$\begin{aligned} \exp\{(1-\varepsilon)\delta(Q, \theta)r^n\} &\leq |A_0 e^Q| \\ &\leq \exp\{r^{\beta+\varepsilon}\} + r^{\rho(f)-1+\varepsilon} \exp\{(1+\varepsilon)c\delta(Q, \theta)r^n\} + r^{2(\rho(f)-1+\varepsilon)}. \end{aligned} \tag{2.16}$$

By $\varepsilon$ $\left(0 < \varepsilon < \min\left(\frac{1-c}{2(1+c)}, n - \beta\right)\right)$, we have as $r \to +\infty$

$$\frac{\exp\{r^{\beta+\varepsilon}\}}{\exp\{(1-\varepsilon)\delta(Q, \theta)r^n\}} \to 0, \tag{2.17}$$

$$\frac{r^{\rho(f)-1+\varepsilon} \exp\{(1+\varepsilon)c\delta(Q, \theta)r^n\}}{\exp\{(1-\varepsilon)\delta(Q, \theta)r^n\}} \to 0, \tag{2.18}$$

$$\frac{r^{2(\rho(f)-1+\varepsilon)}}{\exp\{(1-\varepsilon)\delta(Q, \theta)r^n\}} \to 0. \tag{2.19}$$

By (2.16)–(2.19), we get $1 \leq 0$. This is a contradiction. Hence $\rho(L_f) \geq n$.

## 3. Proof of Theorem 1.1

Assume that $f$ is a solution of equation (1.3). We prove that $f$ is of infinite order. We suppose the contrary $\rho(f) < \infty$. By Lemma 2.5, we have

$n \le \rho(L_f) = \rho(F) < n$ and this is a contradiction. Hence, every solution $f$ of equation (1.3) is of infinite order. By Lemma 2.3, every solution $f$ of equation (1.3) satisfies (1.4).

## 4. Proof of Theorem 1.2

Assume that $f_0$ is a solution of (1.3) with $\rho(f_0) = \rho < \infty$. If $f_1$ is a second finite order solution of (1.3), then $\rho(f_1 - f_0) < \infty$, and $f_1 - f_0$ is a solution of the corresponding homogeneous equation (1.2) of (1.3), but $\rho(f_1 - f_0) = \infty$ from Theorem A, this is a contradiction. Hence (1.3) has at most one finite order solution $f_0$ and all other solutions $f_1$ of (1.3) satisfy (1.4) by Lemma 2.3.

## 5. Proof of Theorem 1.3

We first prove $\rho(g_f) = \rho(d_2 f'' + d_1 f' + d_0 f) = \infty$. Suppose that $f$ is a solution of equation (1.3). Then by Theorem 1.1, we have $\rho(f) = \infty$. First we suppose that $d_2 \not\equiv 0$. Substituting $f'' = F - A_1 e^P f' - A_0 e^Q f$ into $g_f$, we get

$$g_f - d_2 F = \big(d_1 - d_2 A_1 e^P\big) f' + \big(d_0 - d_2 A_0 e^Q\big) f. \tag{3.1}$$

Differentiating both sides of equation (3.1) and replacing $f''$ with $f'' = F - A_1 e^P f' - A_0 e^Q f$, we obtain

$$\begin{aligned} &g_f' - (d_2 F)' - \big(d_1 - d_2 A_1 e^P\big) F \\ &\quad = \big[d_2 A_1^2 e^{2P} - \big((d_2 A_1)' + P' d_2 A_1 + d_1 A_1\big) e^P - d_2 A_0 e^Q + d_0 + d_1'\big] f' \\ &\qquad + \big[d_2 A_0 A_1 e^{P+Q} - \big((d_2 A_0)' + Q' d_2 A_0 + d_1 A_0\big) e^Q + d_0'\big] f. \end{aligned} \tag{3.2}$$

Then, by (1.5)–(1.7), (3.1) and (3.2), we have

$$\alpha_1 f' + \alpha_0 f = g_f - d_2 F, \tag{3.3}$$

$$\beta_1 f' + \beta_0 f = g_f' - (d_2 F)' - \big(d_1 - d_2 A_1 e^P\big) F. \tag{3.4}$$

Set

$$\begin{aligned} h &= \alpha_1\beta_0 - \alpha_0\beta_1 \\ &= \big(d_1 - d_2A_1e^P\big)\big[d_2A_0A_1e^{P+Q} - \big((d_2A_0)' + Q'd_2A_0 + d_1A_0\big)e^Q + d_0'\big] \\ &\quad - \big(d_0 - d_2A_0e^Q\big)\big[d_2A_1^2e^{2P} - \big((d_2A_1)' + P'd_2A_1 + d_1A_1\big)e^P \\ &\qquad - d_2A_0e^Q + d_0 + d_1'\big]. \end{aligned} \tag{3.5}$$

Now check all the terms of $h$. Since the term $d_2^2A_1^2A_0e^{2P+Q}$ is eliminated, by (3.5) we can write $h = \Psi_2(z) - d_2^2A_0^2e^{2Q}$, where $\Psi_2(z)$ is defined as in Lemma 2.4 (ii). By $d_2 \not\equiv 0$, $A_0 \not\equiv 0$ and Lemma 2.4 (ii), we see that $h \not\equiv 0$. By (3.3), (3.4) and (3.5), we obtain

$$f = \frac{\alpha_1\big(g_f' - (d_2F)' - \alpha_1F\big) - \beta_1(g_f - d_2F)}{h}. \tag{3.6}$$

If $\rho(g_f) < \infty$, then by (3.6) we get $\rho(f) < \infty$ and this is a contradiction. Hence $\rho(g_f) = \infty$.

Set $w(z) = d_2f'' + d_1f' + d_0f - \varphi$. Then, by $\rho(\varphi) < \infty$, we have $\rho(w) = \rho(g_f) = \rho(f) = \infty$. In order to prove $\overline{\lambda}(g_f - \varphi) = \infty$, we need to prove only $\overline{\lambda}(w) = \infty$. Using $g_f = w + \varphi$, we get from (3.6)

$$f = \frac{\alpha_1\big(w' + \varphi' - (d_2F)' - \alpha_1F\big) - \beta_1(w + \varphi - d_2F)}{h}. \tag{3.7}$$

So,

$$f = \frac{\alpha_1w' - \beta_1w}{h} + \psi, \tag{3.8}$$

where $\psi$ is defined in (1.9). Substituting (3.8) into equation (1.3), we obtain

$$\begin{aligned} &\frac{\alpha_1}{h}w''' + \phi_2w'' + \phi_1w' + \phi_0w \\ &\quad = F - \big(\psi'' + A_1(z)e^{P(z)}\psi' + A_0(z)e^{Q(z)}\psi\big) = A, \end{aligned} \tag{3.9}$$

where $\phi_j$ $(j = 0,1,2)$ are meromorphic functions with $\rho(\phi_j) < \infty$ $(j = 0,1,2)$. Since $\rho(\psi) < \infty$, it follows that $A \not\equiv 0$ by Theorem 1.1. By $\alpha_1 \not\equiv 0$, $h \not\equiv 0$ and Lemma 2.3, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, i.e., $\overline{\lambda}(g_f - \varphi) = \infty$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$ or $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$. Using a similar reasoning to that above we get $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, i.e., $\overline{\lambda}(g_f - \varphi) = \infty$.

Setting now $\varphi(z) = z$, we obtain that $\overline{\lambda}(g_f - z) = \overline{\tau}(g_f) = \infty$.

## 6. Proof of Theorem 1.4

By hypothesis of Theorem 1.4, $\psi(z)$ is not a solution of equation (1.3). Then

$$F - \left(\psi'' + A_1(z)e^{P(z)}\psi' + A_0(z)e^{Q(z)}\psi\right) \not\equiv 0.$$

By using Theorem 1.2 and similar reasoning to that in the proof of Theorem 1.3, we can prove Theorem 1.4.

## 7. Proof of Theorem 1.5

Suppose that $f_1$ is a solution of equation (1.10) and $f_2$ is a solution of equation (1.11). Set $w = f_1 - Cf_2$. Then $w$ is a solution of equation

$$w'' + A_1(z)e^{P(z)}w' + A_0(z)e^{Q(z)}w = F_1 - CF_2.$$

By $\rho(F_1 - CF_2) < n$, $F_1 - CF_2 \not\equiv 0$ and Theorem 1.1, we have $\rho(w) = \infty$. Thus, by using Theorem 1.3, we obtain Theorem 1.5.

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