

## A conjugate system and tangential derivative norms on parabolic Bergman spaces

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**Abstract.** The  $\alpha$ -parabolic Bergman space  $\mathbf{b}_\alpha^p(\lambda)$  is the Banach space of solutions of the parabolic equation  $L^{(\alpha)} = \partial/\partial t + (-\Delta_x)^\alpha$  on the upper half space  $H$  which have finite  $L^p(H, t^\lambda dV)$  norms, where  $t^\lambda dV$  is the weighted Lebesgue volume measure on  $H$ . It is known that  $\mathbf{b}_{1/2}^p(\lambda)$  coincide with the harmonic Bergman spaces. In this paper, we introduce the extension of notion of conjugate functions of  $\mathbf{b}_\alpha^p(\lambda)$ -functions and study their properties. As an application, we give estimates of tangential derivative norms on  $\mathbf{b}_\alpha^p(\lambda)$ .

*Key words:* conjugate function, tangential derivative, heat equation, parabolic operator of fractional order, Bergman space.

### 1. Introduction

Let  $H$  be the upper half space of  $\mathbb{R}^{n+1}$  ( $n \geq 1$ ), that is,  $H = \{X = (x, t); x \in \mathbb{R}^n, t > 0\}$ . For  $0 < \alpha \leq 1$ , the parabolic operator  $L^{(\alpha)}$  is defined by

$$L^{(\alpha)} := \frac{\partial}{\partial t} + (-\Delta_x)^\alpha,$$

where  $\Delta_x := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$  is the Laplacian on the  $x$ -space  $\mathbb{R}^n$ . A real-valued continuous function  $u$  on  $H$  is said to be  $L^{(\alpha)}$ -harmonic if  $u$  satisfies  $L^{(\alpha)}u = 0$  in the sense of distributions. (The explicit definition of the  $L^{(\alpha)}$ -harmonic function is described in Section 3.) For  $\lambda > -1$  and  $1 \leq p < \infty$ , the  $\alpha$ -parabolic Bergman space  $\mathbf{b}_\alpha^p(\lambda)$  is the set of all  $L^{(\alpha)}$ -harmonic functions  $u$  on  $H$  with

$$\|u\|_{L^p(\lambda)} := \left( \int_H |u(x, t)|^p t^\lambda dV(x, t) \right)^{1/p} < \infty,$$

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where  $dV$  is the Lebesgue volume measure on  $H$  and  $L^p(\lambda) := L^p(H, t^\lambda dV)$ . In particular, we may write  $L^p = L^p(0)$  and  $\mathbf{b}_\alpha^p = \mathbf{b}_\alpha^p(0)$ , respectively.

Our aim of this paper is to study conjugate systems on  $\alpha$ -parabolic Bergman spaces. The  $\alpha$ -parabolic Bergman spaces  $\mathbf{b}_\alpha^p$  were introduced and studied by Nishio, Shimomura, and Suzuki [7]. It was shown in [7] that  $\mathbf{b}_{1/2}^p$  coincide with the usual harmonic Bergman spaces of Ramey and Yi [11]. Accordingly, usual harmonic Bergman spaces are the classes of  $L^p$ -solutions of the parabolic equation  $L^{(\alpha)}u = 0$  with  $\alpha = 1/2$ . In this paper, we extend the notion of conjugacy of harmonic functions to  $\alpha$ -parabolic Bergman spaces. In [12], Stein and Weiss studied properties of systems of conjugate harmonic functions on the harmonic Hardy spaces. In the theory of the harmonic Bergman spaces, properties of conjugate functions were also studied by Ramey and Yi [11], and as an application, estimates of tangential derivative norms of harmonic Bergman functions were given. However, the suitable notion of conjugacy are not extended to  $\alpha$ -parabolic Bergman spaces. (For instance, although Yamada [13] gave an extension of the notion of conjugacy, it seems that the extension of Yamada is not suitable.) In this paper, we introduce a suitable extension of conjugacy to  $\alpha$ -parabolic Bergman spaces and study their properties. We also give estimates of tangential derivative norms of  $\alpha$ -parabolic Bergman functions.

Now, we introduce the extension of conjugacy to  $\alpha$ -parabolic Bergman spaces. Let  $\partial_j = \partial/\partial x_j$  ( $1 \leq j \leq n$ ) and  $\partial_t = \partial/\partial t$ . Let  $C(\Omega)$  be the set of all real-valued continuous functions on a region  $\Omega$ , and for a positive integer  $k$ ,  $C^k(\Omega) \subset C(\Omega)$  denotes the set of all  $k$  times continuously differentiable functions on  $\Omega$ , and put  $C^\infty(\Omega) = \bigcap_k C^k(\Omega)$ . Furthermore, for a real number  $\kappa$ , let  $\mathcal{D}_t^\kappa = (-\partial_t)^\kappa$  be the fractional differential operator with respect to  $t$ . (The definition of the fractional differential operator and the fundamental properties of fractional calculus for  $\alpha$ -parabolic Bergman functions are described in Section 2.)

**Definition 1** For a function  $u \in \mathbf{b}_\alpha^p(\lambda)$ , we shall say that a vector-valued function  $V = (v_1, \dots, v_n)$  on  $H$  is an  $\alpha$ -parabolic conjugate function of  $u$  if  $v_j \in C^1(H)$  and  $V$  satisfies the equations

$$\nabla_x u = -\mathcal{D}_t V, \quad \nabla_x v_j = \partial_j V \quad (1 \leq j \leq n), \quad (\text{C.1})$$

and

$$\mathcal{D}_t^{\frac{1}{\alpha}-1}u = \nabla_x \cdot V, \quad (\text{C.2})$$

where  $\nabla_x = (\partial_1, \dots, \partial_n)$  and  $\nabla_x \cdot V$  is the divergence of  $V$ .

We remark that the fractional derivative  $\mathcal{D}_t^{\frac{1}{\alpha}-1}u$  is well defined whenever  $u \in \mathbf{b}_\alpha^p(\lambda)$  with  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ , and  $\lambda > -1$  (see Section 2). Our formulation of the extension of conjugacy is based on the Cauchy-Riemann equations  $u_x = v_t$  and  $-u_t = v_x$  on a region of the two-dimensional Euclidean space. Evidently, when  $\alpha = 1/2$ , the equations (C.1) and (C.2) coincide with the generalized Cauchy-Riemann equations for harmonic functions in [12];

$$\partial_j u = \partial_t v_j, \quad \partial_k v_j = \partial_j v_k, \quad 1 \leq j, k \leq n, \quad (\text{1.1})$$

and

$$\partial_t u + \sum_{j=1}^n \partial_j v_j = 0. \quad (\text{1.2})$$

Particularly, an  $(n+1)$ -tuple  $(v_1, \dots, v_n, u)$  which satisfies (1.1) and (1.2) is said to be a system of conjugate harmonic functions on  $H$ . We present results of Ramey and Yi [11] concerning with conjugate functions of harmonic Bergman functions.

**Theorem A** (Theorem 6.1 of [11]) *Let  $1 \leq p < \infty$  and  $u \in \mathbf{b}_{1/2}^p$ . Then, there exists a unique 1/2-parabolic conjugate function  $V = (v_1, \dots, v_n)$  of  $u$  such that  $v_j \in \mathbf{b}_{1/2}^p$ . Also, there exists a constant  $C = C(n, p) > 0$  independent of  $u$  such that*

$$C^{-1}\|u\|_{L^p} \leq \| |V| \|_{L^p} \leq C\|u\|_{L^p},$$

where  $|V| := \{v_1^2 + \dots + v_n^2\}^{1/2}$ .

For a multi-index  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n$ , let  $\partial_x^\gamma := \partial_1^{\gamma_1} \dots \partial_n^{\gamma_n}$ , where  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . The following theorem gives estimates of tangential derivative norms of harmonic Bergman functions.

**Theorem B** (Theorem 6.2 of [11]) *Let  $1 \leq p < \infty$  and  $u \in \mathbf{b}_{1/2}^p$ . Then, for each  $m \in \mathbb{N}_0$ , there exists a constant  $C = C(n, p, m) > 0$  independent of  $u$  such that*

$$C^{-1}\|u\|_{L^p} \leq \sum_{|\gamma|=m} \|t^m \partial_x^\gamma u\|_{L^p} \leq C\|u\|_{L^p}.$$

We describe the main results of this paper. We remark that the condition  $p(\frac{1}{2\alpha} - 1) + \lambda > -1$  in Theorem 1 below holds for all  $1 \leq p < \infty$  and  $\lambda > -1$  whenever  $0 < \alpha \leq 1/2$ .

**Theorem 1** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ , and  $u \in \mathbf{b}_\alpha^p(\lambda)$ . If  $\alpha$ ,  $p$ , and  $\lambda$  satisfy the condition  $\eta = p(\frac{1}{2\alpha} - 1) + \lambda > -1$ , then there exists a unique  $\alpha$ -parabolic conjugate function  $V = (v_1, \dots, v_n)$  of  $u$  such that  $v_j \in \mathbf{b}_\alpha^p(\eta)$ . Also, there exists a constant  $C = C(n, p, \alpha, \lambda) > 0$  independent of  $u$  such that*

$$C^{-1}\|u\|_{L^p(\lambda)} \leq \|V\|_{L^p(\eta)} \leq C\|u\|_{L^p(\lambda)}. \quad (1.3)$$

In Section 4, we show that  $\mathbf{b}_\alpha^p(\lambda) = \{0\}$  when  $\lambda \leq -1$ . Therefore, similar statements in Theorem 1 can not hold for the case  $\eta = p(\frac{1}{2\alpha} - 1) + \lambda \leq -1$ . We do not know whether Theorem A is extended to the full range  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ , and  $\lambda > -1$ . However, we can give estimates of tangential derivative norms of  $\mathbf{b}_\alpha^p(\lambda)$ -functions.

**Theorem 2** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ , and  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Then, for each  $m \in \mathbb{N}_0$ , there exists a constant  $C = C(n, p, \alpha, \lambda, m) > 0$  independent of  $u$  such that*

$$C^{-1}\|u\|_{L^p(\lambda)} \leq \sum_{|\gamma|=m} \|t^{\frac{m}{2\alpha}} \partial_x^\gamma u\|_{L^p(\lambda)} \leq C\|u\|_{L^p(\lambda)}. \quad (1.4)$$

We display the plan of this paper. In Section 2, we describe basic properties of fractional calculus on  $\mathbf{b}_\alpha^p(\lambda)$ . In Section 3, we define integral operators induced by the fundamental solution of the parabolic operator  $L^{(\alpha)}$  and investigate their properties, which are useful for studying  $\alpha$ -parabolic conjugate functions. In Section 4, we give the proof of Theorem 1. Moreover, we show a decomposition theorem for  $\alpha$ -parabolic conjugate functions when  $\eta = p(\frac{1}{2\alpha} - 1) + \lambda > -1$ . In Section 5, we give the proof of Theorem 2. More properties of  $\alpha$ -parabolic conjugate functions are studied in Section 6.

Throughout this paper,  $C$  will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

## 2. Fractional calculus on $b_\alpha^p(\lambda)$

In order to extend conjugacy to  $\alpha$ -parabolic Bergman spaces, we need fractional calculus on  $b_\alpha^p(\lambda)$ . First, we describe fractional differential operators for functions on  $\mathbb{R}_+ = (0, \infty)$ . For a real number  $\kappa > 0$ , let

$$\mathcal{FC}^{-\kappa} := \{\varphi \in C(\mathbb{R}_+); \exists \varepsilon > 0, \exists C > 0 \text{ s.t. } |\varphi(t)| \leq Ct^{-\kappa-\varepsilon}, \forall t \in \mathbb{R}_+\}. \quad (2.1)$$

For a function  $\varphi \in \mathcal{FC}^{-\kappa}$ , we can define the fractional integral  $\mathcal{D}_t^{-\kappa}\varphi$  of  $\varphi$  by

$$\mathcal{D}_t^{-\kappa}\varphi(t) := \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1} \varphi(\tau+t) d\tau = \frac{1}{\Gamma(\kappa)} \int_t^\infty (\tau-t)^{\kappa-1} \varphi(\tau) d\tau, \quad t \in \mathbb{R}_+, \quad (2.2)$$

where  $\Gamma$  is the gamma function. Moreover, let

$$\mathcal{FC}^\kappa := \{\varphi; d_t^{[\kappa]}\varphi \in \mathcal{FC}^{-([\kappa]-\kappa)}\}, \quad (2.3)$$

where  $d_t = d/dt$ ,  $[\kappa]$  is the smallest integer greater than or equal to  $\kappa$ , and we will write  $\mathcal{FC}^0 := C(\mathbb{R}_+)$ . We can also define the fractional derivative  $\mathcal{D}_t^\kappa\varphi$  of  $\varphi \in \mathcal{FC}^\kappa$  by

$$\mathcal{D}_t^\kappa\varphi(t) := \mathcal{D}_t^{-([\kappa]-\kappa)}((-d_t)^{[\kappa]}\varphi)(t), \quad t \in \mathbb{R}_+. \quad (2.4)$$

In particular, we will write  $\mathcal{D}_t^0\varphi = \varphi$ . For a real number  $\kappa$ , we may call both (2.2) and (2.4) *the fractional derivatives of  $\varphi$  with order  $\kappa$* . And, we call  $\mathcal{D}_t^\kappa$  *the fractional differential operator with order  $\kappa$* . Some basic properties of the fractional differential operators are the following.

**Lemma 2.1** (Proposition 2.1 of [4]) *For real numbers  $\kappa, \nu > 0$ , the following statements hold.*

- (1) *If  $\varphi \in \mathcal{FC}^{-\kappa}$ , then  $\mathcal{D}_t^{-\kappa}\varphi \in C(\mathbb{R}_+)$ .*
- (2) *If  $\varphi \in \mathcal{FC}^{-\kappa-\nu}$ , then  $\mathcal{D}_t^{-\kappa}\mathcal{D}_t^{-\nu}\varphi = \mathcal{D}_t^{-\kappa-\nu}\varphi$ .*
- (3) *If  $d_t^k\varphi \in \mathcal{FC}^{-\nu}$  for all integers  $0 \leq k \leq [\kappa] - 1$  and  $d_t^{[\kappa]}\varphi \in \mathcal{FC}^{-([\kappa]-\kappa)-\nu}$ , then  $\mathcal{D}_t^\kappa\mathcal{D}_t^{-\nu}\varphi = \mathcal{D}_t^{-\nu}\mathcal{D}_t^\kappa\varphi = \mathcal{D}_t^{\kappa-\nu}\varphi$ .*

- (4) If  $d_t^{k+\lceil\nu\rceil}\varphi \in \mathcal{FC}^{-(\lceil\nu\rceil-\nu)}$  for all integers  $0 \leq k \leq \lceil\kappa\rceil - 1$ ,  $d_t^{\lceil\kappa\rceil+\ell}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)}$  for all integers  $0 \leq \ell \leq \lceil\nu\rceil - 1$ , and  $d_t^{\lceil\kappa\rceil+\lceil\nu\rceil}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)-(\lceil\nu\rceil-\nu)}$ , then  $\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \varphi = \mathcal{D}_t^{\kappa+\nu} \varphi$ .

Here, we give some examples of fractional derivatives of elementary functions.

**Example 2.2** Let  $\kappa > 0$  and  $\nu$  be real numbers. Then, we have the following.

- (1)  $\mathcal{D}_t^\nu e^{-\kappa t} = \kappa^\nu e^{-\kappa t}$ .  
(2) If  $-\kappa < \nu$ , then  $\mathcal{D}_t^\nu t^{-\kappa} = \frac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)} t^{-\kappa-\nu}$ .

Next, we also describe some basic results concerning with the fundamental solution of  $L^{(\alpha)}$ . For  $x \in \mathbb{R}^n$ , let

$$W^{(\alpha)}(x, t) := \begin{cases} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i x \cdot \xi) d\xi & (t > 0) \\ 0 & (t \leq 0), \end{cases} \quad (2.5)$$

where  $x \cdot \xi$  denotes the inner product on  $\mathbb{R}^n$  and  $|\xi| = (\xi \cdot \xi)^{1/2}$ . The function  $W^{(\alpha)}$  is the fundamental solution of  $L^{(\alpha)}$  and it is  $L^{(\alpha)}$ -harmonic on  $H$ . We note that  $W^{(\alpha)} \geq 0$  on  $H$  and  $\int_{\mathbb{R}^n} W^{(\alpha)}(x, t) dx = 1$  for all  $0 < t < \infty$ . Furthermore,  $W^{(\alpha)} \in C^\infty(H)$ . Let  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n$  be a multi-index and  $k \in \mathbb{N}_0$ . The following estimate is Lemma 1 of [9]: there exists a constant  $C = C(n, \alpha, \gamma, k) > 0$  such that

$$|\partial_x^\gamma \partial_t^k W^{(\alpha)}(x, t)| \leq C(t + |x|^{2\alpha})^{-\left(\frac{n+|\gamma|}{2\alpha} + k\right)} \quad (2.6)$$

for all  $(x, t) \in H$ . In particular, by (2.6), we note that for each  $x \in \mathbb{R}^n$ , the function  $\varphi(\cdot) = W^{(\alpha)}(x, \cdot)$  belongs to  $\mathcal{FC}^\kappa$  for  $\kappa > -\frac{n}{2\alpha}$ . The statements in the following lemma are consequences of [4].

**Lemma 2.3** (Theorem 3.1 of [4]) *Let  $0 < \alpha \leq 1$ ,  $\gamma \in \mathbb{N}_0^n$  be a multi-index, and  $\kappa$  be a real number such that  $\kappa > -\frac{n}{2\alpha}$ . Then, the following statements hold.*

- (1) *The derivatives  $\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t)$  and  $\mathcal{D}_t^\kappa \partial_x^\gamma W^{(\alpha)}(x, t)$  can be defined, and the equation  $\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t) = \mathcal{D}_t^\kappa \partial_x^\gamma W^{(\alpha)}(x, t)$  holds. Furthermore,*

there exists a constant  $C = C(n, \alpha, \gamma, \kappa) > 0$  such that

$$|\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t)| \leq C(t + |x|^{2\alpha})^{-\left(\frac{n+|\gamma|}{2\alpha} + \kappa\right)}$$

for all  $(x, t) \in H$ .

- (2) If a real number  $\nu$  satisfies the condition  $\nu + \kappa > -\frac{n}{2\alpha}$ , then the derivative  $\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t)$  is well defined, and

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t) = \partial_x^\gamma \mathcal{D}_t^{\nu+\kappa} W^{(\alpha)}(x, t)$$

for all  $(x, t) \in H$ .

- (3) The derivative  $\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t)$  is  $L^{(\alpha)}$ -harmonic on  $H$ .

We present basic properties of fractional derivatives of  $\mathbf{b}_\alpha^p(\lambda)$ -functions. We begin with describing estimates of ordinary derivatives of  $\mathbf{b}_\alpha^p(\lambda)$ -functions. Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ ,  $\gamma \in \mathbb{N}_0^n$  be a multi-index, and  $k \in \mathbb{N}_0$ . Then, it is known that  $\mathbf{b}_\alpha^p(\lambda) \subset C^\infty(H)$  (see [13]) and the following estimate is given by Lemma 3.4 of [13]: there exists a constant  $C = C(n, \alpha, p, \lambda, \gamma, k) > 0$  such that

$$|\partial_x^\gamma \partial_t^k u(x, t)| \leq C t^{-\left(\frac{|\gamma|}{2\alpha} + k\right) - \left(\frac{n}{2\alpha} + \lambda + 1\right) \frac{1}{p}} \|u\|_{L^p(\lambda)} \quad (2.7)$$

for all  $u \in \mathbf{b}_\alpha^p(\lambda)$  and  $(x, t) \in H$ . The estimate (2.7) implies that the point evaluation is a bounded linear functional on  $\mathbf{b}_\alpha^p(\lambda)$ . Furthermore, the estimate (2.7) also shows that a function  $\varphi(\cdot) = u(x, \cdot)$  belongs to  $\mathcal{FC}^\kappa$  for  $u \in \mathbf{b}_\alpha^p(\lambda)$  and  $\kappa > -\left(\frac{n}{2\alpha} + \lambda + 1\right) \frac{1}{p}$ , so we can define fractional derivatives of  $\mathbf{b}_\alpha^p(\lambda)$ -functions. Some properties of fractional derivatives of  $\mathbf{b}_\alpha^p(\lambda)$ -functions are given in the following.

**Lemma 2.4** (Proposition 4.1 of [4]) *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ ,  $\gamma \in \mathbb{N}_0^n$  be a multi-index, and  $\kappa$  be a real number such that  $\kappa > -\left(\frac{n}{2\alpha} + \lambda + 1\right) \frac{1}{p}$ . If  $u \in \mathbf{b}_\alpha^p(\lambda)$ , then the following statements hold.*

- (1) The derivatives  $\partial_x^\gamma \mathcal{D}_t^\kappa u(x, t)$  and  $\mathcal{D}_t^\kappa \partial_x^\gamma u(x, t)$  can be defined, and the equation  $\partial_x^\gamma \mathcal{D}_t^\kappa u(x, t) = \mathcal{D}_t^\kappa \partial_x^\gamma u(x, t)$  holds. Furthermore, there exists a constant  $C = C(n, \alpha, p, \lambda, \gamma, \kappa) > 0$  independent of  $u$  such that

$$|\partial_x^\gamma \mathcal{D}_t^\kappa u(x, t)| \leq C t^{-\left(\frac{|\gamma|}{2\alpha} + \kappa\right) - \left(\frac{n}{2\alpha} + \lambda + 1\right) \frac{1}{p}} \|u\|_{L^p(\lambda)}$$

for all  $(x, t) \in H$ .

- (2) If a real number  $\nu$  satisfies the condition  $\nu + \kappa > -\left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}$ , then the derivative  $\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa u(x, t)$  is well defined, and

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa u(x, t) = \partial_x^\gamma \mathcal{D}_t^{\nu+\kappa} u(x, t)$$

for all  $(x, t) \in H$ .

- (3) The derivative  $\partial_x^\gamma \mathcal{D}_t^\kappa u(x, t)$  is  $L^{(\alpha)}$ -harmonic on  $H$ .

For a real number  $\kappa > 0$ , let  $C_\kappa = 2^\kappa / \Gamma(\kappa)$ . The following lemma is also a consequence of [4], and (2.8) is the reproducing formula for  $\mathbf{b}_\alpha^p(\lambda)$ -functions.

**Lemma 2.5** (Theorem 5.2 of [4]) *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ , and  $\lambda > -1$ . Suppose that  $\nu$  and  $\kappa$  are real numbers such that  $\nu > -\frac{\lambda+1}{p}$  and  $\kappa > \frac{\lambda+1}{p}$ . Then,*

$$u(x, t) = C_{\nu+\kappa} \int_H \mathcal{D}_t^\nu u(y, s) \mathcal{D}_t^\kappa W^{(\alpha)}(x-y, t+s) s^{\nu+\kappa-1} dV(y, s) \quad (2.8)$$

holds for all  $u \in \mathbf{b}_\alpha^p(\lambda)$  and  $(x, t) \in H$ . Furthermore, (2.8) also holds for  $\kappa = \lambda + 1$  when  $p = 1$ .

In our later arguments, we use the following lemma frequently. By (1) of Lemma 2.3 and the following lemma, if  $1 < p \leq \infty$ , then we have  $\mathcal{D}_t^{\lambda+1} W^{(\alpha)}(x - \cdot, t + \cdot) \in L^p(\lambda)$  for each  $(x, t) \in H$ . Therefore, it follows that  $C_{\lambda+1} \mathcal{D}_t^{\lambda+1} W^{(\alpha)}(x - \cdot, t + \cdot)$  is the reproducing kernel for the Hilbert space  $\mathbf{b}_\alpha^2(\lambda)$ .

**Lemma 2.6** (Lemma 5 of [9]) *Let  $\theta, c \in \mathbb{R}$ . If  $\theta > -1$  and  $\frac{n}{2\alpha} + \theta + 1 - c < 0$ , then there exists a constant  $C = C(n, \alpha, \theta, c) > 0$  such that*

$$\int_H \frac{s^\theta}{(t+s+|x-y|^{2\alpha})^c} dV(y, s) = Ct^{\frac{n}{2\alpha} + \theta + 1 - c}$$

for all  $(x, t) \in H$ .

### 3. Integral operators induced by the fundamental solution

In this section, we define integral operators induced by the fundamental solution  $W^{(\alpha)}$  and investigate their properties. These investigations are useful for studying  $\alpha$ -parabolic conjugate functions of  $\mathbf{b}_\alpha^p(\lambda)$ -functions in Section 4.

First, we recall the definition of  $L^{(\alpha)}$ -harmonic functions. (For details, see Section 2 of [7].) We describe the operator  $(-\Delta_x)^\alpha$ . Since the case  $\alpha = 1$  is trivial, we only describe the case  $0 < \alpha < 1$ . Let  $C_c^\infty(H) \subset C(H)$  be the set of all infinitely differentiable functions on  $H$  with compact support. Then,  $(-\Delta_x)^\alpha$  is the convolution operator defined by

$$(-\Delta_x)^\alpha \psi(x, t) := -c_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y| > \delta} (\psi(x + y, t) - \psi(x, t)) |y|^{-n-2\alpha} dy \quad (3.1)$$

for all  $\psi \in C_c^\infty(H)$  and  $(x, t) \in H$ , where  $c_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma((n + 2\alpha)/2) / \Gamma(-\alpha) > 0$ . Let  $\tilde{L}^{(\alpha)} := -\partial t + (-\Delta_x)^\alpha$  be the adjoint operator of  $L^{(\alpha)}$ . Then, a function  $u \in C(H)$  is said to be  $L^{(\alpha)}$ -harmonic if  $u$  satisfies  $L^{(\alpha)}u = 0$  in the sense of distributions, that is,  $\int_H |u \tilde{L}^{(\alpha)} \psi| dV < \infty$  and  $\int_H u \tilde{L}^{(\alpha)} \psi dV = 0$  for all  $\psi \in C_c^\infty(H)$ . By (3.1) and the compactness of  $\text{supp}(\psi)$  (the support of  $\psi$ ), there exist  $0 < t_1 < t_2 < \infty$  and a constant  $C > 0$  such that

$$\begin{aligned} \text{supp}(\tilde{L}^{(\alpha)} \psi) \subset S = \mathbb{R}^n \times [t_1, t_2] \text{ and } |\tilde{L}^{(\alpha)} \psi(x, t)| \leq C(1 + |x|)^{-n-2\alpha} \\ \text{for } (x, t) \in S. \end{aligned} \quad (3.2)$$

Hence, the condition  $\int_H |u \tilde{L}^{(\alpha)} \psi| dV < \infty$  for all  $\psi \in C_c^\infty(H)$  is equivalent to the following: for any  $0 < t_1 < t_2 < \infty$ ,

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u(x, t)| (1 + |x|)^{-n-2\alpha} dx dt < \infty.$$

Next, we define integral operators induced by the fundamental solution  $W^{(\alpha)}$ . Let  $\gamma \in \mathbb{N}_0^n$  be a multi-index and  $\kappa, \rho \in \mathbb{R}$  with  $\kappa > -\frac{n}{2\alpha}$ . Then, we define the integral operator  $P_\alpha^{\gamma, \kappa, \rho}$  by

$$P_\alpha^{\gamma, \kappa, \rho} f(x, t) := \int_H f(y, s) \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x - y, t + s) s^\rho dV(y, s), \quad (3.3)$$

whenever the integral is well defined. Some properties of  $P_\alpha^{\gamma, \kappa, \rho}$  are given in the following theorem.

**Theorem 3.1** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ , and  $\sigma \in \mathbb{R}$ . Suppose that a multi-index  $\gamma \in \mathbb{N}_0^n$  and  $\kappa, \rho \in \mathbb{R}$  with  $\kappa > -\frac{n}{2\alpha}$  satisfy*

$$\sigma - \rho p < p - 1 < \left( \frac{|\gamma|}{2\alpha} + \kappa \right) p + \sigma - \rho p. \quad (3.4)$$

Then, for every  $f \in L^p(\sigma)$ , the following assertions hold.

- (1) *The function  $P_\alpha^{\gamma, \kappa, \rho} f(x, t)$  is well defined for every  $(x, t) \in H$  and there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|P_\alpha^{\gamma, \kappa, \rho} f\|_{L^p(\eta)} \leq C \|f\|_{L^p(\sigma)}, \quad (3.5)$$

where  $\eta = \left( \frac{|\gamma|}{2\alpha} + \kappa - \rho - 1 \right) p + \sigma$ . Moreover,  $P_\alpha^{\gamma, \kappa, \rho} f$  is  $L^{(\alpha)}$ -harmonic on  $H$ . Consequently,  $P_\alpha^{\gamma, \kappa, \rho} f \in \mathfrak{b}_\alpha^p(\eta)$ .

- (2) *Furthermore, let  $\beta \in \mathbb{N}_0^n$  be a multi-index and  $\nu \in \mathbb{R}$ . If  $\nu$  satisfies*

$$\nu + \kappa > -\frac{n}{2\alpha} \quad \text{and} \quad p - 1 < \left( \frac{|\gamma|}{2\alpha} + \nu + \kappa \right) p + \sigma - \rho p, \quad (3.6)$$

then the derivative  $\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f(x, t)$  is well defined for every  $(x, t) \in H$  and  $\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f = P_\alpha^{\beta+\gamma, \nu+\kappa, \rho} f$ , that is,

$$\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f(x, t) = \int_H f(y, s) \partial_x^{\beta+\gamma} \mathcal{D}_t^{\nu+\kappa} W^{(\alpha)}(x-y, t+s) s^\rho dV(y, s). \quad (3.7)$$

Consequently, put  $\eta = \left( \frac{|\beta|+|\gamma|}{2\alpha} + \nu + \kappa - \rho - 1 \right) p + \sigma$ , then there exists a constant  $C > 0$  independent of  $f$  such that

$$\|\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f\|_{L^p(\eta)} \leq C \|f\|_{L^p(\sigma)} \quad (3.8)$$

and  $\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f \in \mathfrak{b}_\alpha^p(\eta)$ .

*Proof.* Let  $f \in L^p(\sigma)$  and put

$$\Psi_{\alpha}^{\gamma, \kappa, \rho} f(x, t) := \int_H |f(y, s)| b_{\alpha}^{\gamma, \kappa}(x, t; y, s) s^{\rho} dV(y, s), \quad (3.9)$$

where  $b_{\alpha}^{\gamma, \kappa}(x, t; y, s) := (t + s + |x - y|^{2\alpha})^{-\left(\frac{n+|\gamma|}{2\alpha} + \kappa\right)}$ . We remark that  $|P_{\alpha}^{\gamma, \kappa, \rho} f(x, t)| \leq C \Psi_{\alpha}^{\gamma, \kappa, \rho} f(x, t)$  by (1) of Lemma 2.3. Suppose that  $p > 1$  and let  $q$  be the exponent conjugate to  $p$ .

(1) Put  $m_1 = -1$ ,  $M_1 = \frac{|\gamma|}{2\alpha} + \kappa - 1$ ,  $m_2 = -\left(\frac{|\gamma|}{2\alpha} + \kappa\right) \frac{1}{p-1} - \frac{\sigma - \rho p}{p-1}$ , and  $M_2 = -\frac{\sigma - \rho p}{p-1}$ . Then, (3.4) implies that  $m_1 < M_2$  and  $m_2 < M_1$ . Thus, there exists a real number  $\theta$  such that  $\theta \in (m_1, M_1) \cap (m_2, M_2)$ . Therefore, the Hölder inequality implies that

$$\begin{aligned} \Psi_{\alpha}^{\gamma, \kappa, \rho} f(x, t) &= \int_H |f(y, s)| s^{\rho - \frac{\theta}{q}} s^{\frac{\theta}{q}} b_{\alpha}^{\gamma, \kappa}(x, t; y, s) dV(y, s) \\ &\leq \left( \int_H |f(y, s)|^p s^{(\rho - \frac{\theta}{q})p} b_{\alpha}^{\gamma, \kappa}(x, t; y, s) dV(y, s) \right)^{1/p} \\ &\quad \times \left( \int_H s^{\theta} b_{\alpha}^{\gamma, \kappa}(x, t; y, s) dV(y, s) \right)^{1/q}. \end{aligned}$$

Since  $\theta \in (m_1, M_1)$ , Lemma 2.6 implies that

$$\begin{aligned} &\left\{ \Psi_{\alpha}^{\gamma, \kappa, \rho} f(x, t) \right\}^p \\ &\leq C t^{(\theta + 1 - \frac{|\gamma|}{2\alpha} - \kappa) \frac{p}{q}} \int_H |f(y, s)|^p s^{(\rho - \frac{\theta}{q})p} b_{\alpha}^{\gamma, \kappa}(x, t; y, s) dV(y, s). \quad (3.10) \end{aligned}$$

We show that the function  $P_{\alpha}^{\gamma, \kappa, \rho} f(x, t)$  is well defined for every  $(x, t) \in H$ . Since  $\theta \in (m_2, M_2)$ , we have

$$\sigma - \left( \rho - \frac{\theta}{q} \right) p = \sigma - \rho p + (p-1)\theta = (p-1) \left( \frac{\sigma - \rho p}{p-1} + \theta \right) < 0$$

and

$$-\left( \frac{|\gamma|}{2\alpha} + \kappa \right) - \left\{ \sigma - \left( \rho - \frac{\theta}{q} \right) p \right\} = -\left( \frac{|\gamma|}{2\alpha} + \kappa \right) - (p-1) \left( \frac{\sigma - \rho p}{p-1} + \theta \right) < 0.$$

It follows from above inequalities that

$$\begin{aligned} b_{\alpha}^{\gamma, \kappa}(x, t; y, s) &\leq C(t+s)^{-\left(\frac{n+|\gamma|}{2\alpha}+\kappa\right)-\left\{\sigma-\left(\rho-\frac{\theta}{q}\right)p\right\}+\sigma-\left(\rho-\frac{\theta}{q}\right)p} \\ &\leq Ct^{-\left(\frac{n+|\gamma|}{2\alpha}+\kappa\right)-\left\{\sigma-\left(\rho-\frac{\theta}{q}\right)p\right\}}s^{\sigma-\left(\rho-\frac{\theta}{q}\right)p}. \end{aligned}$$

Therefore, (3.10) implies that

$$\begin{aligned} \left\{\Psi_{\alpha}^{\gamma, \kappa, \rho} f(x, t)\right\}^p &\leq Ct^{\left(\theta+1-\frac{|\gamma|}{2\alpha}-\kappa\right)\frac{p}{q}-\left(\frac{n+|\gamma|}{2\alpha}+\kappa\right)-\left\{\sigma-\left(\rho-\frac{\theta}{q}\right)p\right\}}\|f\|_{L^p(\sigma)}^p \\ &\leq Ct^{-\left(\frac{n}{2\alpha}+\sigma+1\right)-\left(\frac{|\gamma|}{2\alpha}+\kappa-\rho-1\right)p}\|f\|_{L^p(\sigma)}^p \end{aligned} \quad (3.11)$$

for all  $(x, t) \in H$ . Thus,  $P_{\alpha}^{\gamma, \kappa, \rho} f(x, t)$  is well defined for every  $(x, t) \in H$ . We show the inequality (3.5). By (3.10) and the Fubini theorem, we have

$$\begin{aligned} &\int_H \left\{\Psi_{\alpha}^{\gamma, \kappa, \rho} f(x, t)\right\}^p t^{\eta} dV(x, t) \\ &\leq C \int_H |f(y, s)|^p s^{\left(\rho-\frac{\theta}{q}\right)p} \int_H t^{\left(\theta+1-\frac{|\gamma|}{2\alpha}-\kappa\right)\frac{p}{q}+\eta} b_{\alpha}^{\gamma, \kappa}(x, t; y, s) dV(x, t) dV(y, s). \end{aligned}$$

Since  $\theta \in (m_2, M_2)$ , we also have

$$\left(\theta+1-\frac{|\gamma|}{2\alpha}-\kappa\right)\frac{p}{q}+\eta=(p-1)\theta+\frac{|\gamma|}{2\alpha}+\kappa+\sigma-\rho p-1>-1$$

and

$$\left(\theta+1-\frac{|\gamma|}{2\alpha}-\kappa\right)\frac{p}{q}+\eta+1-\frac{|\gamma|}{2\alpha}-\kappa=(p-1)\theta+\sigma-\rho p<0.$$

Therefore, Lemma 2.6 implies that  $\|P_{\alpha}^{\gamma, \kappa, \rho} f\|_{L^p(\eta)} \leq C\|\Psi_{\alpha}^{\gamma, \kappa, \rho} f\|_{L^p(\eta)} \leq C\|f\|_{L^p(\sigma)}$ .

We show that  $P_{\alpha}^{\gamma, \kappa, \rho} f$  is  $L^{(\alpha)}$ -harmonic on  $H$ . First, we claim that

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Psi_{\alpha}^{\gamma, \kappa, \rho} f(x, t)(1+|x|)^{-n-2\alpha} dx dt < \infty \quad (3.12)$$

for all  $0 < t_1 < t_2 < \infty$ . In fact, the Hölder inequality implies that

$$\begin{aligned}
 & \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Psi_{\alpha}^{\gamma, \kappa, \rho} f(x, t) (1 + |x|)^{-n-2\alpha} dx dt \\
 & \leq \left( \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \{ \Psi_{\alpha}^{\gamma, \kappa, \rho} f(x, t) \}^p t^n dx dt \right)^{1/p} \left( \int_{t_1}^{t_2} \int_{\mathbb{R}^n} t^{-\frac{nq}{p}} (1 + |x|)^{-n-2\alpha} dx dt \right)^{1/q} \\
 & \leq C \|f\|_{L^p(\sigma)} \left( \int_{\mathbb{R}^n} (1 + |x|)^{-n-2\alpha} dx \right)^{1/q} < \infty.
 \end{aligned}$$

Thus, (3.12) is obtained. Since  $\partial_x^{\gamma} \mathcal{D}_t^{\kappa} W^{(\alpha)}(x-y, t+s)$  is  $L^{(\alpha)}$ -harmonic with respect to  $(x, t)$ , the Fubini theorem implies that  $P_{\alpha}^{\gamma, \kappa, \rho} f$  is  $L^{(\alpha)}$ -harmonic on  $H$ .

(2) Suppose that a real number  $\nu$  satisfies the conditions of (3.6). First, we show that  $\mathcal{D}_t^{\nu} P_{\alpha}^{\gamma, \kappa, \rho} f(x, t) = P_{\alpha}^{\gamma, \nu+\kappa, \rho} f(x, t)$ , that is,

$$\mathcal{D}_t^{\nu} P_{\alpha}^{\gamma, \kappa, \rho} f(x, t) = \int_H f(y, s) \partial_x^{\gamma} \mathcal{D}_t^{\nu+\kappa} W^{(\alpha)}(x-y, t+s) s^{\rho} dV(y, s). \quad (3.13)$$

Let  $\nu$  be a nonnegative integer. Then, as in the proof of (3.11), we have

$$\Psi_{\alpha}^{\gamma, \nu+\kappa, \rho} f(x, t) \leq C t^{-\left(\frac{n}{2\alpha} + \sigma + 1\right) \frac{1}{p} - \left(\frac{|\gamma|}{2\alpha} + \nu + \kappa - \rho - 1\right)} \|f\|_{L^p(\sigma)} \quad (3.14)$$

for all  $(x, t) \in H$ . Thus, we can differentiate through the integral (3.3) with respect to  $t$ . Therefore, we obtain  $\mathcal{D}_t^{\nu} P_{\alpha}^{\gamma, \kappa, \rho} f(x, t) = P_{\alpha}^{\gamma, \nu+\kappa, \rho} f(x, t)$ . Let  $\nu$  be a real number. Put

$$\omega(\nu) := \begin{cases} \lceil \nu \rceil & \nu \geq 0 \\ 0 & \nu < 0. \end{cases}$$

We claim that

$$\int_0^{\infty} \tau^{\omega(\nu) - \nu - 1} \Psi_{\alpha}^{\gamma, \omega(\nu) + \kappa, \rho} f(x, \tau + t) d\tau < \infty \quad (3.15)$$

for all  $(x, t) \in H$ . Indeed, the second condition of (3.6) implies that

$$\begin{aligned} & \left( \frac{n}{2\alpha} + \sigma + 1 \right) \frac{1}{p} + \left( \frac{|\gamma|}{2\alpha} + \omega(\nu) + \kappa - \rho - 1 \right) \\ & > \omega(\nu) - \nu + \left\{ \frac{n}{2\alpha} + \left( \frac{|\gamma|}{2\alpha} + \nu + \kappa \right) p + \sigma - \rho p - p + 1 \right\} \frac{1}{p} > \omega(\nu) - \nu. \end{aligned}$$

Therefore, by (3.14), we have  $\Psi_\alpha^{\gamma, \omega(\nu) + \kappa, \rho} f(x, \cdot) \in \mathcal{FC}^{-(\omega(\nu) - \nu)}$  for every  $x \in \mathbb{R}^n$ , so that (3.15) is obtained. Hence, the Fubini theorem and (2) of Lemma 2.3 show that

$$\begin{aligned} \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f(x, t) &= \mathcal{D}_t^{-(\omega(\nu) - \nu)} \mathcal{D}_t^{\omega(\nu)} P_\alpha^{\gamma, \kappa, \rho} f(x, t) \\ &= \mathcal{D}_t^{-(\omega(\nu) - \nu)} P_\alpha^{\gamma, \omega(\nu) + \kappa, \rho} f(x, t) \\ &= \frac{1}{\Gamma(\omega(\nu) - \nu)} \int_0^\infty \tau^{\omega(\nu) - \nu - 1} \\ & \quad \cdot \int_H f(y, s) \partial_x^\gamma \mathcal{D}_t^{\omega(\nu) + \kappa} W^{(\alpha)}(x - y, \tau + t + s) s^\rho dV(y, s) d\tau \\ &= \int_H f(y, s) \frac{1}{\Gamma(\omega(\nu) - \nu)} \\ & \quad \cdot \int_0^\infty \tau^{\omega(\nu) - \nu - 1} \partial_x^\gamma \mathcal{D}_t^{\omega(\nu) + \kappa} W^{(\alpha)}(x - y, \tau + t + s) d\tau s^\rho dV(y, s) \\ &= \int_H f(y, s) \mathcal{D}_t^{-(\omega(\nu) - \nu)} \partial_x^\gamma \mathcal{D}_t^{\omega(\nu) + \kappa} W^{(\alpha)}(x - y, t + s) s^\rho dV(y, s) \\ &= \int_H f(y, s) \partial_x^\gamma \mathcal{D}_t^{\nu + \kappa} W^{(\alpha)}(x - y, t + s) s^\rho dV(y, s) = P_\alpha^{\gamma, \nu + \kappa, \rho} f(x, t). \end{aligned}$$

Let  $\beta \in \mathbb{N}_0^n$  be a multi-index and  $\nu \in \mathbb{R}$ . As in the proof of (3.11), (3.4) and the second condition of (3.6) imply that

$$\Psi_\alpha^{\beta + \gamma, \nu + \kappa, \rho} f(x, t) \leq C t^{-\left(\frac{n}{2\alpha} + \sigma + 1\right) \frac{1}{p} - \left(\frac{|\beta| + |\gamma|}{2\alpha} + \nu + \kappa - \rho - 1\right)} \|f\|_{L^p(\sigma)}$$

for all  $(x, t) \in H$ . Therefore, we can differentiate through the integral (3.13) with respect to  $x$ . Hence, we obtain  $\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f(x, t) = \partial_x^\beta P_\alpha^{\gamma, \nu + \kappa, \rho} f(x, t) = P_\alpha^{\beta + \gamma, \nu + \kappa, \rho} f(x, t)$ .

Since the proofs of (1) and (2) for the case  $p = 1$  are easier, we omit the proofs.  $\square$

We have the following corollary.

**Corollary 3.2** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ , and  $\lambda > -1$ . Then, the following assertions hold.*

- (1) *If a real number  $\kappa$  satisfies  $\kappa > \frac{\lambda+1}{p}$ , then the operator  $C_\kappa P_\alpha^{0,\kappa,\kappa-1}$  is a bounded projection from  $L^p(\lambda)$  onto  $\mathbf{b}_\alpha^p(\lambda)$ .*
- (2) *Let  $1 < p < \infty$  and  $q$  be the exponent conjugate to  $p$ . Then,  $(\mathbf{b}_\alpha^p(\lambda))^* \cong \mathbf{b}_\alpha^q(\lambda)$  under the pairing*

$$\langle u, v \rangle = \int_H u(x, t)v(x, t)t^\lambda dV(x, t), \quad u \in \mathbf{b}_\alpha^p(\lambda), \quad v \in \mathbf{b}_\alpha^q(\lambda).$$

- (3) *For a real number  $\nu > -\frac{\lambda+1}{p}$ , there exists a constant  $C = C(n, p, \alpha, \lambda, \nu) > 0$  such that*

$$\begin{aligned} C^{-1}\|u\|_{L^p(\lambda)} &\leq \|t^\nu \mathcal{D}_t^\nu u\|_{L^p(\lambda)} \leq \sum_{|\gamma| < \nu + \frac{\lambda+1}{p}} \|t^{\frac{|\gamma|}{2\alpha} + \nu - |\gamma|} \partial_x^\gamma \mathcal{D}_t^{\nu - |\gamma|} u\|_{L^p(\lambda)} \\ &\leq C\|u\|_{L^p(\lambda)} \end{aligned}$$

for all  $u \in \mathbf{b}_\alpha^p(\lambda)$ , where  $\gamma \in \mathbb{N}_0^n$  denotes a multi-index.

*Proof.* (1) Let  $\kappa$  be a real number such that  $\kappa > \frac{\lambda+1}{p}$ . Then, (1) of Theorem 3.1 implies that  $P_\alpha^{0,\kappa,\kappa-1} f \in \mathbf{b}_\alpha^p(\lambda)$  and  $\|P_\alpha^{0,\kappa,\kappa-1} f\|_{L^p(\lambda)} \leq C\|f\|_{L^p(\lambda)}$  for all  $f \in L^p(\lambda)$ . Also, by Lemma 2.5, we have  $C_\kappa P_\alpha^{0,\kappa,\kappa-1} u = u$  for all  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Therefore, the operator  $C_\kappa P_\alpha^{0,\kappa,\kappa-1}$  is a bounded projection from  $L^p(\lambda)$  onto  $\mathbf{b}_\alpha^p(\lambda)$ .

(2) Since  $\mathcal{D}_t^{\lambda+1} W^{(\alpha)}$  is symmetric, the Hahn-Banach theorem and (1) of Corollary 3.2 with  $\kappa = \lambda + 1$  imply that  $(\mathbf{b}_\alpha^p(\lambda))^* \cong \mathbf{b}_\alpha^q(\lambda)$ . (The proof is similar to that of Theorem 8.1 of [7].)

(3) Let  $\nu$  be a real number such that  $\nu > -\frac{\lambda+1}{p}$ . First, we show the last inequality of (3). It suffices to show that for every  $\gamma \in \mathbb{N}_0^n$  with  $|\gamma| < \nu + \frac{\lambda+1}{p}$ , there exists a constant  $C > 0$  such that

$$\|t^{\frac{|\gamma|}{2\alpha} + \nu - |\gamma|} \partial_x^\gamma \mathcal{D}_t^{\nu - |\gamma|} u\|_{L^p(\lambda)} \leq C\|u\|_{L^p(\lambda)}.$$

By Lemma 2.5, we have

$$\begin{aligned} u(x, t) &= C_{\lambda+2} \int_H u(y, s) \mathcal{D}_t^{\lambda+2} W^{(\alpha)}(x-y, t+s) s^{\lambda+1} dV(y, s) \\ &= C_{\lambda+2} P_\alpha^{0, \lambda+2, \lambda+1} u(x, t) \end{aligned}$$

for all  $u \in \mathbf{b}_\alpha^p(\lambda)$  and  $(x, t) \in H$ . Since  $\nu > -\frac{\lambda+1}{p}$ , (2) of Theorem 3.1 implies that

$$\begin{aligned} \partial_x^\gamma \mathcal{D}_t^{\nu-|\gamma|} u(x, t) &= C_{\lambda+2} \partial_x^\gamma \mathcal{D}_t^{\nu-|\gamma|} P_\alpha^{0, \lambda+2, \lambda+1} u(x, t) \\ &= C_{\lambda+2} P_\alpha^{\gamma, \nu-|\gamma|+\lambda+2, \lambda+1} u(x, t), \end{aligned}$$

and there exists a constant  $C > 0$  such that

$$\|\partial_x^\gamma \mathcal{D}_t^{\nu-|\gamma|} u\|_{L^p(\eta)} = C_{\lambda+2} \|P_\alpha^{\gamma, \nu-|\gamma|+\lambda+2, \lambda+1} u\|_{L^p(\eta)} \leq C \|u\|_{L^p(\lambda)},$$

where  $\eta = (\frac{|\gamma|}{2\alpha} + \nu - |\gamma|)p + \lambda$ . Therefore, the last inequality of (3) is obtained.

Since the second inequality of (3) is trivial, we show the first inequality of (3). By the last inequality of (3),  $f = t^\nu \mathcal{D}_t^\nu u$  belongs to  $L^p(\lambda)$ . Therefore, by Lemma 2.5, we have

$$\begin{aligned} u(x, t) &= C_{\nu+\lambda+2} \int_H s^\nu \mathcal{D}_t^\nu u(y, s) \mathcal{D}_t^{\lambda+2} W^{(\alpha)}(x-y, t+s) s^{\lambda+1} dV(y, s) \\ &= C_{\nu+\lambda+2} (P_\alpha^{0, \lambda+2, \lambda+1} (t^\nu \mathcal{D}_t^\nu u))(x, t) \end{aligned}$$

for all  $u \in \mathbf{b}_\alpha^p(\lambda)$  and  $(x, t) \in H$ . Hence, (1) of Theorem 3.1 implies that

$$\|u\|_{L^p(\lambda)} = C_{\nu+\lambda+2} \|P_\alpha^{0, \lambda+2, \lambda+1} (t^\nu \mathcal{D}_t^\nu u)\|_{L^p(\lambda)} \leq C \|t^\nu \mathcal{D}_t^\nu u\|_{L^p(\lambda)}.$$

Therefore, the first inequality of (3) is obtained.  $\square$

#### 4. Uniqueness of $\alpha$ -parabolic conjugate functions

In this section, we give the proof of Theorem 1, and also give a decomposition theorem for  $\alpha$ -parabolic conjugate functions. First, we need the following lemma.

**Lemma 4.1** *Let  $0 < \alpha \leq 1$ . Then,*

$$\left( \mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x \right) W^{(\alpha)}(x, t) = 0$$

for all  $(x, t) \in H$ .

*Proof.* Differentiating through the integral (2.5) with respect to  $x$ , we have

$$\Delta_x W^{(\alpha)}(x, t) = \frac{-1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^2 \exp(-t|\xi|^{2\alpha} + i x \cdot \xi) d\xi.$$

Also, since

$$\int_0^\infty \int_{\mathbb{R}^n} \tau^{\lceil \frac{1}{\alpha} \rceil - \frac{1}{\alpha} - 1} |\xi|^{2\alpha \lceil \frac{1}{\alpha} \rceil} \exp(-(\tau + t)|\xi|^{2\alpha}) d\xi d\tau < \infty,$$

differentiating through the integral (2.5) with respect to  $t$ , the Fubini theorem and (2) of Example 2.2 imply that

$$\begin{aligned} \mathcal{D}_t^{\frac{1}{\alpha}} W^{(\alpha)}(x, t) &= \frac{1}{(2\pi)^n} \frac{1}{\Gamma(\lceil \frac{1}{\alpha} \rceil - \frac{1}{\alpha})} \int_0^\infty \tau^{\lceil \frac{1}{\alpha} \rceil - \frac{1}{\alpha} - 1} \\ &\quad \cdot \int_{\mathbb{R}^n} \mathcal{D}_t^{\lceil \frac{1}{\alpha} \rceil} \exp(- (t + \tau)|\xi|^{2\alpha} + i x \cdot \xi) d\xi d\tau \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{\Gamma(\lceil \frac{1}{\alpha} \rceil - \frac{1}{\alpha})} \\ &\quad \cdot \int_0^\infty \tau^{\lceil \frac{1}{\alpha} \rceil - \frac{1}{\alpha} - 1} \mathcal{D}_t^{\lceil \frac{1}{\alpha} \rceil} \exp(- (t + \tau)|\xi|^{2\alpha} + i x \cdot \xi) d\tau d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{D}_t^{\frac{1}{\alpha}} \exp(-t|\xi|^{2\alpha} + i x \cdot \xi) d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^2 \exp(-t|\xi|^{2\alpha} + i x \cdot \xi) d\xi. \end{aligned}$$

Hence, this completes the proof.  $\square$

Now, we give the proof of Theorem 1.

*Proof of Theorem 1.* We put  $\eta = p(\frac{1}{2\alpha} - 1) + \lambda$ , and suppose that  $\eta > -1$ .

Then, for  $u \in \mathbf{b}_\alpha^p(\lambda)$ , we define a vector-valued function  $V$  on  $H$  by

$$V(x, t) := -C_{\lambda+2} \int_H u(y, s) \nabla_x \mathcal{D}_t^{\lambda+1} W^{(\alpha)}(x - y, t + s) s^{\lambda+1} dV(y, s), \quad (4.1)$$

that is,  $v_j(x, t) = -C_{\lambda+2} P_\alpha^{\gamma(j), \lambda+1, \lambda+1} u(x, t)$  for each  $1 \leq j \leq n$  and  $V = (v_1, \dots, v_n)$ , where  $\gamma(j) = (\delta_{j1}, \dots, \delta_{jn}) \in \mathbb{N}_0^n$  and  $\delta_{j\ell}$  is the Kronecker  $\delta$ . By (1) of Theorem 3.1, each  $v_j$  is  $L^{(\alpha)}$ -harmonic on  $H$ , and there exists a constant  $C > 0$  such that

$$\|v_j\|_{L^p(\eta)} = C_{\lambda+2} \|P_\alpha^{\gamma(j), \lambda+1, \lambda+1} u\|_{L^p(\eta)} \leq C \|u\|_{L^p(\lambda)}, \quad (4.2)$$

that is,  $|V| \in L^p(\eta)$ .

We show that the functions  $u$  and  $V$  satisfy the equations (C.1) and (C.2). By Lemma 2.5, (2) of Theorem 3.1, and (4.1), we have

$$\begin{aligned} \nabla_x u &= C_{\lambda+2} \nabla_x P_\alpha^{0, \lambda+2, \lambda+1} u \\ &= C_{\lambda+2} (P_\alpha^{\gamma(1), \lambda+2, \lambda+1} u, \dots, P_\alpha^{\gamma(n), \lambda+2, \lambda+1} u) = -\mathcal{D}_t V \end{aligned}$$

and

$$\nabla_x v_j = -C_{\lambda+2} \nabla_x P_\alpha^{\gamma(j), \lambda+1, \lambda+1} u = \partial_j V$$

for all  $1 \leq j \leq n$ . Furthermore, by Lemma 2.5, (2) of Theorem 3.1, and (2) of Lemma 2.3, we have

$$\begin{aligned} &\mathcal{D}_t^{\frac{1}{\alpha}-1} u(x, t) - \nabla_x \cdot V(x, t) \\ &= C_{\lambda+2} P_\alpha^{0, \lambda+1+\frac{1}{\alpha}, \lambda+1} u(x, t) + C_{\lambda+2} \sum_{j=1}^n P_\alpha^{2\gamma(j), \lambda+1, \lambda+1} u(x, t) \\ &= C_{\lambda+2} \int_H u(y, s) \mathcal{D}_t^{\lambda+1} \left( \mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x \right) W^{(\alpha)}(x - y, t + s) s^{\lambda+1} dV(y, s). \end{aligned}$$

Therefore, by Lemma 4.1, the functions  $u$  and  $V$  satisfy the equations (C.1) and (C.2).

Since the second inequality of (1.3) has already obtained by (4.2), we show the first inequality of (1.3). By the first inequality of (3) of Corollary

3.2 and the equation (C.2), we have

$$\|u\|_{L^p(\lambda)} \leq C \|t^{\frac{1}{\alpha}-1} \mathcal{D}_t^{\frac{1}{\alpha}-1} u\|_{L^p(\lambda)} \leq C \sum_{j=1}^n \|t^{\frac{1}{\alpha}-1} \partial_j v_j\|_{L^p(\lambda)}.$$

Since  $v_j$  belongs to  $\mathbf{b}_\alpha^p(\eta)$ , the last inequality of (3) of Corollary 3.2 with  $\nu = 1 > -\frac{\eta+1}{p}$  implies that

$$\|t^{\frac{1}{\alpha}-1} \partial_j v_j\|_{L^p(\lambda)} = \|t^{\frac{1}{2\alpha}} \partial_j v_j\|_{L^p(\eta)} \leq C \|v_j\|_{L^p(\eta)}.$$

Hence, we obtain the inequalities of (1.3).

We suppose that  $U = (u_1, \dots, u_n)$  is an  $\alpha$ -parabolic conjugate function of  $u$  such that  $u_j \in \mathbf{b}_\alpha^p(\eta)$ . Then, we will show  $U = V$ . In fact, for each  $1 \leq j \leq n$ , by the first inequality of (3) of Corollary 3.2 and the equation (C.1), we have

$$\|v_j - u_j\|_{L^p(\eta)} \leq C \|t \mathcal{D}_t(v_j - u_j)\|_{L^p(\eta)} = C \|t \partial_j(u - u)\|_{L^p(\eta)} = 0.$$

Since  $u_j$  and  $v_j$  are continuous on  $H$ , we obtain  $u_j(x, t) = v_j(x, t)$  for all  $(x, t) \in H$ . Hence, this completes the proof of Theorem 1.  $\square$

By Theorem 1, we can extend Theorem A to the case  $\eta = p(\frac{1}{2\alpha}-1)+\lambda > -1$ . Here, as a remark, we show that  $\mathbf{b}_\alpha^p(\lambda) = \{0\}$  whenever  $\lambda \leq -1$ .

**Lemma 4.2** *Let  $0 < \alpha \leq 1$  and  $1 \leq p < \infty$ . If a function  $u$  is  $L^{(\alpha)}$ -harmonic on  $H$  and  $\int_S |u|^p dV < \infty$  for each closed strip  $S = \mathbb{R}^n \times [t_1, t_2] \subset H$ , then the function  $I_\alpha^p(t) = \int_{\mathbb{R}^n} |u(x, t)|^p dx$  is decreasing on  $(0, \infty)$ .*

*Proof.* By the proof of Theorem 4.1 of [7], if an  $L^{(\alpha)}$ -harmonic function  $u$  satisfies  $\int_S |u|^p dV < \infty$  for each closed strip  $S = \mathbb{R}^n \times [t_1, t_2] \subset H$ , then  $u$  holds the Huygens property, that is,

$$u(x, s+t) = \int_{\mathbb{R}^n} u(y, s) W^{(\alpha)}(x-y, t) dy$$

for all  $x \in \mathbb{R}^n$ ,  $0 < s < \infty$  and  $0 < t < \infty$ . Since  $W^{(\alpha)}(x-y, t) \geq 0$  and  $\int_{\mathbb{R}^n} W^{(\alpha)}(x-y, t) dy = 1$ , the Jensen inequality and the Fubini theorem imply that

$$\int_{\mathbb{R}^n} |u(x, s+t)|^p dx \leq \int_{\mathbb{R}^n} |u(y, s)|^p \int_{\mathbb{R}^n} W^{(\alpha)}(x-y, t) dx dy = \int_{\mathbb{R}^n} |u(y, s)|^p dy.$$

This completes the proof  $\square$

**Proposition 4.3** *Let  $0 < \alpha \leq 1$  and  $1 \leq p < \infty$ . If  $\lambda \leq -1$ , then  $\mathbf{b}_\alpha^p(\lambda) = \{0\}$ .*

*Proof.* Let  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Then,  $u$  belongs to  $L^p(\lambda)$ , and it follows that  $\int_S |u|^p dV < \infty$  for each closed strip  $S = \mathbb{R}^n \times [t_1, t_2] \subset H$ . Thus, Lemma 4.2 implies that

$$\infty > \|u\|_{L^p(\lambda)}^p \geq \int_0^s t^\lambda \int_{\mathbb{R}^n} |u(x, t)|^p dx dt \geq I_u^p(s) \int_0^s t^\lambda dt$$

for all  $0 < s < \infty$ . Hence, we have  $I_u^p(s) = 0$  for all  $0 < s < \infty$ , because  $\lambda \leq -1$ . Since  $u$  is continuous on  $H$ , we obtain  $u(x, t) = 0$  for all  $(x, t) \in H$ .  $\square$

We can give a decomposition theorem for  $\alpha$ -parabolic conjugate functions. We begin with showing the following lemma. We can not prove whether every  $u \in \mathbf{b}_\alpha^p(\eta)$  satisfies the equation  $\mathcal{D}_t^{-1} \mathcal{D}_t u = u$ . However, the following lemma holds.

**Lemma 4.4** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ , and  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Suppose  $\alpha$ ,  $p$ , and  $\lambda$  satisfy the condition  $\eta = p(\frac{1}{2\alpha} - 1) + \lambda > -1$ . Then, for every  $\alpha$ -parabolic conjugate function  $U = (u_1, \dots, u_n)$  of  $u$ , the function  $\mathcal{D}_t^{-1} \mathcal{D}_t u_j$  on  $H$  is well defined and belongs to  $\mathbf{b}_\alpha^p(\eta)$  for all  $1 \leq j \leq n$ .*

*Proof.* Let  $U = (u_1, \dots, u_n)$  be an  $\alpha$ -parabolic conjugate function of  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Then, by the equation (C.1) and (1) of Lemma 2.4, there exists a constant  $C > 0$  such that  $|\mathcal{D}_t u_j(x, t)| = |\partial_j u(x, t)| \leq C t^{-\frac{1}{2\alpha} - (\frac{n}{2\alpha} + \lambda + 1)\frac{1}{p}}$  for all  $(x, t) \in H$ . Therefore, the hypothesis  $\eta = p(\frac{1}{2\alpha} - 1) + \lambda > -1$  implies that  $\mathcal{D}_t u_j(x, \cdot) \in \mathcal{FC}^{-1}$  for every  $x \in \mathbb{R}^n$ . Thus, we can define a function  $\mathcal{D}_t^{-1} \mathcal{D}_t u_j$  on  $H$ .

We show that  $\mathcal{D}_t^{-1} \mathcal{D}_t u_j \in \mathbf{b}_\alpha^p(\eta)$ . By (3) of Lemma 2.4 and (3) of Corollary 3.2, the derivative  $\mathcal{D}_t u_j = \partial_j u$  is  $L^{(\alpha)}$ -harmonic and

$$\|\mathcal{D}_t u_j\|_{L^p(\sigma)} = \|t^{\frac{1}{2\alpha}} \mathcal{D}_t u_j\|_{L^p(\lambda)} = \|t^{\frac{1}{2\alpha}} \partial_j u\|_{L^p(\lambda)} \leq C \|u\|_{L^p(\lambda)} < \infty,$$

where  $\sigma = \frac{p}{2\alpha} + \lambda$ . Thus, we obtain  $\mathcal{D}_t u_j \in \mathbf{b}_\alpha^p(\sigma)$ . Again, by (3) of Lemma 2.4 and (3) of Corollary 3.2, the function  $\mathcal{D}_t^{-1} \mathcal{D}_t u_j = \mathcal{D}_t^{-1}(\mathcal{D}_t u_j)$  is also  $L^{(\alpha)}$ -harmonic and

$$\|\mathcal{D}_t^{-1}(\mathcal{D}_t u_j)\|_{L^p(\eta)} = \|t^{-1} \mathcal{D}_t^{-1}(\mathcal{D}_t u_j)\|_{L^p(\sigma)} \leq C \|\mathcal{D}_t u_j\|_{L^p(\sigma)} < \infty.$$

Hence, we get  $\mathcal{D}_t^{-1} \mathcal{D}_t u_j \in \mathbf{b}_\alpha^p(\eta)$ . □

**Theorem 4.5** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ , and  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Suppose  $\alpha$ ,  $p$ , and  $\lambda$  satisfy the condition  $\eta = p(\frac{1}{2\alpha} - 1) + \lambda > -1$ . Then, every  $\alpha$ -parabolic conjugate function  $U = (u_1, \dots, u_n)$  of  $u$  can be uniquely expressed in the form*

$$U(x, t) = V(x, t) + F(x), \quad (x, t) \in H, \quad (4.3)$$

where  $V = (v_1, \dots, v_n)$  is the unique  $\alpha$ -parabolic conjugate function of  $u$  with  $v_j \in \mathbf{b}_\alpha^p(\eta)$  in Theorem 1 and  $F = (f_1, \dots, f_n)$  is an  $n$ -tuple of harmonic functions on  $\mathbb{R}^n$  with  $\partial_k f_j = \partial_j f_k$ ,  $1 \leq j, k \leq n$  and  $\sum_{j=1}^n \partial_j f_j = 0$  (that is,  $F = (f_1, \dots, f_n)$  is a system of conjugate harmonic functions on  $\mathbb{R}^n$ , consequently each  $u_j$  belongs to  $C^\infty(H)$ ). Conversely, every function  $U$  of the form (4.3) is an  $\alpha$ -parabolic conjugate function of  $u$ .

*Proof.* Let  $U = (u_1, \dots, u_n)$  be an  $\alpha$ -parabolic conjugate function of  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Then, by Lemma 4.4, we can define a function  $v_j \in \mathbf{b}_\alpha^p(\eta)$  by

$$v_j(x, t) := \mathcal{D}_t^{-1} \mathcal{D}_t u_j(x, t) = \int_0^\infty \mathcal{D}_t u_j(x, \tau + t) d\tau, \quad (x, t) \in H. \quad (4.4)$$

Since  $u_j \in C^1(H)$  and the infinite integral (4.4) converges for every  $(x, t) \in H$ , the limit function  $f_j(x) := \lim_{\tau \rightarrow \infty} u_j(x, \tau)$  exists for every  $x \in \mathbb{R}^n$ , and so we have

$$v_j(x, t) = u_j(x, t) - f_j(x), \quad (x, t) \in H. \quad (4.5)$$

We show that  $F = (f_1, \dots, f_n)$  is a system of conjugate harmonic functions on  $\mathbb{R}^n$ . Let  $x \in \mathbb{R}^n$  be fixed. By (4.5), we have

$$\partial_k f_j(x) = \partial_k u_j(x, \tau) - \partial_k v_j(x, \tau)$$

for all  $\tau \in \mathbb{R}_+$  and  $f_j$  belongs to  $C^1(\mathbb{R}^n)$ . Since  $v_j \in \mathbf{b}_\alpha^p(\eta)$ , (1) of Lemma 2.4 implies that

$$\partial_k f_j(x) = \lim_{\tau \rightarrow \infty} (\partial_k u_j(x, \tau) - \partial_k v_j(x, \tau)) = \lim_{\tau \rightarrow \infty} \partial_k u_j(x, \tau). \quad (4.6)$$

Hence, (4.6) and the equation (C.1) show that

$$\partial_k f_j(x) = \lim_{\tau \rightarrow \infty} \partial_k u_j(x, \tau) = \lim_{\tau \rightarrow \infty} \partial_j u_k(x, \tau) = \partial_j f_k(x).$$

Also, (4.6), the equation (C.2), and (1) of Lemma 2.4 imply that

$$\begin{aligned} \sum_{j=1}^n \partial_j f_j(x) &= \sum_{j=1}^n \left( \lim_{\tau \rightarrow \infty} \partial_j u_j(x, \tau) \right) = \lim_{\tau \rightarrow \infty} \sum_{j=1}^n \partial_j u_j(x, \tau) \\ &= \lim_{\tau \rightarrow \infty} \mathcal{D}_t^{\frac{1}{\alpha}-1} u(x, \tau) = 0. \end{aligned}$$

Therefore,  $F = (f_1, \dots, f_n)$  is a system of conjugate harmonic functions on  $\mathbb{R}^n$ , thus  $f_j$  is harmonic on  $\mathbb{R}^n$  and  $f_j \in C^\infty(\mathbb{R}^n)$ .

Since we see that  $F = (f_1, \dots, f_n)$  is a system of conjugate harmonic functions on  $\mathbb{R}^n$ , we clearly have  $V = (v_1, \dots, v_n)$  is an  $\alpha$ -parabolic conjugate function of  $u$ . Hence, every  $\alpha$ -parabolic conjugate function of  $u$  is expressed in the form (4.3) and the expression is unique by Theorem 1. Conversely, it is obvious that every function  $U$  of the form (4.3) is an  $\alpha$ -parabolic conjugate function of  $u$ . This completes the proof of theorem.  $\square$

## 5. Estimates of tangential derivative norms

In this section, we estimate tangential derivative norms of  $\mathbf{b}_\alpha^p(\lambda)$ -functions and give the proof of Theorem 2. By Lemma 4.1, we also have the following lemma.

**Lemma 5.1** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ , and  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Then,*

$$\left( \mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x \right) u(x, t) = 0$$

for all  $(x, t) \in H$ .

*Proof.* By Lemma 2.5 with  $\nu = 0$  and  $\kappa = \lambda + 2$ , we have  $u(x, t) =$

$C_{\lambda+2}P_\alpha^{0,\lambda+2,\lambda+1}u(x,t)$ . Therefore, (2) of Theorem 3.1 implies that

$$\begin{aligned} & \left(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x\right)u(x,t) \\ &= C_{\lambda+2} \int_H u(y,s) \mathcal{D}_t^{\lambda+2} \left(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x\right) W^{(\alpha)}(x-y,t+s) s^{\lambda+1} dV(y,s). \end{aligned}$$

Hence, Lemma 5.1 follows from Lemma 4.1.  $\square$

As an application of Theorem 1, we give the first inequality of (1.4) in Theorem 2.

**Proposition 5.2** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ , and  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Then, for each  $m \in \mathbb{N}_0$ , there exists a constant  $C = C(n,p,\alpha,\lambda,m) > 0$  independent of  $u$  such that*

$$\|u\|_{L^p(\lambda)} \leq C \sum_{|\gamma|=m} \|t^{\frac{m}{2\alpha}} \partial_x^\gamma u\|_{L^p(\lambda)}.$$

*Proof.* Let  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Suppose that  $m$  is even, that is, there exists  $k \in \mathbb{N}$  such that  $m = 2k$ . Then, by (2) of Lemma 2.4 and Lemma 5.1, we have

$$\mathcal{D}_t^{\frac{m}{2\alpha}} u = \mathcal{D}_t^{\frac{k}{\alpha}} u = \left(\mathcal{D}_t^{\frac{1}{\alpha}}\right)^k u = (-1)^k \Delta_x^k u = (-1)^k \sum_{j_1, \dots, j_k=1}^n \partial_{j_1}^2 \dots \partial_{j_k}^2 u. \quad (5.1)$$

Therefore, (3) of Corollary 3.2 implies that

$$\begin{aligned} \|u\|_{L^p(\lambda)} &\leq C \|t^{\frac{m}{2\alpha}} \mathcal{D}_t^{\frac{m}{2\alpha}} u\|_{L^p(\lambda)} \leq C \sum_{j_1, \dots, j_k=1}^n \|t^{\frac{m}{2\alpha}} \partial_{j_1}^2 \dots \partial_{j_k}^2 u\|_{L^p(\lambda)} \\ &\leq C \sum_{|\gamma|=m} \|t^{\frac{m}{2\alpha}} \partial_x^\gamma u\|_{L^p(\lambda)}. \end{aligned}$$

Suppose that  $m$  is odd, that is, there exists  $k \in \mathbb{N}_0$  such that  $m = 2k+1$ . Put  $v = \mathcal{D}_t u$ . Then, (3) of Lemma 2.4 and (3) of Corollary 3.2 imply that  $v$  belongs to  $\mathbf{b}_\alpha^p(\eta)$ , where  $\eta = p + \lambda$ . Therefore, Theorem 1 implies that there exists an  $\alpha$ -parabolic conjugate function  $V = (v_1, \dots, v_n)$  of  $v$  such that  $v_j \in \mathbf{b}_\alpha^p(\sigma)$ , where  $\sigma = p(\frac{1}{2\alpha} - 1) + \eta = \frac{p}{2\alpha} + \lambda > -1$ . Thus, by (2) of

Lemma 2.4 and the equation (C.2), we have

$$\mathcal{D}_t^{\frac{m+1}{2\alpha}-1} v = \mathcal{D}_t^{\frac{k}{\alpha}} \mathcal{D}_t^{\frac{1}{\alpha}-1} v = \sum_{j=1}^n \mathcal{D}_t^{\frac{k}{\alpha}} \partial_j v_j.$$

Therefore, (3) of Corollary 3.2 implies that

$$\begin{aligned} \|v\|_{L^p(\eta)} &\leq C \|t^{\frac{m+1}{2\alpha}-1} \mathcal{D}_t^{\frac{m+1}{2\alpha}-1} v\|_{L^p(\eta)} \leq C \sum_{j=1}^n \|t^{\frac{m+1}{2\alpha}-1} \mathcal{D}_t^{\frac{k}{\alpha}} \partial_j v_j\|_{L^p(\eta)} \\ &= C \sum_{j=1}^n \|t^{\frac{m}{2\alpha}} \mathcal{D}_t^{\frac{k}{\alpha}} \partial_j v_j\|_{L^p(\sigma)} \leq C \sum_{j=1}^n \|t^{\frac{k}{\alpha}+1} \mathcal{D}_t^{\frac{k}{\alpha}+1} v_j\|_{L^p(\sigma)}. \end{aligned}$$

Since the equation (C.1) implies that  $\mathcal{D}_t v_j = -\partial_j v$ , (2) of Lemma 2.4 and (5.1) show that

$$\begin{aligned} \|v\|_{L^p(\eta)} &\leq C \sum_{j=1}^n \|t^{\frac{k}{\alpha}+1} \mathcal{D}_t^{\frac{k}{\alpha}} \partial_j v\|_{L^p(\sigma)} \\ &\leq C \sum_{j=1}^n \sum_{j_1, \dots, j_k=1}^n \|t^{\frac{k}{\alpha}+1} \partial_{j_1}^2 \dots \partial_{j_k}^2 \partial_j v\|_{L^p(\sigma)} \\ &\leq C \sum_{|\gamma|=m} \|t^{\frac{k}{\alpha}+1} \partial_x^\gamma v\|_{L^p(\sigma)} \leq C \sum_{|\gamma|=m} \|t \partial_x^\gamma v\|_{L^p(\rho)}, \end{aligned}$$

where  $\rho = \frac{m}{2\alpha} p + \lambda$ . Since  $\partial_x^\gamma u$  with  $|\gamma| = m$  belongs to  $\mathbf{b}_\alpha^p(\rho)$  by (3) of Lemma 2.4 and (3) of Corollary 3.2, thus (2) of Lemma 2.4 and (3) of Corollary 3.2 imply that

$$\|t \partial_x^\gamma v\|_{L^p(\rho)} = \|t \mathcal{D}_t(\partial_x^\gamma u)\|_{L^p(\rho)} \leq C \|\partial_x^\gamma u\|_{L^p(\rho)} = C \|t^{\frac{m}{2\alpha}} \partial_x^\gamma u\|_{L^p(\lambda)}.$$

Also, by (3) of Corollary 3.2, we have  $\|u\|_{L^p(\lambda)} \leq C \|t \mathcal{D}_t u\|_{L^p(\lambda)} = C \|v\|_{L^p(\eta)}$ . Hence, this completes the proof of Proposition 5.2.  $\square$

*Proof of Theorem 2.* Proposition 5.2 shows the first inequality of Theorem 2. By the last inequality of (3) of Corollary 3.2 with  $\nu = m \in \mathbb{N}_0$ , we also have the second inequality of Theorem 2.  $\square$

## 6. More properties of $\alpha$ -parabolic conjugate functions

In this section, we study more properties of  $\alpha$ -parabolic conjugate functions. Given a harmonic function  $u$  on  $H$ , it is well known that a vector-valued function  $V = (v_1, \dots, v_n)$  on  $H$  with  $v_j \in C^1(H)$  satisfies the equations (1.1) and (1.2) if and only if there exists a function  $g \in C^2(H)$  such that

$$g \text{ is harmonic on } H \text{ and } \nabla_{(x,t)} g = (v_1, \dots, v_n, u), \quad (6.1)$$

where  $\nabla_{(x,t)} = (\partial_1, \dots, \partial_n, \partial_t)$ . First, we give an analogous equivalence for our case. We show the following lemma.

**Lemma 6.1** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ , and  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Suppose that there exists a function  $g \in C^1(H)$  such that  $\mathcal{D}_t g = u$ . Then, for a real number  $\kappa \geq 0$ , the function  $g(x, \cdot)$  belongs to  $\mathcal{FC}^{\kappa+1}$  for each  $x \in \mathbb{R}^n$  and  $\mathcal{D}_t^{\kappa+1} g = \mathcal{D}_t^\kappa u$ .*

*Proof.* It suffices to show the case  $\kappa > 0$ , thus we let  $\kappa > 0$ . We remark that the derivative  $\mathcal{D}_t^\kappa u$  is well defined by (1) of Lemma 2.4. We show  $g(x, \cdot) \in \mathcal{FC}^{\kappa+1}$  for each  $x \in \mathbb{R}^n$ . In fact, since  $\kappa > 0$ , there exists an integer  $k \in \mathbb{N}$  such that  $\lceil \kappa + 1 \rceil = k + 1$ . Thus, we have  $\mathcal{D}_t^{\lceil \kappa + 1 \rceil} g = \mathcal{D}_t^{k+1} g = \mathcal{D}_t^k u$ , and so (1) of Lemma 2.4 implies that

$$|\mathcal{D}_t^{\lceil \kappa + 1 \rceil} g(x, t)| = |\mathcal{D}_t^k u(x, t)| \leq C t^{-k - (\frac{n}{2\alpha} + \lambda + 1) \frac{1}{p}} \quad (6.2)$$

for all  $(x, t) \in H$ . Furthermore, since

$$k + \left( \frac{n}{2\alpha} + \lambda + 1 \right) \frac{1}{p} > k = \lceil \kappa + 1 \rceil - 1 > \lceil \kappa + 1 \rceil - (\kappa + 1),$$

we have  $\mathcal{D}_t^{\lceil \kappa + 1 \rceil} g(x, \cdot) \in \mathcal{FC}^{-(\lceil \kappa + 1 \rceil - (\kappa + 1))}$  for each  $x \in \mathbb{R}^n$ .

Moreover, since  $\lceil \kappa + 1 \rceil - (\kappa + 1) = \lceil \kappa \rceil - \kappa$  and  $\lceil \kappa + 1 \rceil = \lceil \kappa \rceil + 1$ , we have

$$\mathcal{D}_t^{\kappa+1} g = \mathcal{D}_t^{-(\lceil \kappa + 1 \rceil - (\kappa + 1))} (\mathcal{D}_t^{\lceil \kappa + 1 \rceil} g) = \mathcal{D}_t^{-(\lceil \kappa \rceil - \kappa)} (\mathcal{D}_t^{\lceil \kappa \rceil} (\mathcal{D}_t g)) = \mathcal{D}_t^\kappa u$$

This completes the proof.  $\square$

**Theorem 6.2** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ , and  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Then, a vector-valued function  $V = (v_1, \dots, v_n)$  on  $H$  is an  $\alpha$ -parabolic conjugate function of  $u$  if and only if there exists a function  $g \in C^2(H)$  such that  $g(x, \cdot) \in \mathcal{FC}^{\frac{1}{\alpha}}$  for each  $x \in \mathbb{R}^n$  and*

$$(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x)g = 0 \text{ on } H \text{ and } \nabla_{(x,t)}g = (v_1, \dots, v_n, u). \quad (6.3)$$

*Proof.* We show the “if” part. Suppose that there exists a function  $g \in C^2(H)$  such that  $g(x, \cdot) \in \mathcal{FC}^{\frac{1}{\alpha}}$  for each  $x \in \mathbb{R}^n$  and  $g$  satisfies (6.3). Then, since  $g \in C^2(H)$ , the function  $v_j = \partial_j g$  belongs to  $C^1(H)$  and

$$\begin{aligned} \nabla_x u &= \nabla_x \partial_t g = \partial_t \nabla_x g = -\mathcal{D}_t V, \\ \nabla_x v_j &= \nabla_x \partial_j g = \partial_j \nabla_x g = \partial_j V \quad (1 \leq j \leq n). \end{aligned}$$

Furthermore, Lemma 6.1 with  $\kappa = \frac{1}{\alpha} - 1 \geq 0$  implies that

$$\nabla_x \cdot V = \nabla_x \cdot \nabla_x g = \Delta_x g = -\mathcal{D}_t^{\frac{1}{\alpha}} g = -\mathcal{D}_t^{(\frac{1}{\alpha}-1)+1} g = \mathcal{D}_t^{\frac{1}{\alpha}-1} u.$$

We show the “only if” part. Suppose that  $V = (v_1, \dots, v_n)$  is an  $\alpha$ -parabolic conjugate function of  $u$ . Let  $(x, t) = (x_1, \dots, x_n, t) \in H$  be fixed. We put

$$g(x, t) := \sum_{k=1}^n \int_0^{x_k} v_k(\xi_k(\tau), t) d\tau + \int_1^t u(0, \tau) d\tau,$$

where  $\xi_1(\tau) = (\tau, x_2, \dots, x_n)$  and  $\xi_k(\tau) = (0, \dots, 0, \tau, x_{k+1}, \dots, x_n)$  for  $2 \leq k \leq n$ . Then, since  $v_k \in C^1(H)$ , we have

$$\begin{aligned} \partial_j g(x, t) &= \sum_{k=1}^{j-1} \partial_j \left( \int_0^{x_k} v_k(\xi_k(\tau), t) d\tau \right) + \partial_j \left( \int_0^{x_j} v_j(\xi_j(\tau), t) d\tau \right) \\ &= \sum_{k=1}^{j-1} \int_0^{x_k} \partial_j v_k(\xi_k(\tau), t) d\tau + v_j(\xi_j(x_j), t). \end{aligned}$$

Also, since  $V$  satisfies the equation (C.1) and  $\xi_k(0) = \xi_{k+1}(x_{k+1})$ , we obtain

$$\begin{aligned}
\partial_j g(x, t) &= \sum_{k=1}^{j-1} \int_0^{x_k} \partial_k v_j(\xi_k(\tau), t) d\tau + v_j(\xi_j(x_j), t) \\
&= \sum_{k=1}^{j-1} (v_j(\xi_k(x_k), t) - v_j(\xi_k(0), t)) + v_j(\xi_j(x_j), t) \\
&= v_j(x, t).
\end{aligned}$$

Similarly, we have  $\partial_t g(x, t) = u(x, t)$ . Therefore,  $\nabla_{(x,t)} g = (v_1, \dots, v_n, u)$  and  $g \in C^2(H)$ . Furthermore, Lemma 6.1 with  $\kappa = \frac{1}{\alpha} - 1$  and the equation (C.2) imply that  $g(x, \cdot) \in \mathcal{FC}^{\frac{1}{\alpha}}$  for each  $x \in \mathbb{R}^n$  and

$$(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x)g = \mathcal{D}_t^{(\frac{1}{\alpha}-1)+1}g + \nabla_x \cdot \nabla_x g = -\mathcal{D}_t^{\frac{1}{\alpha}-1}u + \nabla_x \cdot V = 0.$$

This completes the proof of theorem.  $\square$

We also have the following proposition.

**Proposition 6.3** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ ,  $\lambda > -1$ , and  $u \in \mathbf{b}_\alpha^p(\lambda)$ . Let  $1 \leq j \leq n$  be fixed. Suppose that a vector-valued function  $V = (v_1, \dots, v_n)$  on  $H$  is an  $\alpha$ -parabolic conjugate function of  $u$ . Then,  $v_j(x, \cdot) \in \mathcal{FC}^{\frac{1}{\alpha}}$  for each  $x \in \mathbb{R}^n$ . Furthermore, if  $v_k \in C^2(H)$  for all  $1 \leq k \leq n$ , then  $(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x)v_j = 0$  on  $H$ .*

*Proof.* Let  $1 \leq j \leq n$  be fixed and put  $u' = -\partial_j u$ . Then, (3) of Lemma 2.4 and (3) of Corollary 3.2 imply that  $u' \in \mathbf{b}_\alpha^p(\sigma)$ , where  $\sigma = \frac{p}{2\alpha} + \lambda$ . Since  $\mathcal{D}_t v_j = -\partial_j u = u'$ , Lemma 6.1 with  $\kappa = \frac{1}{\alpha} - 1$  shows that  $v_j(x, \cdot) \in \mathcal{FC}^{\frac{1}{\alpha}}$  for each  $x \in \mathbb{R}^n$  and  $\mathcal{D}_t^{\frac{1}{\alpha}} v_j = \mathcal{D}_t^{\frac{1}{\alpha}-1} u'$ . Furthermore, if  $v_j \in C^2(H)$  for all  $1 \leq j \leq n$ , then the equations (C.1), (C.2), and (1) of Lemma 2.4 imply that

$$\begin{aligned}
\Delta_x v_j &= \sum_{k=1}^n \partial_k^2 v_j = \sum_{k=1}^n \partial_k \partial_j v_k = \partial_j \left( \sum_{k=1}^n \partial_k v_k \right) \\
&= \partial_j \mathcal{D}_t^{\frac{1}{\alpha}-1} u = \mathcal{D}_t^{\frac{1}{\alpha}-1} \partial_j u = -\mathcal{D}_t^{\frac{1}{\alpha}-1} u'.
\end{aligned}$$

Hence, we obtain  $\Delta_x v_j = -\mathcal{D}_t^{\frac{1}{\alpha}} v_j$ .  $\square$

Finally, we give an inversion theorem, that is, for a vector-valued function  $V = (v_1, \dots, v_n)$  on  $H$  we construct a function  $u \in \mathbf{b}_\alpha^p(\lambda)$  such that  $V$  is an  $\alpha$ -parabolic conjugate function of  $u$ .

**Theorem 6.4** *Let  $0 < \alpha \leq 1$ ,  $1 \leq p < \infty$ , and  $\eta > -1$ . Suppose that a vector-valued function  $V = (v_1, \dots, v_n)$  on  $H$  satisfies  $v_j \in \mathbf{b}_\alpha^p(\eta)$  and  $\nabla_x v_j = \partial_j V$  for all  $1 \leq j \leq n$ . If  $\alpha$ ,  $p$ , and  $\eta$  satisfy the condition  $\lambda = p(1 - \frac{1}{2\alpha}) + \eta > -1$ , then there exists a unique function  $u$  on  $H$  such that  $u \in \mathbf{b}_\alpha^p(\lambda)$  and  $V$  is an  $\alpha$ -parabolic conjugate function of  $u$ . Also, there exists a constant  $C = C(n, p, \alpha, \eta) > 0$  independent of  $V$  such that*

$$C^{-1} \| \|V\| \|_{L^p(\eta)} \leq \|u\|_{L^p(\lambda)} \leq C \| \|V\| \|_{L^p(\eta)}. \quad (6.4)$$

*Proof.* We put  $\lambda = p(1 - \frac{1}{2\alpha}) + \eta$  and suppose that  $\lambda > -1$ . Then, we can define a function  $u \in \mathbf{b}_\alpha^p(\lambda)$  by

$$u(x, t) := \mathcal{D}_t^{1-\frac{1}{\alpha}} \nabla_x \cdot V(x, t), \quad (x, t) \in H. \quad (6.5)$$

In fact, (3) of Lemma 2.4 and (3) of Corollary 3.2 imply that  $\nabla_x \cdot V \in \mathbf{b}_\alpha^p(\sigma)$ , where  $\sigma = \frac{p}{2\alpha} + \eta$ . Therefore, again (3) of Lemma 2.4 and (3) of Corollary 3.2 show that  $u$  is  $L^{(\alpha)}$ -harmonic on  $H$  and there exists a constant  $C = C(n, p, \alpha, \eta) > 0$  independent of  $V$  such that

$$\|u\|_{L^p(\lambda)} = \| \mathcal{D}_t^{1-\frac{1}{\alpha}} \nabla_x \cdot V \|_{L^p(\lambda)} \leq C \| \nabla_x \cdot V \|_{L^p(\sigma)} \leq C \| \|V\| \|_{L^p(\eta)}.$$

Thus we obtain  $u \in \mathbf{b}_\alpha^p(\lambda)$  and the second inequality of (6.4).

We show that the functions  $u$  and  $V$  satisfy the equations (C.1) and (C.2). By (1) of Lemma 2.4, the hypothesis  $\nabla_x v_j = \partial_j V$ , and Lemma 5.1, we have

$$\partial_j u = \partial_j \mathcal{D}_t^{1-\frac{1}{\alpha}} \nabla_x \cdot V = \mathcal{D}_t^{1-\frac{1}{\alpha}} \partial_j \nabla_x \cdot V = \mathcal{D}_t^{1-\frac{1}{\alpha}} \Delta_x v_j = -\mathcal{D}_t^{1-\frac{1}{\alpha}} \mathcal{D}_t^{\frac{1}{\alpha}} v_j.$$

Hence, by (2) of Lemma 2.4, we obtain  $\partial_j u = -\mathcal{D}_t v_j$ , so the equation (C.1) is satisfied. Furthermore, (6.5) and (2) of Lemma 2.4 imply that  $\mathcal{D}_t^{\frac{1}{\alpha}-1} u = \nabla_x \cdot V$ , and thus the equation (C.2) is also satisfied. Furthermore, by (3) of Corollary 3.2 and the equation (C.1), we have

$$\begin{aligned} \|v_j\|_{L^p(\eta)} &\leq C \|t\mathcal{D}_t v_j\|_{L^p(\eta)} = C \|t\partial_j u\|_{L^p(\eta)} = C \|t^{\frac{1}{2\alpha}} \partial_j u\|_{L^p(\lambda)} \\ &\leq C \|u\|_{L^p(\lambda)}, \end{aligned}$$

thus we also obtain the first inequality of (6.4).

We suppose that a function  $v$  on  $H$  belongs to  $\mathbf{b}_\alpha^p(\lambda)$  and  $V$  is an  $\alpha$ -parabolic conjugate function of  $v$ . Then, (3) of Corollary 3.2 and the equation (C.2) imply that

$$\|u - v\|_{L^p(\lambda)} \leq C \|t^{\frac{1}{\alpha}-1} \mathcal{D}_t^{\frac{1}{\alpha}-1}(u - v)\|_{L^p(\lambda)} = C \|\nabla_x \cdot V - \nabla_x \cdot V\|_{L^p(\sigma)} = 0.$$

Hence, we obtain  $u = v$  on  $H$ . This completes the proof.  $\square$

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