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Verified numerical computation of solutions for the stationary Navier-Stokes equation in nonconvex polygonal domains

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Abstract. We propose a method to enclose solutions for the stationary Navier-Stokes equation in nonconvex polygonal domains. Our method is based on an infinite dimensional Newton-type formulation by using the finite element method with constructive error estimates and fixed point theorems. Numerical examples related to the step flow problems in L-shape domain are presented.

Key words: Navier-Stokes equation, nonconvex polygonal domains, step flow.

1. Introduction

In the present paper, we consider a numerical method to verify the existence and the local uniqueness of solutions for the following stationary Navier-Stokes equations:

$$\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u + \nabla p &= 0 \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where u and p are the velocity vector and the pressure, respectively. Assume that Ω is a nonconvex polygonal domain in $\mathbf{R}^2$. In addition, g is a given boundary vector function and $\nu > 0$ is a viscosity coefficient.

1.1. Motivation

The problem (1.1) is considered in [1]. For L-shaped domains, the equation (1.1) is known as a mathematical model for the step flow problems. From the theoretical point of view on the reliability of numerical computations, it is important to give a mathematically rigorous a posteriori error analysis for the approximate solutions of the flow. However, the equation (1.1) is also known as the difficult problem because of the singularity which is influenced by the reentrant corner. Thus, our purpose in this

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paper is to find an exact solution of (1.1) and clarify its behavior using a computer-asisted proof and some mathematical techniques.

In [11], there already exists a similar work for the convex domain in which the error estimates are more easily given. They use a method, so-called *Nakao's method* (see, e.g., [3], [4], [10] for more details), that consists of two kinds of iterative process; one is a finite dimensional Newton-like iterations, the other is the successive computations of the error caused by the gap between the finite and infinite dimension in each iterative procedure (see, e.g., [3], [4], [10] for more details). However, in the original Nakao's method, it has been recently observed ([6]), that for the second order problem having a first order derivative ∇u, the computational process of verification is not necessary efficient but sometimes diverges due to the property of interval computations. In order to overcome such a difficulty, in [6], some improvements are considered by using a technique with estimation of the norm for the inverse of a matrix corresponding to the linearlized operator, instead of direct solving an interval system of equations. Moreover, in [5], some further extended techniques are considered to develop a verification method by using an infinite dimensional Newton-like method for the second order elliptic problems.

In this paper, according to the analogous arguments to that in [5], which is a modified version of one of the authors' method (*cf.* [3] [4] etc.), we present a guaranteed estimates of the inverse of linearized operator for the Navier-Stokes equation (1.1) to get a verification condition based on the infinite dimensional Newton-like procedure. On the other hand, Plum's method which is also well known to verify the solutions for nonlinear elliptic boundary value problems [8] [9], would also be applicable, if it is possible to bound the eigenvalues for linearlized operator corresponding to (1.1). However, this eigenvalue bounding process for the present case seems to be quite complicated.

In order to apply the method in [5], in general to use Nakao's method, it is necessary to obtain the constructive a priori error estimate between a function and its appropriate projections. Namely, for example, when we denote the H_0^1-projection as P_h, it is necessary to determine the constant C numerically in the a priori error estimate of the form:

$$\|v - P_h v\|_{H_0^1} \leq C\|\Delta v\|_{L^2},$$

where C depends on the mesh size h of the finite element space such that

$C \to 0$ as $h \to 0$. This constant is naturally dependent on the regularity of solutions for the Poisson equation with homogeneous boundary conditions. For example, it implies that $C = O(h)$, if Ω is a convex domain. However, the order of magnitude decreases for nonconvex polygonal domains, that is, $C \approx O(h^{2/3})$, if Ω is the L-shaped domain. When we apply our method, it is essential and important to determine the above constant as small as possible. However, for nonconvex polygons, this task is usually not so easy but very hard by only theoretical considerations. As one of the computational approaches by some guaranteed numerical computations, Yamamoto and one of authors presented a computational method to get the explicit constant [12], which will be used in Section 4 in this paper.

In the following section, we define the Stokes projection and describe its constructive error estimates. The invertibility conditions of linearized operator and the norm estimation procedure for its inverse are considered in Section 3, which play an essential role in the verification by the infinite dimensional Newton-like method. In Section 4, we mention about the actual verification procedure for solutions of the nonlinear Navier-Stokes problem (1.1). Some verification examples of the step flow problem are presented in the last section.

1.2. Notations

We denote the usual k-th order Sobolev space on Ω by $H^k(\Omega)$ and define $(\,\cdot\,,\,\cdot\,)_0$ as the L^2 inner product. We also define the following Sobolev spaces as usual:

$$H_0^1(\Omega) \equiv \{v \in H^1(\Omega)\,;\, v = 0 \quad \text{on } \partial\Omega\},$$
$$L_0^2(\Omega) \equiv \{q \in L^2(\Omega)\,;\, (q,\, 1)_0 = 0\},$$

and set $X \equiv \big(H_0^1(\Omega)\big)^2$, $Y \equiv L_0^2(\Omega)$, $X(\Delta) \equiv \{v \in X\,;\, \Delta v \in \big(L^2(\Omega)\big)^2\}$. Moreover, we denote that

$$V_0 = \{v \in X\,;\, \operatorname{div} v = 0\},$$
$$V_\perp = \{v_\perp \in X\,;\, (\nabla v_\perp,\, \nabla v)_0 = 0,\ \forall v \in V_0\}.$$

Here, we used the same notation $(\,\cdot\,,\,\cdot\,)_0$ as the natural extension to L^2 inner product on vector functions. Then, we have $X = V_0 \oplus V_\perp$, where the orthogonality means in H_0^1 sense.

For $v \in (H_0^1(\Omega))^2$, we also define the H_0^1-norm by $\|v\|_{H_0^1} \equiv (\nabla v,\, \nabla v)_0^{1/2}$. Then, the norm on X will be straightforward. And, $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality pairing between X and X' which is the dual space of X. Moreover, $X_h \subset X$ and $Y_h \subset Y$ denote finite element subspaces which depend on the mesh size h.

2. The constructive a priori and a posteriori error estimations

In this section, we show the constructive a priori and a posteriori error estimations for the Stokes equation. These estimates are essentially presented in [7]. But, for our current purpose, we need some modification for the basic error estimates of the H_0^1-projection due to the nonconvexity of the domain, as well as it is necessary to get addtional estimates, e.g., in H^{-1} sense.

For each $v \in X$, we define the H_0^1-projection $P_h v \in X_h$ by

$$(\nabla(v - P_h v),\, \nabla\phi_h)_0 = 0, \quad \forall \phi_h \in X_h, \tag{2.1}$$

Further, we assume the following a priori error estimates.

Assumption 1 For an arbitrary $v \in X(\Delta)$, there exists a constant $C(h)$ depending on h such that

$$\|v - P_h v\|_{H_0^1} \le C(h)\|\Delta v\|_{L^2}.$$

Here, $C(h)$ has to be numerically determined.

Notice that Assumption 1 is equivalent to the following inequality:

$$\|v - P_h v\|_{L^2} \le C(h)\|v - P_h v\|_{H_0^1}.$$

We first refer the following well known result.

Lemma 2 (Babuš̃ka-Aziz [2]) *For all $q \in Y$, there exists a unique $v_\perp \in V_\perp$ such that*

$$\operatorname{div} v_\perp = q, \quad \|v_\perp\|_{H_0^1} \le \beta\|q\|_{L^2},$$

where $\beta > 0$ is a constant depending on Ω.

Now, we define the following functionals.

$$\begin{aligned} \mathcal{X}(u,\, p) &\equiv \sup_{v \in X} \frac{\nu(\nabla u,\, \nabla v)_0 - (p,\, \operatorname{div} v)_0}{\|v\|_{H_0^1}}, \\ \mathcal{Y}(u) &\equiv \sup_{q \in Y} \frac{(q,\, \operatorname{div} u)_0}{\|q\|_{L^2}}. \end{aligned} \tag{2.2}$$

Then, we have the following result.

Theorem 3 *For an arbitrary* $(u,\, p) \in X \times Y$*, it implies that*

$$\|u\|_{H_0^1} \le \frac{1}{\nu}\left[\left(\mathcal{X}(u,\, p)\right)^2 + \left(\nu\beta\mathcal{Y}(u)\right)^2\right]^{1/2},$$
$$\|p\|_{L^2} \le \beta\mathcal{X}(u, p) + \nu\beta^2\mathcal{Y}(u).$$

Proof. First, for an arbitrary $u \in X$, we decompose it as $u = u_0 \oplus u_\perp \in V_0 \oplus V_\perp$. Then, we have

$$\mathcal{X}(u,\, p) \ge \sup_{v \in V_0} \frac{\nu(\nabla u,\, \nabla v)_0 - (p,\, \operatorname{div} v)_0}{\|v\|_{H_0^1}} = \sup_{v \in V_0} \frac{\nu(\nabla u_0,\, \nabla v)_0}{\|v\|_{H_0^1}} = \nu\|u_0\|_{H_0^1}.$$

Also by Lemma 2, we have

$$\mathcal{Y}(u) \ge \frac{1}{\beta}\|u_\perp\|_{H_0^1}.$$

Thus the first part of the theorem is obtained.

Next, for $(u,\, p) \in X \times Y$, from Lemma 2, there exists $v_\perp \in V_\perp$ satisfying $\operatorname{div} v_\perp = -p$. Setting $q \in Y$ as $q = K \cdot \operatorname{div} u_\perp$, where $K = \nu(\nabla u_\perp,\, \nabla v_\perp)_0 / \|\operatorname{div} u_\perp\|_{L^2}^2$, it implies that

$$\begin{aligned}\|p\|_{L^2}^2 &= \nu(\nabla u_\perp,\, \nabla v_\perp)_0 + \|p\|_{L^2}^2 - (q,\, \operatorname{div} u_\perp)_0 \\ &= \|v_\perp\|_{H_0^1} \frac{\nu(\nabla u_\perp,\, \nabla v_\perp)_0 - (p,\, \operatorname{div} v_\perp)_0}{\|v_\perp\|_{H_0^1}} - \|q\|_{L^2} \frac{(q,\, \operatorname{div} u_\perp)_0}{\|q\|_{L^2}} \\ &\le \|v_\perp\|_{H_0^1}\mathcal{X}(u,\, p) + \|q\|_{L^2}\mathcal{Y}(u).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\|q\|_{L^2} = K\|\operatorname{div} u_\perp\|_{L^2} &= \frac{\nu(\nabla u_\perp,\, \nabla v_\perp)_0}{\|\operatorname{div} u_\perp\|_{L^2}} \\ &\le \frac{\nu\|u_\perp\|_{H_0^1}\|v_\perp\|_{H_0^1}}{\|\operatorname{div} u_\perp\|_{L^2}} \\ &\le \nu\beta^2\|p\|_{L^2}.\end{aligned}$$

From $\|v_\perp\|_{H_0^1} \le \beta\|\operatorname{div} v_\perp\|_{L^2}$, we obtain the second result. Therefore, this proof is completed. □

Now, let define the map $\mathcal{B}\colon X \times Y \longrightarrow X' \times Y$ by

$$\mathcal{B}(u,\, p) \equiv \big(S(u,\, p),\, -\operatorname{div} u\big), \tag{2.3}$$

where $S(u,\, p) \equiv -\nu\Delta u + \nabla p$ for $(u,\, p) \in X \times Y$. Then, for an arbitrary $(u,\, p) \in X \times Y$, we define the $\mathcal{Q}_h$-projection $\mathcal{Q}_h(u,\, p) \equiv (u_h,\, p_h) \in X_h \times Y_h$ by

$$\begin{aligned} \nu(\nabla(u-u_h),\, \nabla v_h)_0 - (p-p_h,\, \operatorname{div} v_h)_0 &= 0, \quad \forall v_h \in X_h, \\ -(\operatorname{div}(u-u_h),\, q_h)_0 &= 0, \quad \forall q_h \in Y_h. \end{aligned} \tag{2.4}$$

Then, we have the following main result of this section.

Theorem 4 *Let $(u,\, p) \in V_0 \times Y$ and let $(u_h,\, p_h) \in X_h \times Y_h$ be the $\mathcal{Q}_h$-projection of $(u,\, p)$. We assume that $S(u,\, p) \in \big(L^2(\Omega)\big)^2$ and that there exist constants η and σ independent of $(u,\, p)$ satisfying*

$$\begin{aligned} \|\nabla p_h\|_{L^2} &\le \eta \|S(u,\, p)\|_{L^2}, \\ \|\operatorname{div} u_h\|_{L^2} &\le \sigma \|S(u,\, p)\|_{L^2}. \end{aligned}$$

Then, we have the following a priori error estimations.

$$\begin{aligned} \|u-u_h\|_{H_0^1} &\le \nu^{-1} E_u(h) \|S(u,\, p)\|_{L^2}, \\ \|p-p_h\|_{L^2} &\le \qquad E_p(h) \|S(u,\, p)\|_{L^2}, \end{aligned}$$

where $E_u(h) := \big[\big(C(h)(1+\eta)\big)^2 + (\nu\beta\sigma)^2\big]^{1/2}$ and $E_p(h) := C(h)(1+\eta)\beta + \nu\beta^2\sigma$. Here, the constant β is defined in Lemma 2.
Moreover, define as in [7], $\overline{\nabla} u_h \in (X_h)^2$ *and* $\overline{\Delta} u_h \equiv \nabla \cdot \overline{\nabla} u_h$, *where* $\overline{\nabla} u_h$ *is determined by*

$$(\overline{\nabla} u_h,\, \mathbf{v}_h)_0 = (\nabla u_h,\, \mathbf{v}_h)_0, \quad \textit{for all } \mathbf{v}_h \in (X_h)^2.$$

Then, we have the following a posteriori error estimations.

$$\begin{aligned} \|u-u_h\|_{H_0^1} &\le \frac{1}{\nu}\big[\big(C(h)K_1 + \nu K_2\big)^2 + (\nu\beta K_3)^2\big]^{1/2}, \\ \|p-p_h\|_{L^2} &\le \beta\big(C(h)K_1 + \nu K_2\big) + \nu\beta^2 K_3, \end{aligned}$$

and

$$\|u-u_h\|_{L^2} \le E(h)\|u-u_h\|_{H_0^1} + \sigma\|p-p_h\|_{L^2}, \tag{2.5}$$

where $E(h) := E_u(h) + E_p(h)$ and the constants K_i, $(1 \le i \le 3)$ are defined *as*

$$K_1 \equiv \|S(u,\,p) + \nu\overline{\Delta}u_h - \nabla p_h\|_{L^2}, \quad K_2 \equiv \|\overline{\nabla}u_h - \nabla u_h\|_{H_0^1},$$
$$K_3 \equiv \|\operatorname{div} u_h\|_{L^2}.$$

Proof. First, by the definition of $\mathcal{X}$ and the property of the $\mathcal{Q}_h$-projection, i.e., $\nu(\nabla(u-u_h),\,\nabla v_h)_0 - (p-p_h,\,\operatorname{div} v_h)_0 = 0$ for all $v_h \in X_h$, it implies that

$$\begin{aligned}
&\mathcal{X}(u-u_h,\,p-p_h)\\
&= \sup_{v\in X} \frac{\nu(\nabla(u-u_h),\,\nabla(v-P_hv))_0 - (p-p_h,\,\operatorname{div}(v-P_hv))_0}{\|v\|_{H_0^1}}\\
&= \sup_{v\in X} \frac{(-\nu\Delta u + \nabla p - \nabla p_h,\,v-P_hv)_0}{\|v\|_{H_0^1}}\\
&\le C(h)\|S(u,\,p) - \nabla p_h\|_{L^2},
\end{aligned} \tag{2.6}$$

where we have used the fact $\|v-P_hv\|_{L^2} \le C(h)\|v-P_hv\|_{H_0^1} \le C(h)\|v\|_{H_0^1}$.

Next, we have

$$\begin{aligned}
\mathcal{Y}(u-u_h) &= \sup_{q\in Y} \frac{(q,\,\operatorname{div} u_h)_0}{\|q\|_{L^2}}\\
&\le \|\operatorname{div} u_h\|_{L^2}.
\end{aligned} \tag{2.7}$$

Hence, using assumptions of this theorem, we have the following estimations.

$$\begin{aligned}
\mathcal{X}(u-u_h,p-p_h) &\le C(h)(1+\eta)\|S(u,p)\|_{L^2},\\
\mathcal{Y}(u-u_h) &\le \sigma\|S(u,p)\|_{L^2}.
\end{aligned}$$

Combining these inequalities with Theorem 3, we obtain the desired a priori estimates.

Now, using the second equality of (2.6), from the fact that $(\nabla u_h,\,\nabla(v-P_hv))_0 = 0$ and $(\overline{\nabla}u_h,\,\nabla\phi)_0 = (-\overline{\Delta}u_h,\,\phi)_0$ for $\phi \in X$, we have

$$\begin{aligned}
&\mathcal{X}(u-u_h,p-p_h)\\
&=\sup_{v\in X} \frac{(-\nu\Delta u+\nabla p-\nabla p_h,v-P_hv)_0-\nu(\nabla u_h,\nabla(v-P_hv))_0}{\|v\|_{H_0^1}}\\
&=\sup_{v\in X} \frac{(S(u,p)+\nu\overline{\Delta}u_h-\nabla p_h,v-P_hv)_0+\nu(\overline{\nabla}u_h-\nabla u_h,\nabla(v-P_hv))_0}{\|v\|_{H_0^1}}\\
&\le C(h)\|S(u,p)+\nu\overline{\Delta}u_h-\nabla p_h\|_{L^2}+\nu\|\overline{\nabla}u_h-\nabla u_h\|_{H_0^1}.
\end{aligned} \tag{2.8}$$

Thus, we obtain the a posteriori error estimates for the Q_h-projection by (2.7) and (2.8).

We now finally present the L^2-estimation of $u - u_h$.
For $(u - u_h,\, 0) \in X \times L^2(\Omega)$, we consider the following Stokes equation.

$$\text{Find } (v,\, q) \in X \times Y \text{ such that } \mathcal{B}(v,\, q) = (u - u_h,\, 0) \quad \text{in } \Omega.$$

From the property of the $\mathcal{Q}_h$-projection, setting $(v_h,\, q_h) := \mathcal{Q}_h(v,\, q)$, we have

$$\begin{aligned}
\|u - u_h\|_{L^2}^2 &= (u - u_h,\, u - u_h)_0 \\
&= (-\nu\Delta v + \nabla q,\, u - u_h)_0 \\
&= \nu(\nabla v,\, \nabla(u - u_h))_0 - (q,\, \operatorname{div}(u - u_h))_0 \\
&= \nu(\nabla(v - v_h),\, \nabla(u - u_h))_0 + (p - p_h,\, \operatorname{div} v_h)_0 \\
&\quad - (q - q_h,\, \operatorname{div}(u - u_h))_0 \\
&\le \nu\|v - v_h\|_{H_0^1}\|u - u_h\|_{H_0^1} + \|p - p_h\|_{L^2}\|\operatorname{div} v_h\|_{L^2} \\
&\quad + \|q - q_h\|_{L^2}\|\operatorname{div}(u - u_h)\|_{L^2}.
\end{aligned}$$

Therefore, using the a priori error estimation and the assumption of this theorem, this proof is completed from the former part of the theorem and the fact that $\|\operatorname{div}(u - u_h)\|_{L^2} \le \|u - u_h\|_{H_0^1}$. □

If $S(u,\, p)$ does not belong to L^2 space, then we have the following estimates, which is readily seen by the similar arguments in the above theorem.

Corollary 5 *Let $(u,\, p) \in V_0 \times Y$ and let $(u_h,\, p_h) \in X_h \times Y_h$ be $\mathcal{Q}_h$-projection of $(u,\, p)$. We assume that $S(u,\, p) \in X'$ and there exist constants $\hat{\eta}$ and $\hat{\sigma}$ satisfying*

$$\begin{aligned}
\|\nabla p_h\|_{L^2} &\le \hat{\eta}\|S(u,\, p)\|_{H^{-1}}, \\
\|\operatorname{div} u_h\|_{L^2} &\le \hat{\sigma}\|S(u,\, p)\|_{H^{-1}}.
\end{aligned}$$

Then, we have the following estimations.

$$\begin{aligned}
\|u - u_h\|_{H_0^1} &\le \nu^{-1} e_u\|S(u,\, p)\|_{H^{-1}}, \\
\|p - p_h\|_{L^2} &\le \quad\;\; e_p\|S(u,\, p)\|_{H^{-1}},
\end{aligned}$$

where $e_u = \left[\left(1 + C(h)\hat{\eta}\right)^2 + (\nu\beta\hat{\sigma})^2\right]^{1/2}$ and $e_p = \left(1 + C(h)\hat{\eta}\right)\beta + \nu\beta^2\hat{\sigma}$. Here, we define the H^{-1}-norm by

$$\|S(u,\,p)\|_{H^{-1}} \equiv \sup_{\phi\in X} \frac{\langle S(u,\,p),\,\phi\rangle}{\|\phi\|_{H_0^1}}.$$

Notice that by some simple calculations, in Corollary 5, it is always taken as $e_u = 2$, because of $\|u\|_{H_0^1} \le \nu^{-1}\|S(u,\,p)\|_{H^{-1}}$ and $\|u_h\|_{H_0^1} \le \nu^{-1}\|S(u,\,p)\|_{H^{-1}}$ if $(u,\,p) \in V_0 \times Y$.

3. Computable verification method for the inverse of the linearized operator

In this section, we describe a numerical method to prove the invertibility of the following linear operator and estimate the norm of the inverse.

The linearized Navier-Stokes equation with homogeneous Dirichlet boundary conditions can be written as

$$\begin{aligned}&\text{Find } (u,\,p) \in X \times Y \text{ such that}\\ &\qquad \mathcal{L}(u,\,p) \equiv \mathcal{B}(u,\,p) + \Psi(u,\,p) = (f,\,0) \quad \text{in } \Omega,\end{aligned} \tag{3.1}$$

where $(f,\,0) \in X' \times L^2(\Omega)$ and the linear map Ψ is defined as

$$\begin{aligned}&\Psi(u,\,p) := (\Phi u,\,0) \quad \text{for each } (u,\,p) \in X \times Y\\ &\text{with } \Phi u := (c\cdot\nabla)u + (u\cdot\nabla)c.\end{aligned} \tag{3.2}$$

Here, $c \in \left(W^1_\infty(\Omega)\right)^2$, the coefficient vector function.

3.1. The invertibility condition of the operator $\mathcal{L}$

First, note that the invertibility of a linear operator $\mathcal{L}$ defined in (3.1) is equivalent to the unique solvability of the fixed point equation:

$$\begin{aligned}z &= \mathcal{A}z\\ &\equiv \mathcal{B}^{-1}\Psi z,\end{aligned} \tag{3.3}$$

where $z = (u,\,p)$ and $\mathcal{A}$ a compact operator on $X \times Y$.

Now, according to the verification principle presented in [5], we formulate a sufficient invertibility condition in numerically. As the preliminary, we define the several matrices as follows:
Namely, $N \times N$ matrices $\mathbf{F} = (\mathbf{F}_{i,j})$, $\mathbf{A} = (\mathbf{A}_{i,j})$, $M \times N$ matrix $\mathbf{B} = (\mathbf{B}_{i,j})$ and $M \times M$ matrix $\mathbf{C} = (\mathbf{C}_{i,j})$ are defined by

$$\begin{aligned}
\mathbf{F}_{i,j} &= \nu(\nabla\phi_j,\, \nabla\phi_i)_0 + (\Phi\phi_j,\, \phi_i)_0 && \text{for } 1 \le i,\, j \le N,\\
\mathbf{A}_{i,j} &= \quad (\nabla\phi_j,\, \nabla\phi_i)_0 && \text{for } 1 \le i,\, j \le N,\\
\mathbf{B}_{i,j} &= -(\operatorname{div}\phi_j,\, \psi_i)_0 && \text{for } 1 \le i \le M,\ 1 \le j \le N,\\
\mathbf{C}_{i,j} &= \quad (\psi_j,\, \psi_i)_0 && \text{for } 1 \le i,\, j \le M,
\end{aligned}$$

where $\{\phi_k\}_{k=1}^N$ and $\{\psi_k\}_{k=1}^M$ are basis of X_h and Y_h, respectively.
Next, supposing that $N \ge M$, we define the $N+M$ square matrix $\mathbf{G}$ by:

$$\mathbf{G} = \begin{bmatrix} \mathbf{F} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{bmatrix}, \tag{3.4}$$

Notice that if $\mathbf{G}$ is nonsingular then it implies that $\mathbf{F}$ and $\mathbf{S} := \mathbf{B}\mathbf{F}^{-1}\mathbf{B}^T$ are also nonsingular and we can write an inverse matrix by

$$\begin{aligned}
\begin{bmatrix} \mathbf{F} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{bmatrix}^{-1} &= \begin{bmatrix} \mathbf{F}^{-1} - \mathbf{F}^{-1}\mathbf{B}^T\mathbf{S}^{-1}\mathbf{B}\mathbf{F}^{-1} & \mathbf{F}^{-1}\mathbf{B}^T\mathbf{S}^{-1} \\ \mathbf{S}^{-1}\mathbf{B}\mathbf{F}^{-1} & -\mathbf{S}^{-1} \end{bmatrix}\\
&=: \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_3 \\ \mathbf{G}_2 & \mathbf{G}_4 \end{bmatrix}.
\end{aligned}$$

Let $\mathbf{L}$ and $\mathbf{M}$ be lower triangular matrices satisfying the Cholesky decompositions:

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T \quad \text{and} \quad \mathbf{C} = \mathbf{M}\mathbf{M}^T, \tag{3.5}$$

respectively. And, we denote the matrix norm induced from the Euclidean 2-norm by $|\cdot|_E$. Also, we define the following constants:

$$\begin{aligned}
&K_c := \big\| |c|_E \big\|_{L^\infty}, \quad K_{\operatorname{div} c} := \big\|\operatorname{div} c\big\|_{L^\infty},\\
&K_{\nabla c} := \big\| |\nabla c|_E \big\|_{L^\infty}, \quad K_{\partial c} := \big(\big\|\partial_i c\cdot\partial_j c\big\|_{L^2}^2\big)_F^{1/4},
\end{aligned}$$

where $\big\| |\nabla c|_E \big\|_{L^\infty}$ and matrix $\|\partial_i c\cdot\partial_j c\|_{L^2}$ mean that $\big\| \big(\sum_i |\nabla c_i|_E^2\big)^{1/2} \big\|_{L^\infty}$ and $\|\partial c/\partial x_i \cdot \partial c/\partial x_j\|_{L^2}$, respectively. Here, $\|\cdot\|_{L^\infty}$ and $(\cdot)_F$ denote the L^∞-norm on Ω and the matrix Frobenius norm, respectively.
By some simple calculations, we have the following lemma.

Lemma 6 *For $u,\, v,\, w \in X$, it implies that*

$$\begin{aligned}
\|(u\cdot\nabla)v\|_{L^2} &\le \big\| |u|_E \big\|_{L^\infty} \|v\|_{H_0^1} && \textit{if } \ u \in X_h,\\
\|(u\cdot\nabla)v\|_{L^2} &\le \|u\|_{L^2} \big\| |\nabla v|_E \big\|_{L^\infty} && \textit{if } \ v \in X_h,\\
\|(u\cdot\nabla)v\|_{L^2} &\le C_{L^4} \|u\|_{H_0^1} \big(\|\partial_i v\cdot\partial_j v\|_{L^2}^2\big)_F^{1/4} && \textit{if } \ v \in X_h.
\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
&((u\cdot\nabla)v,\, w)_0\\
&\qquad \le \big(\big\||u|_E\big\|_{L^\infty}\|w\|_{H_0^1} + \|\operatorname{div} u\|_{L^\infty}\|w\|_{L^2}\big)\|v\|_{L^2} \quad \text{if } u\in X_h,\\
&((u\cdot\nabla)v, w)_0\\
&\qquad \le \big(\|u\|_{H_0^1}\|w\|_{L^2} + \|u\|_{L^2}\|w\|_{H_0^1}\big)\big\||v|_E\big\|_{L^\infty} \quad \text{if } v\in X_h,\\
&\langle (u\cdot\nabla)v,\, w\rangle \le C_{L^4}^2\|u\|_{H_0^1}\|v\|_{H_0^1}\|w\|_{H_0^1},
\end{aligned}
$$

where C_{L^4} *is a constant such that* $\|\phi\|_{L^4}\le C_{L^4}\|\phi\|_{H_0^1}$ *for all* $\phi\in H_0^1(\Omega)$.

We now have the following main result of this paper.

Theorem 7 *For the constants defined above, if* $\mathbf{G}$ *is nonsingular and*

$$
\kappa \equiv \frac{1}{\nu}E_u(h)\big(M_uC_1C_2 + C_2\big) < 1
$$

holds then the operator $\mathcal{L}$ *defined in* (3.1) *is invertible.*
Here, $M_u \equiv \|\mathbf{L}^T\mathbf{G}_1\mathbf{L}\|_E$ *and* $E_u(h)$ *is the a priori constant in* Theorem 4.
And, the constants C_1 *and* C_2 *are given by*

$$
C_1 = 3C_{L^2}K_c, \quad C_2 = K_c + C_{L^4}K_{\partial c},
$$

where C_{L^2} *is a Poincaré constant such that* $\|\phi\|_{L^2}\le C_{L^2}\|\phi\|_{H_0^1}$ *for all* $\phi\in H_0^1(\Omega)$.

Proof. First, as in [3], [4], [10] etc., we decompose the equation $u=\mathcal{A}u$ into two parts as follows:

$$
\begin{aligned}
\mathcal{Q}_h z &= \mathcal{Q}_h\mathcal{A}z\\
(I-\mathcal{Q}_h)z &= (I-\mathcal{Q}_h)\mathcal{A}z
\end{aligned}
$$

where I implies the identity map on $X\times Y$.
Next, according to the similar formulation to that in [5], we define two operators by

$$
\mathcal{N}_h z \equiv \mathcal{Q}_h z - [I-\mathcal{A}]_h^{-1}\mathcal{Q}_h(I-\mathcal{A})z
$$

and

$$
\mathcal{T}z \equiv \mathcal{N}_h z + (I-\mathcal{Q}_h)\mathcal{A}z,
$$

respectively, where $[I-\mathcal{A}]_h^{-1}$ means the inverse of $\mathcal{Q}_h(I-\mathcal{A})|_{X_h\times Y_h} : X_h\times Y_h \longrightarrow X_h\times Y_h$.

Now, for two dimensional positive vectors $\alpha = (\alpha_u, \alpha_p)$ and $\gamma = (\gamma_u, \gamma_p)$, we define *the candidate set* $Z = Z_h \oplus Z_* \subset X \times Y$ which possibly encloses the solution of (3.3). Here, Z_h and Z_* are taken as

$$Z_h := \{ z_h \in X_h \times Y_h \,;\, [\|z_h\|] \le \gamma \},$$
$$Z_* := \{ z_* \in (X_h \times Y_h)^\perp \,;\, [\|z_*\|] \le \alpha \},$$

where $(\ )^\perp$ means the orthogonal complement in the sense of Q_h-projection, that is $z_* \in Z_* \Rightarrow Q_h z_* = 0$. Also denote $[\|z\|] \equiv (\|u\|_{H_0^1}, \|p\|_{L^2})$ for $z = (u, p) \in X \times Y$ and the inequality stands for elementwise.

Then, by the fact that $z = \mathcal{A}z$ is equivalent to $z = \mathcal{T}z$. In order to prove the unique existence of a solution to (3.3) in the set Z, it suffices to show $|||\mathcal{T}||| < 1$ for any kind of norm $||| \cdot |||$ in $X \times Y$. This fact follows by Banach's fixed point theorem from the linearity of the equation.

Further notice that a sufficient condition can be written as

$$[\|\mathcal{N}_h Z\|] \equiv \sup_{z \in Z} [\|\mathcal{N}_h z\|] < \gamma \tag{3.6}$$

and

$$[\|(I - \mathcal{Q}_h)\mathcal{A}Z\|] \equiv \sup_{z \in Z} [\|(I - \mathcal{Q}_h)\mathcal{A}u\|] < \alpha. \tag{3.7}$$

Therefore, by using constants defined above, we try to estimate norms $[\|\mathcal{N}_h z\|]$ and $[\|(I - \mathcal{Q}_h)\mathcal{A}z\|]$ in (3.6) and (3.7), respectively.

First, for an arbitrary $z = z_h + z_* \in Z_h + Z_*$, we have

$$\begin{aligned} \mathcal{N}_h z &= z_h - [I - \mathcal{A}]_h^{-1} \mathcal{Q}_h (I - \mathcal{A})(z_h + z_*) \\ &= [I - \mathcal{A}]_h^{-1} \mathcal{Q}_h \mathcal{A} z_*. \end{aligned} \tag{3.8}$$

We now set $(w_h^u, w_h^p) := \mathcal{N}_h z$, which means

$$\begin{aligned} \nu(\nabla w_h^u, \nabla v_h)_0 + (\Phi w_h^u, v_h)_0 - (w_h^p, \operatorname{div} v_h)_0 &= (-\Phi u_*, v_h)_0, \\ -(\operatorname{div} w_h^u, q_h)_0 &= 0, \end{aligned} \tag{3.9}$$

for all $v_h \in X_h$, $q_h \in Y_h$. Here, we choose $w := \Delta^{-1}\Phi u_* \in X$. Since the right-hand side of (3.9) satisfies

$$(-\Phi u_*, v_h)_0 = (\nabla w, \nabla v_h)_0 = (\nabla P_h w, \nabla v_h)_0,$$

we can obtain the following matrix linear equation:

$$\begin{bmatrix} \mathbf{F} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{bmatrix} \begin{bmatrix} \overline{w_h^u} \\ \overline{w_h^p} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{w_h} \\ 0 \end{bmatrix},$$

where $\overline{w_h^u} = (w_1^u, w_2^u, \ldots, w_N^u)^T$, $\overline{w_h^p} = (w_1^p, w_2^p, \ldots, w_M^p)^T$ and $\overline{w_h} = (w_1, w_2, \ldots, w_N)^T$ are coefficient vectors of w_h^u, w_h^p and $w_h \equiv P_h w$, respectively, which are set as

$$w_h^u := \sum_{i=1}^{N} w_i^u \phi_i, \quad w_h^p := \sum_{i=1}^{M} w_i^p \psi_i, \quad w_h := \sum_{i=1}^{N} w_i \phi_i.$$

Therefore, it implies that

$$\begin{aligned} \begin{bmatrix} \|w_h^u\|_{H_0^1} \\ \|w_h^p\|_{L^2} \end{bmatrix} = \begin{bmatrix} \|\mathbf{L}^T\ \overline{w_h^u}\|_E \\ \|\mathbf{M}^T \overline{w_h^p}\|_E \end{bmatrix} &= \begin{bmatrix} \|(\mathbf{L}^T\ \mathbf{G}_1\mathbf{L})(\mathbf{L}^T\overline{w_h})\|_E \\ \|(\mathbf{M}^T\mathbf{G}_2\mathbf{L})(\mathbf{L}^T\overline{w_h})\|_E \end{bmatrix} \\ &\le \begin{bmatrix} \|\mathbf{L}^T\ \mathbf{G}_1\mathbf{L}\|_E \|\mathbf{L}^T\overline{w_h}\|_E \\ \|\mathbf{M}^T\mathbf{G}_2\mathbf{L}\|_E \|\mathbf{L}^T\overline{w_h}\|_E \end{bmatrix} \\ &= \begin{bmatrix} \|\mathbf{L}^T\ \mathbf{G}_1\mathbf{L}\|_E \|w_h\|_{H_0^1} \\ \|\mathbf{M}^T\mathbf{G}_2\mathbf{L}\|_E \|w_h\|_{H_0^1} \end{bmatrix}. \end{aligned}$$

So, we can obtain the following estimations.

$$\|w_h^u\|_{H_0^1} \le M_u \|w_h\|_{H_0^1}, \qquad \|w_h^p\|_{L^2} \le M_p \|w_h\|_{H_0^1}, \tag{3.10}$$

where $M_u = \|\mathbf{L}^T\mathbf{G}_1\mathbf{L}\|_E$ and $M_p = \|\mathbf{M}^T\mathbf{G}_2\mathbf{L}\|_E$.
From the property of the H_0^1-projection, we have

$$\begin{aligned} \|w_h\|_{H_0^1} \equiv \|P_h w\|_{H_0^1} \le \|w\|_{H_0^1} &= \|\Delta^{-1}\Phi u_*\|_{H_0^1} \\ &\le \|\Delta^{-1}(c \cdot \nabla)u_*\|_{H_0^1} + \|\Delta^{-1}(u_* \cdot \nabla)c\|_{H_0^1}. \end{aligned}$$

Hence, we now estimate the H_0^1-norm of $w_1 := \Delta^{-1}(c \cdot \nabla)u_*$ and $w_2 := \Delta^{-1}(u_* \cdot \nabla)c$.

For the estimation of $\|w_1\|_{H_0^1}$, some simple calculations yields from Lemma 6 that

$$\begin{aligned} \|w_1\|_{H_0^1}^2 = (\nabla w_1, \nabla w_1)_0 &= (-\Delta w_1, w_1)_0 \\ &= (-(c \cdot \nabla)u_*, w_1)_0 \\ &\le C_{L^2} \big\| |c|_E \big\|_{L^\infty} \|u_*\|_{H_0^1} \|w_1\|_{H_0^1}. \end{aligned} \tag{3.11}$$

Furthermore, for the estimation of $\|w_2\|_{H_0^1}$, by applying the similar argument to the above and using Lemma 6, we have

$$\|w_2\|_{H_0^1} \le 2C_{L^2} \big\| |c|_E \big\|_{L^\infty} \|u_*\|_{H_0^1}. \tag{3.12}$$

Thus, by (3.10)–(3.12), we obtain the following estimate for the finite dimensional part

$$[\|\mathcal{N}_h Z\|] \le \begin{bmatrix} M_u \\ M_p \end{bmatrix} C_1 \alpha_u, \tag{3.13}$$

where $C_1 \equiv 3C_{L^2} K_c$.

For $z \in Z$, from Theorem 4 and Lemma 6, it implies that

$$[\|(I - \mathcal{Q}_h)\mathcal{A}z\|] \le \begin{bmatrix} \nu^{-1} E_u(h) \\ E_p(h) \end{bmatrix} C_2(\gamma_u + \alpha_u),$$

where $C_2 \equiv K_c + C_{L^4} K_{\partial c}$.

Therefore, the invertibility condition follows:

$$\begin{aligned} M_u C_1 \alpha_u &< \gamma_u, \\ M_p C_1 \alpha_u &< \gamma_p, \\ \nu^{-1} E_u(h) C_2(\gamma_u + \alpha_u) &< \alpha_u, \\ E_p(h) C_2(\gamma_u + \alpha_u) &< \alpha_p. \end{aligned}$$

Here, the second and fourth conditions of the above can always be valid provided that γ_p and α_p are suitable chosen. Therefore, we only consider the condition:

$$\begin{aligned} M_u C_1 \alpha_u &< \gamma_u, \\ \nu^{-1} E_u(h) C_2(\gamma_u + \alpha_u) &< \alpha_u. \end{aligned}$$

And, it is readily seen that this inequality is equivalent to

$$\frac{1}{\nu} E_u(h) C_2 (M_u C_1 C_2 + C_2) < 1.$$

Thus, the proof is completed. □

3.2. The norm estimation

In this subsection, we show the a priori estimates for the solution of the linear equation (3.1).

Theorem 8 *Under the same assumptions in* Theorem 7, *provided that* $\kappa < 1$ *and let* $z = (u, p) \in X \times Y$ *be a unique solution for the linear* *equation* (3.1)*, that is,* $\mathcal{L}z = (f,\, 0)$ *for* $(f,\, 0) \in X' \times L^2(\Omega)$*. Then, we have the following estimations*:

$$\begin{aligned} \|u\|_{H_0^1} &\le \mathcal{M}_u^* \|f\|_{H^{-1}}, \\ \|p\|_{L^2} &\le \mathcal{M}_p^* \|f\|_{H^{-1}}, \end{aligned}$$

where $\mathcal{M}_u^* \equiv \tau_1^* + \tau_2^*$*,* $\mathcal{M}_p^* \equiv \tau_3^* + \tau_4^*$ *and the constants* τ_i^* $(1 \le i \le 4)$ *are given by*

$$\begin{aligned} \tau_1^* &= \frac{1}{\nu} \frac{M_u E_u(h) C_2 + e_u}{1-\kappa}, & \tau_2^* &= M_u (C_1 \tau_1^* + 1), \\ \tau_3^* &= M_p (C_1 \tau_1^* + 1), & \tau_4^* &= E_p(h) C_2 (\tau_1^* + \tau_2^*) + e_p. \end{aligned}$$

Moreover, if $f \in \big(L^2(\Omega)\big)^2$*, then*

$$\begin{aligned} \|u\|_{H_0^1} &\le \mathcal{M}_u \|f\|_{L^2}, \\ \|p\|_{L^2} &\le \mathcal{M}_p \|f\|_{L^2}, \end{aligned}$$

where $\mathcal{M}_u \equiv \tau_1 + \tau_2$*,* $\mathcal{M}_p \equiv \tau_3 + \tau_4$ *and the constants* τ_i $(1 \le i \le 4)$ *are given by*

$$\begin{aligned} \tau_1 &= \frac{1}{\nu} \frac{E_u(h)(M_u C_2 C_{L^2} + 1)}{1-\kappa}, & \tau_2 &= M_u (C_1 \tau_1 + C_{L^2}), \\ \tau_3 &= M_p (C_1 \tau_1 + C_{L^2}), & \tau_4 &= E_p(h)\big(C_2(\tau_1 + \tau_2) + 1\big). \end{aligned}$$

Proof. For any $f \in X'$, define $(\varphi_u,\, \varphi_p) \equiv \mathcal{B}^{-1}(f,\, 0) \in X \times Y$. Then, by the Fredholm alternative theorem, the invertibility of $(I - \mathcal{A})$ implies that there exists a unique element $z \in X \times Y$ satisfying $(I - \mathcal{A})z = (\varphi_u,\, \varphi_p)$. When we set

$$\begin{aligned} \mathcal{N}_h z &:= \mathcal{Q}_h z - [I - \mathcal{A}]_h^{-1} \mathcal{Q}_h((I - \mathcal{A})z - (\varphi_u,\, \varphi_p)), \\ \mathcal{T} z &:= \mathcal{N}_h z + (I - \mathcal{Q}_h)(\mathcal{A}z + (\varphi_u,\, \varphi_p)), \end{aligned}$$

notice that $(I - \mathcal{A})z = (\varphi_u,\, \varphi_p)$ is equivalent to $\mathcal{T}z = z$. Using the decomposition $z = z_h + z_*$ with $z_h \equiv \mathcal{Q}_h z$ and $z_* \equiv z - \mathcal{Q}_h z$, by some simple calculations, we have

$$\begin{aligned} z_h &= [I - \mathcal{A}]_h^{-1} (\mathcal{Q}_h \mathcal{A} z_* + \mathcal{Q}_h(\varphi_u,\, \varphi_p)), \\ z_* &= (I - \mathcal{Q}_h)\mathcal{A}(z_h + z_*) + (I - \mathcal{Q}_h)(\varphi_u,\, \varphi_p). \end{aligned} \tag{3.14}$$

Hence, taking the estimates in the proof of Theorem 7 and letting $\varphi =$

$\Delta^{-1}f$, we have by (3.14)

$$\begin{aligned}\begin{bmatrix}\|u_h\|_{H_0^1}\\ \|p_h\|_{L^2}\end{bmatrix} &\le \begin{bmatrix}M_u\\ M_p\end{bmatrix}(C_1\|u_*\|_{H_0^1}+\|P_h\varphi\|_{H_0^1})\\ &\le \begin{bmatrix}M_u\\ M_p\end{bmatrix}(C_1\|u_*\|_{H_0^1}+\|f\|_{H^{-1}}),\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\begin{bmatrix}\|u_*\|_{H_0^1}\\ \|p_*\|_{L^2}\end{bmatrix} &\le \begin{bmatrix}\nu^{-1}E_u(h)\\ E_p(h)\end{bmatrix}C_2(\|u_h\|_{H_0^1}+\|u_*\|_{H_0^1})\\ &\quad + [\|(I-\mathcal{Q}_h)\mathcal{B}^{-1}(f,0)\|]\\ &\le \begin{bmatrix}\nu^{-1}E_u(h)\\ E_p(h)\end{bmatrix}C_2(\|u_h\|_{H_0^1}+\|u_*\|_{H_0^1})\\ &\quad + \begin{bmatrix}\nu^{-1}e_u\\ e_p\end{bmatrix}\|f\|_{H^{-1}}.\end{aligned} \tag{3.16}$$

Substituting the estimate of $\|u_h\|_{H_0^1}$ in (3.15) into the last right-hand side of (3.16) and solving it with respect to $\|u_*\|_{H_0^1}$, we get

$$\begin{aligned}\|u_*\|_{H_0^1} &= \frac{1}{\nu}\frac{\big(M_uE_u(h)C_2+e_u\big)\|f\|_{H^{-1}}}{1-\kappa}\\ &= \tau_1^*\|f\|_{H^{-1}}.\end{aligned} \tag{3.17}$$

Thus, we also have by (3.15)

$$\begin{bmatrix}\|u_h\|_{H_0^1}\\ \|p_h\|_{L^2}\end{bmatrix} \le \begin{bmatrix}M_u\\ M_p\end{bmatrix}(C_1\tau_1^*+1)\|f\|_{H^{-1}} = \begin{bmatrix}\tau_2^*\\ \tau_3^*\end{bmatrix}\|f\|_{H^{-1}}. \tag{3.18}$$

Hence, it implies that

$$\begin{aligned}\|p_*\|_{L^2} &= \big(E_p(h)C_2(\tau_1^*+\tau_2^*)+e_p\big)\|f\|_{H^{-1}}\\ &= \tau_4^*\|f\|_{H^{-1}}.\end{aligned} \tag{3.19}$$

Therefore, from (3.17)–(3.19) and $\|u\|_{H_0^1} \le \|u_h\|_{H_0^1}+\|u_*\|_{H_0^1}$, $\|p\|_{L^2} \le \|p_h\|_{L^2}+\|p_*\|_{L^2}$, the proof of the former part is completed. Also, for the case that $f \in \big(L^2(\Omega)\big)^2$, one can easily derive the results in the latter part by the similar arguments above. □

4. Applications to nonlinear problems

In this section, we describe the actual applications of the results obtained in the previous section to the verification of solutions for the stationary Navier-Stokes equation (1.1). We assume that a function $\mathbf{g} \in X(\Delta)$ satisfies $\mathbf{g} = g$ on $\partial\Omega$ and $\operatorname{div} \mathbf{g} = 0$ in Ω. Then, our original problem can be written as

$$\begin{aligned} -\nu\Delta u + ((u+\mathbf{g})\cdot\nabla)(u+\mathbf{g}) + \nabla p &= \nu\Delta\mathbf{g} \quad \text{in } \Omega, \\ -\operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

We transform the original stationary Navier-Stokes problem (4.1) into the so-called residual equation by using an approximate solution $(\tilde{u}_h, \tilde{p}_h) \in X_h \times Y_h$ defined by

$$\begin{aligned} \nu(\nabla\tilde{u}_h, \nabla v_h)_0 - (\tilde{p}_h, \operatorname{div} v_h)_0 &= (\nu\Delta\mathbf{g} - f(\tilde{u}_h + \mathbf{g}), v_h)_0, \\ (-\operatorname{div}\tilde{u}_h, q_h)_0 &= 0, \end{aligned} \tag{4.2}$$

for all $v_h \in X_h$, $q_h \in Y_h$, where $f(u) := (u\cdot\nabla)u$.

For the effective computation of the solution for (4.2) with guaranteed accuracy, refer, for example, [11] etc.
Next, we define $(\bar{u}, \bar{p}) \in X \times Y$ by the solution of the Stokes equation:

$$\mathcal{B}(\bar{u}, \bar{p}) = (\nu\Delta\mathbf{g} - f(\tilde{u}_h + \mathbf{g}), 0).$$

Further, let define residues by

$$\begin{aligned} u - \tilde{u}_h &= w_u + v_0, \quad \text{where } w_u := u - \bar{u},\ v_0 := \bar{u} - \tilde{u}_h, \\ p - \tilde{p}_h &= w_p + q_0, \quad \text{where } w_p := p - \bar{p},\ q_0 := \bar{p} - \tilde{p}_h. \end{aligned} \tag{4.3}$$

Note that v_0 and q_0 are unknown functions but its norm can be computed by an a priori and a posteriori techniques (e.g., see [7] [11] [12]). Thus, concerned problem is reduced to the following residual form

$$\begin{aligned} &\text{Find } (w_u, w_p) \in X \times Y \text{ such that} \\ &\quad \mathcal{B}(w_u, w_p) = (f(\tilde{u}_h + \mathbf{g}) - f(w_u + v_0 + \tilde{u}_h + \mathbf{g}), 0) \quad \text{in } \Omega. \end{aligned} \tag{4.4}$$

In this case, the coefficient vector function in (3.2) is given by $c := \tilde{u}_h + \mathbf{g}$. By using the map Φ defined in the previous section, we have

$$f(\tilde{u}_h + \mathbf{g}) - f(w_u + v_0 + \tilde{u}_h + \mathbf{g}) = -\Phi(w_u + v_0) - f(w_u + v_0).$$

Hence, as in (3.1), the Newton-type residual equation for (4.4) is written as:

Find $w = (w_u, w_p) \in X \times Y$ such that
$$\mathcal{L}w \equiv \mathcal{B}w + \Psi w = (-\Phi v_0 - f(w_u + v_0),\, 0) \quad \text{in } \Omega. \tag{4.5}$$

If $\mathcal{L}$ is invertible, then (4.5) is rewritten as the fixed point form
$$w = F(w) \quad \big(\equiv \mathcal{L}^{-1}(-\Phi v_0 - f(w_u + v_0),\, 0)\big). \tag{4.6}$$

Note that, from the above definition, the nonlinear map F in (4.6) means a Newton-like operator and is compact on $X \times Y$ by the property of the nonlinear map f, and it is expected to be a contraction map on some neighborhood of zero. Therefore, we consider the *candidate set* $W_\alpha = W_u \times W_p$ for $\alpha = (\alpha_u, \alpha_p)$ of the form
$$\begin{aligned} W_u &\equiv \{w_u \in X\, ;\, \|w_u\|_{H_0^1} \le \alpha_u\}, \\ W_p &\equiv \{w_p \in Y\, ;\, \|w_p\|_{L^2} \le \alpha_p\}. \end{aligned}$$

First, for the existential condition of solutions, based on the Schauder fixed point theorem, we need to choose the set W_α so that:
$$F(W_\alpha) \subset W_\alpha. \tag{4.7}$$

And next, for the proof of local uniqueness within W_α, the following contraction property is needed:
$$[\|F(w_1) - F(w_2)\|] \le \lambda [\|w_1 - w_2\|], \quad \forall w_1,\, w_2 \in W_\alpha, \tag{4.8}$$

for some constant $0 < \lambda < 1$.
Taking account that $f(w_u + v_0) \in X'$, by Theorem 8, a sufficient condition for (4.7) can be written as
$$\begin{aligned} [\|F(W_\alpha)\|] &\equiv \sup_{w \in W_\alpha} [\|F(w)\|] \\ &\le \begin{bmatrix} \mathcal{M}_u \\ \mathcal{M}_p \end{bmatrix} \sup_{w_u \in W_u} \|\Phi v_0\|_{L^2} \\ &\quad + \begin{bmatrix} \mathcal{M}_u^* \\ \mathcal{M}_p^* \end{bmatrix} \sup_{w_u \in W_u} \|f(w_u + v_0)\|_{H^{-1}} \\ &\le \alpha, \end{aligned} \tag{4.9}$$

where $(\mathcal{M}_u,\, \mathcal{M}_u)$ and $(\mathcal{M}_u^*,\, \mathcal{M}_u^*)$ are the constants defined in Theorem 8.

Further we have the following estimates

$$\begin{aligned}
\|\Phi v_0\|_{L^2} &= \|(c\cdot\nabla)v_0 + (v_0\cdot\nabla)c\|_{L^2} \\
&\le (K_c + C_{L^4}K_{\partial c})\,\|v_0\|_{H_0^1}, \\
\|f(w_u+v_0)\|_{H^{-1}} &= \|(v_0\cdot\nabla)v_0 + (w_u\cdot\nabla)v_0 \\
&\qquad + (v_0\cdot\nabla)w_u + (v_0\cdot\nabla)v_0\|_{H^{-1}} \\
&\le C_{L^4}^2\big(\|w_u\|_{H_0^1} + \|v_0\|_{H_0^1}\big)^2 \\
&\le C_{L^4}^2\big(\alpha_u + \|v_0\|_{H_0^1}\big)^2.
\end{aligned}$$

Hence, we can rewrite the existential condition (4.9) as

$$\begin{bmatrix} \mathcal{M}_u^* C_{L^4}^2\big(\alpha_u + \|v_0\|_{H_0^1}\big)^2 + \mathcal{M}_u\big(K_c + C_{L^4}K_{\partial c}\big)\|v_0\|_{H_0^1} \\ \mathcal{M}_p^* C_{L^4}^2\big(\alpha_u + \|v_0\|_{H_0^1}\big)^2 + \mathcal{M}_p\big(K_c + C_{L^4}K_{\partial c}\big)\|v_0\|_{H_0^1} \end{bmatrix} < \begin{bmatrix} \alpha_u \\ \alpha_p \end{bmatrix}.$$

From above, we obtain the local uniqueness condition (4.8) with λ by

$$\lambda \equiv 2\mathcal{M}_u^* C_{L^4}^2\big(\alpha_u + \|v_0\|_{H_0^1}\big) < 1.$$

5. Numerical examples

In this section, we present numerical examples for the stationary Navier-Stokes equation related to a mathematical model of the step flow problem. In such a case, it should be natural to take a domain as $\Omega = (0, A) \times (0, B) \setminus [0, a] \times [0, b]$, where the constants A, B, a and b satisfy $0 < a < A$ and $0 < b < B$.

The boundary vector function $g = (g_1, g_2)$ is given as

$$g_1 \equiv g_1(x,y) = \begin{cases} \dfrac{(B-y)(y-b)}{(B-b)^3} & \text{if } x = 0, \\ \dfrac{(B-y)(y-0)}{(B-0)^3} & \text{if } x = A, \\ 0 & \text{otherwise}, \end{cases} \tag{5.1}$$

$g_2 \equiv g_2(x, y) = 0$ on $\partial\Omega$, respectively. In particular, we choose that $A = 2$, $B = 1$ and $a = b = 0.5$.

Notice that the function g_1 satisfies the following relation which corresponding to the incompressibility condition.

$$\int_b^B g_1(0, y)dy = \int_0^B g_1(A, y)dy.$$

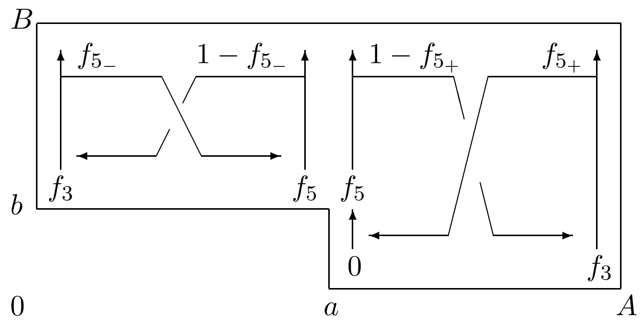

Fig. 1. Image of ψ

For this example, we can present a C^3-class stream function ψ such that $\mathbf{g} = (\psi_y, -\psi_x)$ in (4.1) for the boundary vector function g in (5.1). Namely, setting functions f_{5_-}, f_{5_+}, f_5 and f_3 which are defined by

$$
\begin{aligned}
f_3 &\equiv f_3(y, k) = -\frac{1}{6k^3}(2y - 3k)y^2, \\
f_{5_+} &\equiv f_{5_+}(x, k) = -\frac{1}{k^5}(4x - 5k)x^4, \\
f_5 &\equiv f_5(y, k) = -\frac{1}{6k^5}(4y - 5k)y^4, \\
f_{5_-} &\equiv f_{5_-}(x, k) = \frac{1}{k^5}(4x + k)(x - k)^4,
\end{aligned}
$$

the stream function $\psi \equiv \psi(x, y)$ is given by (see Fig. 1)

$$
\psi(x, y) = \begin{cases}
f_{5_+}(x - a, A - a) f_3(y, B) & \\
\qquad + (1 - f_{5_+}(x - a, A - a)) f_5(y - b, B - b) & \text{in } \Omega_1 \\
f_{5_+}(x - a, A - a) f_3(y, B) & \text{in } \Omega_2 \\
f_{5_-}(x, a) f_3(y - b, B - b) & \\
\qquad + (1 - f_{5_-}(x, a)) f_5(y - b, B - b) & \text{in } \Omega_3
\end{cases}
$$

where $\Omega_1 = [a, A] \times [b, B]$, $\Omega_2 = [a, A] \times [0, b]$ and $\Omega_3 = [0, a] \times [b, B]$.

In the below, as the finite element subspaces, we used the bi-quadratic C^0 element for the velocity, the bi-linear C^0 element for the pressure. And note that the Poincaré constant can be computed by $C_{L^2} = \sqrt{AB - ab}/\pi = \sqrt{1.75}/\pi$ in the present case.

We show several computational results for the constructive a priori constants in Theorem 4 and Corollary 5 by Table 1 in which the constant β is calculated by the method in [7].

$1/h$	$E_u(h)$	$E_p(h)$	η	σ	$C(h)$
20	1.6760e-1	2.2878e-0	1.9870	1.4721e-2/ν	0.5069$\cdot h$
40	9.2139e-2	1.2945e-0	2.2181	7.6094e-3/ν	0.6234$\cdot h$
60	7.0026e-2	9.9109e-1	2.4079	5.6365e-3/ν	0.7099$\cdot h$
$1/h$	$e_u(h)$	$e_p(h)$	$\hat{\eta}$	$\hat{\sigma}$	β
20	$\min(15.1208, 2)$	200.07	184.88	1.3793/ν	10.1572
40	$\min(15.7029, 2)$	213.68	382.56	1.3856/ν	10.1572
60	$\min(16.0338, 2)$	220.91	563.59	1.3863/ν	10.1572

Table 1. Numerical results for the a priori constant

Notice that the a priori constant $C(h)$ for the H_0^1-projection in Assumption 1 is obtained by the procedure which is presented in [12]. Due to the noncovexity of the domain, as shown in Table 1, the rate of convergence in the a priori constant $C(h)$ seems to be less than 1, i.e., worse than $O(h)$. Also, the constant β is much bigger compared with regular domains such as the rectangle in [7]. Table 2 shows the verification results for the stationary Navier-Stokes equation (1.1) with the boundary condition (5.1). As shown in this table, we could verify the invertibility of the linearlized operator at the approximate solution as well as the verification of solutions for the nonlinear problem with rather rough mesh size, for example $h = 1/20$. However, we would need more finer mesh for smaller elasticity constants. Fig. 2 illustrates the contour of stream lines of approximate solution for this problem with $h = 1/60$.

$1/h$	$\mathcal{M}_u^*$	$\mathcal{M}_u$	M_u	κ	$\|v_0\|_{H_0^1}$	α_u
20	0.2932	0.0678	0.1416	6.4642e-2	9.0835e-1	2.4632e-1
40	0.2701	0.0621	0.1417	2.6443e-2	6.4212e-1	1.2143e-1
60	0.2646	0.0611	0.1417	1.8879e-2	5.5259e-1	9.6571e-2

Table 2. Numerical results for $\nu = 10$

All computations in tables are carried out on the Dell Precision 650 Workstation Intel Xeon Dual CPU 3.20GHz by MATLAB.

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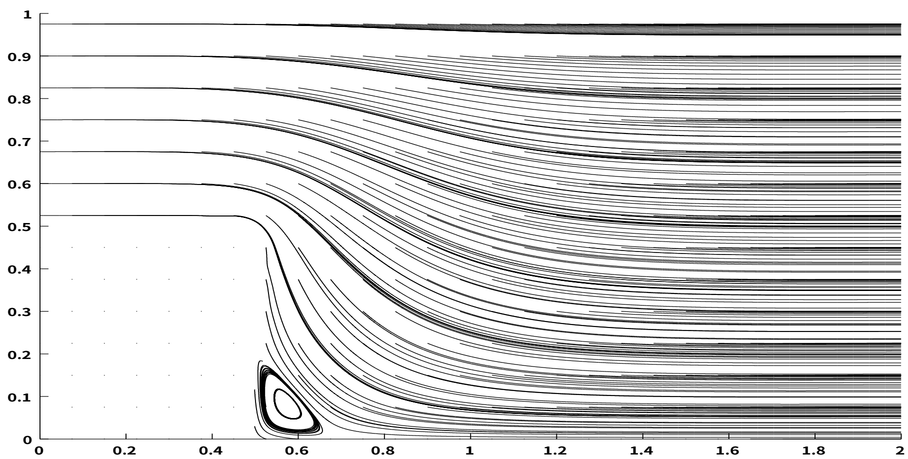

Fig. 2. Approximate contour of solution

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