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On nilpotent injectors of Fischer group $M(22)$

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Abstract. The aim of this paper is to prove the following theorem:

Theorem 1 *The nilpotent injectors of $M(22)$ are Sylow 2-subgroups.*

Key words: nilpotent injectors, generalized fitting group.

1. Introduction

A finite group G is said to be of type $M(22)$ if G possesses an involution d such that $H = C_G(d)$ is quasisimple with $H/\langle d\rangle \cong U_6(2)$ and d is not weakly closed in H with respect to G. For more information one is referred to [2]. The notion of N-injectors in a finite group G was first introduced by B. Fischer in [9] and defined as follows: A subgroup A of G is an N-injector of G, if for each $H \lhd\lhd G$, $A \cap H$ is a maximal nilpotent subgroup of H. In [12] it has been proved that if $C(F(G)) \subseteq F(G)$, then G contains N-injectors, they form a conjugacy class, and they can be characterized as the maximal nilpotent subgroups which contain $F(G)$ where $F(G)$ denotes the Fitting group of G. If G is solvable then N-injectors exist and any two of them are conjugate [9]. N-injectors of finite solvable groups, Symmetric groups S_n, and alternating group A_n were studied in [5], [6] and [7]. The notion of nilpotent injectors was introduced by A. Bialostocki in [6].

A nilpotent injector in a finite group G is any maximal nilpotent subgroup B of G satisfying $d_2(B) = d_2(G)$, where $d_2(X)$ is defined as $Max\{|A|$, $A \leq X$ and A is nilpotent of class at most $2\}$. Also we define $d_{2,p}(G)$, $m_k(G)$, and $Om_k(G)$ as follows: Let p be a prime, $d_{2,p}(G) = \max\{|P| \mid P$ is a p-subgroup of G of class at most $2\}$, $m_k(G) = \max\{|C_G(x)| \mid x \in G$, $o(x) = p_1p_2\dots p_k$, p_i's are distinct primes$\}$, where $o(x)$ is the order of x in G. Let $g \in G$ such that $o(g) = p_1p_2\dots p_k$, $p_i \neq p_j$ if $i \neq j$ for $i = 1,\dots,k$, and let $|C_G(g)| = p_1^{a_1}p_1^{a_2}\dots p_k^{a_k}.m$ where $p_i \nmid m$ for $i = 1,\dots,k$, define $Om_k(G) = \max\{|C_G(x)| \mid o(x) = p_1p_2\dots p_k, 2 < p_1 < p_2 \dots < p_k\}$. So we

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get the following criterion: If $H \leq G$ such that H is nilpotent and $|H|$ has at least k-prime divisors different from 2, then $|H| \leq Om_k(G) \leq m_k(G)$ [1]. Nilpotent injectors are sometimes called B-injectors. The B-injectors of some sporadic groups have been determined in [1]. A. Neumann [13] studied the nilpotent injectors in finite groups, proving that nilpotent injectors are really N-injectors. The motivation behind this work is that B-injectors will lead to theorems similar to Glauberman's ZJ-Theorem and it is hoped that they provide tools and arguments for a modified and shortened proof of the classification theorem of finite simple groups. The Fischer group $M(22)$ among others turn out to be critical in answering the question whether the B-injectors are conjugate or not.

2. Preliminaries and notation

Let $\tilde{F}$ denote the group $M(22)$. Then $\tilde{F} = \langle D \rangle$ where D is a class of involutions (3-transpositions) with the property, t_1, $t_2 \in D$ implies that $o(t_1t_2) = 1, 2$ or 3, where $o(t_1t_2)$ denotes the order of t_1t_2. So if $t_1 \neq t_2$, then $\langle t_1, t_2 \rangle$ is a group of order 4, i.e. $\langle t_1, t_2 \rangle \cong 2^2$ or $\langle t_1, t_2 \rangle \cong S_3$, the symmetric group of degree 3. There are 3-classes of involutions j in $M(22)$ with the following representatives

(i) $j = d \in D$ such that $C_{\tilde{F}}(d) = 2U_6(2)$.
(ii) $j = d_1d_2 = d_2d_1$, where d_1, d_2 are uniquely determined by j and if $g \in C_{\tilde{F}}(j)$, one obtains $d_1^g = d_1$, $d_2^g = d_2$ or $d_1^g = d_2$, $d_2^g = d_1$. So $C_{\tilde{F}}(d_1) \cap C_{\tilde{F}}(d_2)$ is a normal subgroup of $C_{\tilde{F}}(j)$ of index 2.
(iii) $j = d_1d_2d_3$ where $d_id_j = d_jd_i$.

Lemma 1 ([2])
(a) $C_{\tilde{F}}(d_1) = 2 \cdot U_6(2)$.
(b) $C_{\tilde{F}}(d_1d_2) = 2.2^{1+8} : U_4(2).2$.
(c) $C_{\tilde{F}}(d_1d_2d_3) \leq K.(A_6 \times S_3)$, *where K is a 2-group isomorphic to* 2^{5+8}.
(d) *Let $C^* = C_{\tilde{F}}(d_1d_2d_3)$, it holds that $O_2(C^*)$ is a special group of shape 2^{5+8}, and $C^*/O_2(C^*)$, is isomorphic to $S_3 \times 3^2 : 4$, where $O_p(G)$ is the unique maximal normal p-subgroup of G. Moreover for $M = N_{\tilde{F}}(O_2(C^*))$ it holds that $M/O_2(C^*)$ is isomorphic to $S_3 \times A_6$, where M is maximal in $\tilde{F}$ and it is 2-constrained.*

Corollary 1 $d_2(\tilde{F}) \geq d_{2,2}(\tilde{F}) \geq 2^{13} > d_{2,3}(\tilde{F})$.

Proof. From Lemma 1 we get $d_{2,2}(\tilde{F}) \geq 2^{13}$. The Sylow 3-subgroups of $\tilde{F}$ have order 3^9, and are isomorphic to Sylow 3-subgroups of $O(7,3)$. As $\tilde{F}$ contains subgroups isomorphic to $O(7,3)$, it can be easily verified that Sylow 3-subgroups of $O(7,3)$ are of class greater than 2. Hence $d_{2,3}(\tilde{F}) \leq 3^8 \leq 2^{13}$, and the claim follows. □

3. Definitions and results

Our notation is fairly standard. Throughout, all groups are finite. If G is a group, the generalized Fitting group $F^*(G)$ is defined by $F^*(G) = F(G)E(G)$ where $E(G) = \langle L/L \triangleleft\triangleleft G$ and L is quasisimple$\rangle$ is a subgroup of G. A group L is called quasisimple iff $L' = L$ where L' is the derived group of L and $L'/Z(L)$ is a non abelian simple group. Let $Z(G)$ denote the center of G. If H and X are subsets of $G, C_H(X)$ and $N_H(X)$ denote respectively the centralizer and normalizer of X in H. The components of a group X are its subnormal quasisimple subgroups.

Lemma 2

(a) *If K is a quasi-simple group, and $M \trianglelefteq K$, then $M = K$ or $M \subseteq Z(K)$.*
(b) *If $N \trianglelefteq G$ and G/N is solvable, then $E(G) = E(N)$.*

Proof.

(a) Since $M \trianglelefteq K$, $MZ(K)/Z(K) \trianglelefteq K/Z(K)$. As $K/Z(K)$ is simple, it follows that $MZ(K) = K$ or $MZ(K) = Z(K)$. If $MZ(K) = K$, then $K = K' = (MZ(K))' = M' \subseteq M$, so $K \subseteq M \subseteq K$, and thus $K = M$.
(b) If K is a component of G, then $N \cap K \trianglelefteq K$ and $K/K \cap N \cong KN/N \leq G/N$. As G/N is solvable, $K/K \cap N$ is solvable, and by (a), $K \cap N = K$ or $K \cap N \leq Z(K)$. If $K \cap N = K$, then $K \subseteq N$ and if $K \cap N \leq Z(K)$, it follows that $K/Z(K) \cong K/K \cap N/Z(K)/K \cap N$. So $K/Z(K)$ is a factor group of $K/K \cap N$ which implies that $K/Z(K)$ is solvable, a contradiction. □

Lemma 3 ([1]) *Let H be a nilpotent injector of a group G. If there exists a subgroup $M \leq G$ such that*:

(i) *$H \leq M \leq G$.*
(ii) *$F^*(M) = O_p(M)$, then H is a Sylow p-subgroup of G.*

Proposition 1 *Let H be a finite group such that $H/O_2(H)$ is a non-abelian simple group, then $F^*(H) = O_2(H)$ or any element of odd order in*

H centralizes $O_2(H)$.

Proof. Since $O_2(H) \leq F(H) \leq F^*(H) \trianglelefteq H$, $F(H)/O_2(H) \trianglelefteq H/O_2(H)$ and $F^*(H)/O_2(H) \trianglelefteq H/O_2(H)$. As $H/O_2(H)$ is simple, we see that $F(H) = O_2(H)$ and $F^*(H) = O_2(H)$ or $F^*(H) = H$. So, assume that $F^*(H) = H$. Thus $H = F^*(H) = F(H)E(H) = O_2(H)E(H)$ and $[E(H), O_2(H)] = 1$. If p is an odd prime divisor of $|H|$, then the Sylow p-subgroups of $E(H)$ are also Sylow p-subgroups of H. So let P be a Sylow p-subgroup of H, this implies that there exists $x \in H$ such that $P^x \leq E(H) \leq C_H(O_2(H)) \trianglelefteq H$, so $P \leq (C_H(O_2(H)))^{x^{-1}} = C_H(O_2(H))$. Hence for any odd prime p, all the Sylow p-subgroups of H are contained in $C_H(O_2(H))$. Thus $H/C_H(O_2(H))$ is a 2-group. So, any element of odd order centralizes $O_2(H)$. □

Proposition 2 *If M, K are two normal subgroups of H, such that $M < K \leq H$, M is a 2-group, K/M is a non abelian simple group, $H/K \cong S_3$ and $F^*(K) = O_2(K)$, then $F^*(H) = O_2(H)$ or there exists an element $t \in H$, such that $o(t) = 3$ and t centralizes K.*

Proof. As $H/K \cong S_3$ is solvable, and $F^*(K) = O_2(K)$, it follows that $E(H) = E(K) = 1$. So $F^*(H) = F(H)$ and $O_p(H) \cap K \subset O_p(K) \subseteq F^*(K) = O_2(K)$. Hence if $p \neq 2$, then $O_p(H) \cap K = 1$, and $O_p(H) = O_p(H)/O_p(H) \cap K = O_p(H)K/K \leq H/K \cong S_3$. This implies that $p = 3$, and $|O_3(H)| \leq 3$. Assume that $O_3(H) \neq 1$, and let $t \in O_3(H)$ such that $o(t) = 3$. It follows that t centralizes M as $M \subseteq O_2(H)$ and $[M, \langle t \rangle] \subseteq [O_2(H), O_3(H)] = 1$, and $M = O_2(K) = F^*(K)$ as K/M is simple.

Now let $x \in K$. Since $x^{-1}x^t = x^{-1}t^{-1}xt = (t^{-1})^x t \in K \cap C_H(M)$ since $K \trianglelefteq H$ and $C_H(M)$ are normal subgroups of H. It follows that $x^{-1}x^t \in K \cap C_H(M) = C_K(M) = C_K(F^*(K)) \subseteq F^*(K) = M$, thus $x^{-1}x^t = z \in Z(M)$ or $x^t = xz$. This implies $x^{t^2} = xz^2$ and $x^{t^3} = xz^3$, but $t^3 = 1$. Hence $z = 1$ and so; $x^t = x$ for all $x \in K$. Thus t centralizes K and the proposition is proved. □

Theorem 1

(i) *$F^*(C_{\tilde{F}}(j_2)) = O_2(C_{\tilde{F}}(j_2))$, where $j_2 = d_1d_2 = d_2d_1$, $d_i \in D$, $i = 1, 2$.*

(ii) *$F^*(C_{\tilde{F}}(j_3)) = O_2(C_{\tilde{F}}(j_3))$, where $j_3 = d_1d_2d_3$, $d_i \in D$, $i = 1, 2, 3$ and*

(iii) *If B is a nilpotent-injector of $M(22)$ containing an involution of type $j_1 = d_1 \in D$ in its center, then there exists a subgroup $X \leq C_{\tilde{F}}(j_1) = 2 \cdot U_6(2)$ such that $B \leq X \leq 2 \cdot U_6(2)$ with $F^*(X) = O_2(X)$.*

Proof.

(i) Let $C_{\tilde{F}}(j_2) = (M.U_4(2))$: 2, where M is a 2-group of order 2^{10}, and let $H = K.2$ where $K = M.U_4(2)$. If $F^*(H) \neq O_2(H)$, then $F^*(K) \neq O_2(K)$. As $5 \mid |U_4(2)|$, then by Proposition 1, there exists an element of order 5 in $\tilde{F}$ centralizes a group of order 2^{10}, this is a contradiction as $\tilde{F}$ contains only one element of order 5 with centralizer $Z_5 \times S_5$. See [2]. So $F^*(C_{\tilde{F}}(j_2)) = O_2(C_{\tilde{F}}(j_2))$.

(ii) $C_{\tilde{F}}(j_3)$ is contained in a subgroup $H = 2^{5+8} \cdot (A_6 \times S_3)$. Let $K \trianglelefteq H$ such that $H/K \cong S_3$ and $K/O_2(K) \cong A_6$ where $|O_2(K)| = 2^{13}$. As $5 \mid |A_6|$, by Proposition 1, it follows that $F^*(K) = O_2(K)$, otherwise there would exist an element of order 5 whose centralizer is divisible by 2^{13}, so $F^*(K) = O_2(K)$. If $F^*(H) \neq O_2(H)$, then there exists an element of order 3 in H centralizing K, this is a contradiction, compare the centralizers of elements of order 3 in $\tilde{F}$ see [2]. So $F^*(C_{\tilde{F}}(j_3)) = O_2(C_{\tilde{F}}(j_3))$.

(iii) If B is a nilpotent injector of $M(22)$ such that B contains an involution of type j_1 in its centre, then $B \leq C_{\tilde{F}}(j_1) = 2 \cdot U_6(2) = H$. As $C_{\tilde{F}}(j_3)$ contains a special group of order 2^{5+8}, in particular it contains a 2-groups of class ≤ 2 and of order $\geq 2^{13}$, then $d_2(\tilde{F}) \geq 2^{13} > 2.3^6$. So $d_2(B) > 2.3^6$. As B is a nilpotent injector of $\tilde{F}$, there exists $A \leq B$, class $(A) \leq 2$ and of order $d_2(B)$, so $Z(H) \leq A$ and $A/Z(H) \leq H/Z(H) = U_6(2)$. Thus $3^6 < \frac{1}{2}d_2(B) = \frac{1}{2}|A| = |A/Z(H)|$. This implies that $2 \mid |A/Z(H)|$, as otherwise by Flavell's bound [10] we would have that $|A/Z(H)| \leq 3^6$. Also $Z(H) \leq B$ and $B/Z(H) \leq H/Z(H) = U_6(2)$. This implies that $2 \mid |B/Z(H)|$ as $A/Z(H) \leq B/Z(H)$. Consider $\bar{B} \leq H/Z(H) = U_6(2)$, and let $\bar{t} \in Z(\bar{B})$ be an involution such that $t \in B$. Hence $\bar{t} = tZ(H)$ and $C_{H/Z(H)}(tZ(H)) = X/Z(H)$ for $Z(H) \leq X \leq H$ and $B \leq X$. As $U_6(2)$ has characteristic 2, then by (Proposition 1.29, [11]), it follows that $F^*(X/Z(H)) = O_2(X/Z(H))$.

Since $|Z(H)| = 2$, $F^*(X) = O_2(X)$. Hence the claim follows. □

Corollary 2 *Under the assumption of Theorem* 2 (iii)*, B is a Sylow* 2-*subgroup.*

Proof. By Lemma 3 and Theorem 2 (iii), it follows that B is a nilpotent injector of X, and hence a Sylow 2-subgroup. Now we are in a position to prove Theorem 1. □

Proof of Theorem 1. Let B be a nilpotent injector of $\tilde{F}$. In particular we have $d_2(B) = d_2(\tilde{F}) \geq d_{2,2}(\tilde{F}) \geq 2^{13}$. As $d_2(B) > 30 = m_3(\tilde{F})$, the order of B can have at most 2 prime divisors. As also $d_2(B) > 21 = Om_2(\tilde{F})$, we find that B is either a p-group or 2 divides its order. □

Case 1 B is a p-group. Then $p = 2$, and B is a Sylow 2-subgroup

Proof. As $d_2(\tilde{F}) = d_2(B) = d_{2,p}(\tilde{F})$, one obtains $2^{13} \leq d_2(\tilde{F}) = d_{2,p}(B) \leq |B|$. As $|B| \geq 2^{13}$, it follows that p is either 2 or 3. As $d_{2,3}(\tilde{F}) < 2^{13}$ by Corollary 1, we have $p = 2$. Hence the claim follows. □

Case 2 If 2 divides the order of B, then B is a Sylow 2-subgroup.

Proof. If 2 divides the order of B, then there exists an involution j in $Z(B)$, and B is a nilpotent injector of $H = C_{\tilde{F}}(j)$. If H is 2-constrained i.e. $F^*(H) = O_2(H)$, then B is a Sylow 2-subgroup by Lemma 3, Theorem 2 and Corollary 2, or $j \in D$ and H is a quasi-simple of shape $2 \cdot U_6(2)$. It is possible to treat the case $2 \cdot U_6(2)$ using by Theorem 2 (iii) and Lemma 3. So B in fact is a Sylow 2-subgroup. □

This completes the proof of Theorem 1. □

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