Hokkaido Mathematical Journal Vol. 39 (2010) p. 1–55

Complex surfaces of constant mean curvature fibered by minimal surfaces

Josef Dorfmeister, Shimpei Kobayashi and Franz Pedit

(Received September 24, 2008)

Abstract. We define complex constant mean curvature immersions in complex three space using a natural extension of constant mean curvature immersions in Euclidean three space via loop group techniques. We then discuss the fundamental properties of these complex constant mean curvature immersions. In particular, we prove that these immersions are doubly ruled by holomorphic null curves. We present a construction of minimal immersions from constant mean curvature immersions in Euclidean three space via the associated complex constant mean curvature immersions.

Key words: constant mean curvature, minimal surfaces.

1. Introduction

Over the past two decades the theory of constant mean curvature (CMC) surfaces in 3-dimensional space forms has been studied by a variety of techniques ranging from non-linear elliptic analysis to integrable systems methods. The latter has been largely motivated by the realization that the integrability equation for CMC surfaces, the elliptic sinh–Gordon equation, is a special real form of the complex sine–Gordon equation, a well known completely integrable evolution equation for which solutions can be found by finite gap integration. This viewpoint has been particularly successful in the classification of CMC tori [28], [20], [3] and in the description of all simply connected CMC surfaces by a Weierstraß representation using loop group factorization techniques [12]. In this representation the CMC surface M in 3-space is constructed from a loop algebra valued meromorphic connection over M, the Weierstrass data, by integration, and the loop group factorization can be interpreted as a kind of "prolongation" of the solution from M to the diagonal in $M \times \bar{M}$. All of these methods use the fact that a CMC surface comes with a circle's worth of CMC surfaces, the associated family, which is the geometric analog of "inserting a spectral parameter into the

---

2000 Mathematics Subject Classification : 53A10.

equation". The complex sine–Gordon equation naturally is the integrability condition for a complex surface $\Psi\colon M \times \bar{M} \to \mathbb{C}^3$ of constant complex mean curvature $H \neq 0$, defined analogously as in the real case. Such surfaces can be constructed from a decoupled meromorphic connection $d + \eta + \tau$ with η and τ meromorphic forms on M and $\bar{M}$ respectively. Then a double loop group factorization produces the complex CMC surface. If $\tau = -\bar{\eta}^t$ is real the corresponding CMC surface will be real and vice versa. In this way every real CMC surface gives rise to a complex CMC surface. This fits into a more general concept, in which the various real forms of the loop group correspond to the various real forms of the complex sine–Gordon equation, which in turn correspond to the various constant (mean) curvature surfaces in $\mathbb{R}^3$ (also with Lorentzian signature). From this perspective the complex CMC surface is a kind of master surface, which contains all the various real constant (mean) curvature surfaces [24].

Finally, there is an interesting relationship to minimal surface theory in $\mathbb{R}^3$. Any complex CMC surface $\Psi : M \times \bar{M} \to \mathbb{C}^3$ is doubly ruled by minimal surfaces: restricting Ψ to one of the factors gives for each $p \in \bar{M}$ a holomorphic null curve $\Psi(-, p) : M \to \mathbb{C}^3$ whose real part thus is a minimal surface in $\mathbb{R}^3$. In this way we get a natural correspondence between CMC surfaces and minimal surfaces in $\mathbb{R}^3$.

In Section 2 we consider holomorphic immersions Ψ from a simply connected domain $\mathbb{D}^2 \subset \mathbb{C}^2$ into $\mathbb{C}^3$. We introduce a quadratic bilinear form (holomorphic metric) on $\mathbb{D}^2$ induced by a complex bilinear form on $\mathbb{C}^3$ and prove the existence of special coordinates for holomorphic immersions Ψ, which we call "null" (Theorem 2.2). These coordinates correspond to conformal coordinates for a surface in $\mathbb{R}^3$. We then consider, for holomorphic immersions Ψ, the moving frame and the structure equations, which we call "complex Gauß-Codazzi" equations. We prove the fundamental theorem for holomorphic immersions in $\mathbb{C}^3$ (Theorem 2.9) by Cartan's method of moving frames. The notion of complex mean curvature for holomorphic immersions in $\mathbb{C}^3$ will be defined analogously to the mean curvature for surfaces in $\mathbb{R}^3$. This allows us to consider complex constant mean curvature (complex CMC) immersions in $\mathbb{C}^3$. We give necessary and sufficient conditions for the constancy of the complex mean curvature of a holomorphic null immersion (Lemma 2.10). The case where the complex mean curvature vanishes corresponds to a pair of classical Weierstraß representation formulas for minimal surfaces in separate variables. Therefore, from this point on, we only consider the case of complex constant mean curvature $H \neq 0$. We finally introduce a loop parameter into the complex Gauß-Codazzi equations of complex CMC-immersions. This loop parameter does not change the complex Gauß-Codazzi equations for complex CMC-immersions and we obtain a one parameter family of complex CMC-immersions. Theorem 2.16 then gives a representation formula for complex CMC-immersions.

Section 3 discusses a construction of complex CMC-immersions via loop group methods analogous [12] to the "generalized Weierstraß type representation for CMC-immersions in $\mathbb{R}^3$". We show that to every complex CMC-immersion there exists a pair of holomorphic 1-forms (η, τ), called a pair of "holomorphic potentials", defined on some simply connected Stein manifold M (Theorem 3.1). The converse procedure then constructs complex CMC-immersions from a pair of holomorphic potentials (Theorem 3.4). We prove that every complex CMC-immersion can be obtained via this construction. Allowing for poles in the potentials, we construct a pair of meromorphic potentials of a special type, called "a pair of normalized potentials". We provide a sufficient condition for the occurrence of singularities in a complex CMC-immersion. This section concludes with simple examples of complex CMC-immersions. These examples are complexifications of CMC-immersions in $\mathbb{R}^3$ in the sense of Theorem A.4.

Section 4 presents some relations with minimal surfaces in $\mathbb{R}^3$ and provides more examples. First we show that each complex CMC-immersion yields two (transversal) families of minimal surfaces in $\mathbb{R}^3$. In fact, every minimal surface in $\mathbb{R}^3$ without umbilical points is contained in such a family of minimal surfaces produced from a complex CMC-immersion. We single out the real CMC-immersions by the relation $\tau = -\bar{\eta}^t$ for the pair of potentials (η, τ) defining the CMC-immerson. Using this characterization of real CMC-immersions among the complex ones we sharpen Theorem 4.6 and prove that every minmal surface in $\mathbb{R}^3$ without umbilical points is contained in the family of minimal surfaces in $\mathbb{R}^3$ produced from some complex CMC-immersion $\Psi(z, w)$, for which $\Psi(z, \bar{z})$ is a CMC-immerson into $\mathbb{R}^3$. Finally, we discuss some examples of minimal surfaces obtained from real CMC-immersions following the discussion of the previous sections. It is not surprising that very simple CMC-immersions yield very simple minimal surfaces. It seems remarkable though that the minmal surfaces associated with (real) Delaunay surfaces have a fairly complicated structure (there seem to be two catenoidal ends and infinitely many flat ends) which, however, is not unfamiliar from minimal surface theory, e.g. [26].

The Appendix recalls basic definitions and notations for loop groups. We state two decomposition theorems, the so-called Birkhoff and Iwasawa decomposition theorems (Theorems A.1 and A.2). We introduce double loop groups [13], [10] and the Iwasawa decomposition for double loop groups (Theorem A.3). We then give a brief explanation for the "generalized Weierstraß type representation for CMC-immersions in $\mathbb{R}^3$ in the double loop group picture" [12], [10]. Moreover, we also explain the existence of a meromorphic extension of the extended framing of CMC-immersions in $\mathbb{R}^3$ (Theorem A.4, see [10]). This existence theorem is the starting point for considering complex CMC-immersions in $\mathbb{C}^3$.

2. Complex CMC-immersions in $\mathbb{C}^3$

Let $\mathbb{D} \subset \mathbb{C}$ be an open and simply connected domain, and let $\varPsi : \mathbb{D} \to \mathbb{R}^3$ be a CMC-immersion. Let $F(z, \bar{z}, \lambda)$ be the extended framing of $\varPsi$, and let $l(z, \bar{z})$ be a diagonal matrix in $SL(2, \mathbb{C})$ which is independent of λ. From [10] (see also Theorem A.4), it is known that for a proper choice of $l(z, \bar{z})$ the mapping $F(z, \bar{z}, \lambda)l(z, \bar{z}) : \mathbb{D} \to \Lambda SU(2)_\sigma$ can be extended meromorphically to $U(z, w, \lambda) : \mathbb{D} \times \bar{\mathbb{D}} \to \Lambda SL(2, \mathbb{C})_\sigma$ (See Section Appendix A for the definitions of loop groups $\Lambda SL(2, \mathbb{C})_\sigma$ and $\Lambda SU(2)_\sigma$). Thus it is natural to consider holomorphic immersions associated with the meromorphic extensions $U(z, w, \lambda)$ of extended framings $F(z, \bar{z}, \lambda)$. In this section, we start from holomorphic immersions into $\mathbb{C}^3$ in a general setting. And we derive the fundamental equations for these holomorphic immersions.

2.1. Holomorphic null immersions in $\mathbb{C}^3$

Let M be a connected 2-dimensional Stein manifold, and let

$$\varPsi : M \to \mathbb{C}^3 \tag{2.1}$$

be a holomorphic immersion into $\mathbb{C}^3$, i.e. $\varPsi$ is holomorphic and the complex rank of $d\varPsi$ is 2. We consider the standard symmetric bilinear form $\langle , \rangle$ on $\mathbb{C}^3$ given by

$$\langle a, b \rangle = \sum_{i=1}^{3} a_i b_i \quad \text{for} \quad a = (a_1, a_2, a_3)^t, \; b = (b_1, b_2, b_3)^t \in \mathbb{C}^3. \tag{2.2}$$

Clearly, this is a $\mathbb{C}$-bilinear extension of the natural inner product in $\mathbb{R}^3$. This symmetric bilinear form induces the holomorphic form g on M given by

$$g = \langle \Psi_z, \Psi_z \rangle dz^2 + 2\langle \Psi_z, \Psi_w \rangle dzdw + \langle \Psi_w, \Psi_w \rangle dw^2,$$

where $(z, w) \in \mathbb{D}^2 \subset \mathbb{C}^2$ are local coordinates, g is holomorphic in both variables z and w. If the induced holomorphic form g is non-degenerate on M, i.e. $\langle \Psi_z, \Psi_z \rangle \langle \Psi_w, \Psi_w \rangle - \langle \Psi_z, \Psi_w \rangle^2 \neq 0$ for all $(z, w) \in \mathbb{D}^2$, g is called a *holomorphic metric* on M. We note that, in general, g restricted to TM is not non-degenerate. Therefore g does not induce a pseudo-Riemannian metric on the smooth manifold M. As a consequence, by considering immersions (M, Ψ) with holomorphic metric we restrict to a special class of immersions. We refer the reader to [23], [25] for more details concerning complex manifolds with a holomorphic metric.

It is well known that for every surface in $\mathbb{R}^3$ there always exist conformal coordinates [6]. Analogous to this we show that for every holomorphic immersion there exist special coordinates. We first define the notion of a holomorphic null immersion in $\mathbb{C}^3$.

Definition 2.1 Let $\Psi : M \to \mathbb{C}^3$ be a holomorphic immersion with a holomorphic metric g, and let $(z, w) \in \mathbb{D}^2 \subset \mathbb{C}^2$ be local coordinates in a neighborhood of some point $p \in M$. We call the coordinates (z, w) *null coordinates* if, in these coordinates, g has the form:

$$g = \rho(z, w) dzdw, \tag{2.3}$$

where $\rho : \mathbb{D}^2 \to \mathbb{C}$ is some holomorphic function never vanishing on $\mathbb{D}^2$. Moreover, we call Ψ *a holomorphic null immersion* if there exist null coordinates for each point in M.

The following theorem will imply that there exist null coordinates for every holomorphic immersion $\Psi : M \to \mathbb{C}^3$ which induces a holomorphic metric on M (see also [23]).

Theorem 2.2 *Let $\mathbb{D}^2$ be a simply connected domain in $\mathbb{C}^2$, and let $\Psi : \mathbb{D}^2 \to \mathbb{C}^3$ be a holomorphic immersion with induced holomorphic metric g. For every point (z_0, w_0) in $\mathbb{D}^2$, there exists an open subset $\check{\mathbb{D}}^2 \subset \mathbb{D}^2$ around (z_0, w_0) such that $\Psi : \check{\mathbb{D}}^2 \to \mathbb{C}^3$ is a null immersion.*

Proof. Let $(z,w) \in \mathbb{D}^2 \subset \mathbb{C}^2$ be local coordinates, and let $\langle \cdot,\cdot \rangle$ denote the symmetric bilinear form defined in (2.2). Moreover let $g = \langle \Psi_z, \Psi_z\rangle dz^2 + 2\langle \Psi_z, \Psi_w\rangle dzdw + \langle \Psi_w, \Psi_w\rangle dw^2$ be the holomorphic metric induced on $\mathbb{D}^2$ by Ψ. If $\langle \Psi_z, \Psi_z\rangle = \langle \Psi_w, \Psi_w\rangle = 0$ for some neighborhood $\tilde{\mathbb{D}}^2 \subset \mathbb{D}^2$ around $(z_0, w_0) \in \mathbb{D}^2$, then Ψ is already a null immersion on $\tilde{\mathbb{D}}^2$. If $\langle \Psi_w, \Psi_w\rangle = 0$ at $(z_0, w_0) \in \mathbb{D}^2$, then $\langle \Psi_z, \Psi_w\rangle \neq 0$ for some neighborhood $\tilde{\mathbb{D}}^2 \subset \mathbb{D}^2$ around (z_0, w_0) since Ψ is an immersion at $(z_0, w_0) \in \mathbb{D}^2$. Moreover $\langle \Psi_z, \Psi_z\rangle\langle \Psi_w, \Psi_w\rangle - \langle \Psi_z, \Psi_w\rangle^2 \neq 0$ on a possibly smaller neighborhood of (z_0, w_0). Thus there exists a holomorphic change of coordinates from (z,w) to (ξ, η) such that $\langle \Psi_\eta, \Psi_\eta\rangle \neq 0$ for some neighborhood $\check{\mathbb{D}}^2 \subset \mathbb{C}^2$ around $(\xi_0, \eta_0) = (\xi(z_0, w_0), \eta(z_0, w_0))$. Therefore, without loss of generality, we can assume that $\langle \Psi_w, \Psi_w\rangle \neq 0$ on some neighborhood $\tilde{\mathbb{D}}^2 \subset \mathbb{D}^2$ around (z_0, w_0).

Under this assumption it is straightforward to verify that g has a representation of the form

$$g = (\mu_1 dz + \mu_2 dw)(\mu_3 dz + dw), \tag{2.4}$$

where

$$\begin{aligned}
\mu_1 &= \langle \Psi_z, \Psi_w\rangle + \sqrt{\langle \Psi_z, \Psi_w\rangle^2 - \langle \Psi_z, \Psi_z\rangle\langle \Psi_w, \Psi_w\rangle},\\
\mu_2 &= \langle \Psi_w, \Psi_w\rangle,\\
\mu_3 &= \frac{\langle \Psi_z, \Psi_w\rangle - \sqrt{\langle \Psi_z, \Psi_w\rangle^2 - \langle \Psi_z, \Psi_z\rangle\langle \Psi_w, \Psi_w\rangle}}{\langle \Psi_w, \Psi_w\rangle}.
\end{aligned} \tag{2.5}$$

We note that the non-degeneracy condition $\langle \Psi_z, \Psi_z\rangle\langle \Psi_w, \Psi_w\rangle - \langle \Psi_z, \Psi_w\rangle^2 \neq 0$ implies that the square roots in μ_1 and μ_3 are well-defined. Since $\langle \Psi_w, \Psi_w\rangle \neq 0$ for some open neighborhood $\tilde{\mathbb{D}}^2$, μ_3 is holomorphic on $\tilde{\mathbb{D}}^2$. Thus μ_1, μ_2 and μ_3 are holomorphic and well-defined on $\tilde{\mathbb{D}}^2$.

Next we show that there exists a holomorphic change of coordinates from (z,w) to $(\tilde{z}, \tilde{w})$ satisfying:

$$\begin{aligned}
d\tilde{z} &= \tilde{z}_z dz + \tilde{z}_w dw = \ell_1(\mu_1 dz + \mu_2 dw),\\
d\tilde{w} &= \tilde{w}_z dz + \tilde{w}_w dw = \ell_2(\mu_3 dz + dw),
\end{aligned} \tag{2.6}$$

where ℓ_j, $j = 1, 2$, are some scalar functions of z and w. One should observe that the equations above are equivalent to

$$\begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix}_z = \begin{pmatrix} \frac{\mu_1}{\mu_2} & 0 \\ 0 & \mu_3 \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix}_w . \tag{2.7}$$

We note that for $w = \bar{z}$, this equation is the well known Beltrami equation [1].

We will show now that the system (2.7) has a solution. Since μ_1/μ_2 and μ_3 are holomorphic on $\tilde{\mathbb{D}}^2$, we can apply the Theorem of Cauchy-Kovalevskaya for the case of holomorphic partial differential equations (see [33], Theorem 17.2 and the last paragraph on page 150). Thus there exists a locally unique holomorphic solution $(\tilde{z}, \tilde{w})$ to (2.7) in an open neighborhood $\check{\mathbb{D}}^2 \subset \tilde{\mathbb{D}}^2$ of the point $(z_0, w_0) \in \tilde{\mathbb{D}}^2$. We can even assume $(z_0, w_0) = (\tilde{z}(z_0, w_0), \tilde{w}(z_0, w_0))$. Therefore, there exists a holomorphic change of coordinates for $\check{\mathbb{D}}^2 \subset \tilde{\mathbb{D}}^2 \subset \mathbb{C}^2$ such that

$$g = (\ell_1 \ell_2)^{-1} d\tilde{z} d\tilde{w}. \tag{2.8}$$

This completes the proof. □

Corollary 2.3 *From Theorem* 2.2*, it follows that a holomorphic immersion $\Psi : M \to \mathbb{C}^3$ which induces a holomorphic metric is a holomorphic null immersion.*

We define $\partial f := \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial w} dw$ for a complex function f on $\mathbb{D}^2 \subset \mathbb{C}^2$ and $\partial \mathbf{f} := \left(\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial w}\right) dz \wedge dw$ for a 1-form $\mathbf{f} = f_1 dz + f_2 dw$ on $\mathbb{D}^2 \subset \mathbb{C}^2$. Moreover, we will need the notion of a "holomorphically convex domain". For a precise definition in full generality we refer to [17]. For this paper it will suffice to consider open balls. From [17], we quote the following result:

Lemma 2.4 ([17], Theorem 1 on page 77, and page 78) *Let $\mathbb{D}^2 \subset \mathbb{C}^2$ be a simply connected holomorphically convex domain. If a* 1*-form $\mathbf{f} = f_1 dz + f_2 dw$ is ∂-closed on $\mathbb{D}^2$, i.e. $\partial \mathbf{f} = 0$, then $\mathbf{f}$ is ∂-exact on $\mathbb{D}^2$, i.e. $\mathbf{f} = \partial h$, where $h : \mathbb{D}^2 \to \mathbb{C}$ is some complex function.*

From Lemma 2.4, we have the following Corollary:

Corollary 2.5 *Let $\mathbb{D}^2 \subset \mathbb{C}^2$ be a simply connected holomorphically convex domain and let $\tilde{g}$ be a non-zero holomorphic function on $\mathbb{D}^2$. Then there exists a holomorphic function $u : \mathbb{D}^2 \to \mathbb{C}$ such that $\tilde{g} = \exp(u)$.*

Proof. Since $\tilde{g}$ is non-zero and $(\partial \tilde{g})/\tilde{g}$ is ∂-closed on $\mathbb{D}^2$, the 1-form $(\partial \tilde{g})/\tilde{g}$ is ∂-exact on $\mathbb{D}^2$ by Lemma 2.4. If we set $p = \exp(\tilde{u})$ with a holomorphic function $\tilde{u} : \mathbb{D}^2 \to \mathbb{C}$ such that $(\partial \tilde{g})/\tilde{g} = \partial \tilde{u}$, then we have $(\partial p)/p = \partial \tilde{u}$, whence $\partial(\tilde{g}^{-1} p) = 0$ and $\tilde{g} = c \exp(\tilde{u})$, where c is some constant. We set $\tilde{g} = \exp(u) = \exp(\tilde{u} + c_1)$, where $c_1 = \log c$. □

Corollary 2.6 *Let $\mathbb{D}^2 \subset \mathbb{C}^2$ be a simply connected holomorphically convex domain and let $\tilde{g}$ be a non-zero holomorphic function on $\mathbb{D}^2$. Then there exists a holomorphic function $\sqrt{\tilde{g}}$, i.e. a holomorphic function h satisfying $h^2 = \tilde{g}$.*

Proof. Let h be $\exp(u/2)$, where u is defined in Corollary 2.5. Then h is holomorphic and $h^2 = \tilde{g}$. □

Let $\tilde{g} = (\ell_1 \ell_2)^{-1}$ be the coefficient function of g in (2.8). Since $\tilde{g}$ is non-zero by the non-degeneracy condition, we can apply Corollary 2.5 to $\tilde{g}$, thus we have $\tilde{g} = \exp(u)$. From now on, we will always assume that the holomorphic metric g has locally the form

$$g = e^{u(z,w)} dz dw. \tag{2.9}$$

2.2. Holomorphic null immersions and Cartan's method of moving frames

In this subsection, we prove the complex version of the fundamental theorem for submanifolds, i.e. for a given holomorphic metric g and a second fundamental form II satisfying the complex Gauß-Codazzi equations, there exists a holomorphic immersion into $\mathbb{C}^3$, which induces the metric g and has the second fundamental form II. To prove this, we use Cartan's method of moving frames to investigate the existence and uniqueness of maps into Lie groups.

Let $\mathbb{D}^2$ be some simply connected domain in $\mathbb{C}^2$, and let $\Psi : \mathbb{D}^2 \to \mathbb{C}^3$ be a holomorphic immersion with a holomorphic metric g. We note that the symmetric bilinear form defined in (2.2) is $SO(3, \mathbb{C})$ invariant.

From [18], we first quote the following Lemma for the existence and uniqueness of maps into Lie groups.

Lemma 2.7 (Cartan [5], Griffiths [18], page 780) *Let M be a simply connected manifold, and let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. Moreover let ω be the Maurer-Cartan form of G. Let ψ be a $\mathfrak{g}$-valued 1-form* *on M. Then there exists a smooth map $\Psi : M \to G$ with $\Psi^*\omega = \psi$, if and only if,*

$$d\psi + \frac{1}{2}[\psi \wedge \psi] = 0. \tag{2.10}$$

Moreover, Ψ is unique up to left translation.

We recall the structure equations for a holomorphic null immersion $\Psi : \mathbb{D}^2 \to \mathbb{C}^3$ using the Cartan method of moving frames. Its ingredients are the holomorphic metric g and the "moving frame" $\mathcal{F}$:

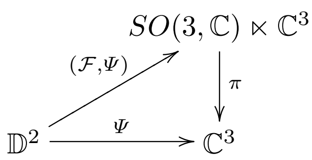
$$\begin{cases} \mathcal{F}(z,w) := \big(e^{-u/2}(\Psi_z + \Psi_w), ie^{-u/2}(\Psi_z - \Psi_w), N\big), \\ g = \langle d\Psi, d\Psi \rangle = e^{u(z,w)} dzdw, \end{cases} \tag{2.11}$$

where

$$N := \big(e^{-u/2}(\Psi_z + \Psi_w) \times ie^{-u/2}(\Psi_z - \Psi_w)\big) = 2ie^{-u}(\Psi_w \times \Psi_z). \tag{2.12}$$

We call $N(= e_3)$ *the Gauß map* of Ψ. It is easy to verify that for each point (z,w), the columns of $\mathcal{F}(z,w)$ form an orthonormal basis of $\mathbb{C}^3$, which we will abbreviate by $\mathcal{F}(z,w) = (e_1, e_2, e_3)$. Note, if a and b are two vectors in $\mathbb{C}^3$ satisfying $\langle a,a \rangle = \langle b,b \rangle = 1$ and $\langle a,b \rangle = 0$, then there exists a unique vector $c \in \mathbb{C}^3$ such that $(a,b,c) \in SO(3,\mathbb{C})$, and this vector can be determined by $c = a \times b$.

Since $SO(3,\mathbb{C})$ acts naturally on $\mathbb{C}^3$, we have the following commutative diagram:



where $(\mathcal{F}, \Psi)$ is called *the Cartan lift* of the immersion Ψ and $SO(3,\mathbb{C}) \ltimes \mathbb{C}^3$ is the group of affine transformations for which the linear part is in $SO(3,\mathbb{C})$. This group will be called the "group of rigid motions in $\mathbb{C}^3$". Since $\mathbb{D}^2$ is a simply connected domain in $\mathbb{C}^2$, the Cartan lift always exists.

Next we compute the moving frame equations for Ψ. By a direct computation (like for Lorentzian surfaces in $\mathbb{R}^3$ in null coordinates [34]), we obtain the following equations:

$$\begin{cases} \mathcal{F}_z = \mathcal{F}A, \\ \mathcal{F}_w = \mathcal{F}B, \end{cases} \tag{2.13}$$

where

$$A = \begin{pmatrix} 0 & \frac{i}{2}u_z & -\big(Q + \frac{1}{2}e^u H\big)e^{-u/2} \\ -\frac{i}{2}u_z & 0 & -i\big(Q - \frac{1}{2}e^u H\big)e^{-u/2} \\ \big(Q + \frac{1}{2}e^u H\big)e^{-u/2} & i\big(Q - \frac{1}{2}e^u H\big)e^{-u/2} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -\frac{i}{2}u_w & -\big(R + \frac{1}{2}e^u H\big)e^{-u/2} \\ \frac{i}{2}u_w & 0 & i\big(R - \frac{1}{2}e^u H\big)e^{-u/2} \\ \big(R + \frac{1}{2}e^u H\big)e^{-u/2} & -i\big(R - \frac{1}{2}e^u H\big)e^{-u/2} & 0 \end{pmatrix},$$

with $Q := \langle \Psi_{zz}, N\rangle$, $R := \langle \Psi_{ww}, N\rangle$ and $H = 2e^{-u}\langle \Psi_{zw}, N\rangle$. Then the compatibility condition for (2.13) is:

$$A_w - B_z + [B, A] = 0. \tag{2.14}$$

We note that, setting $\hat{\alpha} = Adz + Bdw$, this is equivalent to $d\hat{\alpha} + 1/2[\hat{\alpha} \wedge \hat{\alpha}] = 0$.

Then (2.14) can be rephrased as follows:

$$\begin{cases} u_{zw} - 2RQe^{-u} + \dfrac{1}{2}H^2 e^u = 0, \\ Q_w - \dfrac{1}{2}H_z e^u = 0, \\ R_z - \dfrac{1}{2}H_w e^u = 0. \end{cases} \tag{2.15}$$

The first equation in (2.15) will be called *the complex Gauß equation*, and to the second and third equation in (2.15), we will refer to as *the complex Codazzi equations.* Note that the symmetric quadratic form $II := -\langle d\Psi, dN\rangle$, the "second fundamental form" of Ψ, can be rephrased in terms of u, Q, R and H as follows:

$$II := -\langle d\Psi, dN\rangle = Qdz^2 + e^u H dzdw + Rdw^2. \qquad (2.16)$$

Finally we define the mean curvature for a holomorphic null immersion Ψ.

Definition 2.8 Let $\Psi : M \to \mathbb{C}^3$ be a holomorphic null immersion. Then we call the function $H = 2e^{-u}\langle \Psi_{zw}, N\rangle$ *the complex mean curvature of* Ψ.

From the above discussion we know that all holomorphic null immersions $\Psi : M \to \mathbb{C}^3$ inducing a holomorphic metric satisfy the complex Gauß-Codazzi equations.

Conversely, we can show

Theorem 2.9 *Let u, Q, R and H satisfy the complex Gauß-Codazzi equations* (2.15) *on some simply connected domain $\mathbb{D}^2 \subset \mathbb{C}^2$. And let g be a holomorphic metric of the form $g = e^{u(z,w)}dzdw$, and let II be the symmetric quadratic form* (2.16) *in terms of u, Q, R and H. Then there exists a holomorphic null immersion $\Psi : \mathbb{D}^2 \to \mathbb{C}^3$, such that the holomorphic metric and the second fundamental form induced by Ψ are given by g and II respectively. Moreover, the immersion Ψ is unique up to a rigid motion in $\mathbb{C}^3$.*

Proof. Let u, Q, R and H be solutions of the complex Gauß-Codazzi equations (2.15). Let A and B be the $so(3,\mathbb{C})$ matrices defined by (2.13) in terms of u, Q, R, and H. Set $\hat{\alpha} = (Adz + Bdw)$. Then $d\hat{\alpha} + 1/2[\hat{\alpha}\wedge\hat{\alpha}] = 0$ is equivalent to the complex Gauß-Codazzi equations (2.15). Thus by Lemma 2.7, we obtain $\mathcal{F} = (e_1, e_2, e_3) \in SO(3,\mathbb{C})$ such that $\hat{\alpha} = (\mathcal{F}^{-1}d\mathcal{F})$. Set

$$\alpha = \left(Adz + Bdw,\ \frac{1}{2}\big(e^{u/2}e_1 - ie^{u/2}e_2\big)dz + \frac{1}{2}\big(e^{u/2}e_1 + ie^{u/2}e_2\big)dw\right). \qquad (2.17)$$

Then α is an $so(3,\mathbb{C}) \ltimes \mathbb{C}^3$-valued 1-form. Moreover $d\alpha + 1/2[\alpha\wedge\alpha] = 0$, since this equation is equivalent to the complex Gauß-Codazzi equations (2.15). Therefore, again from Lemma 2.7, we obtain that there exists a pair $(\mathcal{F}, \Psi) \in SO(3,\mathbb{C}) \ltimes \mathbb{C}^3$ such that the Maurer-Cartan form of $(\mathcal{F}, \Psi)$ is $\alpha = (\mathcal{F}^{-1}d\mathcal{F}, d\Psi)$ as given in (2.17). A direct computation shows that $\langle d\Psi, d\Psi\rangle$ (resp. $-\langle d\Psi, dN\rangle$) is equal to the given g (resp. II). Finally, the uniqueness of Lemma 2.7 implies that Ψ is unique up to a rigid motion in $\mathbb{C}^3$. □

2.3. Holomorphic null immersions in terms of 2 by 2 matrices

In this subsection, we reformulate the discussion of Section 2.2 in terms of $SL(2,\mathbb{C})$ and its Lie algebra $sl(2,\mathbb{C})$. This is mainly done to simplify computations and the presentation.

We first note that $\mathbb{C}^3$ can be identified with $sl(2,\mathbb{C})$ as follows:

$$(a \;\; b \;\; c)^t \in \mathbb{C}^3 \leftrightarrow -\frac{ia}{2}\sigma_1 - \frac{ib}{2}\sigma_2 - \frac{ic}{2}\sigma_3 \in sl(2,\mathbb{C}), \tag{2.18}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.19}$$

The symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^3$ defined in (2.2) can be rephrased on $sl(2,\mathbb{C})$ by:

$$\langle x, y \rangle = -2\mathrm{Tr}\, \tilde{x}\tilde{y}, \tag{2.20}$$

where $x, y \in \mathbb{C}^3$, and $\tilde{x}, \tilde{y} \in sl(2,\mathbb{C})$ are the corresponding matrices by the identification (2.18). Therefore we can consider holomorphic immersions into $sl(2,\mathbb{C})$ instead of holomorphic immersions into $\mathbb{C}^3$.

It is well known [19] that $SL(2,\mathbb{C})$ is the double cover of $SO(3,\mathbb{C})$. Therefore we can replace the moving frame $\mathcal{F}$ of a holomorphic immersion $\Psi : M \to \mathbb{C}^3$ in $SO(3,\mathbb{C})$ by the moving frame F in $SL(2,\mathbb{C})$ if M is simply connected. For F we have the following Lax pair equations [9] corresponding to the moving frame equations (2.13):

$$\begin{aligned} F_z &= FU, \\ F_w &= FV, \end{aligned} \tag{2.21}$$

where

$$\begin{cases} U = \begin{pmatrix} \frac{1}{4}u_z & -\frac{1}{2}He^{u/2} \\ Qe^{-u/2} & -\frac{1}{4}u_z \end{pmatrix}, \\ V = \begin{pmatrix} -\frac{1}{4}u_w & -Re^{-u/2} \\ \frac{1}{2}He^{u/2} & \frac{1}{4}u_w \end{pmatrix}, \end{cases} \tag{2.22}$$

with u, Q, R and H defined in (2.13). We will also call $F : \mathbb{D}^2 \to SL(2, \mathbb{C})$ *the moving frame* of Ψ.

From the construction (2.11) of the moving frame we know that the Gauß map N is in the $\mathbb{C}^3$ picture obtained by applying $\mathcal{F}$ to the unit vector in positive z-direction, where $(x, y, z) \in \mathbb{C}^3$. In the 2×2-picture this means

$$N(z, w) = \mathrm{Ad}(F) k_0, \tag{2.23}$$

where k_0 is the diagonal matrix $(i/2)\sigma_3$.

In general, we can consider immersions from a simply connected domain $\mathbb{D}^2 \subset \mathbb{C}^2$ into any Lie algebra, and derive the structure equations for such immersions. We refer the reader to [15] for such a theory of immersions from $\mathbb{R}^2$ into any Lie algebra.

2.4. Complex CMC-immersions in $\mathbb{C}^3$

From the complex Gauß-Codazzi equations in (2.15), we have the following characterization of complex CMC-immersions.

Lemma 2.10 *Let M be a connected 2-dimensional Stein manifold, and let $\Psi : M \to \mathbb{C}^3$ be a holomorphic null immersion into $\mathbb{C}^3$. Further let Q, R, H and N be the complex functions defined in* (2.22) *and the Gauß map defined in* (2.23) *respectively. Then the following statements are equivalent*:

(1) *H is a constant*;
(2) *Q depends only on z and R depends only on w*;
(3) *$N_{zw} = \rho N$, for some function $\rho : M \to \mathbb{C}$.*

Proof. Let us parametrize M locally by null coordinates (z, w). Then from the complex Codazzi equations (2.15), the equivalence of (1) and (2) is clear. Let us write

$$N_z = a\Psi_z + b\Psi_w + cN, \tag{2.24}$$

where a, b and c are some functions. Then using $\langle \Psi_{zz}, N \rangle = Q$, $\langle \Psi_{zw}, N \rangle = 1/2He^u$ and $\langle N, N \rangle = 1$, we have

$$N_z = -H\Psi_z - 2Qe^{-u}\Psi_w. \tag{2.25}$$

Differentiating the above equation for w, we obtain

$$N_{zw} = -H_w \Psi_z - H\Psi_{zw} - 2Q_w e^{-u}\Psi_w + 2u_w Q e^{-u}\Psi_w - 2Qe^{-u}\Psi_{ww}.$$

But $\langle \Psi_z, \Psi_w \rangle = e^u/2$ and $\langle \Psi_z, \Psi_z \rangle = \langle \Psi_w, \Psi_w \rangle = 0$ imply that

$$-H\Psi_{zw} + 2u_w Q e^{-u}\Psi_w - 2Qe^{-u}\Psi_{ww} \tag{2.26}$$

is a multiple of the Gauß map N. Since N never vanishes, this defines a holomorphic function $\rho : M \to \mathbb{C}$. Therefore, we can write N_{zw} in the form

$$N_{zw} = -H_w \Psi_z - 2Q_w e^{-u}\Psi_w + \rho N.$$

Now the equivalence of (2) and (3) is clear in view of (2.15). This completes the proof. □

From Lemma 2.10, we obtain a natural definition of the notation of a "complex CMC-immersion", which is analogous to the notion of a CMC-immersion into $\mathbb{R}^3$.

Definition 2.11 Let Ψ be a holomorphic null immersion, and let H be its complex mean curvature. Then Ψ is called *a complex constant mean curvature* (CMC) *immersion* if H is constant.

Remark 2.12 The case of mean curvature $H = 0$ can be carried out in close analogy with the classical (real) case. To keep the presentation simple and to avoid repeated distinction of cases we assume from now on that the complex mean curvature of holomorphic null immersions does not vanish, i.e. $H \in \mathbb{C}^*$.

2.5. Complex CMC-immersions and loop groups

In this subsection, we introduce the "spectral parameter λ". Let $\Psi : \mathbb{D}^2 \to sl(2, \mathbb{C})$ be a complex CMC-immersion, and let $F : \mathbb{D}^2 \subset \mathbb{C}^2 \to SL(2, \mathbb{C})$ be the moving frame defined in (2.21). We introduce the following deformations for Q and R:

$$Q \to \lambda^{-2} Q \text{ and } R \to \lambda^2 R \text{ for } \lambda \in \mathbb{C}^*. \tag{2.27}$$

Since $\lambda \in \mathbb{C}^*$, the deformations (2.27) do not change the complex Gauß equation (2.15). Since H is constant, the complex Codazzi equation with parameter $\lambda \in \mathbb{C}^*$ in (2.15) also holds. We call the parameter $\lambda \in \mathbb{C}^*$ *the spectral parameter*. Therefore Theorem 2.9 (the fundamental theorem for complex submanifolds) implies that there exists a one parameter family of complex CMC-immersions $\Psi_\lambda : \mathbb{D}^2 \to sl(2,\mathbb{C})$ such that $g = 2\langle \Psi_{\lambda,z}, \Psi_{\lambda,w}\rangle dzdw = e^u dzdw$, $\langle \Psi_{\lambda,z}, \Psi_{\lambda,z}\rangle = \langle \Psi_{\lambda,w}, \Psi_{\lambda,w}\rangle = 0$, $\langle \Psi_{\lambda,zz}, N_\lambda\rangle = \lambda^{-2}Q$ and $\langle \Psi_{\lambda,ww}, N_\lambda\rangle = \lambda^2 R$. And the "Lax pair" equations of Ψ_λ take the form:

$$\begin{matrix} F_{\lambda,z} = F_\lambda U_\lambda \\ F_{\lambda,w} = F_\lambda V_\lambda \end{matrix}, \quad \text{where} \quad \begin{cases} U_\lambda = \begin{pmatrix} \frac{1}{4}u_z & -\frac{1}{2}He^{u/2} \\ \lambda^{-2}Qe^{-u/2} & -\frac{1}{4}u_z \end{pmatrix} \\ V_\lambda = \begin{pmatrix} -\frac{1}{4}u_w & -\lambda^2 Re^{-u/2} \\ \frac{1}{2}He^{u/2} & \frac{1}{4}u_w \end{pmatrix} \end{cases}. \tag{2.28}$$

We set

$$L = \begin{pmatrix} \sqrt{\lambda^{-1}} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}. \tag{2.29}$$

Then $\tilde{F}_\lambda = \mathrm{Ad}(L)F_\lambda$ is defined for $\lambda \in \mathbb{C}^*$ and satisfies the following equations:

$$\begin{matrix} \tilde{F}_{\lambda,z} = \tilde{F}_\lambda \tilde{U}_\lambda \\ \tilde{F}_{\lambda,w} = \tilde{F}_\lambda \tilde{V}_\lambda \end{matrix}, \quad \text{where} \quad \begin{cases} \tilde{U}_\lambda = \begin{pmatrix} \frac{1}{4}u_z & -\frac{1}{2}\lambda^{-1}He^{u/2} \\ \lambda^{-1}Qe^{-u/2} & -\frac{1}{4}u_z \end{pmatrix} \\ \tilde{V}_\lambda = \begin{pmatrix} -\frac{1}{4}u_w & -\lambda Re^{-u/2} \\ \frac{1}{2}\lambda He^{u/2} & \frac{1}{4}u_w \end{pmatrix} \end{cases}. \tag{2.30}$$

An argument similar to the one given above for the existence of Ψ_λ shows that there exists a holomorphic immersion $\tilde{\Psi}_\lambda$ corresponding to the moving frame $\tilde{F}_\lambda$. From Theorem 2.9, the immersions Ψ_λ and $\tilde{\Psi}_\lambda$ are the same up to a rigid motion in $sl(2,\mathbb{C})$. Thus from now on, we will use the symbol F_λ (resp. U_λ, V_λ and Ψ_λ) instead of $\tilde{F}_\lambda$ (resp. $\tilde{U}_\lambda, \tilde{V}_\lambda$ and $\tilde{\Psi}_\lambda$). Since U_λ and V_λ are in $\Lambda sl(2,\mathbb{C})_\sigma$, it is clear that F_λ is in $\in \Lambda SL(2,\mathbb{C})_\sigma$. We will also use the notation $F(z,w,\lambda) = F_\lambda(z,w)$, $U(z,w,\lambda) = U_\lambda(z,w)$ and $V(z,w,\lambda) = V_\lambda(z,w)$ respectively.

2.6. Complex Gauß maps and the Sym formula

In this subsection, we present another characterization of complex

CMC-immersions in $sl(2,\mathbb{C})$. This is based on the fact that the Gauß maps of complex CMC-immersions can be characterized similar to the Gauß maps of CMC-immersions in $\mathbb{R}^3$. Moreover, we will also obtain a Sym-type formula for complex CMC-immersions.

Let N be the Gauß map of some holomorphic null immersion Ψ defined by (2.23). Then, clearly, N is a map into $SL(2,\mathbb{C})/K$, where K is the diagonal subgroup of $SL(2,\mathbb{C})$. We note the identification $SL(2,\mathbb{C})/K \cong \mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta$, where $\Delta = (q,q) \in \mathbb{CP}^1 \times \mathbb{CP}^1$. Let $F : \mathbb{D}^2 \to SL(2,\mathbb{C})$ be the moving frame defined by (2.21), and let $\alpha = F^{-1}dF$ be the Maurer-Cartan form of F. Then α satisfies the Maurer-Cartan equation $d\alpha + 1/2[\alpha \wedge \alpha] = 0$ (see Lemma 2.7) and we have the following commutative diagram:

$$\begin{array}{ccc} & & SL(2,\mathbb{C}) \\ & \overset{F}{\nearrow} & \downarrow \pi \\ \mathbb{D}^2 \subset \mathbb{C}^2 & \xrightarrow{\quad N \quad} & SL(2,\mathbb{C})/K = \mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta \end{array}$$

We denote the Lie algebra of $SL(2,\mathbb{C})$ by $\mathfrak{g}$ and the Lie algebra of K by $\mathfrak{k}$. It is well known that $\mathfrak{g}$ can be decomposed into $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (Cartan decomposition, see [19]). Correspondingly we decompose α as follows:

$$\alpha = \alpha' dz + \alpha'' dw, \ \alpha' = \alpha_0' + \alpha_1' \text{ and } \alpha'' = \alpha_0'' + \alpha_1'', \tag{2.31}$$

where

$$\alpha_0', \ \alpha_0'' \in \mathfrak{k} \text{ and } \alpha_1', \ \alpha_1'' \in \mathfrak{p}. \tag{2.32}$$

We note that α_0', α_0'', α_1' and α_1'' can be represented by the diagonal part of U, the diagonal part of V, the off-diagonal part of U and the off-diagonal part of V respectively in (2.21). A comparison with (2.30) shows that we want the spectral parameter $\lambda \in \mathbb{C}^*$ to be inserted into the Maurer-Cartan form α as follows:

$$\alpha^\lambda := \alpha_0' dz + \alpha_0'' dw + \lambda^{-1}\alpha_1' dz + \lambda \alpha_1'' dw \text{ with } \lambda \in \mathbb{C}^*. \tag{2.33}$$

Now we are in a position to prove the following theorem:

Theorem 2.13 *Let* $N : \mathbb{D}^2 \subset \mathbb{C}^2 \to SL(2,\mathbb{C})/K$ *be the complex Gauß* *map of a holomorphic null immersion* $\Psi : \mathbb{D}^2 \to \mathbb{C}^3$. *Then the following statements are equivalent*:

(1) Ψ_λ *is a complex* CMC-*immersion in* $\mathbb{C}^3$;
(2) $N_{zw} = \rho N$, *where* $\rho : \mathbb{D}^2 \to \mathbb{C}$ *is some holomorphic function*;
(3) $\nabla^{\alpha^\lambda} := d + \alpha^\lambda$ *is a flat connection*;
(4) $d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0$.

Proof. From Lemma 2.10, we already know the equivalence of (1) and (2). The equivalence of (3) and (4) is trivial. Next we show the equivalence of (2) and (3). In preparation for this, we first rewrite $N = \mathrm{Ad}(F)k_0$, where $k_0 \in \mathfrak{k}$ (Lie algebra of K, see also (2.23)). Then we have

$$\begin{cases} N_z = \mathrm{Ad}(F)\big[\alpha', k_0\big], \\ N_w = \mathrm{Ad}(F)\big[\alpha'', k_0\big]. \end{cases}$$

Using $[\mathfrak{k}, \mathfrak{k}] = 0$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $N_{zw} = \rho N$ implies

$$\big[\alpha'', [\alpha', k_0]\big] + \big[\alpha'_w, k_0\big] = \rho k_0.$$

One obtains this by differentiating N_z for w. Of course, one could also have differentiated N_w for z. This is

$$\big[\alpha', [\alpha'', k_0]\big] + \big[\alpha''_z, k_0\big] = \rho k_0.$$

These two equations are equivalent, as follows from the integrability of α together with the Jacobi identity. Since $\mathfrak{k}$ is abelian, for the $\mathfrak{k}$-parts and $\mathfrak{p}$-parts we obtain

$$\begin{cases} \big[\alpha''_1, [\alpha'_1, k_0]\big] = \rho k_0, \\ \big[\alpha''_0, [\alpha'_1, k_0]\big] + \big[\alpha'_{1w}, k_0\big] = 0. \end{cases} \tag{2.34}$$

and

$$\begin{cases} \big[\alpha'_1, [\alpha''_1, k_0]\big] = \rho k_0, \\ \big[\alpha'_0, [\alpha''_1, k_0]\big] + \big[\alpha''_{1z}, k_0\big] = 0. \end{cases} \tag{2.35}$$

Using the Jacobi identity, $[\mathfrak{k}, \mathfrak{k}] = 0$, and the fact that $\mathrm{ad}(k_0)$ is injective on

$\mathfrak{p}$, we obtain equivalently from the second equation in (2.34) and the second equation in (2.35):

$$-\alpha'_{1w} + \left[\alpha'_1, \alpha''_0\right] = 0, \tag{2.36}$$

$$\alpha''_{1z} + \left[\alpha'_0, \alpha''_1\right] = 0. \tag{2.37}$$

Finally, again by the fact that $\mathfrak{k}$ is abelian, it is easy to see that the first equation in (2.34) and the first equation in (2.35) are equivalent and follow from the integrability of α. Thus $N_{zw} = \rho N$ is equivalent with (2.36) and (2.37), where we assume the integrability of α is true anyway.

Next we note that $d\alpha^\lambda + 1/2[\alpha^\lambda \wedge \alpha^\lambda] = 0$ for all $\lambda \in \mathbb{C}^*$ is equivalent with the fact that the coefficients of each power of λ vanish. This yields the equations

$$\alpha''_{0z} - \alpha'_{0w} + \left[\alpha'_1, \alpha''_1\right] = 0, \tag{2.38}$$

$$-\alpha'_{1w} + \left[\alpha'_1, \alpha''_0\right] = 0, \tag{2.39}$$

$$\alpha''_{1z} + \left[\alpha'_0, \alpha''_1\right] = 0. \tag{2.40}$$

Moreover, the $\mathfrak{k}$ part of $d\alpha + 1/2[\alpha \wedge \alpha] = 0$ is equivalent to (2.38), and therefore satisfied. This proves (2) $\Leftrightarrow$ (3). This completes the proof. □

Remark 2.14 If Ψ is a real CMC-immersion, i.e. $\Psi : \mathbb{D} \to \mathbb{R}^3$, then the second statement of Theorem 2.13 is equivalent to the statement that the Gauß map N is harmonic [30].

Definition 2.15 Let Ψ_λ be a complex CMC-immersion defined by the above procedure, and let $F(z, w, \lambda) = F_\lambda$ be the moving frame of Ψ_λ. Then we call $F(z, w, \lambda)$ (resp. Ψ_λ) *the complex extended framing* (resp. *the associated family*) of Ψ.

Next, we give a useful formula representing complex CMC-immersions. This is a generalization of the Sym formula [10] for CMC-immersions in $\mathbb{R}^3$.

Theorem 2.16 *Let $F(z, w, \lambda)$, $(z, w) \in \mathbb{D}^2$, $\lambda \in \mathbb{C}^*$ be the complex extended framing of a complex CMC-immersion Ψ as defined in* (2.30) *and assume $H \in \mathbb{C}^*$. We set*

$$\Psi_\lambda = -\frac{1}{2H}\left(i\lambda\partial_\lambda F(z,w,\lambda)\cdot F(z,w,\lambda)^{-1} + \frac{i}{2}F(z,w,\lambda)\sigma_3 F(z,w,\lambda)^{-1}\right), \tag{2.41}$$

where σ_3 has been defined in (2.19)*. Then Ψ_λ is for every $\lambda \in \mathbb{C}^*$ a complex constant mean curvature immersion in $\mathbb{C}^3$ with mean curvature $H \in \mathbb{C}^*$, and the Gauß map of Ψ_λ can be described by $N_\lambda = \frac{i}{2}F(z,w,\lambda)\sigma_3 F(z,w,\lambda)^{-1}$. Moreover, the moving frame for Ψ_λ is equal to $F(z,w,\lambda)$. In particular, the moving frame of $\Psi_{\lambda=1}$ is $F(z,w,\lambda=1) = F(z,w)$.*

Proof. Let Ψ_λ be the map defined by the Sym formula (2.41). We denote by (z,w) local coordinates for $\mathbb{D}^2$. We now consider derivatives of Ψ_λ for z and w:

$$\begin{cases} \Psi_{\lambda,z} = \dfrac{-i}{2}\lambda^{-1}e^{u/2}\mathrm{Ad}(F(z,w,\lambda))\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \\ \Psi_{\lambda,w} = \dfrac{-i}{2}\lambda e^{u/2}\mathrm{Ad}(F(z,w,\lambda))\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}. \end{cases} \tag{2.42}$$

Since the cross product in $\mathbb{C}^3$ corresponds to the Lie bracket in $sl(2,\mathbb{C})$, we obtain (see (2.12)) for the Gauß map N_λ of Ψ_λ

$$\begin{aligned} N_\lambda &= e^{-u}\left[\Psi_{\lambda,z} + \Psi_{\lambda,w}, i(\Psi_{\lambda,z} - \Psi_{\lambda,w})\right] \\ &= 2ie^{-u}[\Psi_{\lambda,w}, \Psi_{\lambda,z}] \\ &= -\frac{i}{2}\mathrm{Ad}(F(z,w,\lambda))\left[\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\right] \\ &= \frac{i}{2}\mathrm{Ad}(F(z,w,\lambda))\sigma_3. \end{aligned} \tag{2.43}$$

We can compute $\langle \Psi_{\lambda,z}, \Psi_{\lambda,w}\rangle = e^u/2$, $\langle \Psi_{\lambda,z}, \Psi_{\lambda,z}\rangle = \langle \Psi_{\lambda,w}, \Psi_{\lambda,w}\rangle = 0$, $\langle \Psi_{\lambda,zz}, N_\lambda\rangle = \lambda^{-2}Q$, $\langle \Psi_{\lambda,ww}, N_\lambda\rangle = \lambda^2 R$ and $\langle \Psi_{\lambda,zw}, N_\lambda\rangle = He^u/2$. Therefore, Ψ_λ is a complex CMC-immersion. This completes the proof. □

From the above theorem, we have the following corollary:

Corollary 2.17 *We retain the assumptions of Theorem* 2.16*. Let l be a λ-independent $SL(2,\mathbb{C})$ diagonal matrix with entries $l_0 \neq 0$ and $l_0^{-1} \neq 0$.*

We set

$$\tilde{\Psi}_\lambda = -\frac{1}{2H}\left(i\lambda\partial_\lambda(Fl)\cdot(Fl)^{-1} + \frac{i}{2}(Fl)\sigma_3(Fl)^{-1}\right), \text{ with } \lambda \in \mathbb{C}^*.$$

Then $\tilde{\Psi}_\lambda = \Psi_\lambda$.

3. Complex CMC-immersions via generalized Weierstraß type representations

In this section, we present a "generalized Weierstraß type representation" for complex CMC-immersions which is analogous to the "generalized Weierstraß type representation for CMC-immersions in the double loop group picture" as presented in Section A.3. We also give examples of complex CMC-immersions using this representation.

3.1. The pair of holomorphic potentials associated with complex CMC-immersions

In this subsection, we produce for every complex CMC-immersion a pair of $\Lambda sl(2,\mathbb{C})_\sigma$-valued holomorphic 1-forms, called "the pair of holomorphic potentials".

Theorem 3.1 *Let M be a connected 2-dimensional Stein manifold. Let $\Psi_\lambda : M \to \mathbb{C}^3$ be an associated family of complex CMC-immersions with mean curvature $H \in \mathbb{C}^*$. Let $F(z,w,\lambda)$ be the extended framing of Ψ_λ. Then there exist $V_+ : M \to \Lambda^+ SL(2,\mathbb{C})_\sigma$ and $V_- : M \to \Lambda^- SL(2,\mathbb{C})_\sigma$ such that*

$$C(z,\lambda) = F(z,w,\lambda)V_+(z,w,\lambda), \tag{3.1}$$

$$L(w,\lambda) = F(z,w,\lambda)V_-(z,w,\lambda). \tag{3.2}$$

In particular we have $\partial_w C = 0$ and $\partial_z L = 0$. Moreover, setting $\eta = C^{-1}dC$ and $\tau = L^{-1}dL$ we obtain a pair of $\Lambda sl(2,\mathbb{C})_\sigma$-valued 1-forms of the form

$$\check{\eta} = (\eta(z,\lambda),\ \tau(w,\mu)) = \left(\sum_{k=-1}^{\infty}\eta_k(z)\lambda^k,\ \sum_{m=-\infty}^{1}\tau_m(w)\mu^m\right). \tag{3.3}$$

The upper right entry of η_{-1} and the lower left entry of τ_1 do not vanish on M.

Proof. To find the pair of holomorphic potentials we are looking for, we look for matrix valued functions V_+ and V_- satisfying

$$C(z,\lambda) = F(z,w,\lambda)V_+(z,w,\lambda), \quad L(w,\lambda) = F(z,w,\lambda)V_-(z,w,\lambda), \tag{3.4}$$

where $V_+(z,w,\lambda) \in \Lambda^+SL(2,\mathbb{C})_\sigma$ and $V_-(z,w,\lambda) \in \Lambda^-SL(2,\mathbb{C})_\sigma$.

Let (U_α) be an open cover of M, where $U_\alpha = \{(z,w) \in \mathbb{C}^2 \mid |z-z_\alpha| \leq \epsilon, |w-w_\alpha| \leq \epsilon\}$. We consider the first equation in (3.4) on U_α. This is equivalent to

$$V_{+\alpha,w}(z,w,\lambda) = -V(z,w,\lambda)V_{+\alpha}(z,w,\lambda), \tag{3.5}$$

where the subscript w means the partial derivative with respect to w. Thus we look for a solution $V_{+\alpha} \in \Lambda^+SL(2,\mathbb{C})$ to equation (3.5). Since the frame F is holomorphic, $V = F^{-1}F_w$ is also holomorphic. So, we can write

$$V = \sum_{j,k\in\mathbb{Z}} V_{j,k}(z-z_\alpha)^j(w-w_\alpha)^k,$$

for all $|z-z_\alpha| \leq \epsilon$, $|w-w_\alpha| \leq \epsilon$, where ϵ is sufficient small and V as in (2.22). Now for every fixed $z, \partial_w V_{+\alpha} = -VV_{+\alpha}$ is an ordinary differential equation. We solve this ODE with initial condition $V_{+\alpha}(z_\alpha, w_\alpha) = \mathrm{Id}$. Note that $\det V_{+\alpha}(z,w) \equiv 1$, since $\mathrm{Tr}(V(z,w)) \equiv 0$. Note as well that $V_{+\alpha}(z,w)$ is defined for all (z,w) satisfying $|z-z_\alpha| \leq \epsilon$ and $|w-w_\alpha| \leq \epsilon$, because the differential equation is linear. Note that $V_{+\alpha}(z,w)$ is twisted, that is, $V_{+\alpha}(-\lambda) = \sigma_3 V_{+\alpha}(\lambda)\sigma_3$, because $V_{+\alpha}|_{z=z_\alpha} = \mathrm{Id}$ and V is twisted.

Thus, for all α, we can find holomorphic solutions $V_{+\alpha} : U_\alpha \to \Lambda^+SL(2,\mathbb{C})_\sigma$ to (3.5). On $U_\alpha \cap U_\beta$, we define

$$h_{\alpha\beta} = V_{+\alpha}^{-1}V_{+\beta} : U_\alpha \cap U_\beta \to \Lambda^+SL(2,\mathbb{C})_\sigma.$$

Then $h_{\alpha\beta}$ satisfies the co-cycle condition, i.e. $h_{\alpha\beta}h_{\beta\gamma} = h_{\alpha\gamma}$. Moreover, $h_{\alpha\beta}$ is holomorphic. Thus there exists a holomorphic $\Lambda^+SL(2,\mathbb{C})_\sigma$-principal fiber bundle $P \to M$ whose holomorphic sections are described by (3.5). Since M is a Stein manifold, by a generalization of Grauert's theorem [4], such a holomorphic bundle is trivial. Thus we have $h_{\alpha\beta} = h_\alpha h_\beta^{-1}$ for all α and β, where $h_\alpha : U_\alpha \to \Lambda^+SL(2,\mathbb{C})_\sigma$ is holomorphic. Then $V_+ := V_{+\alpha}h_\alpha$ is well-defined on M and (3.1) holds on M. Analogously one can prove (3.2).

Note that this implies obviously $\partial_w C = 0$ and $\partial_z L = 0$. To verify the last statement we note that, since $\eta = V_+^{-1}(F^{-1}dF)V_+ + V_+^{-1}dV_+$ and $V_+|_{\lambda=0}$ has no zeros on M, the λ^{-1}-term of the upper right entry of η does not vanish on M. Analogously one can show that λ-term of the left lower entry of τ does not vanish. This completes the proof. □

Definition 3.2 Let $\check{\eta}$ be the pair of holomorphic 1-forms defined in Theorem 3.1. We call $\check{\eta}$ *the pair of holomorphic potentials* associated with Ψ_λ.

Remark 3.3

(1) The first part of Theorem 3.1, which is the existence of $C(z,\lambda)$ and $L(w,\lambda)$, also holds in the case $H = 0$.
(2) For a pair of potentials $\check{\eta} = (\eta, \tau)$, the condition $H \in \mathbb{C}^*$ is equivalent with saying that the right upper entry of η_{-1} and the left lower entry of τ_1 never vanish. We will refer to this property as "Property $(*)$".
(3) In case the two Hopf differentials Q and R never vanish property $(*)$ can be expressed in a simple way. In this case: property $(*)$ $\Leftrightarrow$ η_{-1} and τ_1 are semisimple on M $\Leftrightarrow$ $\det \eta_{-1} \neq 0$, $\det \tau_1 \neq 0$ on M.

3.2. Complex CMC-immersions from a pair of holomorphic potentials

In this subsection, we explain how one can construct a complex CMC-immersion from a pair of holomorphic potentials. Let $\check{\eta} = (\eta(z,\lambda), \tau(w,\lambda))$ be a pair of holomorphic potentials

$$\check{\eta} = (\eta(z,\lambda), \tau(w,\lambda)) = \left(\sum_{k=-1}^{\infty} \eta_k(z)\lambda^k, \ \sum_{m=-\infty}^{1} \tau_m(w)\lambda^m \right), \tag{3.6}$$

where $(z,w) \in \mathbb{D}^2$, $\mathbb{D}^2$ is some simply connected holomorphically convex domain in $\mathbb{C}^2$, $\lambda \in \mathbb{C}^*$ [17], and η_k and τ_m are $sl(2,\mathbb{C})$-valued holomorphic differential 1-forms. Moreover, $\check{\eta}$ is twisted. We assume that the upper right entry of $\eta_{-1}(z)$ and the lower left entry $\tau_1(w)$ do not vanish for all $(z,w) \in \mathbb{D}^2$.

Let C and L denote the solutions to the differential equations

$$\begin{cases} dC = C\eta, & C(z_0,\lambda) = \mathrm{Id}, \\ dL = L\tau, & L(w_0,\mu) = \mathrm{Id}, \end{cases} \tag{3.7}$$

where $(z_0, w_0) \in \mathbb{D}^2$ is a fixed base point. We consider the generalized Iwasawa decomposition of Theorem A.3:

$$(C, L) = (F, F)(\mathrm{Id}, W)(V_+, V_-). \tag{3.8}$$

In view of the initial conditions and the fact that the big cell in the double loop group is open, we can assume $W = \mathrm{Id}$, if (z, w) is sufficiently close to (z_0, w_0). Thus

$$C = FV_+, \quad L = FV_-, \tag{3.9}$$

and F is holomorphic in the two complex variables z and w [21]. After these preparations we can prove:

Theorem 3.4 *With the assumptions above, let $F(z, w, \lambda)$ be defined by the generalized Iwasawa decomposition* (3.8)*. Then there exists a λ-independent diagonal matrix $l(z, w)$ such that $F(z, w, \lambda)l(z, w)$ is a complex extended framing of the associated family of complex* CMC*-immersions $\Psi_\lambda : \tilde{\mathbb{D}}^2 \to \mathbb{C}^3$ defined by the Sym formula* (2.41)*, where $\tilde{\mathbb{D}}^2$ is an open neighborhood of $(z_0, w_0) \in \mathbb{D}^2$. Moreover, $l(z, w)$ can be determined from V_+ and V_- of* (3.8) *as stated in the proof below.*

Proof. Let us compute the Maurer-Cartan form $\alpha = F^{-1}dF$. From the first equation in (3.9), we have

$$\alpha = V_+ \eta V_+^{-1} - dV_+ V_+^{-1} = \sum_{j=-1}^{\infty} \lambda^j \hat{\alpha}_j, \tag{3.10}$$

where $\hat{\alpha}_j$ are $sl(2, \mathbb{C})$-valued 1-forms. On the other hand, from the second equation in (3.9), we have

$$\alpha = V_- \tau V_-^{-1} - dV_- V_-^{-1} = \sum_{j=-\infty}^{1} \lambda^j \tilde{\alpha}_j, \tag{3.11}$$

where $\tilde{\alpha}_j$ are $sl(2, \mathbb{C})$-valued 1-forms. Thus, we obtain $\alpha = \sum_{j=-1}^{1} \lambda^j \alpha_j$. Since F is in $\Lambda SL(2, \mathbb{C})_\sigma$, α_j is diagonal (resp. off-diagonal) if j is even (resp. odd), and $\mathrm{Tr}(\alpha_j) = 0$. We set

$$\alpha_{-1} = \begin{pmatrix} 0 & \alpha_{-112} \\ \alpha_{-121} & 0 \end{pmatrix} \quad \text{and} \quad \alpha_1 = \begin{pmatrix} 0 & \alpha_{112} \\ \alpha_{121} & 0 \end{pmatrix}.$$

Let us consider the constant coefficient of the Fourier expansion of V_+ (resp. V_-), and the λ^{-1} (resp. λ) coefficient η_{-1} (resp. τ_1) of η (resp. τ). These matrices we write respectively in the form:

$$\begin{gathered} V_+(z, w, \lambda = 0) = \begin{pmatrix} v_+ & 0 \\ 0 & v_+^{-1} \end{pmatrix}, \quad V_-(z, w, \mu = \infty) = \begin{pmatrix} v_- & 0 \\ 0 & v_-^{-1} \end{pmatrix}, \\ \eta_{-1} = \begin{pmatrix} 0 & \eta_{12} \\ \eta_{21} & 0 \end{pmatrix} \quad \text{and} \quad \tau_1 = \begin{pmatrix} 0 & \tau_{12} \\ \tau_{21} & 0 \end{pmatrix}. \end{gathered} \tag{3.12}$$

With this notation we obtain

$$\alpha = \lambda^{-1} \begin{pmatrix} 0 & v_+^2 \eta_{12} \\ v_+^{-2} \eta_{21} & 0 \end{pmatrix} dz + \alpha_0 + \lambda \begin{pmatrix} 0 & v_-^2 \tau_{12} \\ v_-^{-2} \tau_{21} & 0 \end{pmatrix} dw. \tag{3.13}$$

From the coefficients of the dw (resp. the dz) part of α in (3.10) (resp. (3.11)), we have

$$\alpha_0 = \begin{pmatrix} -(\log v_+)_w dw - (\log v_-)_z dz & 0 \\ 0 & (\log v_+)_w dw + (\log v_-)_z dz \end{pmatrix}.$$

Since η_{12} and τ_{21} never vanish on $(z, w) \in \mathbb{D}^2$, we consider the following holomorphic change of coordinates:

$$(\tilde{z}, \tilde{w}) = \left(\frac{-2}{H} \int \eta_{12} dz, \frac{2}{H} \int \tau_{21} dw \right). \tag{3.14}$$

Noting that $v_- \neq 0$ and $v_+ \neq 0$, we set

$$l = \begin{pmatrix} \sqrt{v_+ v_-} & 0 \\ 0 & 1/\sqrt{v_+ v_-} \end{pmatrix}. \tag{3.15}$$

Then the Maurer-Cartan equation for $F \cdot l$ under the holomorphic change of coordinates (3.14) is

$$\check{\alpha} = (F\cdot l)^{-1} d(F\cdot l)$$

$$= \lambda^{-1}\begin{pmatrix} 0 & -\frac{Hv_+v_-^{-1}}{2} \\ -\frac{H\eta_{21}v_+^{-1}v_-}{2\eta_{12}} & 0 \end{pmatrix} d\tilde{z} + \check{\alpha}_0 + \lambda \begin{pmatrix} 0 & \frac{H\tau_{12}v_+^{-1}v_-}{2\tau_{21}} \\ \frac{Hv_+v_-^{-1}}{2} & 0 \end{pmatrix} d\tilde{w},$$

where

$$\check{\alpha}_0 = \begin{pmatrix} \frac{1}{2}(\log v_+v_-^{-1})_{\tilde{z}}d\tilde{z} \\ \quad -\frac{1}{2}(\log v_+v_-^{-1})_{\tilde{w}}d\tilde{w} & 0 \\ 0 & -\frac{1}{2}(\log v_+v_-^{-1})_{\tilde{z}}d\tilde{z} \\ & \quad +\frac{1}{2}(\log v_+v_-^{-1})_{\tilde{w}}d\tilde{w} \end{pmatrix}.$$

Using Corollary 2.5, we set $v_+v_-^{-1} = e^{u/2}$, and we also set $Q = -(H/2)\eta_{21}\eta_{12}^{-1}$ and $R = -(H/2)\tau_{12}\tau_{21}^{-1}$. Then $\check{\alpha} = \tilde{U}_\lambda dz + \tilde{V}_\lambda dw$, where $\tilde{U}_\lambda$ and $\tilde{V}_\lambda$ have the form stated in (2.30). Therefore, $F\cdot l$ is a complex extended framing of some complex CMC-immersion. This completes the proof. □

Remark 3.5
(1) By Corollary 2.17, F and $F\cdot l$ in Theorem 3.4 define the same complex CMC-immersions.
(2) We can also use a pair of meromorphic potentials instead of a pair of holomorphic potentials in the above procedure.

Next we show that the pair of holomorphic potentials is not uniquely defined for the complex CMC-immersion. Let $\check{\eta} = (\eta(z,\lambda), \tau(w,\lambda))$ be a pair of holomorphic potentials defined in (3.6), and let $\check{\eta}_2$ be the pair of holomorphic potentials defined as follows:

$$\check{\eta}_2 = (\eta_2, \tau_2) = \big(\tilde{V}_+\eta\tilde{V}_+^{-1} + \tilde{V}_+^{-1}d\tilde{V}_+,\ \tilde{V}_-\tau\tilde{V}_-^{-1} + \tilde{V}_-^{-1}d\tilde{V}_-\big), \qquad (3.16)$$

where $\tilde{V}_+ : \mathbb{D}^2 \to \Lambda^+SL(2,\mathbb{C})_\sigma$ (resp. $\tilde{V}_- : \mathbb{D}^2 \to \Lambda^-SL(2,\mathbb{C})_\sigma$) is some positive (resp. negative) loop depending only on z (resp. w) with the initial condition $\tilde{V}_+(z_0) = \mathrm{Id}$ (resp. $\tilde{V}_-(w_0) = \mathrm{Id}$). We note that $\check{\eta}_2$ is a pair of holomorphic potentials, since $V_+ \in \Lambda^+SL(2,\mathbb{C})_\sigma$ and $V_- \in \Lambda^-SL(2,\mathbb{C})_\sigma$. Then the solutions to the differential equations (3.7) with the pair of holomorphic potentials $\check{\eta}_2$ can be written in $(C\tilde{V}_+, L\tilde{V}_-)$, where C and L are the solutions of the differential equations (3.7) with the pair of holomorphic potentials $\check{\eta}$. By the generalized Iwasawa decomposition of Theorem A.3, we have

$$\big(C\tilde{V}_+(z), L\tilde{V}_-(w)\big) = (F,F)(\mathrm{Id},W)\big(V_+\tilde{V}_+(z), V_-\tilde{V}_-(w)\big), \tag{3.17}$$

where $(C,L) = (F,F)(\mathrm{Id},W)(V_+,V_-)$ is the generalized Iwasawa decomposition for (C,L).

From the above discussion, we have

Remark 3.6 If the pairs of holomorphic potentials $\check{\eta}$ and $\check{\eta}_2$ are related via a pair $(\tilde{V}_+(z), \tilde{V}_-(w))$, then we say that $\check{\eta}_2$ is obtained from $\check{\eta}$ via *a holomorphic gauge* (or by *a pair of holomorphic gauges*). Setting $\tilde{h}_+ = \tilde{V}_+(z_0)^{-1}$ and $\tilde{h}_- = \tilde{V}_-(w_0)^{-1}$ we consider $(\tilde{h}_+F, \tilde{h}_-F) = (\tilde{F},\tilde{F})(\tilde{L}_+,\tilde{L}_-)$ and obtain

$$\big(\tilde{h}_+C\tilde{V}_+(z), \tilde{h}_-L\tilde{V}_-(w)\big) = (\tilde{F},\tilde{F})\big(\tilde{L}_+V_+\tilde{V}_+(z), \tilde{L}_-V_-\tilde{V}_-(w)\big)$$

in a sufficiently small neighborhood of (z_0,w_0). Therefore, the surface associated with $\check{\eta}_2$ is the one associated with $\tilde{F}$. We say that the surface associated with $\check{\eta}_2$ has been obtained from the surface associated with $\check{\eta}$ by *dressing*. Clearly, if $\tilde{V}_+(z_0) = \mathrm{Id}$ and $\tilde{V}_-(w_0) = \mathrm{Id}$, then $F = \tilde{F}$ and the holomorphic gauge $(\tilde{V}_+(z), \tilde{V}_-(w))$ does not change the surface.

Remark 3.7 Due to the fact that the generalized Iwasawa decomposition is not global, the extended framing obtained by (3.8) is only meromorphic. Thus the immersion Ψ defined from F by the Sym-Formula is meromorphic in (z,w) and thus will have in general singularities where F has singularities. We will sometimes refer to such a Ψ as a *meromorphic immersion*.

3.3. The pair of normalized potentials for a complex CMC-immersion

In this subsection we show that for every complex CMC-immersion there exists a pair $\check{\eta} = (\eta,\tau)$ of meromorphic potentials, where η only has a λ^{-1}-term and τ only has a λ-term.

Let M be a connected 2-dimensional Stein manifold, and $\widetilde{M}$ be its simply connected universal cover. For our purpose it is sufficient to assume $\widetilde{M}$ is a domain in $\mathbb{C}^2$. We note that $\widetilde{M}$ is again a Stein manifold (see [32], [17] Chapter V, page 126). Let $F(z,w,\lambda) : \widetilde{M} \to \Lambda SL(2,\mathbb{C})_\sigma$ be the extended framing of some associated family of complex CMC-immersions $\Psi_\lambda : M \to \mathbb{C}^3$, and (z_0, w_0) be some fixed point in $\widetilde{M}$ such that $F(z_0, w_0, \lambda) = \mathrm{Id}$. Let $C(z, \lambda)$ and $L(w, \lambda)$ be $\Lambda SL(2, \mathbb{C})_\sigma$ loops as defined in (3.1) and (3.2). First we apply the Birkhoff decomposition Theorem A.1 to $F(z, w, \lambda)$ for (z, w) sufficiently close to $(z_0, w_0) \in \mathbb{D}^2$. We obtain:

$$\begin{cases} C(z,\lambda) = F_-(z,w,\lambda) \cdot (F_+(z,w,\lambda)V_+(z,w,\lambda)), \\ L(w,\lambda) = \tilde{F}_+(z,w,\lambda) \cdot (\tilde{F}_-(z,w,\lambda)V_-(z,w,\lambda)), \end{cases} \tag{3.18}$$

where F_- and $\tilde{F}_-$ (resp. F_+ and $\tilde{F}_+$) are loops in $\Lambda_*^- SL(2, \mathbb{C})_\sigma$ and $\Lambda^- SL(2, \mathbb{C})_\sigma$ (resp. $\Lambda^+ SL(2, \mathbb{C})_\sigma$ and $\Lambda_*^+ SL(2, \mathbb{C})_\sigma$). Since (3.18) describes the Birkhoff decompositions of $C(z, \lambda)$ and $L(w, \lambda)$, it follows that F_- (resp. $\tilde{F}_+$) depends only on z (resp. w).

Remark 3.8 In general, we have diagonal terms w_n in the Birkhoff decomposition as in Theorem A.1. We denote $S = \{p \in \widetilde{M} \mid F_-(p)$ is not contained in the big cell$\} \subset \widetilde{M}$. Then by Lemma 2.6 in [12], F_- extends meromorphically to S.

We fix w as w_0 and z as z_0 in the first and second equations in (3.18) respectively. Then we have:

$$F_-(z,\lambda) = C(z,\lambda)(F_+(z,w_0,\lambda)V_+(z,w_0,\lambda))^{-1},$$
$$\tilde{F}_+(w,\lambda) = L(w,\lambda)(\tilde{F}_-(z_0,w,\lambda)V_-(z_0,w,\lambda))^{-1}.$$

From (3.6), we have $(C(z)^{-1}dC(z), L(w)^{-1}dL(w)) = \big(\sum_{j=-1}^{\infty} \eta_j \lambda^j, \sum_{j=-\infty}^{1} \tau_j \lambda^j\big)$, and we note that $(F_+V_+)^{-1} \in \Lambda^+ SL(2, \mathbb{C})_\sigma$ and $(\tilde{F}_-V_-)^{-1} \in \Lambda^- SL(2, \mathbb{C})_\sigma$. From this we infer

$$\begin{cases} \xi(z,\lambda) := F_-(z,\lambda)^{-1} dF_-(z,\lambda) = \lambda^{-1} \begin{pmatrix} 0 & f \\ \hat{Q}/f & 0 \end{pmatrix} dz, \\ \nu(z,\lambda) := \tilde{F}_+(w,\lambda)^{-1} d\tilde{F}_+(w,\lambda) = \lambda \begin{pmatrix} 0 & \hat{R}/g \\ g & 0 \end{pmatrix} dw, \end{cases}$$

where $\hat{R}$ (resp. $\hat{Q}$) is a holomorphic function on $\widetilde{M}$ with respect to w (resp. z) and g (resp. f) is a meromorphic function with respect to w (resp. z).

We summarize the discussion above in the following theorem [35]:

Theorem 3.9 *Let M be a connected 2-dimensional Stein manifold, and let $\widetilde{M} \subset \mathbb{C}^2$ be its simply connected universal cover. Let $F(z, w, \lambda) : \widetilde{M} \to \Lambda SL(2, \mathbb{C})_\sigma$ be the extended framing of some associated family of complex* CMC*-immersions $\Psi_\lambda : M \to \mathbb{C}^3$ with mean curvature $H \in \mathbb{C}^*$. Then Ψ_λ is produced by a pair of meromorphic* 1*-forms $\check{\xi}$ on $\widetilde{M}$ of the form*

$$\check{\xi} = (\xi(z, \lambda), \nu(w, \lambda)) = \left(\lambda^{-1} \begin{pmatrix} 0 & f \\ \hat{Q}/f & 0 \end{pmatrix} dz,\ \lambda \begin{pmatrix} 0 & \hat{R}/g \\ g & 0 \end{pmatrix} dw\right), \tag{3.19}$$

where $\hat{R}$ and $\hat{Q}$ are holomorphic functions on $\widetilde{M}$ with respect to w (resp. z) and g and f are meromorphic functions with respect to w (resp. z). Moreover, f, g, Q and R are obtained as follows:

$$\begin{cases} f = -\dfrac{H}{2} \exp(u(z, w_0) - u(z_0, w_0)/2), \\ g = \dfrac{H}{2} \exp(u(z_0, w) - u(z_0, w_0)/2), \\ \hat{Q} = -\dfrac{H}{2} \langle \Psi_{zz}, N \rangle = -\dfrac{H}{2} Q \ \textit{and}\ \hat{R} = -\dfrac{H}{2} \langle \Psi_{ww}, N \rangle = -\dfrac{H}{2} R, \end{cases} \tag{3.20}$$

where $e^{u(z,w)}$ is the holomorphic metric defined in (2.9) *for the complex* CMC*-immersions Ψ_λ, (z_0, w_0) is some fixed point in $\widetilde{M}$ such that $F(z_0, w_0, \lambda) = \mathrm{Id}$, H is the complex mean curvature of Ψ_λ, which is a non-zero constant, and Q and R are defined in* (2.13).

Proof. We only need to show that f, g, $\hat{Q}$ and $\hat{R}$ have the form in (3.20). Since $F(z, w, \lambda) = F_-(z, \lambda) F_+(z, w, \lambda)$ by the Birkhoff decomposition Theorem A.1, we obtain

$$\begin{aligned} \xi(z, \lambda) &= F_-(z, \lambda)^{-1} dF_-(z, \lambda) \\ &= \mathrm{Ad}(F_+(z, w_0, \lambda))\big(F(z, w_0, \lambda)^{-1} dF(z, w_0, \lambda)\big) \\ &\quad - dF_+(z, w_0, \lambda) F_+(z, w_0, \lambda)^{-1}. \end{aligned}$$

From (2.30), we have

$$F(z,w_0,\lambda)^{-1}\partial_z F(z,w_0,\lambda) = \begin{pmatrix} \frac{1}{4}u_z(z,w_0) & -\frac{1}{2}\lambda^{-1}He^{u(z,w_0)/2} \\ \lambda^{-1}Qe^{-u(z,w_0)/2} & -\frac{1}{4}u_z(z,w_0) \end{pmatrix}.$$

Since $F_- \in \Lambda_*^- SL(2,\mathbb{C})$ and $F(z_0,w_0,\lambda) = \mathrm{Id}$, we have $F_-(z_0,\lambda) = \mathrm{Id}$. Noting $F_- \in \Lambda_*^- SL(2,\mathbb{C})_\sigma$ and comparing the diagonal part of $F(z,w_0,\lambda)^{-1}dF(z,w_0,\lambda)$ to $\xi(z,\lambda)$, we obtain

$$F_+(z,w_0,\lambda=0) = \begin{pmatrix} e^{u(z,w_0)/4-u(z_0,w_0)/4} & 0 \\ 0 & e^{-u(z,w_0)/4+u(z_0,w_0)/4} \end{pmatrix}.$$

Therefore f and $\hat{Q}$ are the functions defined in (3.20). A similar argument applies to $\nu(w,\lambda)$. This completes the proof. □

Definition 3.10 The potential $\check{\xi} = (\xi,\nu)$ just defined will be called *the pair of normalized potentials* for the complex CMC-immersion Ψ.

3.4. Singular points of complex CMC-immersions

In Section 3.2 we started form a pair of holomorphic potentials and obtained a complex CMC-immersion which may only be meromorphic as a function of (z,w) due to the fact that generalized Iwasawa decomposition is not global. In this section, we consider a different type of singularity.

Lemma 3.11 *Let $\check{\eta} = (\eta,\tau)$ be a pair of holomorphic potentials of the form* (3.6) *for which the upper right entry of η_{-1} and the lower left entry of τ_1 does not vanish identically. Then the resulting map Ψ_λ obtained in Theorem* 3.3 *is not an immersion at $(z_p,w_p) \in \mathbb{D}^2$ if the upper right entry of η_{-1} has a zero at z_p or the lower left entry of τ_1 has a zero at w_p and if the extended framing F associated with $\check{\eta}$ is defined at (z_p,w_p).*

Proof. We note that on the connected open set $\Omega \subset \mathbb{D}^2$, where the right upper entry of η_{-1} and the left lower entry of τ_1 do not vanish we can apply the results of the previous sections and obtain (at least locally) complex CMC-immersions with mean curvature $H \neq 0$. However, the group decomposition results also apply to points outside of Ω. Let F be the extended framing obtained from the pair of potentials $\check{\eta} = (\eta,\tau)$ by the procedure (of Theorem 3.3). Then the Maurer-Cartan form $\alpha = F^{-1}dF$ has the form (2.30). The Maurer-Cartan form α also can be written as follows:

$$\begin{aligned}\alpha &= V_+ C^{-1} dC V_+^{-1} - dV_+ V_+^{-1} = V_- L^{-1} dL V_-^{-1} - dV_- V_-^{-1} \\ &= V_+ \eta V_+^{-1} - dV_+ V_+^{-1} = V_- \tau V_-^{-1} - dV_- V_-^{-1}.\end{aligned} \tag{3.21}$$

where C and L are defined by (3.7), V_+ and V_- are defined by (3.8). We denote the upper right entry of η_{-1} by $\eta_{12}(z)$ and the lower left entry of τ_1 by $\tau_{21}(w)$. By (3.13), the λ^{-1} coefficient of the upper right entry of α is $v_+^2 \eta_{12}$ and the λ^{-1} coefficient of the lower left entry of α is $v_-^2 \tau_{21}$. Using the λ-independent diagonal matrix l with entries l_0 and l_0^{-1} as in (3.15), we have $\hat{\alpha} = (F \cdot l)^{-1} d(F \cdot l) = \mathrm{Ad}(l^{-1})\alpha + l^{-1} dl$. Comparing the Maurer-Cartan form $\hat{\alpha}$ to (2.30), we obtain

$$\begin{cases} l_0(z,w)^{-2} v_+(z,w)^2 \eta_{12}(z) = -\dfrac{1}{2} H e^{u(z,w)/2}, \\ l_0(z,w)^2 v_-(z,w)^{-2} \tau_{21}(w) = \dfrac{1}{2} H e^{u(z,w)/2}. \end{cases} \tag{3.22}$$

Note that l_0, v_- and v_+ have no zeros or poles on $\mathbb{D}^2$ by our assumptions. Therefore we have the singularity $e^{u(z_p,w_p)/2} = 0$ if $\eta_{12}(z_p) = 0$ or $\tau_{21}(w_p) = 0$. □

3.5. Examples of complex CMC-immersions in $\mathbb{C}^3$

In this subsection, we present some examples of complex CMC-immersions starting from a pair of holomorphic potentials as defined in Section 3.2. These four examples have the symmetry typical for the complexfication of some real CMC-immersion (see Section 4.3).

Example 3.12 (Complex sphere) We set

$$\check{\eta} = (\eta(z,\lambda), \tau(w,\lambda)) = \left(\begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} dz, \ \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix} dw \right). \tag{3.23}$$

Solutions to the holomorphic differential equations (3.7) with the above pair of holomorphic potentials $\check{\eta}$ are

$$C = \begin{pmatrix} 1 & \lambda^{-1} z \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 1 & 0 \\ -\lambda w & 1 \end{pmatrix}.$$

Then, in this case, the Iwasawa decomposition of (C, L) can be computed explicitly as follows:

$$
\begin{aligned}
(C, L) &= (F, F)(V_+, V_-) \\
&= \left(\begin{pmatrix} \frac{1}{1+zw} & \lambda^{-1} z \\ -\frac{\lambda w}{1+zw} & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{1+zw} & \lambda^{-1} z \\ -\frac{\lambda w}{1+zw} & 1 \end{pmatrix} \right) \\
&\quad \cdot \left(\begin{pmatrix} 1 & 0 \\ \frac{\lambda w}{1+zw} & 1 \end{pmatrix}, \begin{pmatrix} 1+zw & -\lambda^{-1} z \\ 0 & \frac{1}{1+zw} \end{pmatrix} \right).
\end{aligned}
$$

Using the Sym formula (2.41), we obtain

$$
\Psi_\lambda = -\frac{i}{4H(1+zw)} \begin{pmatrix} -3zw+1 & -4\lambda^{-1} z \\ -4\lambda w & 3zw-1 \end{pmatrix}.
$$

The resulting immersion defines a complex sphere centered at $(0,0,-1/(2H))$ with radius $1/H$:

$$
\left(\frac{-(\lambda^{-1} z + \lambda w)}{H(1+zw)} \right)^2 + \left(\frac{i(\lambda^{-1} z - \lambda w)}{H(1+zw)} \right)^2 + \left(\frac{1-zw}{H(1+zw)} \right)^2 = \frac{1}{H^2}. \quad (3.24)
$$

Example 3.13 (Complex cylinder) We set

$$
\check{\eta} = (\eta(z,\lambda), \tau(w,\lambda)) = \left(\lambda^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dz, \ -\lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dw \right). \quad (3.25)
$$

Solutions to the holomorphic differential equation (3.7) with the above pair of holomorphic potentials $\check{\eta}$ are

$$
C = \exp\left(\lambda^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z \right) \ \text{and} \ L = \exp\left(-\lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w \right).
$$

Then the Iwasawa decomposition $(C, L) = (F, F)(V_+, V_-)$ of (C, L) can be computed explicitly as follows:

$$
F = \exp\left((\lambda^{-1} z - \lambda w) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),
$$

$$V_+ = \exp\left(\lambda w \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad \text{and} \quad V_- = \exp\left(-\lambda^{-1} z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

Using the Sym formula (2.41), we obtain for the complex CMC-immersion the explicit formula:

$$\begin{aligned}\Psi_\lambda &= -\frac{i}{2H}\begin{pmatrix} 1/2 + \sinh 2(p) & -\lambda^{-1}z - \lambda w - \cosh(p)\sinh(p) \\ -\lambda^{-1}z - \lambda w + \cosh(p)\sinh(p) & -1/2 - \sinh 2(p) \end{pmatrix} \\ &= -\frac{i}{4H}\begin{pmatrix} \cosh(2p) & -2(\lambda^{-1}z + \lambda w) - \sinh(2p) \\ -2(\lambda^{-1}z + \lambda w) + \sinh(2p) & -\cosh(2p) \end{pmatrix},\end{aligned}$$

where $p = \lambda^{-1}z - \lambda w$. The resulting immersion in $\mathbb{C}^3$ is the surface

$$\left(-\frac{1}{H}(\lambda^{-1}z + \lambda w),\ -\frac{1}{2H}\sin(2i(\lambda^{-1}z - \lambda w)),\ \frac{1}{2H}\cos(2i(\lambda^{-1}z - \lambda w))\right)^t.$$

If $2ip = 0$, then this surface looks like a “complex cylinder” of radius $1/(2H)$ with axis the x-axis and with profile curve

$$\left(-\frac{1}{H}(\lambda^{-1}z + \lambda w),\ 0,\ \frac{1}{2H}\right)^t. \tag{3.26}$$

Example 3.14 (Complex surfaces of revolution) We set

$$\check{\eta} = (\eta(z,\lambda), \tau(w,\lambda)) = (Adz,\ Bdw), \tag{3.27}$$

where

$$A = \begin{pmatrix} 0 & \lambda^{-1}a + \lambda b \\ \lambda^{-1}b + \lambda a & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -\lambda^{-1}a - \lambda b \\ -\lambda^{-1}b - \lambda a & 0 \end{pmatrix},$$

with $a, b \in \mathbb{R}^*$, $a + b = 1/2$ and $|a| > |b|$. Solutions to the holomorphic differential equation (3.7) for the above pair of holomorphic potentials $\check{\eta}$ are

$$C = \exp(Az), \quad L = \exp(Bw). \tag{3.28}$$

Similar to [31], the Iwasawa decomposition of (C, L) can be computed explicitly:

Theorem 3.15 *With the notation above, the Iwasawa decomposition of* (C, L) *is given by the equation*:

$$(C, L) = (F, F)(V_+, V_-), \tag{3.29}$$

where

$$\begin{cases} F(z, w, \lambda) = C \exp(-\mathbf{f} A) B_1^{-1}, \\ V_+(z, w, \lambda) = B_1 \exp(\mathbf{f} A) \quad \text{and} \quad V_- = \overline{V_+(\bar{w}, \bar{z}, 1/\bar{\lambda})}^{t-1}, \end{cases}$$

where the functions $v(\hat{z}) = v(\frac{z+w}{2})$, $\mathbf{f}(\hat{z}) = \mathbf{f}(\frac{z+w}{2})$ *and the matrices* B_0, B_1 *satisfy*

$$(v_{\hat{z}}(\hat{z}))^2 = -\big(v^2(\hat{z}) - 4a^2\big)\big(v^2(\hat{z}) - 4b^2\big) \quad \text{with} \quad v(0) = 2b,$$

$$\mathbf{f}(\hat{z}) = \int_0^{\hat{z}} \frac{2dt}{1 + (4ab\lambda^2)^{-1} v^2(t)}, \tag{3.30}$$

$$B_0 = \begin{pmatrix} 2v(b + a\lambda^2) & -v_{\hat{z}}\lambda \\ 0 & 4ab\lambda^2 + v^2 \end{pmatrix}, \quad B_1 = (\det B_0)^{-1/2} B_0.$$

Corollary 3.16 *We have* $v(\hat{z}) = v(\frac{z+w}{2}) = He^{u(z,w)/2}$, *where* $e^{u(z,w)}$ *is the holomorphic metric of the resulting immersion* Ψ_λ. *Moreover, we have the following explicit form for* $v(\hat{z}) = v(\frac{z+w}{2})$ *using the Jacobi elliptic function*:

$$\begin{aligned} v(\hat{z}) = v\left(\frac{z+w}{2}\right) &= 2b \cdot \operatorname{sn}\left(2ia\left(\frac{z+w}{2}\right) + K, \frac{b^2}{a^2}\right) \\ &= \frac{2b}{\operatorname{dn}\big(2a(\frac{z+w}{2}), 1 - b^2/a^2\big)}, \end{aligned} \tag{3.31}$$

where sn *and* dn *are Jacobi elliptic functions and* $K = K(b^2/a^2)$ *is the quarter period of* $\operatorname{sn}(z, b^2/a^2)$ *with* $0 < b^2/a^2 < 1$.

Analogously to [11], the surface in the associated family of complex CMC-immersions for the parameter $\lambda = 1$, constructed from the pair of holomorphic potentials (3.27) is computed via the Sym formula as:

$$\Psi_{\lambda=1} = -\frac{i}{2H}\Big\{\big(2\sqrt{v(\hat{z})^2}\cos(-i\hat{w}) + (b-a)\big)\sigma_3 + \big(2\sqrt{v(\hat{z})^2}\sin(-i\hat{w})\big)\sigma_2 - \left(\frac{\mathbf{f}_{\lambda,1}(\hat{z})}{2} - \left(\frac{v_{\hat{z}}(\hat{z})}{v(\hat{z}) + 4abv(\hat{z})^{-1}}\right)\right)\sigma_1\Big\}, \tag{3.32}$$

where $\hat{w} = (z-w)/2$, $\hat{z} = (z+w)/2$, σ_j $(j = 1, 2, 3)$ is defined in (2.19) and $\mathbf{f}_{\lambda,1}(\hat{z}) = (\partial_\lambda \mathbf{f}(\hat{z}))|_{\lambda=1}$.

Remark 3.17 In the above examples, the potentials have the special symmetry $\tau(w) = -\overline{\eta(\bar{w}, 1/\bar{\lambda})}^t$. In general, we do not need to impose such a symmetry in the pair of potentials $\check{\eta} = (\eta, \tau)$. However, even for the most simple cases such as the mixture of a sphere potential and a cylinder potential as presented in Examples 3.12 and 3.13, the explicit Iwasawa decomposition is not known, and we do not know the properties of the resulting complex CMC-immersions.

4. Minimal and CMC-immersions in $\mathbb{R}^3$ via Complex CMC-immersions

In this section, we consider fibrations of complex CMC-immersions by holomorphic null curves in $\mathbb{C}^3$, which correspond to minimal immersions in $\mathbb{R}^3$. And we also discuss the converse construction. Finally we show that complex CMC-immersions satisfying a certain symmetry condition restrict to CMC-immersions in $\mathbb{R}^3$.

4.1. Complex CMC-immersions in $\mathbb{C}^3$ fibered by minimal surfaces

In this subsection, we show that minimal immersions in $\mathbb{R}^3$ can be obtained from certain fibrations of complex CMC-immersions in $\mathbb{C}^3$.

Let $\mathbb{D}$ (resp. $\mathbb{D}^2$) be a simply connected domain in $\mathbb{C}$ (resp. simply connected holomorphically convex domain in $\mathbb{C}^2$), and let Ψ be a complex CMC-immersion $\Psi : \mathbb{D}^2 \to \mathbb{C}$ with null coordinates $(z, w) \in \mathbb{D}^2$. Then we consider the following map:

$$\begin{array}{rccc} \varphi_{[w_0]} : & \mathbb{D} & \to & \mathbb{D}^2 \ , \\ & \Cup & & \Cup \\ & z & \mapsto & (z, w_0) \end{array} \tag{4.1}$$

where w_0 is some fixed point in $\mathbb{D}$. Then we consider the composition of maps $\Psi \circ \varphi_{[w_0]} : \mathbb{D} \to \mathbb{C}^3$. Since Ψ is a holomorphic null immersion, each fiber $\Psi \circ \varphi_{[w_0]}$ is locally a holomorphic null curve (see also (2.42)), since

$$\langle \partial_z(\Psi(z, w_0)), \partial_z(\Psi(z, w_0)) \rangle = 0 \ \text{ for all } z \in \mathbb{D} \text{ and } w_0 \in \mathbb{D} \text{ fixed.}$$

We summarize the above discussion as follows:

Proposition 4.1 *Let $\mathbb{D}^2$ be a simply connected holomorphically convex domain in $\mathbb{C}^2$, and let $\check{\eta}$ be a pair of holomorphic potentials as defined in* (3.6) *satisfying property* $(*)$ *in Remark* 3.3*. Let $F(z, w, \lambda)$ be the complex extended framing obtained by the generalized Weierstraß type representation of Section* 3.2*, and let $\Psi_\lambda : \mathbb{D}^2 \to \mathbb{C}^3$ $(\cong sl(2, \mathbb{C}))$ be the corresponding complex* CMC*-immersion defined by the Sym formula* (2.41)*. Then $\Psi_\lambda \circ \varphi_{[w_0]} : \mathbb{D} \to \mathbb{C}^3$ $(\cong sl(2, \mathbb{C}))$ is a holomorphic null curve in $\mathbb{C}^3$ $(\cong sl(2, \mathbb{C}))$, i.e. $\langle \partial_z(\Psi_\lambda \circ \varphi_{[w_0]}), \partial_z(\Psi_\lambda \circ \varphi_{[w_0]}) \rangle = 0$, where $\varphi_{[w_0]}$ is defined in* (4.1)*.*

It is well known [27] that every minimal immersion in $\mathbb{R}^3$ can be obtained as the real part of a holomorphic null curve in $\mathbb{C}^3$.

Corollary 4.2 (Osserman [27], Lemma 8.2 on page 64) *We retain the assumptions of Proposition* 4.1*. The holomorphic null curve $\Psi_\lambda \circ \varphi_{[w_0]}|_{\lambda \in \mathbb{C}^*}$ induces the minimal immersion in $\mathbb{R}^3$ given by the formula*:

$$z \mapsto \mathrm{Re}\big(\hat{\Psi}_{\lambda,1}(z, w_0), \hat{\Psi}_{\lambda,2}(z, w_0), \hat{\Psi}_{\lambda,3}(z, w_0)\big), \tag{4.2}$$

where $\hat{\Psi}_{\lambda,j}(z, w_0)$ $(j = 1, 2, 3)$ is the j-th component of the immersion $\hat{\Psi}_\lambda$ into $\mathbb{C}^3$ which is obtained from $\Psi_\lambda \circ \varphi_{[w_0]}$ by the identification of $sl(2, \mathbb{C})$ and $\mathbb{C}^3$ by (2.18)*.*

The metric and the Gauß map of the minimal immersion defined in (4.2) is as follows:

Corollary 4.3 (Osserman [27], page 65–66) *We retain the assumptions of Proposition* 4.1 *and Corollary* 4.2*. Then the metric $g_{\min}$ and the Gauß map $N_{\min}$ of the minimal immersion defined by* (4.2) *are given by the formulas*:

$$g_{\min} = \frac{1}{2}\sum_{k=1}^{3}\left|\hat{\Psi}_{\lambda,k,z}(z,w_0)\right|^2 dz d\bar{z},$$

$$N_{\min} = \frac{\mathrm{Im}\left\{\left(\hat{\Psi}_{\lambda,2,z}(z,w_0)\overline{\hat{\Psi}_{\lambda,3,z}(z,w_0)},\ \hat{\Psi}_{\lambda,3,z}(z,w_0)\overline{\hat{\Psi}_{\lambda,1,z}(z,w_0)},\ \hat{\Psi}_{\lambda,1,z}(z,w_0)\overline{\hat{\Psi}_{\lambda,2,z}(z,w_0)}\right)\right\}}{1/2\sum_{k=1}^{3}\left|\hat{\Psi}_{\lambda,k,z}(z,w_0)\right|^2},$$

where $\hat{\Psi}_{\lambda,k,z}(z,w_0)$, $(k \in \{1,2,3\})$ *is the derivative of* $\hat{\Psi}_{\lambda,k}(z,w_0)$ *with respect to* z.

Remark 4.4 One can also consider the following map

$$\begin{array}{cccc} \tilde{\varphi}_{[z_0]}: & \mathbb{D} & \to & \mathbb{D}^2 \ , \\ & \Cup & & \Cup \\ & w & \mapsto & (z_0, w) \end{array} \tag{4.3}$$

where $z_0 \in \mathbb{D}$ is some fixed point. According to the arguments above, also $\mathrm{Re}(\Psi_\lambda \circ \tilde{\varphi}_{[z_0]})$ yields a minimal immersion in $\mathbb{R}^3$.

Remark 4.5 In the above construction of minimal immersions, we have three parameters. The spectral parameter $\lambda \in \mathbb{C}^*$, $w_0 \in \mathbb{D}$ and the mean curvature $H \in \mathbb{C}^*$. In the Examples 4.12 and 4.13, changes in λ, w_0 and H just correspond to transformations in the associated family of the minimal immersions. In general, however, we do not know the role of these parameters.

4.2. Complex CMC-immersions from minimal surfaces

In the last section we have constructed minimal immersions in $\mathbb{R}^3$ from complex CMC-immersions in $\mathbb{C}^3$. In this subsection, we present the converse construction under the condition that the minimal surfaces in this consideration do not have any umbilical points.

Theorem 4.6 *Let* $\mathbb{D} \subset \mathbb{C}$ *be a simply connected domain, and let* $\psi(z) : \mathbb{D} \to su(2) \cong \mathbb{R}^3$ *be a minimal immersion which does not have umbilical points on* $\mathbb{D}$, *and let* $\hat{\psi}(z) : \mathbb{D} \to sl(2,\mathbb{C}) \cong \mathbb{C}^3$ *be the* z*-derivative of* $\psi(z)$. *Then there exists an associated family of complex* CMC*-immersions* $\Psi_\lambda : \check{\mathbb{D}}^2 = \check{\mathbb{D}} \times \check{\mathbb{D}} \to sl(2,\mathbb{C}) \cong \mathbb{C}^3$ *such that* $\partial_z(\Psi_\lambda \circ \varphi_{[w_0]})|_{\lambda=1} = \hat{\psi}(z)$, *where* $\varphi_{[w_0]}$ *is defined in* (4.1) *and* $\check{\mathbb{D}} \subset \mathbb{D}$ *is some simply connected domain in* $\mathbb{C}$.

Proof. We first note that we identify $\mathbb{C}^3$ with $sl(2,\mathbb{C})$ by (2.18). Let $\psi(z)$ be a minimal immersion, and let $\hat{\psi}(z) = \partial_z \psi(z)$ be the z-derivative of $\psi(z)$ (see Lemma 8.2 in [27]). Then

$$\hat{\psi}(z) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in sl(2,\mathbb{C}),$$

where the null condition is equivalent to $\det \hat{\psi}(z) = -a^2 - bc = 0$. Since also $\operatorname{Tr} \hat{\psi}(z) = 0$, it follows that $\hat{\psi}(z)$ is nilpotent. From Lemma 8.1 in [27] and a direct computation, we obtain

$$a = -\frac{i}{2}\mathfrak{f}\mathfrak{g}, \quad b = -\frac{i}{2}\mathfrak{f} \ \text{ and } \ c = \frac{i}{2}\mathfrak{f}\mathfrak{g}^2, \tag{4.4}$$

where $\mathfrak{g}$ is a meromorphic function on $\mathbb{D}$ and $\mathfrak{f}$ is a holomorphic function on $\mathbb{D}$ such that at each point, where $\mathfrak{g}$ has a pole of order m, $\mathfrak{f}$ has a zero of order at least $2m$. Moreover from Lemma 8.2 in [27] we show that the minimal immersion $\psi(z)$ is regular if and only if $\mathfrak{f}$ vanishes only at poles of $\mathfrak{g}$, and the order of vanishing at such a point is exactly twice the order of the pole of $\mathfrak{g}$. Using $\mathfrak{f}$ and $\mathfrak{g}$, we compute the Hopf differential $\mathcal{Q}$ of $\psi(z)$, which is the $(2,0)$-part of the second fundamental form $\psi(z)$, as follows (see page 75 in [27]):

$$\mathcal{Q} = -\frac{1}{2}\mathfrak{f}\mathfrak{g}_z dz^2.$$

Since we assume that the minimal immersion $\psi(z)$ does not have umbilical points on $\mathbb{D}$, i.e. $\mathcal{Q} \neq 0$ on $\mathbb{D}$, and using the properties of $\mathfrak{f}$ and $\mathfrak{g}$, there are only two cases where $\mathfrak{f}$ and $\mathfrak{g}$ have zeros and poles respectively on $\mathbb{D}$; (1) $\mathfrak{g}$ has a pole of first order on $\mathbb{D}$, where $\mathfrak{f}$ has a zero of second order. (2) $\mathfrak{g}$ has a zero of first order on $\mathbb{D}$. Let $z_0 \in \mathbb{D}$ be a point where $\mathfrak{g}$ has a zero or a pole of first order. Since $\mathfrak{g}$ is the stereographic projection of the Gauß map (see page 76 in [27]), we can assume that $\mathfrak{g}$ does not have a zero nor a pole at z_0 after applying an appropriate isometry of $\mathbb{R}^3$. Since the zeros and poles of $\mathfrak{g}$ are isolated, there exists an open neighborhood $\tilde{\mathbb{D}} \subset \mathbb{D}$ around z_0 such that $\mathfrak{g}$ does not have zeros nor poles on $\tilde{\mathbb{D}}$. Then from (4.4), we have $a \neq 0$ on $\tilde{\mathbb{D}}$. Moreover the null condition for $\hat{\psi}(z)$, which is $\det \hat{\psi}(z) = -a^2 - bc = 0$, implies that $b \neq 0$ and $c \neq 0$ on $\tilde{\mathbb{D}}$. Thus we can rewrite $c = -a^2/b$. By a straightforward computation, it is easy to verify that $\hat{\psi}(z)$ can be represented by using an $SL(2, \mathbb{C})$ matrix $\hat{F}$ and holomorphic functions p and q in the form

$$\hat{\psi}(z) = -\frac{i}{2} q \hat{F} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{F}^{-1}, \tag{4.5}$$

where

$$\hat{F} = \begin{pmatrix} -(b/a)\sqrt{a^2/b} & -(b/a^2)\sqrt{a^2/b} \\ \sqrt{a^2/b} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2i/q} & p\sqrt{b/a^2} \\ 0 & \sqrt{q/(2i)} \end{pmatrix}. \tag{4.6}$$

We note that $\sqrt{a^2/b}$ is well-defined, since $a \neq 0$, $b \neq 0$ and $c \neq 0$ on $\tilde{\mathbb{D}}$. To determine p and q, we consider the Maurer-Cartan form $\hat{F}^{-1}\hat{F}_z$ of $\hat{F}$, and assume $\hat{F}^{-1}\hat{F}_z$ has the following form:

$$\hat{F}^{-1}\hat{F}_z = \begin{pmatrix} \frac{1}{2}\partial_z(\log q) & -\frac{1}{2}Hq \\ Qq^{-1} & -\frac{1}{2}\partial_z(\log q) \end{pmatrix}, \tag{4.7}$$

where the subscript z denotes the partial derivative with respect to z, Q is a holomorphic function and H is some complex constant. By a straightforward computation, this is equivalent to the following equations:

$$\begin{cases} Q = 2i(ab_z/b - a_z), \\ Qq_{zz} - Q_z q_z + \left(-\alpha^2 Q - \alpha_z Q + \alpha Q_z - \frac{1}{2} H Q^2 \right) q = 0, \\ Qp = \sqrt{2iq}\sqrt{a^2/b}(\alpha - (\log q)_z), \end{cases} \tag{4.8}$$

where $\alpha = a_z/a - b_z/(2b)$. An other straightforward computation shows that $Q/2$ is the Hopf differential of the minimal immersion $\psi(z)$, i.e. $\mathcal{Q} = Q/2$. Since the second equation in (4.8) is a second order linear ODE [8], there are two uniquely determined linearly independent solutions $q_j(z)$, $j \in \{1, 2\}$, defined around $z_0 \in \tilde{\mathbb{D}}$ for each initial condition $q_j(z_0) \neq 0$ and $\partial_z q_j(z_0) \neq 0$, $j \in \{1, 2\}$. Since we have assumed that the minimal immersion $\Psi(z)$ does not have umbilical points, i.e. $\mathcal{Q} \neq 0$ on $\mathbb{D}$ and α is holomorphic on $\tilde{\mathbb{D}}$, the corresponding solutions $q_1(z), q_2(z)$ are holomorphic functions on $\tilde{\mathbb{D}}$.

Let $q(z)$ be a non-zero solution of the second equation in (4.8) around $z_0 \in \check{\mathbb{D}} \subset \tilde{\mathbb{D}}$, where $\check{\mathbb{D}}$ is an open neighborhood. We note again that we can always choose such a solution satisfying the initial conditions $q(z_0) \neq 0$, $\partial_z q(z_0) \neq 0$. We now set

$$\check{\xi} = \left(\lambda^{-1} \begin{pmatrix} 0 & f(z) \\ \hat{Q}(z)/f(z) & 0 \end{pmatrix} dz,\ \lambda \begin{pmatrix} 0 & \hat{R}(w)/g(w) \\ g(w) & 0 \end{pmatrix} dw \right), \tag{4.9}$$

where $f(z) = -\frac{H}{2} q(z)^2 q(z_0)^{-1}$, $\hat{Q}(z) = -\frac{H}{2} Q$, $g(w)$ is an arbitrary non-zero holomorphic function around $w_0 \in \check{\mathbb{D}}$ and $\hat{R}(w)$ is a holomorphic function around $w_0 \in \check{\mathbb{D}}$. Note that the upper right entry and the lower left entry of (4.9) never vanish. For the above pair of normalized potentials, we apply the generalized Weierstraß type representation formula in Section 3.2 with the base point $(z_0, w_0) \in \check{\mathbb{D}}^2$, then we obtain an extended framing $F(z, w, \lambda)$ and a complex CMC-immersion Ψ_λ. Since $\check{\xi}$ in (4.9) is a pair of normalized potentials as in (3.19), the holomorphic metric $e^{u(z,w)}$ of the resulting complex CMC-immersion Ψ_λ via $\check{\xi}$ satisfies the relation $e^{u(z,w_0) - u(z_0,w_0)/2} = q(z)^2 q(z_0)^{-1}$. This implies $q(z_0) = e^{u(z_0,w_0)/2}$ and thus $q(z) = e^{u(z,w_0)}$. Then clearly $F(z, w_0, \lambda = 1)^{-1} dF(z, w_0, \lambda = 1)$ is the right hand side of (4.7), thus $\hat{F}(z) = AF(z, w_0, \lambda = 1)$, where $A \in SL(2, \mathbb{C})$ is a z-independent matrix. We note that one of the Hopf differentials of the complex CMC-immersion Ψ_λ is $Q = 2\mathcal{Q}$, where $\mathcal{Q}$ is the Hopf differential of the minimal immersion $\psi(z)$. Moreover, (4.6) and (2.42) show that we have $\partial_z(\Psi_\lambda \circ \varphi_{[w_0]})|_{\lambda=1} = A^{-1} \hat{\psi}(z) A$ in (4.5). This completes the proof, since conjugation by some matrix of determinant 1 is a rigid motion in the realm of complex CMC-surfaces. □

Remark 4.7 If the Hopf differential $\mathcal{Q}$ of a minimal immersion, which is also a half of the Hopf differential of the complex CMC-immersion, has a zero at $z_1 \in \mathbb{D}$, then the solution $q(z)$ of the second order linear differential equation (4.8) could have a singularity at z_1. Therefore, the corresponding complex CMC-immersion will in general not be defined globally. We hope to clarify the role of these singularities in a future publication.

4.3. CMC-immersions in $\mathbb{R}^3$ and complex CMC-immersions

In this subsection, we show that every CMC-immersion in $\mathbb{R}^3$ can be obtained from a complex CMC-immersion Ψ_λ in $\mathbb{C}^3$ by a certain restriction.

Let $\check{\eta} = (\eta, \tau)$ be a pair of holomorphic potentials as defined in (3.6) with property $(*)$ in Remark 3.3. We assume in addition the following symmetry for $\check{\eta}$:

$$\check{\eta} = (\eta(z,\lambda), \tau(w,\lambda)) = \left(\eta(z,\lambda), -\overline{\eta(\bar{w}, 1/\bar{\lambda})}^{t}\right). \tag{4.10}$$

The symmetry $\tau(w,\lambda) = -\overline{\eta(\bar{w}, 1/\bar{\lambda})}^{t}$ and the initial condition $(z_0, w_0) = (z_0, \bar{z}_0) \in \mathbb{D}^2, C(z_0) = L(\bar{z}_0) = \mathrm{Id}$ imply the symmetry $L(w,\lambda) = (\overline{C(\bar{w}, 1/\bar{\lambda})}^{t})^{-1}$. In this case, the extended framing $F(z,w,\lambda)$ obtained from the generalized Iwasawa decomposition of Theorem A.3 acquires the symmetry $F(z,w,\lambda) = (\overline{F(\bar{w}, \bar{z}, 1/\bar{\lambda})}^{t})^{-1} k(z,w)$, where $k(z,w)$ is a λ-independent non-zero diagonal matrix satisfying $k(z,w) = \overline{k(\bar{w}, \bar{z})}$ (a precise computation follows from Theorem 3.2 in [10]). Next we consider the following map

$$\begin{array}{rccc} \varphi_{[\bar{z}]}: & \mathbb{D} & \to & \mathbb{D}^2 \\ & \Psi & & \Psi \\ & z & \mapsto & (z, \bar{z}), \end{array} \tag{4.11}$$

where $\bar{z}$ denotes the complex conjugate of z. Since $w = \bar{z}$, $k(z,w) = k(z,\bar{z})$ is a positive definite diagonal matrix with entries $k_0 > 0$ and $1/k_0 > 0$. We now define $\tilde{k}(z,\bar{z})$ as the positive definite diagonal matrix with entries $\sqrt{k_0} > 0$ and $1/\sqrt{k_0} > 0$. Setting $F_{\mathrm{cmc}}(z,\bar{z},\lambda) = F(z,\bar{z},\lambda)\tilde{k}(z,\bar{z},\lambda)^{-1}$, we obtain $F_{\mathrm{cmc}}(z,\bar{z},\lambda) = (\overline{F_{\mathrm{cmc}}(z,\bar{z},1/\bar{\lambda})}^{t})^{-1}$. We note that F and F_{cmc} give the same immersion by Corollary 2.17. The symmetry of $F_{\mathrm{cmc}}(z,\bar{z},\lambda)$ implies the following symmetry for the immersion $\Psi_\lambda \circ \varphi_{[\bar{z}]} : \mathbb{D} \to \mathbb{C}^3$, where $H \in \mathbb{R}^*$ and $\lambda \in S^1$:

$$\Psi_\lambda \circ \varphi_{[\bar{z}]} = -\overline{\Psi_\lambda \circ \varphi_{[\bar{z}]}}^{t}.$$

Thus the image of the map $\Psi_\lambda \circ \varphi_{[\bar{z}]}$ is contained in $su(2) \cong \mathbb{R}^3$. It is easy to see that H, which is a real constant, is the mean curvature of $\Psi_\lambda \circ \varphi_{[\bar{z}]}$. Therefore $\Psi_\lambda \circ \varphi_{[\bar{z}]}$ defines a CMC-immersion into $\mathbb{R}^3$. Moreover, we have the following proposition:

Proposition 4.8 *Let $\Psi_\lambda : \mathbb{D}^2 \to \mathbb{C}^3$ be the complex* CMC-*immersion derived from a pair of potentials as defined in* (4.10) *with $H \in \mathbb{R}^*$ and*

$\lambda \in S^1$. *Then* $\Psi_\lambda \circ \varphi_{[\bar{z}]}$ *is a conformal* CMC*-immersion in* $\mathbb{R}^3$. *Moreover, all conformal* CMC*-immersions can be obtained in this way.*

Proof. Clearly, the above construction gives conformal CMC-immersions in $\mathbb{R}^3$. Conversely, let $\Psi_\lambda : \mathbb{D} \to \mathbb{R}^3$ be an associated family of CMC-immersions in $\mathbb{R}^3$ with extended framing $F_{\mathrm{cmc}}(z, \bar{z}, \lambda)$, where $\mathbb{D}$ is some simply connected domain. From Corollary 4.6 in [12], there exists a holomorphic potential $\eta(z, \lambda) = \sum_{j \geq -1}^{\infty} \lambda^j \eta_j(z)$, where the upper right entry of $\eta_{-1}(z)$ never vanish on $\mathbb{D}$. Let $\check{\eta}(z, \bar{z}, \lambda)$ be the pair of holomorphic potentials $\check{\eta}(z, \bar{z}, \lambda) = (\eta(z, \lambda), -\overline{\eta(z, 1/\bar{\lambda})}^{\,t})$. We now consider $\bar{z}$ as a free variable w. Then we obtain the pair of holomorphic potentials $\check{\eta}(z, w, \lambda)$ on $\mathbb{D} \times \bar{\mathbb{D}}$. From Theorem 3.2 in [10] (see also Appendix A.3), we know that, using the above pair of potentials $\check{\eta}(z, w, \lambda)$, there exists a holomorphic loop $F(z, w, \lambda)$ and a λ-independent diagonal matrix $l(z, \bar{z})$ such that $F(z, \bar{z}, \lambda) = F_{\mathrm{cmc}}(z, \bar{z}, \lambda) l(z, \bar{z})$. From Corollary 2.17, $F(z, \bar{z}, \lambda)$ and $F_{\mathrm{cmc}}(z, \bar{z}, \lambda)$ give the same conformal CMC-immersion. This completes the proof. $\square$

Remark 4.9 In general, we can consider any map of the form

$$\begin{array}{cccc} \varphi_{[w=f(z,\bar{z})]}: & \mathbb{D} & \longrightarrow & \mathbb{D}^2 \\ & \Psi & & \Psi \\ & z & \mapsto & (z, w = f(z, \bar{z})), \end{array}$$

where $f(z, \bar{z})$ is any complex function. Then the map $\Psi_\lambda \circ \varphi_{[w=f(z,\bar{z})]} : \mathbb{D} \to \mathbb{C}^3$ defines an immersion. It would be interesting to investigate for which maps $f(z, \bar{z})$ the resulting immersion $\Psi \circ \varphi_{[w=f(z,\bar{z})]}$ will be a CMC-immersion.

4.4. A construction of minimal immersions from CMC-immersions in $\mathbb{R}^3$

In this subsection, we consider a local construction of minimal immersions from CMC-immersions in $\mathbb{R}^3$. And we also give examples using this construction. The construction is divided into 4 steps as follows:

Step 1: Let $\Psi_\lambda : \mathbb{D} \to \mathbb{R}^3$ be a conformal CMC-immersion with mean curvature $H \neq 0$, and let $F_{\mathrm{cmc}}(z, \bar{z}, \lambda) : \mathbb{D} \to \Lambda SU(2)_\sigma$ be the extended framing of Ψ_λ.

Step 2: From Theorem A.4, we obtain a unique meromorphic extension $F(z,w,\lambda) : \mathbb{D}^2 \to \Lambda SL(2,\mathbb{C})_\sigma$ of $F_{\mathrm{cmc}}(z,\bar{z},\lambda)l(z,\bar{z}) : \mathbb{D} \to \Lambda SU(2)_\sigma$:

$$F_{\mathrm{cmc}}(z,\bar{z},\lambda)l(z,\bar{z}) \to F(z,w,\lambda), \tag{4.12}$$

where $l(z,\bar{z}) \in SL(2,\mathbb{C})$ is some λ-independent diagonal matrix, and $(z,w) \in \mathbb{D}^2 \subset \mathbb{C}^2$. Putting $F(z,w,\lambda)$ into the Sym formula $\Psi_\lambda(z,w)$ in (2.41), we obtain an associated family of complex CMC-immersions $\Psi_\lambda(z,w) : \mathbb{D}^2 \to \mathbb{C}^3$.

Remark 4.10 Assume that $\tilde{F}(z,w,\lambda)$ can also be presented in the form $\tilde{F}(z,w,\lambda) = F(z,w,\lambda)k(z,w)$, where $F(z,w,\lambda)$ is the complex extended framing of some complex CMC-immersion, and $k(z,w) \in SL(2,\mathbb{C})$ is some λ-independent diagonal matrix. Then, by Corollary 2.17, $\tilde{F}(z,w,\lambda)$ and $F(z,w,\lambda)$ define the same complex CMC-immersion.

Step 3: Fix w as $w_0 \in \mathbb{D}$ and consider the restriction $\Psi_\lambda(z,w_0) = \Psi_\lambda \circ \varphi_{[w_0]}$, $z \in \mathbb{D}$. Then, by Proposition 4.1, $\Psi_\lambda(z,w_0) = \Psi_\lambda \circ \varphi_{[w_0]}$ defines a holomorphic null curve in $\mathbb{C}^3$.

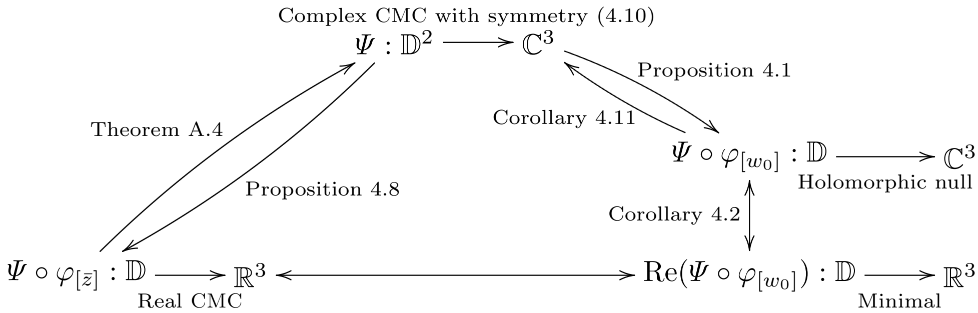
Step 4: The classical Weierstraß representation formula (4.2) associated with $\Psi_\lambda(z,w_0) = \Psi_\lambda \circ \varphi_{[w_0]}$ yields a minimal immersion in $\mathbb{R}^3$.

Moreover, we have the following theorem.

Corollary 4.11 *Let $\mathbb{D} \subset \mathbb{C}$ be a simply connected domain, and let $\psi(z) : \mathbb{D} \to su(2) \cong \mathbb{R}^3$ be a minimal immersion which does not have umbilical points on $\mathbb{D}$, and let $\hat{\psi}(z) : \mathbb{D} \to sl(2,\mathbb{C}) \cong \mathbb{C}^3$ be the z-derivative of $\psi(z)$. Assume that the base point is of the form $(z_0, \bar{z}_0)$. Then there exists a real* CMC*-immersion $\Psi_\lambda : \check{\mathbb{D}} \to su(2) \cong \mathbb{R}^3$ such that for its complex extension we have $\partial_z(\Psi_\lambda \circ \varphi_{[\bar{z}_0]})|_{\lambda=1} = \hat{\psi}(z)$, where $\varphi_{[\bar{z}_0]}$ is defined in* (4.1) *and $\check{\mathbb{D}} \subset \mathbb{D}$ is some simply connected domain in $\mathbb{C}$. In particular, the surface* $\mathrm{Re}(\Psi_\lambda \circ \varphi_{[\bar{z}_0]})|_{\lambda=1}$ *coincides with the original minimal surface $\psi(z)$ up to some rigid motion.*

Proof. We follow the proof of Theorem 4.6. The first point where we proceed differently from the proof just quoted is, when we choose the pair of normalized potentials. While before we had a lot of freedom, now, in view of our goal to obtain a real CMC-immersion we need to take into account the result of the last section and choose the second component of our pair of potentials accordingly. Thus we obtain again the formula $\hat{F}(z) = AF(z, \bar{z}_0, \lambda = 1)$. It is easy to verify (see the computation at the very end of the proof of Theorem 4.6) that it suffices to show A is unitary. To obtain this we observe that in the proof of Theorem 4.6 there was a lot of freedom in the choice of q. For our purposes it will be necessary to fix this freedom: perform a classical Iwasawa decomposition of the matrix $\hat{F}(z_0) = U \cdot L$, where L is upper triangular. Comparing this to (4.6) we see that the proper choice of initial conditions for q will render $\hat{F}(z_0)$ unitary. From this we infer that $AF(z_0, \bar{z}_0, \lambda = 1)$ is unitary and the claim follows. □

Schematically we have



Finally, we give some examples of minimal immersions obtained from CMC-immersions in $\mathbb{R}^3$ by the construction above.

Example 4.12 (Round spheres and planes) We consider Example 3.12. The pair of holomorphic potentials $\check{\eta} = (\eta, \tau)$ has the symmetry $\tau(w, \lambda) = -\overline{\eta(\bar{w}, 1/\bar{\lambda})}^t$. We consider the map $\varphi_{[\bar{z}]}$ defined in (4.11), then we have $\Psi_\lambda \circ \varphi_{[\bar{z}]} = -\overline{\Psi_{1/\lambda} \circ \varphi_{[\bar{z}]}}^t$. The resulting immersion $\Psi \circ \varphi_{[\bar{z}]}$ is

$$\Psi_\lambda \circ \varphi_{[\bar{z}]} = -\frac{i}{4H(1 + z\bar{z})} \begin{pmatrix} -3z\bar{z} + 1 & -4\lambda^{-1}z + 1 \\ -4\lambda\bar{z} & 3z\bar{z} - 1 \end{pmatrix} \in su(2), \qquad (4.13)$$

where $\lambda \in S^1$. Clearly, this is the associated family of a round sphere of radius $1/H$ centered at $(0, 0, -1/(2H))$.

We can also consider the fibration $\varphi^{-1}_{[w_0]}$ defined in (4.1). The corresponding holomorphic null curve $\Psi \circ \varphi_{[w_0]}$ is

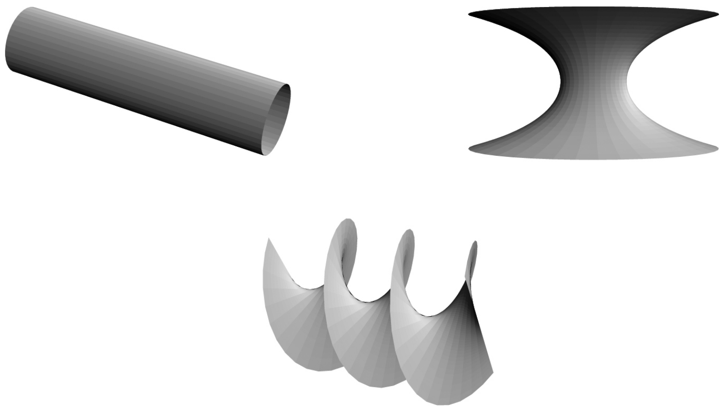


Figure 1. The above two figures are a cylinder and the corresponding minimal surface, which is a catenoid, with parameters $w_0 = 0$, $\lambda = 1$ and $H = 1/2$. The below figure is the corresponding minimal surface with parameters w_0, $\lambda = 1$ and $H = i/2$.

$$\Psi_\lambda \circ \varphi_{[w_0]} = -\frac{i}{4H(1+zw_0)}\begin{pmatrix} -3zw_0+1 & -4\lambda^{-1}z+1 \\ -4\lambda w_0 & 3zw_0-1 \end{pmatrix} \in sl(2,\mathbb{C}). \quad (4.14)$$

Using the Weierstraß type representation (4.2), the resulting minimal immersion is a plane for each fixed $w_0 \in \mathbb{D}$, $\lambda \in \mathbb{C}^*$ and $H \in \mathbb{C}^*$.

Example 4.13 (Cylinders and catenoids) We consider Example 3.13. The pair of holomorphic potentials $\check{\eta} = (\eta, \tau)$ also has the symmetry $\tau(w,\lambda) = -\overline{\eta(\bar{w}, 1/\bar{\lambda})}^{t}$. We consider the map $\varphi_{[\bar{z}]}$ defined in (4.11). The resulting immersion $\Psi_\lambda \circ \varphi_{[\bar{z}]}$ is

$$\Psi_\lambda \circ \varphi_{[\bar{z}]} = -\frac{i}{4H}\begin{pmatrix} \cosh(2p_1) & -2(\lambda^{-1}z+\lambda\bar{z}) - \sinh(2p_1) \\ -2(\lambda^{-1}z+\lambda\bar{z}) + \sinh(2p_1) & -\cosh(2p_1) \end{pmatrix}, \quad (4.15)$$

where $p_1 = \lambda^{-1}z - \lambda\bar{z}$, and $\lambda \in S^1$. Thus the resulting immersion is the associated family of a round cylinder.

We can also consider the fibration $\varphi^{-1}_{[w_0]}$ defined in (4.1). The corresponding holomorphic null curve $\Psi \circ \varphi_{[w_0]}$ is

$$\Psi_\lambda \circ \varphi_{[w_0]} = \\ -\frac{i}{4H}\begin{pmatrix} \cosh(2p_2) & -2(\lambda^{-1}z + \lambda w_0) - \sinh(2p_2) \\ -2(\lambda^{-1}z + \lambda w_0) + \sinh(2p_2) & -\cosh(2p_2) \end{pmatrix},$$

where $p_2 = \lambda^{-1}z - \lambda w_0$ with a fixed point w_0. A simple calculation and the classical Weierstraß representation (4.2) show that the resulting immersion is, for each $w_0 \in \mathbb{D}$, $\lambda \in \mathbb{C}^*$ and $H \in \mathbb{C}^*$, the associated family of a catenoid.

Example 4.14 (Delaunay surfaces and the corresponding minimal surfaces) We consider Example 3.14. The pair of holomorphic potentials $\check{\eta} = (\eta, \tau)$ has again the symmetry $\tau(w, \lambda) = -\overline{\eta(\bar{w}, 1/\bar{\lambda})}^t$. We can consider the map $\varphi_{[\bar{z}]}$ defined in (4.11). Then we have again the symmetry $\Psi_\lambda \circ \varphi_{[\bar{z}]} = -\overline{\Psi_{1/\bar{\lambda}} \circ \varphi_{[\bar{z}]}}^t$. The resulting immersion $\Psi_\lambda \circ \varphi_{[\bar{z}]}$ at $\lambda = 1$ is

$$\big(\Psi_\lambda \circ \varphi_{[\bar{z}]}\big)|_{\lambda=1} = -\frac{i}{2H}\Big\{\big(2\sqrt{v(x)^2}\cos(y) + (b-a)\big)\sigma_3 + \big(2\sqrt{v(x)^2}\sin(y)\big)\sigma_2 \\ -\left(\frac{\mathbf{f}_{\lambda,1}(x)}{2} - \left(\frac{v_x(x)}{v(x) + 4abv(x)^{-1}}\right)\right)\sigma_1\Big\}, \quad (4.16)$$

where $z = x + iy$, σ_j $(j = 1, 2, 3)$ is defined in (2.19) and $\mathbf{f}_{\lambda,1}(x) = (\partial_\lambda \mathbf{f}(x))|_{\lambda=1}$, and $v(x)$ is defined in (3.31). A simple computation shows that the resulting surface is a Delaunay surface [11].

We can consider also in this case the fibration $\varphi_{[w_0]}^{-1}$ defined in (4.1). The corresponding holomorphic null curve $\Psi_\lambda \circ \varphi_{[w_0]}$ is

$$\big(\Psi_\lambda \circ \varphi_{[w_0]}\big)|_{\lambda=1} \\ = -\frac{i}{2H}\Big\{\big(2\sqrt{v(\hat{z})^2}\cos(-i\hat{w}) + (b-a)\big)\sigma_3 + \big(2\sqrt{v(\hat{z})^2}\sin(-i\hat{w})\big)\sigma_2 \\ -\left(\frac{\mathbf{f}_{\lambda,1}(\hat{z})}{2} - \left(\frac{v_{\hat{z}}(\hat{z})}{v(\hat{z}) + 4abv(\hat{z})^{-1}}\right)\right)\sigma_1\Big\}, \quad (4.17)$$

where $\hat{w} = (z - w_0)/2$, $\hat{z} = (z + w_0)/2$, σ_j $(j = 1, 2, 3)$ is defined in (2.19) and $\mathbf{f}_{\lambda,1}(\hat{z}) = (\partial_\lambda \mathbf{f}(\hat{z}))|_{\lambda=1}$, and $u = u(z, w_0)$. Using the Weierstraß representation (4.2), we obtain the minimal immersion corresponding to the Delaunay surface (4.16).

We rewrite z as $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$. From (4.17), we know that $\cos(-i(z-w_0)/2)$ and $\sin(-i(z-w_0)/2)$ are periodic functions with respect to the direction y with period 4π. Moreover, by (3.31), we have

$$v(\hat{z} = (z + w_0)/2) = 2b \cdot \operatorname{sn}\left(2ia\left(\frac{z+w_0}{2}\right) + K(b^2/a^2), \frac{b^2}{a^2}\right). \tag{4.18}$$

Thus $v(\hat{z})$ is an elliptic function with period $(4/a)K'(b^2/a^2)$ (resp. $(4/a)K(b^2/a^2)$) with respect to x-direction (resp. y-direction), where $K(b^2/a^2)$ and $K'(b^2/a^2)$ are the quarter periods of $\operatorname{sn}(z, b^2/a^2)$ in x-direction and y-direction respectively. We note that $v(\hat{z})$ has a simple pole in the parallelogram defined by $K(b^2/a^2)$ and $K'(b^2/a^2)$. Let $\operatorname{period}(a, \ell)$ be the following function:

$$\operatorname{period}(a, \ell) = (4/a)K(b^2/a^2) - \ell\pi, \tag{4.19}$$

where $\ell \in \mathbb{N}$ and $b = 1/2 - a$. We note that $1/4 < a < \infty$ (or equivalently $0 < b^2/a^2 < 1$) and

$$K(b^2/a^2) = (\pi/2)_2F_1(1/2, 1/2, 1, b^2/a^2),$$

where ${}_2F_1(\alpha, \beta, \gamma, x)$ is the Gauß hypergeometric function (see page 318 in [14]). Since ${}_2F_1(1/2, 1/2, 1, b^2/a^2)$ is a real valued and monotonically increasing function on $b^2/a^2 \in [0, 1)$ and ${}_2F_1(1/2, 1/2, 1, b^2/a^2)$ tends to 1 (resp. $+\infty$) if b^2/a^2 goes to 0 (resp. 1). Thus $(4/a)K(b^2/a^2)$ takes values in $(0, \infty)$. Therefore the function $\operatorname{period}(a, \ell)$ defined in (4.19) has, for each $\ell \in \mathbb{N}$, a unique 0 in $a \in (1/4, \infty)$. Therefore $v(\hat{z})$ has the period $\ell\pi$ by the proper choice a, which is a unique solution of $\operatorname{period}(a, \ell) = 0$.

We summarize the above discussion as the following theorem:

Theorem 4.15 *Let $v(\hat{z})$ be the elliptic function defined in* (4.18)*, and let a and $b = 1/2 - a$ be real constants so that $v(\hat{z})$ has the period $(4/a)K = \ell\pi$ in y-direction, where $(1/a)K$ is the quarter period of $v(\hat{z})$ in y-direction. Let $(1/a)K'$ denote the quarter period of $v(\hat{z})$ in x-direction. Moreover let $p_{0,0}$ be the pole of $v(\hat{z})$ in the parallelogram, including the point $0 \in \mathbb{C}$, defined by $(1/a)K$ and $(1/a)K'$. Then the minimal immersion $(\Psi_\lambda \circ \varphi_{[w_0]})|_{\lambda=1}$ defined in* (4.17) *into $su(2) \cong \mathbb{R}^3$ is well-defined on $(\mathbb{C}/m\pi i\mathbb{Z}) \setminus S$, where m is the least common multiple of 4 and ℓ, and $S = \{p_{k,j} \in \mathbb{C} \,|_{k,j}= p_{0,0} + k \cdot (i/a)K + j \cdot (1/a)K'$ for $k \in \mathbb{Z}/m\mathbb{Z}$ and $j \in \mathbb{Z}\}$.*

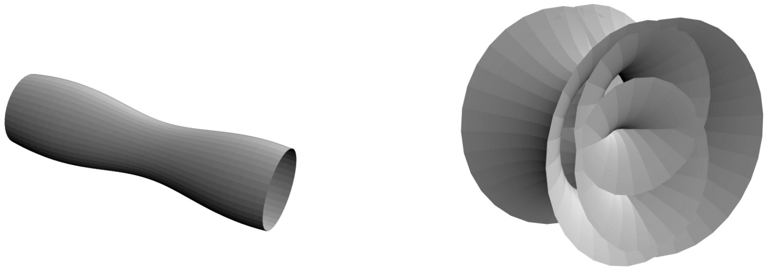


Figure 2. A Delaunay surface (an unduloid) surface with $ab = 0.06$ and the corresponding minimal surface (see Theorem 4.15). This minimal surface is defined on an infinitely punctured strip $\mathbb{C}/8\pi i\mathbb{Z}$, and it seems to have catenoid-type ends on the strip at $x = +\infty$ and at $x = -\infty$ and planer ends at all other punctures.

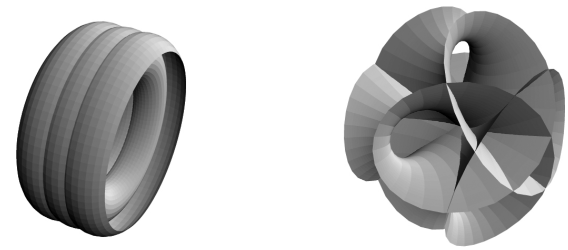


Figure 3. A Delaunay surface (a nodoid) with $ab = -0.64$ and the corresponding minimal surface (see Theorem 4.15). This minimal surface is defined on an infinitely punctured strip $\mathbb{C}/4\pi i\mathbb{Z}$.

Remark 4.16 In general, the minimal immersion corresponding to a Delaunay surface does not have a rotational symmetry. Hence, in general, these surfaces are only defined on the simply connected cover.

Appendix A. Basic notations and results for Loop groups

A.1. Loop groups

In this subsection, we introduce a loop group, a loop algebra and two decomposition theorems. Let $C_r := \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$ be the circle of radius r with $r \in (0, 1]$, and let $D_r := \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$ be the open disk of radius r. We denote the closure of D_r by $\overline{D_r}^{c\ell} := \{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}$. Also, let $A_r = \{\lambda \in \mathbb{C} \mid r < |\lambda| < 1/r\}$. This is an open annulus of containing S^1. Let $\overline{A_r}^{c\ell}$ denote the closure of A_r. Furthermore, let $E_r = \{\lambda \in \mathbb{C} \mid r < |\lambda|\}$

be the exterior of the circle C_r.

For any $r \in (0,1] \subset \mathbb{R}$, we consider the twisted loop algebra and the twisted loop group:

$$\Lambda_r sl(2,\mathbb{C})_\sigma = \big\{\alpha : C_r \to sl(2,\mathbb{C}) \mid \alpha \text{ is continous and } \alpha(-\lambda) = \sigma_3\alpha(\lambda)\sigma_3\big\},$$
$$\Lambda_r SL(2,\mathbb{C})_\sigma = \big\{g : C_r \to SL(2,\mathbb{C}) \mid g \text{ is continous and } g(-\lambda) = \sigma_3 g(\lambda)\sigma_3\big\},$$

where σ_3 is defined in (2.19). We define some subgroups of $\Lambda SL(2,\mathbb{C})_\sigma$: the twisted $SU(2)$-r-loop group is

$$\Lambda_r SU(2)_\sigma = \big\{F(\lambda) \in \Lambda_r SL(2,\mathbb{C})_\sigma \mid F(\lambda) \in SU(2), \text{ for all } \lambda \in S^1, \\ F(\lambda) \text{ extends holomorphically to } A_r\big\}.$$

Note that the definition of $\Lambda_r SU(2)_\sigma$ implies that F is continuous on $\overline{A_r}^{c\ell}$ and holomorphic on A_r. The twisted "plus r-loop group" and "minus r-loop group" are

$$\Lambda^+_{r,B} SL(2,\mathbb{C})_\sigma = \big\{W_+ \in \Lambda_r SL(2,\mathbb{C})_\sigma \mid W_+(\lambda) \text{ extends holomorphically} \\ \text{to } D_r \text{ and } W_+(0) \in \boldsymbol{B}\big\},$$
$$\Lambda^-_{r,B} SL(2,\mathbb{C})_\sigma = \big\{W_- \in \Lambda_r SL(2,\mathbb{C})_\sigma \mid W_-(\lambda) \text{ extends holomorphically} \\ \text{to } E_r \text{ and } W_-(\infty) \in \boldsymbol{B}\big\},$$

where $\boldsymbol{B}$ is a subgroup of $SL(2,\mathbb{C})$. If $\boldsymbol{B} = \{\mathrm{Id}\}$ we write the subscript $*$ instead of $\boldsymbol{B}$, if $\boldsymbol{B} = SL(2,\mathbb{C})$ we abbreviate $\Lambda^+_{r,B} SL(2,\mathbb{C})_\sigma$ and $\Lambda^-_{r,B} SL(2,\mathbb{C})_\sigma$ by $\Lambda^+_r SL(2,\mathbb{C})_\sigma$ and $\Lambda^-_r SL(2,\mathbb{C})_\sigma$ respectively. From now on we will use the subscript $\boldsymbol{B}$ as above only if $\boldsymbol{B} \cap SU(2) = \{\mathrm{Id}\}$ holds. When $r = 1$, we always omit the 1. In order to make the above groups and algebras complex Banach Lie groups and Lie algebras, we restrict the occurring matrix coefficients to the "Wiener algebra"

$$\mathcal{A} = \Big\{f(\lambda) = \sum_{n\in\mathbb{Z}} f_n\lambda^n : C_r \to \mathbb{C}\ ;\ \sum_{n\in\mathbb{Z}} |f_n| < \infty\Big\}. \tag{A.1}$$

We will assume from here on that all matrix coefficients are contained in the Wiener algebra $\mathcal{A}$. It is well known that the Wiener algebra is a Banach algebra relative to the norm $\|f\| = \sum |f_n|$, and that $\mathcal{A}$ consists of continous functions. Moreover, with coefficients in $\mathcal{A}$, the loop groups and loop algebras are Banach Lie groups and Banach Lie algebras.

From [7] and [16] (see also [29]), we quote the following decomposition Theorem:

Theorem A.1 ([7], [16], Birkhoff decomposition) *For any $r \in (0,1]$, we have the disjoint union*

$$\Lambda_r SL(2,\mathbb{C})_\sigma = \bigcup \Lambda_r^- SL(2,\mathbb{C})_\sigma \cdot w_n \cdot \Lambda_r^+ SL(2,\mathbb{C})_\sigma,$$

and

$$\Lambda_r SL(2,\mathbb{C})_\sigma = \bigcup \Lambda_r^+ SL(2,\mathbb{C})_\sigma \cdot w_n \cdot \Lambda_r^- SL(2,\mathbb{C})_\sigma,$$

where $w_n = \left(\begin{smallmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{smallmatrix}\right)$ if $n = 2k$ and $\left(\begin{smallmatrix} 0 & \lambda^n \\ -\lambda^{-n} & 0 \end{smallmatrix}\right)$ if $n = 2k+1$. The loops, for which $n = 0$, form an open dense subset of $\Lambda_r SL(2,\mathbb{C})_\sigma$, which is called "the big cell", and the multiplication maps

$$\Lambda_{r,*}^- SL(2,\mathbb{C})_\sigma \times \Lambda_r^+ SL(2,\mathbb{C})_\sigma \longrightarrow \Lambda_r SL(2,\mathbb{C})_\sigma$$

and

$$\Lambda_{r,*}^+ SL(2,\mathbb{C})_\sigma \times \Lambda_r^- SL(2,\mathbb{C})_\sigma \longrightarrow \Lambda_r SL(2,\mathbb{C})_\sigma$$

are a analytic diffeomorphisms onto their image.

We also quote the following decomposition Theorem [29], [10]:

Theorem A.2 (Iwasawa decomposition) *For any $r \in (0,1]$ and each solvable subgroup $\boldsymbol{B}$ of $SL(2,\mathbb{C})$, which satisfies $SU(2) \cdot \boldsymbol{B} = SL(2,\mathbb{C})$ and $SU(2) \cap \boldsymbol{B} = \{\mathrm{Id}\}$, the multiplication map*

$$\Lambda_r SU(2)_\sigma \times \Lambda_{r,B}^+ SL(2,\mathbb{C})_\sigma \to \Lambda_r SL(2,\mathbb{C})_\sigma$$

is a real analytic diffeomorphism onto.

A.2. Double loop groups and the Iwasawa decompositions

In this subsection, we present the basic notation and the basic theorems for double loop groups according to [13], but interchanging "+" and "−"

and "R"and "r". We set

$$\mathcal{H} = \Lambda_r SL(2,\mathbb{C})_\sigma \times \Lambda_R SL(2,\mathbb{C})_\sigma,$$

where $0 < r < R$. Moreover, we set

$$\mathcal{H}_+ = \Lambda_r^+ SL(2,\mathbb{C})_\sigma \times \Lambda_R^- SL(2,\mathbb{C})_\sigma,$$
$$\mathcal{H}_{+,*} = \Lambda_{r,*}^+ SL(2,\mathbb{C})_\sigma \times \Lambda_R^- SL(2,\mathbb{C})_\sigma,$$

and

$$\mathcal{H}_- = \big\{(g_1, g_2) \in \mathcal{H}; g_1 \text{ and } g_2 \text{ extend holomorphically to } A_r \text{ and } g_1|_{A_r} = g_2|_{A_r}\big\}.$$

We quote Theorem 2.6 in [13] (see also [22]).

Theorem A.3 ([13], [22]) *We have the disjoint union*

$$\mathcal{H} = \bigcup_{n=0}^{\infty} \mathcal{H}_- w_n \mathcal{H}_+,$$

where $w_n = \big(\mathrm{Id}, \left(\begin{smallmatrix}\lambda^n & 0\\ 0 & \lambda^{-n}\end{smallmatrix}\right)\big)$ *if* $n = 2k$ *and* $\big(\mathrm{Id}, \left(\begin{smallmatrix}0 & \lambda^n\\ -\lambda^{-n} & 0\end{smallmatrix}\right)\big)$ *if* $n = 2k+1$. *Moreover the multiplication map* $\mathcal{H}_- \times \mathcal{H}_{+,*} \to \mathcal{H}$ *is an analytic diffeomorphism onto its image, which is open and dense in* $\mathcal{H}$.

We would like to point out that the proof of the theorem above is almost verbatim the proof given in the basic decomposition paper [2] (see also [10]).

A.3. Generalized Weierstraß type representation for CMC-immersions in the double loop group picture and a unique meromorphic extension of an extended framing

In this subsection, we give a brief explanation of a construction of CMC-immersions in $\mathbb{R}^3$ via double loop groups. We refer the reader to [10] for a more detailed discussion.

Let $F(z, \bar{z}, \lambda)$ be the extended framing of some CMC surface with mean curvature $H \neq 0$ is defined on unit circle $|\lambda| = 1$. Let $C = C(z, \lambda) = FW_+$ be a holomorphic extended framing of this CMC-immersion. We consider the embedding:

$$g \to \big(g, (\bar{g}^t)^{-1}\big) \tag{A.2}$$

of $\Lambda SL(2,\mathbb{C})_\sigma$ into $\Lambda SL(2,\mathbb{C})_\sigma \times \Lambda SL(2,\mathbb{C})_\sigma$. If $\eta = \eta(z,\lambda)$ denotes the Maurer-Cartan-Form of C, $\eta = C^{-1}dC$, then the image under the differential of the map (A.2) is given by

$$\big(\eta(z,\lambda),\ \ -\overline{\eta(z,1/\bar{\lambda})}^{\,t}\big). \tag{A.3}$$

We would like to point out

$$\eta(z,\lambda) = \sum_{k=-1}^{\infty} \eta_k(z)\lambda^k \quad \text{and} \quad -\overline{\eta(z,1/\bar{\lambda})}^{\,t} = \sum_{k=-1}^{\infty} -\overline{\eta_k(z)}^{\,t}\lambda^{-k}. \tag{A.4}$$

Thus we obtain a pair of potentials (A.3) for each CMC-immersion in $\mathbb{R}^3$.

We now consider the reverse construction. (A.4) can be rephrased as

$$\tau(w,\lambda) = -\overline{\eta(\bar{w},1/\bar{\lambda})}^{\,t}, \tag{A.5}$$

where $w = \bar{z}$. Thus

$$\tau(w,\lambda) = \sum_{m=-\infty}^{1} \tau_m(w)\lambda^m, \tag{A.6}$$

where $\tau_m(w) = -\overline{\eta_{-m}(\bar{w})}^{\,t}$ and $w = \bar{z}$. Using the equations above and setting $L = (\bar{C}^t)^{-1}$, we obtain

$$\begin{cases} dC = C\eta, & C(z_0,\lambda) = \mathrm{Id}, \\ dL = L\tau, & L(\bar{z}_0,\lambda) = \mathrm{Id}. \end{cases} \tag{A.7}$$

where $\tau = -\overline{\eta(z,1/\bar{\lambda})}^{\,t}$, and z_0 is the base point chosen in $\mathbb{D}$. Applying the generalized Iwasawa decomposition A.3, we obtain

$$(C,L) = (F,F)(\mathrm{Id},W)(V_+,V_-), \tag{A.8}$$

where W is as in Theorem A.1. It is important to note that the symmetry $L = (\bar{C}^t)^{-1}$ implies $W = \mathrm{Id}$. Hence $F(z,\bar{z},\lambda)$ is the extended framing of some CMC-immersion. In the above procedure we can consider the variable $w = \bar{z}$ as a free variable [10]. This implies that there exists an extension of the extended framing $F(z, \bar{z}, \lambda)$. More precisely, from [10], we show that for each CMC-immersion defined on $\mathbb{D} \subset \mathbb{C}$, with extended framing $F(z, \bar{z}, \lambda)$, there exist some λ-independent diagonal matrix $l = l(z, \bar{z})$ such that the extended framing $F(z, \bar{z}, \lambda) l(z, \bar{z})$ has a unique meromorphic extension to $\mathbb{D} \times \bar{\mathbb{D}}$:

Theorem A.4 ([10]) *Let $F(z, \bar{z}, \lambda) : \mathbb{D} \to \Lambda SU(2)_\sigma$ be the extended framing of some* CMC*-immersion. Then there exists a λ-independent diagonal matrix $l(z, \bar{z}) \in SL(2, \mathbb{C})$ with entries $l_0 > 0$ and $l_0^{-1} > 0$ such that $F(z, \bar{z}, \lambda) l(z, \bar{z})$ has a unique meromorphic extension $U(z, w, \lambda) : \mathbb{D} \times \bar{\mathbb{D}} \to \Lambda SL(2, \mathbb{C})_\sigma$.*

References

[1] Ahlfors L. V. and Bers L., *Riemann's mapping theorem for variable metrics.* Ann. Math. (2) **72** (1960), 385–404.

[2] Bergvelt M. J. and Guest M. A., *Actions of loop groups on harmonic maps.* Trans. Amer. Math. Soc. **326**(2) (1991), 861–886.

[3] Bobenko A. I., *All constant mean curvature tori in* $\mathbf{R}^3$, S^3, H^3 *in terms of theta-functions.* Math. Ann. **290**(2) (1991), 209–245.

[4] Bungart L., *On analytic fiber bundles. I: Holomorphic fiber bundles with infinite dimensional fibers.* Topology **7** (1968), 55–68.

[5] Cartan É., *Leçons sur la géométrie projective complexe. La théorie des groupes finis et continus et la géométrie différentielle traitées par la méthode du repère mobile. Leçons sur la théorie des espaces à connexion projective.* Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1992. Reprint of the editions of 1931, 1937 and 1937.

[6] Chern S. S., *An elementary proof of the existence of isothermal parameters on a surface.* Proc. Amer. Math. Soc. **6** (1955), 771–782.

[7] Clancey K. and Gohberg I., *Factorization of matrix functions and singular integral operators*, volume 3 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 1981.

[8] Coddington E. A. and Levinson N., *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[9] Dorfmeister J. and Haak G., *Meromorphic potentials and smooth surfaces of constant mean curvature.* Math. Z. **224**(4) (1997), 603–640.

[10] Dorfmeister J. and Kobayashi S.-P., *Coarse classification of constant mean curvature cylinders.* Trans. Amer. Math. Soc. **359**(6) (2007), 2483–2500 (electronic).

[11] Dorfmeister J., Kobayashi S.-P. and Schuster M., *Delaunay surfaces via integrable system methods.* Preprint, 2006.

[12] Dorfmeister J., Pedit F. and Wu H., *Weierstrass type representation of harmonic maps into symmetric spaces.* Comm. Anal. Geom. **6**(4) (1998), 633–668.

[13] Dorfmeister J. and Wu H., *Constant mean curvature surfaces and loop groups.* J. Reine Angew. Math. **440** (1993), 43–76.

[14] Erdélyi A., Magnus W., Oberhettinger F. and Tricomi F. G., *Higher transcendental functions. Vol. II.* Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, Reprint of the 1953 original.

[15] Fokas A. S. and Gelfand I. M., *Surfaces on Lie groups, on Lie algebras, and their integrability.* Comm. Math. Phys. **177**(1) (1996), 203–220. With an appendix by Juan Carlos Alvarez Paiva.

[16] Gohberg I., *The factorization problem in normed rings, functions of isometric and symmetric operators, and singular integral equations.* Uspehi Mat. Nauk, **19**(1 (115)) (1964), 71–124.

[17] Grauert H. and Remmert R., *Theory of Stein spaces.* Classics in Mathematics. Springer-Verlag, Berlin, 2004. Translated from the German by Alan Huckleberry, Reprint of the 1979 translation.

[18] Griffiths P., *On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry.* Duke Math. J. **41** (1974), 775–814.

[19] Helgason S., *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[20] Hitchin N. J., *Harmonic maps from a* 2*-torus to the* 3*-sphere.* J. Differential Geom. **31**(3) (1990), 627–710.

[21] Kaup L. and Kaup B., *Holomorphic functions of several variables*, volume 3 of *de Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 1983. An introduction to the fundamental theory, With the assistance of Gottfried Barthel, Translated from the German by Michael Bridgland.

[22] Kellersch P., *Eine Verallgemeinerung der Iwasawa Zerlegung in Loop Gruppen*, volume 4 of *DGDS. Differential Geometry–Dynamical Systems. Monographs.* Geometry Balkan Press, Bucharest, 2004. Dissertation, Technische Universität München, Munich, 1999, Available electronically at

http://vectron.mathem.pub.ro/dgds/mono/dgdsmono.htm.

[23] Kobayashi S. and Ochiai T., *Holomorphic structures modeled after hyperquadrics*. Tôhoku Math. J. (2) **34**(4) (1982), 587–629.

[24] Kobayashi S.-P., *Real forms of complex surfaces of constant mean curvature*. Trans. Amer. Math. Soc. to appear.

[25] LeBrun C., *Spaces of complex null geodesics in complex-Riemannian geometry*. Trans. Amer. Math. Soc. **278**(1) (1983), 209–231.

[26] Meeks W. H., III, Pérez J. and Ros A., *Properly embedded minimal planar domains*. Preprint, 2007.

[27] Osserman R., *A survey of minimal surfaces*. Dover Publications Inc., New York, second edition, 1986.

[28] Pinkall U. and Sterling I., *On the classification of constant mean curvature tori*. Ann. of Math. (2) **130**(2) (1989), 407–451.

[29] Pressley A. and Segal G., *Loop groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications.

[30] Ruh E. A. and Vilms J., *The tension field of the Gauss map*. Trans. Amer. Math. Soc. **149** (1970), 569–573.

[31] Schmitt N., Kilian M., Kobayashi S.-P. and Rossman W., *Unitarization of monodromy representations and constant mean curvature trinoids in 3-dimensional space forms*. J. London Math. Soc. (2) **75**(2) (2007), 563–581.

[32] Stein K., *Überlagerungen holomorph-vollständiger komplexer Räume*. Arch. Math. **7** (1956), 354–361.

[33] Trèves F., *Basic linear partial differential equations*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 62.

[34] Weinstein T., *An introduction to Lorentz surfaces*, volume 22 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1996.

[35] Wu H., *A simple way for determining the normalized potentials for harmonic maps*. Ann. Global Anal. Geom. **17**(2) (1999), 189–199.

J. Dorfmeister
Zentrum Mathematik der Technischen Universität München
Lehrstuhl Scheurle M 8 Boltzmannstr
3 D-85747 Garching Germany
E-mail: dorfm@ma.tum.de

S.-P. Kobayashi
Graduate School of Science and Technology
Hirosaki University
Bunkyocho 3 Aomori 036-8561 Japan
E-mail: shimpei@cc.hirosaki-u.ac.jp

F. Pedit
Mathematisches Institut der Universität Tübingen
Auf der Morgenstelle 10 72076 Tübingen Germany
Department of Mathematics
University of Massachusetts
Amherst, MA 01003, USA
E-mail: pedit@mathematik.uni-tuebingen.de