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On the prolongation of 2-jet space of 2 independent and 1 dependent variables

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Abstract. We will formulate the Monster Goursat manifolds for multi independent variables cases. We will classify the singularities which appears in the prolongation of 2-jet space of 2 independent and 1 dependent variables.

Key words: differential system, jet space, prolongation.

1. Introduction

In this paper, we will consider the extension of "Monster Goursat manifolds" in [MZ] to multi independent variables case.

Let m, n be positive integers and M a manifold of dimension $m+n$. We denote $J^k(M,n)$ the k-jet space over M with n-independent variables and by C^k the canonical system on it (see Section 2).

We define, for $x \in J^k(M,n)$, the set Σ_x of n-dimensional integral elements of C^k through x;

$$\Sigma_x = \bigl\{n\text{-dim. integral elements of } (J^k(M,n), C^k)\bigr\},$$

and the subset

$$\Sigma(J^k(M,n)) := \bigcup_{x\in J^k(M,n)} \Sigma_x$$

of the Grassmannian $J(C^k,n) = \mathrm{Gr}(C^k,n)$ of n-dimensional linear subspaces of the distribution C^k;

$$J(C^k,n) = \bigcup_{x\in J^k} C_x, \qquad C_x = \mathrm{Gr}(C^k(x),n).$$

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Here the integral elements of a differential system on a manifold are generally defined as follows;

Let (R, D) be a differential system, i.e., R is a manifold and D is a subbundle of TR. We take a system of local defining 1-forms $\{\varpi_1, \dots, \varpi_s\}$ of D. An n-dimensional integral element of D at $x \in R$ is an n-dimensional subspace v of T_xR such that

$$\varpi_i|_v = d\varpi_i|_v = 0 \quad (i = 1, \dots, s).$$

That is, n-dimensional integral elements are candidates for the tangent spaces at x of n-dimensional integral manifolds of D.

By definition,

$$J^{k+1}(M, n) \subset \Sigma(J^k(M, n)) \subset J(C^k, n).$$

The set $\Sigma(J^k(M, n))$ of integral elements is the candidate for the extension of the notion "Monster Goursat manifolds" introduced in [MZ] to the case of several independent variables. However the subset $\Sigma(J^k(M, n))$ of $J(C^k, n) = \mathrm{Gr}(C^k, n)$ may not be a submanifold of $J(C^k, n)$. This situation is quite different from the case of 1 independent variable. One of main purpose of this paper is to check when the set $\Sigma(J^k(M, n))$ of integral elements of C^k becomes a submanifold of $J(C^k, n)$ or not in the case $n \geq 2$. If $\Sigma(J^k(M, n))$ is a submanifold of $J(C^k, n)$, then we define the canonical differential system D on $\Sigma(J^k(M, n))$. In this case, we regard $\Sigma(J^k(M, n))$ endowed with the canonical differential system as an extension of procedure to construct "Monster Goursat manifolds" or the procedure of "prolongation" of the jet space.

When $n = 1$, $\Sigma(J^k(M, 1))$ are called "rank 1 prolongation" of $J^k(M, 1)$ in [SY]. Note that

$$\Sigma(J^k(M, 1)) = J(C^k, 1).$$

We can repeat the procedure of "rank 1 prolongation", starting from any differential system. We can define "k-th rank 1 prolongation" inductively. Moreover, when $n = m = 1$, "k-th rank 1 prolongation" of $(J(M, 1), C)$ are called "Monster Goursat manifold" in [MZ].

Generally $\Sigma(J^k(M, n))$ is a variety and is not a submanifold in $J(C^k, n)$.

Then we have the following result as one of main theorems in this paper;

Theorem 3.1 *The set $\Sigma(J^k(M^{m+n}, n))$ of integral elements of the canonical system C^k on the jet space $J^k(M^{m+n}, n)$ over the $m+n$-dimensional manifold M with n-independent variables is a submanifold of the Grassmannian $J(C^k, n) = Gr(C^k, n)$ if and only if $(k, n, m) = (2, 2, 1)$, $(k, 1, m)$, $(1, n, 1)$.*

It is well known that $\Sigma(J^k(M^{m+n}, n))$ is a submanifold in the cases $n = 1$ or $k = m = 1$. In the case $n = 1$, $\Sigma(J^k(M^{m+1}, 1))$ is nothing but a rank 1 prolongation. As for the case $k = m = 1$, $\Sigma(J^1(M^{1+n}, n))$ is a Lagrange-Grassmann bundle $L(J)$, by definition. Hence it is known that $\Sigma(J^1(M^{1+n}, n))$ is a submanifold of $J(C^1, n)$. We call these cases *trivial cases.*

Now, we will be concerned with the local equivalence problem of $\Sigma(J^2(M^{1+2}, 2))$. (We denote it $\Sigma(J^2)$ for short.) We distinguish any point on $\Sigma(J^2)$ where the canonical system $(\Sigma(J^2), D)$ is not locally isomorphic to the generic model $(J^3(\mathbb{R}^3, 2), C^3)$ for the 3-jet space. Here, we call φ an isomorphism from a differential system (R_1, D_1) to (R_2, D_2), if $\varphi : R_1 \to R_2$ is a diffeomorphism such that $\varphi_*(D_1) = D_2$. The equivalence problem of "Monster Goursat manifolds" of 1 independent variable cases is studied in [M1] and [M3].

First, the points in $\Sigma(J^2)$ are classified in two types according to singularities of the canonical differential system D in the sense of Tanaka theory (Proposition 5.2):

$$\Sigma_1 = \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = 1\}$$

$$\Sigma_2 = \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = 2\}$$

where we mean by the fiber, the fiber of $\pi_* : T(J^2(M^{1+2}, 2)) \supset C^2 \to T(J^1(M^{1+2}, 2))$. Then, along Σ_1, we have the following normal form by constructing local isomorphisms, directly:

We define the differential system $\hat{D}$ on $\mathbb{R}^{12}$ with coordinate $(x, y, z, p, q, r, s, t, a, B, c, e)$ by

$$\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = \varpi_s = 0\}$$

where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt. \end{cases}$$

Then we have

Theorem 5.7 (normal form) *For any $w \in \Sigma_1$, the differential system $(\Sigma(J^2(M^{1+2}, 2), D)$ around w is locally isomorphic to the germ at the origin of $(\mathbb{R}^{12}, \hat{D})$.*

It turns out that the classification along Σ_2 is more complicated. In fact the local isomorphism classes of $(\Sigma(J^2), D)$ along Σ_2 are divided into 3 types by using graded Lie algebras, namely, the hyperbolic type, the elliptic type, the parabolic type (Remark 5.12).

To describe the classification of $(\Sigma(J^2), D)$ along Σ_2, we need to introduce another normal form: We define the differential system $\bar{D}$ on $\mathbb{R}^{12}$ with coordinate $(x, y, z, p, q, r, s, t, B, D, E, F)$ by

$$\bar{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = \varpi_t = 0\}$$

where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE - BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

Theorem 5.13 *There exists a decomposition of Σ_2*

$$\Sigma_2 = \Sigma_h \cup \Sigma_e \cup \Sigma_p$$

into disjoint three subsets such that, if $w \in \Sigma_h$, then $(\Sigma(J^2(M^{1+2}, 2), D)$ around w is locally isomorphic to the germ of $(\mathbb{R}^{12}, \bar{D})$ at $(0, \ldots, 0, 1, 0)$, if $w \in \Sigma_e$, then, $(0, \ldots, 0, -1, 0)$, and if $w \in \Sigma_p$, then, $(0, \ldots, 0, 0, 0)$.

In Section 2, we briefly review the geometric construction of jet bundles in general, following [Y1] and [Y2], which is our basis for the later considerations. In Section 3, we extend the procedure of "Monster Goursat manifolds" to the case of $n \geq 2$. Actually we consider the geometric construction of jet spaces without the transversality conditions which are candidates for the generalization of "Monster Goursat manifold". We will give a criteria for the generalization of "Monster Goursat manifold" to be a manifold (Theorem 3.1). In Section 4, we review the Tanaka theory to consider the equivalence problem of the canonical system on $\Sigma(J^2)$ in Section 5. In Section 5, we give the proofs of Theorem 5.7, 5.13, and, summarizing the results obtained in this section, we give the complete classification of the canonical distribution on $\Sigma(J^2)$ (Corollary 5.15).

2. Geometric construction of Jet Spaces

Let M be a manifold of dimension $m+n$. Fixing the number n, we form the space of n-dimensional *contact elements* to M, i.e., the Grassmann bundle $J(M,n) = \mathrm{Gr}(TM,n)$ over M consisting of n-dimensional subspaces of tangent spaces to M. Namely, $J(M,n)$ is defined by

$$J(M,n) = \bigcup_{x \in M} J_x, \qquad J_x = \mathrm{Gr}(T_x(M),n),$$

where $\mathrm{Gr}(T_x(M),n)$ denotes the Grassmann manifold of n-dimensional subspaces in $T_x(M)$. Let $\pi : J(M,n) \to M$ be the bundle projection. The *canonical system* C on $J(M,n)$ is, by definition, the differential system of codimension m on $J(M,n)$ defined by

$$C(u) = \pi_*^{-1}(u) = \{v \in T_u(J(M,n)) \mid \pi_*(v) \in u\} \subset T_u(J(M,n)) \xrightarrow{\pi_*} T_x(M),$$

where $\pi(u) = x$ for $u \in J(M,n)$.

Let us describe C in terms of a canonical coordinate system in $J(M,n)$. Let $u_o \in J(M,n)$. Let $(x_1,\ldots,x_n,z^1,\ldots,z^m)$ be a coordinate system on a neighborhood U' of $x_o = \pi(u_o)$ such that $dx_1,\ldots,dx_n$ are linearly independent when restricted to $u_o \subset T_{x_o}(M)$. We put $U = \{\, u \in \pi^{-1}(U') | dx_1|_u,\ldots,dx_n|_u$ are linearly independent$\}$. Then U is a neighborhood of u_o in $J(M,n)$. Here $dz^\alpha|_u$ is a linear combination of $dx_i|_u$'s, i.e., $dz^\alpha|_u = \sum_{i=1}^n p_i^\alpha(u) dx_i|_u$. Thus, there exist unique functions p_i^α on U such that C is defined on U by the following 1-forms;

$$\varpi^\alpha = dz^\alpha - \sum_{i=1}^{n} p_i^\alpha dx_i \quad (\alpha = 1, \ldots, m),$$

where we identify z^α and x_i on U' with their lifts on U. The system of functions $(x_i, z^\alpha, p_i^\alpha)$ $(\alpha = 1, \ldots, m,\ i = 1, \ldots, n)$ on U is called a *canonical coordinate system* of $J(M, n)$ subordinate to (x_i, z^α).

The space $(J(M, n), C)$ is called the (geometric) 1-jet space and especially, in case $m = 1$, is the so-called contact manifold. Let M, $\hat{M}$ be manifolds of dimension $m + n$ and $\varphi : M \to \hat{M}$ be a diffeomorphism. Then φ induces the isomorphism $\varphi_* : (J(M, n), C) \to (J(\hat{M}, n), \hat{C})$, i.e., the differential map $\varphi_* : J(M, n) \to J(\hat{M}, n)$ is a diffeomorphism sending C onto $\hat{C}$. The reason why the case $m = 1$ is special is explained by the following theorem of Bäcklund.

Theorem (Bäcklund) *Let M and $\hat{M}$ be manifolds of dimension $m+n$. Assume $m \geq 2$. Then, for an isomorphism $\Phi : (J(M, n), C) \to (J(\hat{M}, n), \hat{C})$, there exists a diffeomorphism $\varphi : M \to \hat{M}$ such that $\Phi = \varphi_*$.*

The essential part of this theorem is to show that $F = \operatorname{Ker} \pi_*$ is the covariant system of $(J(M, n), C)$ for $m \geq 2$. Namely an isomorphism Φ sends F onto $\hat{F} = \operatorname{Ker} \hat{\pi}_*$ for $m \geq 2$. (For the proof, see [Y2] Theorem 1.4.)

In case $m = 1$, it is a well known fact that the group of isomorphisms of $(J(M, n), C)$, i.e., the group of contact transformations, is larger than the group of diffeomorphisms of M. Therefore, when we consider the geometric 2-jet spaces, the situation differs according to whether the number m of dependent variables is 1 or greater.

(1) Case $m = 1$. We should start from a contact manifold (J, C) of dimension $2n+1$, which is locally a space of 1-jet for one dependent variable by Darboux's theorem. Then we can construct the geometric second order jet space $(L(J), E)$ as follows: We consider the Lagrange-Grassmann bundle $L(J)$ over J consisting of all n-dimensional integral elements of (J, C);

$$L(J) = \bigcup_{u \in J} L_u \subset J(J, n),$$

where L_u is the Grassmann manifolds of all Lagrangian (or Legendrian) subspaces of the symplectic vector space $(C(u), d\varpi)$. Here ϖ is a local contact form on J. Namely, $v \in J(J, n)$ is an integral element if and only if $v \subset C(u)$ and $d\varpi|_v = 0$, where $u = \pi(v)$. Let $\pi : L(J) \to J$ be the projection. Then the canonical system E on $L(J)$ is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J),$$

where $\pi(v) = u$ for $v \in L(J)$.

We denote by ∂E the derived system of E. Moreover we denote by $\mathrm{Ch}(C)$ the Cauchy characteristic system of C.

Then we have $\partial E = \pi_*^{-1}(C)$ and $\mathrm{Ch}(C) = \{0\}$ (cf. [Y1]). Hence we get $\mathrm{Ch}(\partial E) = \mathrm{Ker}\,\pi_*$, which implies the Bäcklund theorem for $(L(J), E)$ (cf. [Y1]).

Now we put

$$(J^2(M,n), C^2) = (L(J(M,n)), E),$$

where M is a manifold of dimension $n+1$.

Here recall that the derived system and the Cauchy characteristic system of a differential system (R, D) are generally defined as follows;

The derived system ∂D of D is defined, in terms of sections, by

$$\partial\mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

where $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D. In general ∂D is obtained as a subsheaf of the tangent sheaf of R (for the precise argument, see e.g. [Y1], [BCG3]). Moreover higher derived systems $\partial^i D$ are defined successively by

$$\partial^i D = \partial(\partial^{i-1} D),$$

where we put $\partial^0 D = D$ by convention. D is called regular, if $\partial^i D$ is subbundle for all i.

The Cauchy characteristic system $\mathrm{Ch}(D)$ of a differential system (R, D) is defined by

$$\mathrm{Ch}(D)(x) = \{X \in D(x) \mid X \rfloor d\omega_i \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \quad \text{for } i = 1, \ldots, s\},$$

where $D = \{\omega_1 = \cdots = \omega_s = 0\}$ is defined locally by defining 1-forms $\{\omega_1, \ldots, \omega_s\}$.

(2) Case $m \geq 2$. Since $F = \operatorname{Ker} \pi_*$ is a covariant system of $(J(M,n), C)$, we define $J^2(M,n) \subset J(J(M,n),n)$ by

$$J^2(M,n) = \{n\text{-dim. } \textit{integral elements} \text{ of } (J(M,n),C), \text{ transversal to } F\},$$

C^2 is defined as the restriction to $J^2(M,n)$ of the canonical system on $J(J(M,n),n)$.

Now the higher order (geometric) jet spaces $(J^{k+1}(M,n), C^{k+1})$ for $k \geq 2$ are defined (simultaneously for all m) by induction on k. Namely, for $k \geq 2$, we define $J^{k+1}(M,n) \subset J(J^k(M,n),n)$ and C^{k+1} inductively as follows:

$$J^{k+1}(M,n) = \big\{n\text{-dim. } \textit{integral elements} \text{ of } (J^k(M,n), C^k), \\ \text{transversal to Ker } (\pi^k_{k-1})_*\big\},$$

where $\pi^k_{k-1} : J^k(M,n) \to J^{k-1}(M,n)$ is the projection. Here we have

$$\text{Ker } (\pi^k_{k-1})_* = \operatorname{Ch}(\partial C^k),$$

and C^{k+1} is defined as the restriction to $J^{k+1}(M,n)$ of the canonical system on $J(J^k(M,n),n)$.

Here we observe that, if we drop the transversality condition in our definition of $J^k(M,n)$ and collect all n-dimensional integral elements, we may have some singularities in $J^k(M,n)$ in general. Namely, a set of all n-dimensional integral elements of $(J^k(M,n), C^k)$ may be a variety.

Remark 2.1 In this paper, the notation $J^2(M,n)$ is used for the geometric 2-jet spaces, not for the ordinary 2-jet spaces $J^2(\mathbb{R}^n, \mathbb{R}^m)$. But $J^2(M,n)$ is locally isomorphic to $J^2(\mathbb{R}^n, \mathbb{R}^m)$, that is, local isomorphisms act on $J^2(M,n)$ transitively. Therefore the results of this paper are independent of the difference.

3. Main theorem

Theorem 3.1 *The set $\Sigma(J^k(M^{m+n},n))$ of integral elements of the canonical system C^k on the jet space $J^k(M^{m+n},n)$ over the $m+n$-dimensional manifold M with n-independent variables is a submanifold of the Grass-*

mannian $J(C^k, n) = Gr(C^k, n)$ *if and only if* $(k, n, m) = (2, 2, 1), (k, 1, m),$ $(1, n, 1)$.

Proof. We showed the following theorem in [S];

Theorem 3.2 ([S]) $\Sigma(J^k(M^{m+n}, n))$ *are not manifolds except for* $\Sigma(J^2(M^{1+2}, 2))$ *and trivial cases.*

Therefore, we only prove the following theorem. □

Theorem 3.3 *The set of integral elements* $\Sigma(J^2(M^{1+2}, 2))$ *of* $(J^2(M^{1+2}, 2), C^2)$ *is a submanifold of* $J(C^k, 2) = Gr(C^k, 2)$.

Proof. For $w_0 \in \Sigma(J^2(M^{1+2}, 2))$, let $p(w_0) = v_0 \in J^2(M^{1+2}, 2)$, where $p : \Sigma(J^2(M^{1+2}, 2)) \to J^2(M^{1+2}, 2)$ is the projection. Let $(U, (x, y, z, p, q, r, s, t))$ be a canonical coordinate in $(J^2(M^{1+2}, 2), C^2)$ around v_0. Namely,

$$C^2 = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\},$$

where $\varpi_0 = dz - pdx - qdy$, $\varpi_1 = dp - rdx - sdy$, $\varpi_2 = dq - sdx - tdy$.

Let $\pi : J(C^2, 2) \to J^2(M^{1+2}, 2)$ be the projection. Then $\pi^{-1}(U)$ is covered by 10 open sets in $J(C^2, 2)$:

$$\pi^{-1}(U) = U_{xy} \cup U_{xr} \cup U_{xs} \cup U_{xt} \cup U_{yr} \cup U_{ys} \cup U_{yt} \cup U_{rs} \cup U_{rt} \cup U_{st},$$

where

$$\begin{aligned} U_{xy} &:= \{w \in \pi^{-1}(U) \mid dx \wedge dy|_w \neq 0\} \\ U_{xr} &:= \{w \in \pi^{-1}(U) \mid dx \wedge dr|_w \neq 0\} \\ &\vdots \\ U_{st} &:= \{w \in \pi^{-1}(U) \mid ds \wedge dt|_w \neq 0\}. \end{aligned}$$

In the following, we will explicitly describe the defining equation of $\Sigma(J^2(M^{1+2}, 2))$ in terms of the inhomogeneous Grassmann coordinate of $U_{xy}, \ldots, U_{st}$.

(0) On U_{xy};

In this case, note that $dx \wedge dy|_w \neq 0$ is the transversality condition of the geometric construction of the jet spaces (Section 2). So the defining equation of $\Sigma(J^2(M^{1+2},2))$ in U_{xy} will be that of the third order jet space $J^3(M^{1+2},2)$.

For $w \in U_{xy}$, w is a 2-dimensional subspace of $C^2(v), p(w) = v$. Hence, restricting dr, ds, dt to w, we can introduce the inhomogeneous coordinate p_{ijk} in U_{xy} of $J(C^2,2)$ around w as follows;

$$\begin{cases} dr|_w = p_{111}(w)dx|_w + p_{112}(w)dy|_w \\ ds|_w = p_{121}(w)dx|_w + p_{122}(w)dy|_w \\ dt|_w = p_{221}(w)dx|_w + p_{222}(w)dy|_w. \end{cases}$$

Moreover 2-dimensional integral element w satisfies $d\varpi_1|_w = d\varpi_2|_w = 0$;

$$d\varpi_1|_w = -dr \wedge dx|_w - ds \wedge dy|_w = (p_{112}(w) - p_{121}(w))dx \wedge dy|_w$$
$$d\varpi_2|_w = -ds \wedge dx|_w - dt \wedge dy|_w = (p_{122}(w) - p_{221}(w))dx \wedge dy|_w.$$

In this way, we obtain the defining equations $f_1 = f_2 = 0$ of $\Sigma(J^2(M^{1+2},2))$ in the inhomogeneous coordinate U_{xr} of $J(C^2,2)$, where $f_1 = p_{112} - p_{121}$, $f_2 = p_{122} - p_{221}$;

$$\{f_1 = f_2 = 0\} \subset U_{xy}.$$

Then df_1, df_2 are independent on $\{f_1 = f_2 = 0\}$. Thus, we have

$$\begin{cases} dr|_w = p_{111}(w)dx|_w + p_{112}(w)dy|_w \\ ds|_w = p_{112}(w)dx|_w + p_{122}(w)dy|_w \\ dt|_w = p_{122}(w)dx|_w + p_{222}(w)dy|_w. \end{cases}$$

We see that $(x, y, z, p, q, r, s, t, p_{111}, p_{112}, p_{122}, p_{222})$ is a coordinate system of $\Sigma(J^2(M^{1+2},2))$ in U_{xy}. This coordinate system is called the canonical coordinate system of the 3-jet space $J^3(M^{1+2},2)$ (Section 2, Section 4).

(1) On U_{xr};

For $w \in U_{xr}$, w is a 2-dimensional subspace of $C^2(v), p(w) = v$. Hence, restricting dy, ds, dt to w, we can introduce the inhomogeneous coordinates $(x, y, z, p, q, r, s, t, a, B, c, D, e, F)$ of $J(C^2,2)$ around w as follows;

$$\begin{cases} dy|_w = a(w)dx|_w + B(w)dr|_w \\ ds|_w = c(w)dx|_w + D(w)dr|_w \\ dt|_w = e(w)dx|_w + F(w)dr|_w. \end{cases}$$

Moreover 2-dimensional integral element w satisfies $d\varpi_1|_w = d\varpi_2|_w = 0$;

$$\begin{aligned} d\varpi_1|_w &= -dr \wedge dx|_w - ds \wedge dy|_w \\ &= (-1 - a(w)D(w) + B(w)c(w))dr \wedge dx|_w \\ d\varpi_2|_w &= -ds \wedge dx|_w - dt \wedge dy|_w \\ &= (-D(w) + B(w)e(w) - a(w)F(w))dr \wedge dx|_w. \end{aligned}$$

In this way, we obtain the defining equations $f_1 = f_2 = 0$ of $\Sigma(J^2(M^{1+2}, 2))$ in the inhomogeneous coordinate U_{xr} of $J(C^2, 2)$, where $f_1 = -1 - aD + Bc$, $f_2 = -D + Be - aF$;

$$\{f_1 = f_2 = 0\} \subset U_{xr}.$$

Then df_1, df_2 are independent on $\{f_1 = f_2 = 0\}$.

(2) On U_{xs};

For $w \in U_{xs}$, w is a 2-dimensional subspace of $C^2(v), p(w) = v$. Hence, restricting dy, dr, dt to w, we have

$$\begin{cases} dy|_w = a(w)dx|_w + B(w)ds|_w \\ dr|_w = c(w)dx|_w + D(w)ds|_w \\ dt|_w = e(w)dx|_w + F(w)ds|_w. \end{cases}$$

Moreover 2-dimensional integral element w satisfies $d\varpi_1|_w = d\varpi_2|_w = 0$;

$$\begin{aligned} d\varpi_1|_w &= -dr \wedge dx|_w - ds \wedge dy|_w = (-D(w) - a(w))ds \wedge dx|_w \\ d\varpi_2|_w &= -ds \wedge dx|_w - dt \wedge dy|_w \\ &= (-1 - a(w)F(w) + e(w)B(w))ds \wedge dx|_w. \end{aligned}$$

Then the defining functions of $\Sigma(J^2(M^{1+2}, 2))$ are independent by the same reasoning as in (1).

(3) On U_{xt};

For $w \in U_{xt}$, w is a 2-dimensional subspace of $C^2(v), p(w) = v$. Hence, restricting dy, dr, ds to w, we have

$$\begin{cases} dy|_w = a(w)dx|_w + B(w)dt|_w \\ dr|_w = c(w)dx|_w + D(w)dt|_w \\ ds|_w = e(w)dx|_w + F(w)dt|_w. \end{cases}$$

Moreover 2-dimensional integral element w satisfies $d\varpi_1|_w = d\varpi_2|_w = 0$;

$$d\varpi_1|_w = (-D(w) + e(w)B(w) - a(w)F(w))dt \wedge dx|_w$$
$$d\varpi_2|_w = (-F(w) - a(w))dt \wedge dx|_w.$$

Hence, we have

$$\begin{cases} dy|_w = a(w)dx|_w + B(w)dt|_w \\ dr|_w = c(w)dx|_w + (a^2(w) + e(w)B(w))dt|_w \\ ds|_w = e(w)dx|_w - a(w)dt|_w. \end{cases}$$

$(x, y, z, p, q, r, s, t, a, B, c, e)$ is a coordinate of $\Sigma(J^2(M^{1+2}, 2))$ in U_{xt}.

(4) On U_{yr};

For $w \in U_{yr}$, w is a 2-dimensional subspace of $C^2(v), p(w) = v$. Hence, restricting dx, ds, dt to w, we have

$$\begin{cases} dx|_w = a(w)dy|_w + B(w)dr|_w \\ ds|_w = c(w)dy|_w + D(w)dr|_w \\ dt|_w = e(w)dy|_w + F(w)dr|_w. \end{cases}$$

Moreover 2-dimensional integral element w satisfies $d\varpi_1|_w = d\varpi_2|_w = 0$

$$d\varpi_1|_w = (-a(w) - D(w))dr \wedge dy|_w$$
$$d\varpi_2|_w = (c(w)B(w) - a(w)D(w) - F(w))dr \wedge dy|_w.$$

(5) On U_{ys};

For $w \in U_{ys}$, w is a 2-dimensional subspace of $C^2(v), p(w) = v$. Hence, restricting dx, dr, dt to w, we have

$$\begin{cases} dx|_w = a(w)dy|_w + B(w)ds|_w \\ dr|_w = c(w)dy|_w + D(w)ds|_w \\ dt|_w = e(w)dy|_w + F(w)ds|_w. \end{cases}$$

Moreover 2-dimensional integral element w satisfies $d\varpi_1|_w = d\varpi_2|_w = 0$;

$$\begin{aligned} d\varpi_1|_w &= (c(w)B(w) - a(w)D(w) - 1)ds \wedge dy|_w \\ d\varpi_2|_w &= (-a(w) - F(w))ds \wedge dy|_w. \end{aligned}$$

(6) On U_{yt};

For $w \in U_{yt}$, w is a 2-dimensional subspace of $C^2(v), p(w) = v$. Hence, restricting dx, dr, ds to w, we have

$$\begin{cases} dx|_w = a(w)dy|_w + B(w)dt|_w \\ dr|_w = c(w)dy|_w + D(w)dt|_w \\ ds|_w = e(w)dy|_w + F(w)dt|_w. \end{cases}$$

Moreover 2-dimensional integral element w satisfies $d\varpi_1|_w = d\varpi_2|_w = 0$;

$$\begin{aligned} d\varpi_1|_w &= (c(w)B(w) - a(w)D(w) - F(w))dt \wedge dy|_w \\ d\varpi_2|_w &= (e(w)B(w) - a(w)F(w) - 1)dt \wedge dy|_w. \end{aligned}$$

(7) On U_{rs};

For $w \in U_{rs}$, w is a 2-dimensional subspace of $C^2(v), p(w) = v$. Hence, restricting dx, dy, dt to w, we have

$$\begin{cases} dx|_w = A(w)dr|_w + B(w)ds|_w \\ dy|_w = C(w)dr|_w + D(w)ds|_w \\ dt|_w = E(w)dr|_w + F(w)ds|_w. \end{cases}$$

Moreover 2-dimensional integral element w satisfies $d\varpi_1|_w = d\varpi_2|_w = 0$;

$$d\varpi_1|_w = -dr \wedge dx|_w - ds \wedge dy|_w = (-B(w) + C(w))dr \wedge ds|_w$$
$$\begin{aligned} d\varpi_2|_w &= -ds \wedge dx|_w - dt \wedge dy|_w \\ &= (-A(w) + D(w)E(w) - C(w)F(w))ds \wedge dr|_w. \end{aligned}$$

Hence, we have

$$\begin{cases} dx|_w = (D(w)E(w) - B(w)F(w))dr|_w + B(w)ds|_w \\ dy|_w = B(w)dr|_w + D(w)ds|_w \\ dt|_w = E(w)dr|_w + F(w)ds|_w. \end{cases}$$

$(x, y, z, p, q, r, s, t, B, D, E, F)$ is a coordinate of $\Sigma(J^2(M^{1+2}, 2))$ in U_{rs}.

(8) On U_{rt};

For $w \in U_{rt}$, w is a 2-dimensional subspace of $C^2(v), p(w) = v$. Hence, restricting dx, dy, ds to w, we have

$$\begin{cases} dx|_w = A(w)dr|_w + B(w)dt|_w \\ dy|_w = C(w)dr|_w + D(w)dt|_w \\ ds|_w = E(w)dr|_w + F(w)dt|_w. \end{cases}$$

Moreover 2-dimensional integral element w satisfies $d\varpi_1|_w = d\varpi_2|_w = 0$;

$$d\varpi_1|_w = (-B(w) - D(w)E(w) + C(w)F(w))dr \wedge dt|_w$$
$$d\varpi_2|_w = (-C(w) - A(w)F(w) + B(w)E(w))dt \wedge dr|_w.$$

(9) On U_{st};

For $w \in U_{st}$, w is a 2-dimensional subspace of $C^2(v), p(w) = v$. Hence, restricting dx, dy, dr to w, we have

$$\begin{cases} dx|_w = A(w)ds|_w + B(w)dt|_w \\ dy|_w = C(w)ds|_w + D(w)dt|_w \\ dr|_w = E(w)ds|_w + F(w)dt|_w. \end{cases}$$

Moreover 2-dimensional integral element w satisfies $d\varpi_1|_w = d\varpi_2|_w = 0$;

$$d\varpi_1|_w = (-B(w)E(w) + A(w)F(w) - D(w))ds \wedge dt|_w$$
$$d\varpi_2|_w = (-B(w) + C(w))ds \wedge dt|_w.$$

From $(0), \ldots, (9)$, we conclude $\Sigma(J^2(M^{1+2}, 2))$ is a submanifold in $J(C^2, 2)$. □

Remark 3.4 We have that the projection $p : \Sigma(J^2) \to J^2$ is a submersion with respect to the manifold structure of $\Sigma(J^2)$. This is checked in each case. For instance, on U_{xt},

$$p : (x, y, z, p, q, r, s, t, a, B, c, e) \to (x, y, z, p, q, r, s, t).$$

4. Regularity and symbol algebra of diferential systems

Next we will consider the local equivalence problem of $(\Sigma(J^2(M^{1+2}, 2)), D)$, where D is a canonical system on $\Sigma(J^2(M^{1+2}, 2))$ (see Section 5). To this purpose, we first recall Tanaka theory of weakly regular differential systems in this section (see [T], [Y1]).

4.1. Weak derived system

Let D be a differential system on a manifold R. We denote by $\mathcal{D}$ the sheaf of sections to D. Then we define k-th weak higher derived system $\partial^{(k)}\mathcal{D}$ by;

$$\partial^{(1)}\mathcal{D} = \partial\mathcal{D}, \quad \partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}]$$

where $\mathcal{D} = \Gamma(D)$. A differential system D is called *weakly regular*, if $\partial^{(i)}\mathcal{D}$ is a sheaf of sections for a subbundle $\partial^{(i)}D$, for any i. If D is not weakly regular around $x \in R$, then x is called singular point in the sense of Tanaka theory.

We set $D^{-1} := D$, $D^{-k} := \partial^{(k-1)}D$ $(k \geq 2)$, for a weakly regular differential system D. Then we have;

$(S1)$ There exists a positive integer μ such that
$D^{-1} \subset D^{-2} \subset \cdots \subset D^{-k} \subset \cdots \subset D^{-(\mu-1)} \subset D^{-\mu} = D^{-(\mu+1)} = \cdots$

$(S2)$ $[\mathcal{D}^p, \mathcal{D}^q] \subset D^{p+q}$, for any negative integers p, q,
i.e. $[X, Y] \in \mathcal{D}^{p+q}$, $\quad X \in \mathcal{D}^p$, $Y \in \mathcal{D}^q$.

4.2. Symbol algebra of weakly regular differential system

Let (R, D) be a weakly regular differential system such that

$$T(R) = D^{-\mu} \supset D^{-(\mu-1)} \supset \cdots \supset D^{-1} = D.$$

For all $x \in R$, we put $\mathfrak{g}_{-1}(x) := D^{-1}(x) = D(x)$, $\mathfrak{g}_p(x) := D^p(x)/D^{p+1}(x)$, and put

$$\mathfrak{m}(x) := \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Then $\dim \mathfrak{m}(x) = \dim R$. For $X \in \mathfrak{g}_p(x)$, $Y \in \mathfrak{g}_q(x)$, we take extensions $\tilde{X} \in \mathcal{D}^p$, $\tilde{Y} \in \mathcal{D}^q$ of representatives for X, Y ($\tilde{X}_x, \tilde{Y}_x$ give X, Y in $\mathfrak{g}_p(x), \mathfrak{g}_q(x)$) respectively. Then $[\tilde{X}, \tilde{Y}] \in \mathcal{D}^{p+q}$ and $[\tilde{X}, \tilde{Y}]_x$ does not depend on the choice of extensions up to D_x^{p+q+1} because of the equation

$$[f\tilde{X}, g\tilde{Y}] = fg[\tilde{X}, \tilde{Y}] + f(\tilde{X}g)\tilde{Y} - g(\tilde{Y}f)\tilde{X} \quad (f, g \in C^\infty(R)).$$

Therefore we define $[X, Y] := [\tilde{X}, \tilde{Y}]_x \in \mathfrak{g}_{p+q}(x)$, which makes $\mathfrak{m}(x)$ a graded Lie algebra. We call $(\mathfrak{m}(x), [\ \ ])$ the symbol algebra of (R, D) at x.

Note that the Symbol Algebra $(\mathfrak{m}(x), [\ \ ])$ satisfies the generating conditions

$$[\mathfrak{g}^p, \mathfrak{g}^{-1}] = \mathfrak{g}^{p-1} \quad (p < 0).$$

Later, T. Morimoto introduced the notion of a filtered manifold as generalization of the weakly regular differential system in [M].

We define a filtered manifold (R, F) by a pair of a manifold R and a tangential filtration F. Here, a tangential filtration F on R is a sequence $\{F^p\}_{p<0}$ of subbundles of the tangent bundle TR such that the following conditions are satisfied;

$$(i) \quad TR = F^k = \cdots = F^{-\mu} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset F^0 = 0$$

$$(ii) \quad [\mathcal{F}^p, \mathcal{F}^q] \subset \mathcal{F}^{p+q} \qquad \forall p, q < 0$$

where $\mathcal{F}^p = \Gamma(F^p)$ is the set of sections of F^p.

Let (R, F) be a filtered manifold, for $x \in R$, we put

$$\mathfrak{f}^p(x) := F^p(x)/F^{p+1}(x)$$

and

$$\mathfrak{f}(x) := \bigoplus_{p<0} \mathfrak{f}_p(x).$$

For $X \in \mathfrak{f}_p(x)$, $Y \in \mathfrak{f}_q(x)$, Lie bracket $[X, Y] \in \mathfrak{f}_{p+q}(x)$ is defined by;

Let $\tilde{X} \in \mathcal{F}^p$, $\tilde{Y} \in \mathcal{F}^q$ be extensions ($\tilde{X}_x = X$, $\tilde{Y}_x = Y$), then $[\tilde{X}, \tilde{Y}] \in \mathcal{F}^{p+q}[X, Y] := [\tilde{X}, \tilde{Y}]_x \in \mathfrak{f}_{p+q}(x)$ does not depend on the extensions.

Then we call $(\mathfrak{f}(x), [\ \ ])$ the (associated nilpotent) graded Lie algebra of (R, F) at $x \in R$.

In general $(\mathfrak{f}(x), [\ \ ])$ does not satisfy the generating conditions.

Remark 4.1 Let $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$ be a differential system on a manifold R. We denote by $D^\perp$ the annihilator subbundle of D in T^*R, namely,

$$\begin{aligned} D^\perp(x) &= \{\omega \in T_x^*R \mid \omega(X) = 0 \text{ for any } X \in D(x)\} \\ &= \langle \varpi_1, \ldots, \varpi_s \rangle. \end{aligned}$$

Then the annihilator $(\partial D)^\perp$ of the first derived system of D is given by

$$(\partial D)^\perp = \{\varpi \in D^\perp \mid d\varpi \equiv 0 \pmod{D^\perp}\}.$$

Moreover the annihilator $(\partial^{(k+1)}D)^\perp$ of the $(k+1)$-th weak derived system of D is given by

$$\begin{aligned} (\partial^{(k+1)}D)^\perp = \big\{\varpi \in (\partial^{(k)}D)^\perp \mid d\varpi \equiv 0 \ (\mathrm{mod}\ (\partial^{(k)}D)^\perp, \\ (\partial^{(p)}D)^\perp \wedge (\partial^{(q)}D)^\perp,\ 2 \leq p,\ q \leq k-1)\big\}. \end{aligned}$$

4.3. Example

Example 4.2 Let $J^3(M^{1+2}, 2)$; $(x_1, x_2, y, p_1, p_2, p_{11}, p_{12}, p_{22}, p_{111}, p_{112}, p_{122}, p_{222})$ be a canonical coordinate, then $C^3 = \{\varpi = \varpi_1 = \varpi_2 = \varpi_{11} = \varpi_{12} = \varpi_{22} = 0\}$, where

$$\begin{cases} \varpi = dy - p_1 dx_1 - p_2 dx_2 \\ \varpi_i = dp_i - p_{i1} dx_1 - p_{i2} dx_2 \\ \varpi_{ij} = dp_{ij} - p_{ij1} dx_1 - p_{ij2} dx_2. \end{cases}$$

The structure equation for C^3 is given by

$$\begin{cases} d\varpi \equiv 0 & (\mathrm{mod}\ C^3) \\ d\varpi_i \equiv 0 & (\mathrm{mod}\ C^3) \\ d\varpi_{ij} = -dp_{ij1} \wedge dx_1 - dp_{ij2} \wedge dx_2 & (\mathrm{mod}\ C^3). \end{cases}$$

Therefore $\partial^{(1)} C^3 = \partial C^3 = \{\varpi = \varpi_1 = \varpi_2 = 0\}$. The structure equations for ∂C^3 and $\partial^{(1)} C^3$ are

$$\begin{cases} d\varpi \equiv 0 & (\mathrm{mod}\ \partial C^3) \\ d\varpi_i \equiv -dp_{i1} \wedge dx_1 - dp_{i2} \wedge dx_2 & (\mathrm{mod}\ \partial C^3) \end{cases}$$

$$\begin{cases} d\varpi \equiv 0 & (\mathrm{mod}\ \partial^{(1)} C^3, \varpi_{ij} \wedge \varpi_{kl}) \\ d\varpi_i \equiv -dp_{i1} \wedge dx_1 - dp_{i2} \wedge dx_2 & (\mathrm{mod}\ \partial^{(1)} C^3, \varpi_{ij} \wedge \varpi_{kl}). \end{cases}$$

Thus $\partial^{(2)} C^3 = \partial^2 C^3 = \{\varpi = 0\}$. The structure equations for $\partial^2 C^3$, $\partial^{(2)} C^3$ are

$$\begin{cases} d\varpi \equiv -dp_1 \wedge dx_1 - dp_2 \wedge dx_2 \quad (\mathrm{mod}\ \partial^2 C^3) \\ d\varpi \equiv -dp_1 \wedge dx_1 - dp_2 \wedge dx_2 \\ \qquad\qquad (\mathrm{mod}\ \partial^{(2)} C^3, \varpi_{ij} \wedge \varpi_{kl}, \varpi_i \wedge \varpi_{jk}, \varpi_i \wedge \varpi_j). \end{cases}$$

Therefore $\partial^{(3)} C^3 = \partial^3 C^3 = T(J^3)$. Especially, $(J^3(M,1), C^3)$ is regular and weakly regular.

Symbol algebra of $J^3(M^{1+2}, 2)$;

We take a coframe: $\{\varpi, \varpi_1, \varpi_2, \varpi_{11}, \varpi_{12}, \varpi_{22}, dp_{111}, dp_{112}, dp_{122}, dp_{222}, dx_1, dx_2\}$ and its dual frame $\{X_y, X_1, X_2, X_{11}, X_{12}, X_{22}, X_{111}, X_{112}, X_{122}, X_{222}, X_{x_1}, X_{x_2}\}$, where

$$X_y = \frac{\partial}{\partial y}, \quad X_i = \frac{\partial}{\partial p_i}, \quad X_{ij} = \frac{\partial}{\partial p_{ij}}, \quad X_{ijk} = \frac{\partial}{\partial p_{ijk}},$$

$$X_{x_i} = \frac{d}{dx_i} = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial y} + p_{1i} \frac{\partial}{\partial p_1} + p_{2i} \frac{\partial}{\partial p_2} + p_{11i} \frac{\partial}{\partial p_{11}}$$
$$+ p_{12i} \frac{\partial}{\partial p_{12}} + p_{22i} \frac{\partial}{\partial p_{22}}.$$

Then, at $x \in J^3$,

$$\mathfrak{g}_{-1}(x) := C^3 = \langle X_{111}, X_{112}, X_{122}, X_{222}, X_{x_1}, X_{x_2} \rangle,$$
$$\mathfrak{g}_{-2}(x) := \langle X_{11}, X_{12}, X_{22} \rangle, \quad \mathfrak{g}_{-3}(x) := \langle X_1, X_2 \rangle, \quad \mathfrak{g}_{-4}(x) := \langle X_y \rangle,$$
$$\mathfrak{m}_{jet}(x) = \bigoplus_{p=-1}^{-4} \mathfrak{g}_p(x) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}.$$

The bracket relations are;
$[X_{jkl}, X_{x_i}] = \delta_{il} X_{jk}$, $[X_{jk}, X_{x_i}] = \delta_{ik} X_j$, $[X_j, X_{x_i}] = \delta_{ij} X_y$, the other relations are given by 0. (see the proof of Proposition 5.5 for how to calculate.)

5. Equivalence problem of $(\Sigma(J^2(M^{1+2}, 2)), D)$

Since $\Sigma(J^2)$ is a manifold from Theorem 3.1, we can define the canonical system D on $\Sigma(J^2)$ as follows;

For any $u \in \Sigma(J^2)$ with $p(u) = x \in J^2$, we put

$$D(u) = p_*^{-1}(u) \subset T_u(\Sigma(J^2)) \xrightarrow{p_*} T_x(J^2)$$

where $p : \Sigma(J^2) \to J^2(M^{1+2}, 2)$ is the projection.

In this section, we will consider the equivalence problem of $(\Sigma(J^2), D)$. Namely we will give the orbit decomposition under the action of the $\mathrm{Aut}(\Sigma(J^2), D)$, where

$$\mathrm{Aut}(\Sigma(J^2), D) = \{\varphi : \Sigma(J^2) \to \Sigma(J^2) \mid \varphi :$$
$$\text{local diffeomorpfhism such that } \varphi_*(D) = D\}.$$

Remark 5.1 Let $\varphi : J^2(M, 2) \to J^2(M, 2)$ be an isomorphism, i.e., φ is a diffeomorphism such that $\varphi_*(C^2) = C^2$. Then φ induces the isomorphism $\varphi_* : (\Sigma(J^2), D) \to (\Sigma(J^2), D)$, namely, the differential map $\varphi_* : \Sigma(J^2) \to \Sigma(J^2)$ is a diffeomorphism sending D onto D.

First, we explain geometric meaning of the open covering $U_{xy} \cup \cdots \cup U_{st}$ in the proof of Theorem 3.3. The set $\Sigma(J^2)$ has a geometric decomposition;

$$\Sigma(J^2) = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \text{ (disjoint union)},$$

where $\Sigma_i = \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = i\}$ $(i = 0, 1, 2)$, and the fiber means that of $T(J^2) \supset C^2 \to T(J^1)$. Then, locally,

$$\Sigma_0 = U_{xy}|_{\Sigma(J^2)}$$
$$\Sigma_1 = \{(U_{xr} \cup U_{xs} \cup U_{xt} \cup U_{yr} \cup U_{ys} \cup U_{yt}) \backslash U_{xy}\}|_{\Sigma(J^2)}$$
$$\Sigma_2 = \{(U_{rs} \cup U_{rt} \cup U_{st}) \backslash (U_{xy} \cup U_{xr} \cup U_{xs} \cup U_{xt} \cup U_{yr} \cup U_{ys} \cup U_{yt})\}|_{\Sigma(J^2)}.$$

The set $\Sigma_0 = J^3$ is an open set in $\Sigma(J^2)$. The set Σ_1 is a codimension 1 submanifold in $\Sigma(J^2)$. The set Σ_2 is a codimension 2 submanifold in $\Sigma(J^2)$ and is a $\mathbb{P}^2$-bundle over J^2.

Proposition 5.2 *The differential system D on $\Sigma(J^2) = \Sigma(J^2(M^{1+2}, 2))$ is regular, but is not weakly regular. Precisely we obtain that*

$$D \subset \partial D \subset \partial^2 D \subset \partial^3 D = T\Sigma(J^2).$$

Moreover $\partial^2 D = \partial^{(2)} D$ and

$$\begin{cases} \partial^{(3)} D = T\Sigma(J^2) & \text{on } \Sigma_0 \cup \Sigma_1 \\ \partial^{(3)} D = \partial^{(2)} D & \text{on } \Sigma_2 \end{cases}$$

Remark 5.3 Note that $\Sigma_0 = J^3$ by definition. So the derived system, weak derived system around $w \in \Sigma_0$ and the symbol algebra at $w \in \Sigma_0$ are given as in Example 4.2.

Proof. We take canonical coordinates on J^2 and consider the covering of $\Sigma(J^2)$: $U_{xy} \cup U_{xr} \cup U_{xs} \cup U_{xt} \cup U_{yr} \cup U_{ys} \cup U_{yt} \cup U_{rs} \cup U_{rt} \cup U_{st}$ (see proof of Theorem 3.3). First of all, we show that it is enough to work on three open sets U_{xr}, U_{rs}, U_{rt}.

Lemma 5.4 $p^{-1}(U) = U_{xy} \cup U_{xr} \cup U_{xs} \cup U_{xt} \cup U_{yr} \cup U_{ys} \cup U_{yt} \cup U_{rs} \cup U_{rt} \cup U_{st} = U_{xy} \cup U_{xt} \cup U_{yr} \cup U_{rs} \cup U_{rt} \cup U_{st}$*, under the notation of the proof of Theorem* 3.3.

Proof. First, we prove $U_{xr} \subset U_{xt} \cup U_{xy}$.

For $w \in U_{xr}$,

$$\begin{cases} dy|_w = a(w)dx|_w + B(w)dr|_w \\ ds|_w = c(w)dx|_w + D(w)dr|_w \\ dt|_w = e(w)dx|_w + F(w)dr|_w, \end{cases}$$

and the relations are $f_1 = -1 - aD + Bc = 0$, $f_2 = -D + Be - aF = 0$. Note that $w \in U_{xy}$ if and only if dx and dy are independent at w, i.e., $B(w) \neq 0$. Assume that $B(w) = 0$, then $F(w) \neq 0$ from $f_1 = 0$. So

$$dx \wedge dt|_w = dx \wedge (e(w)dx + F(w)dr)|_w = F(w)dx \wedge dr|_w \neq 0.$$

Therefore $w \in U_{xt} \cup U_{xy}$.

The same argument yeilds $U_{xs} \subset U_{xt} \cup U_{xy}$, $U_{ys} \subset U_{yr} \cup U_{xy}$ and $U_{yt} \subset U_{yr} \cup U_{xy}$. □

From above remark, lemma and natural symmetry, where natural symmetry means the isomorphism induced by $\bar{x} = y$, $\bar{y} = x$, $\bar{p} = q$, $\bar{q} = p$, $\bar{r} = t$, $\bar{t} = r$, it is enough to work on U_{xt}, U_{rs}, U_{rt}, because every germ in U_{yr} appears in U_{xt} and that of U_{st} appears in U_{rs}.

On U_{xt};

We take a coordinate $(x, y, z, p, q, r, s, t, a, B, c, e)$ on U_{xt} (see proof of Theorem 3.3), then $D = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = \varpi_s = 0\}$, where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt. \end{cases}$$

Recall that, for $w = (x, y, z, p, q, r, s, t, a, B, c, e)$, $B \neq 0$ if and only if $w \in \Sigma_0$, therefore it is enough to consider at w in the hypersurface $\{B = 0\} \subset \Sigma(J^2)$. The structure equation at a point in $\{B = 0\}$ is

$$\begin{cases} d\varpi_i \equiv 0 & (i = 0, 1, 2) \\ d\varpi_y = -da \wedge dx - dB \wedge dt \not\equiv 0 & \\ d\varpi_r = -dc \wedge dx - (edB + 2ada) \wedge dt \not\equiv 0 & \\ d\varpi_s = -de \wedge dx + da \wedge dt \not\equiv 0 & (\text{mod } D). \end{cases}$$

Hence $\partial D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$. The structure equation of ∂D at a point in $\{B = 0\}$ is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv (-\varpi_r - a\varpi_s + e\varpi_y) \wedge dx - a\varpi_y \wedge dt \not\equiv 0 \\ d\varpi_2 = -\varpi_s \wedge dx - dt \wedge \varpi_y \not\equiv 0 \quad (\text{mod } \partial D, \varpi_\alpha \wedge \varpi_\beta \ (\alpha, \beta = y, r, s)). \end{cases}$$

Hence $\partial^{(2)} D = \partial^2 D = \{\varpi_0 = 0\}$. The structure equation of $\partial^2 D$ at a point in $\{B = 0\}$ is

$$\begin{cases} d\varpi_0 \equiv -(\varpi_1 + a\varpi_2) \wedge dx \not\equiv 0 \\ \qquad\qquad (\text{mod } \partial^2 D, \varpi_\alpha \wedge \varpi_\beta \ (\alpha, \beta \in \{y, r, s, 1, 2\})). \end{cases}$$

Hence $\partial^{(3)} D = \partial^3 D = T(\Sigma(J^2))$. We conclude

$$\partial^{(3)} D = T\Sigma(J^2) \text{ on } \Sigma_0 \cup \Sigma_1.$$

On U_{rs};

Let $(x, y, z, p, q, r, s, t, B, D, E, F)$ be a coordinate on U_{rs}. Then $D = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = \varpi_t = 0\}$, where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE - BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

$w \in \Sigma_2$ if and only if $dx|_w = dy|_w = 0$. So, in this coordinate, Σ_2 is a $\{B = D = 0\}$: codimension 2 submanifold in $\Sigma(J^2)$.

The structure equation at a point in $\{B = D = 0\}$ is

$$\begin{cases} d\varpi_i \equiv 0 & (i = 0, 1, 2) \\ d\varpi_x = -(EdD - FdB) \wedge dr - dB \wedge ds \not\equiv 0 & \\ d\varpi_y = -dB \wedge dr - dD \wedge ds \not\equiv 0 & \\ d\varpi_t = -dE \wedge dr - dF \wedge ds \not\equiv 0 & (\text{mod } D). \end{cases}$$

Hence $\partial D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$. The structure equation of ∂D at a point in $\{B = D = 0\}$ is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv -dr \wedge \varpi_x - ds \wedge \varpi_y \not\equiv 0 \\ d\varpi_2 \equiv -ds \wedge \varpi_x - (Edr + Fds) \wedge \varpi_y \not\equiv 0 \\ \qquad\qquad (\text{mod } \partial D, \varpi_\alpha \wedge \varpi_\beta \ (\alpha, \beta = x, y, t)). \end{cases}$$

Hence $\partial^{(2)} D = \partial^2 D = \{\varpi_0 = 0\}$. The structure equation of $\partial^2 D$ at a point in $\{B = D = 0\}$ is

$$\begin{cases} d\varpi_0 \equiv -\varpi_1 \wedge \varpi_x - \varpi_2 \wedge \varpi_y \not\equiv 0 \quad (\text{mod } \partial^2 D) \\ d\varpi_0 \equiv 0 \quad (\text{mod } \partial^2 D, \varpi_\alpha \wedge \varpi_\beta \ (\alpha, \beta \in \{x, y, t, 1, 2\})). \end{cases}$$

Therefore $\partial^3 D = T(\Sigma(J^2))$ on U_{rs} and $\partial^{(3)} D = \{\varpi_0 = 0\}$ on $\Sigma_2 \cap U_{rs}$.

On U_{rt};

$$\begin{cases} dx|_w = A(w)dr|_w + B(w)dt|_w \\ dy|_w = C(w)dr|_w + D(w)dt|_w \\ ds|_w = E(w)dr|_w + F(w)dt|_w, \end{cases}$$

where defining equations are $-B - DE + CF = 0$, $-C - AF + BE = 0$. $w \in \Sigma_2$ if and only if $A = B = C = D = 0$. Moreover, if $(E, F) \neq (0, 0)$ then the point is in U_{rs} or U_{st}. So we consider a point $(E, F) = (0, 0)$ and take a coordinate $(x, y, z, p, q, r, s, t, A, D, E, F)$ around $(E, F) = (0, 0)$. Then $D = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = \varpi_s = 0\}$ where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - Adr - \frac{DE+AF^2}{EF-1}dt \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - \frac{DE^2+AF}{EF-1}dr - Ddt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - Edr - Fdt. \end{cases}$$

The structure equation at a point $(A, D, E, F) = 0$ is

$$\begin{cases} d\varpi_i \equiv 0 & (i = 0, 1, 2) \\ d\varpi_x = -dA \wedge dr \not\equiv 0 & \\ d\varpi_y = -dD \wedge dt \not\equiv 0 & \\ d\varpi_s = -dE \wedge dr - dF \wedge dt \not\equiv 0 & \pmod{D}. \end{cases}$$

Hence $\partial D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$. The structure equation of ∂D at a point in $(A, D, E, F) = 0$ is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv -dr \wedge \varpi_x \not\equiv 0 \\ d\varpi_2 \equiv -dt \wedge \varpi_y \not\equiv 0 \quad (\text{mod } \partial D, \varpi_\alpha \wedge \varpi_\beta \ (\alpha, \beta = x, y, s)). \end{cases}$$

Hence $\partial^{(2)} D = \partial^2 D = \{\varpi_0 = 0\}$. The structure equation of $\partial^2 D$ at a point in $(A, D, E, F) = 0$ is

$$\begin{cases} d\varpi_0 \equiv -\varpi_1 \wedge \varpi_x - \varpi_2 \wedge \varpi_y \not\equiv 0 \quad (\text{mod } \partial^2 D) \\ d\varpi_0 \equiv 0 \quad (\text{mod } \partial^2 D, \varpi_\alpha \wedge \varpi_\beta \ (\alpha, \beta \in \{x, y, s, 1, 2\})). \end{cases}$$

We conclude that $(\Sigma(J^2), D)$ is regular and not weakly regular;

$$\partial^{(3)} D = \partial^{(2)} D \text{ on } \Sigma_2. \qquad \square$$

5.1. Classification of Σ_1

From above proposition, $(\Sigma(J^2), D)$ is locally weak regular around $w \in \Sigma_1$. So we can define symbol algebra at $w \in \Sigma_1$ and the following holds;

Proposition 5.5 *For $w \in \Sigma_1$, the symbol algebra $\mathfrak{m}(w)$ is isomorphic to $\mathfrak{m}$, $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}$ and $[,]$ is given by;*

$$X_y = [X_a, X_x] = [X_B, X_t], \quad X_r = [X_c, X_x], \quad X_s = [X_e, X_x] = -[X_a, X_t]$$
$$X_p = [X_r, X_x], \quad X_q = [X_s, X_x] = -[X_y, X_t]$$
$$X_z = [X_p, X_x], \qquad \textit{the other is trivial,}$$

where $\{X_z, X_p, X_q, X_y, X_r, X_s, X_x, X_t, X_a, X_B, X_c, X_e\}$ *are basis, and*

$$\mathfrak{g}_{-1} = \langle \{X_x, X_t, X_a, X_B, X_c, X_e\} \rangle$$
$$\mathfrak{g}_{-2} = \langle \{X_y, X_r, X_s\} \rangle$$
$$\mathfrak{g}_{-3} = \langle \{X_p, X_q\} \rangle$$
$$\mathfrak{g}_{-4} = \langle \{X_z\} \rangle$$

Especially, for $w \in \Sigma_1$*, the symbol algebra* $(\mathfrak{m}(w), [,])$ *is not isomorphic to the jet type symbol algebra* $\mathfrak{m}_{jet}$ *given as in Example* 4.2.

Remark 5.6 If the canonical systems $(\Sigma(J^2), D)$ at $w, w' \in \Sigma(J^2)$ are locally isomorphic, then the symbol algebras $\mathfrak{m}(w)$ and $\mathfrak{m}(w')$ are isomorphic as a graded Lie algebra. The symbol algebra $\mathfrak{m}(w)$ for $w \in \Sigma_0$ is isomorphic to $\mathfrak{m}_{jet}$ as a graded Lie algebra. Hence, by Proposition 5.5, we have that the canonical system D around $w \in \Sigma_1$ is not locally isomorphic to the canonical system D around $w \in \Sigma_0$.

Proof. "On U_{xt}" in the proof of Proposition 5.2, we put $\hat{\varpi}_1 := \varpi_1 + a\varpi_2$, $\hat{\varpi}_r := \varpi_r + 2a\varpi_s - e\varpi_y$, $\varpi_c = dc + 2ade - eda$ and take a coframe:

$$\{\varpi_0, \hat{\varpi}_1, \varpi_2, \varpi_y, \hat{\varpi}_r, \varpi_s, dx, dt, da, dB, \varpi_c, de\},$$

then the structure equations are

$$\begin{cases} d\varpi_i \equiv 0 & (i = 0, 1, 2) \\ d\varpi_y = -da \wedge dx - dB \wedge dt & \\ d\hat{\varpi}_r = -\varpi_c \wedge dx & \\ d\varpi_s = -de \wedge dx + da \wedge dt & (\text{mod } D) \end{cases}$$

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv -\hat{\varpi}_r \wedge dx \\ d\varpi_2 = -\varpi_s \wedge dx - dt \wedge \varpi_y \not\equiv 0 \quad (\mathrm{mod}\ \partial D,\ \varpi_\alpha \wedge \varpi_\beta\ (\alpha, \beta = y, r, s)) \end{cases}$$

$$\begin{cases} d\varpi_0 \equiv -\hat{\varpi}_1 \wedge dx \not\equiv 0 \\ \qquad\qquad (\mathrm{mod}\ \partial^2 D,\ \varpi_\alpha \wedge \varpi_i,\ \varpi_i \wedge \varpi_j\ (\alpha \in \{y, r, s\}, i, j \in \{1, 2\})). \end{cases}$$

We take its dual frame $\{X_z, X_p, X_q, X_y, X_r, X_s, X_x, X_t, X_a, X_B, X_c, X_e\}$ and put

$$[X_c, X_x] = A_y X_y + A_r X_r + A_s X_s \in \mathfrak{g}_{-2} \quad (A_y, A_r, A_s \in \mathbb{R}).$$

Then

$$\begin{aligned} d\hat{\varpi}_r(X_c, X_x) &= X_c(\hat{\varpi}_r(X_x)) - X_x(\hat{\varpi}_r(X_c)) - \hat{\varpi}_r([X_c, X_x]) \\ &= -\hat{\varpi}_r([X_c, X_x]) \\ &= -A_r. \end{aligned}$$

On the other hand

$$\begin{aligned} d\hat{\varpi}_r(X_c, X_x) &= -\varpi_c(X_c)dx(X_x) + dx(X_c)\varpi_c(X_x) \\ &= -1. \end{aligned}$$

Therefore, $A_r = 1$. From the same argument, we get $A_y = A_s = 0$. Hence

$$[X_c, X_x] = X_r.$$

The others are left to the reader. Hence its dual frame satisfies the relation of this proposition.

Finally, we will prove that the graded Lie algebra $\mathfrak{m}$ is not isomorphic to the jet type symbol algebra $\mathfrak{m}_{jet}$ (see Example 4.2).

From the above Lie bracket relations of $\mathfrak{m}$, we have a special direction in $\mathfrak{g}_{-3}$,

$$\big\{\langle X\rangle \mid X \in \mathfrak{g}_{-3}, X \neq 0, [X, \mathfrak{g}_{-1}] = 0\big\} = \langle X_q\rangle.$$

But $\mathfrak{m}_{jet}$ does not have such direction. This completes the proof of proposition. $\square$

Theorem 5.7 (normal form) *For any $w \in \Sigma_1$, the differential system $(\Sigma(J^2(M^{1+2}, 2), D)$ around w is locally isomorphic to the germ at the origin of $(\mathbb{R}^{12}, \hat{D})$ given as in the introduction.*

Proof. We construct the paths from any points to the origin, directly. For $w_0 \in \Sigma_1$, we may assume w_0 is expressed by a germ at $w_0 = (0, \ldots, 0, a_0, 0, c_0, e_0)$,

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt, \end{cases}$$

because of normal form of J^2 and $w_0 \in \Sigma_1$ if and only if $B = 0$. Hence what we have to do is to construct $\varphi \in \mathrm{Aut}(\Sigma(J^2), D))$ sending $(0, \ldots, 0, a_0, 0, c_0, e_0)$ to $(0, \ldots, 0)$. Let φ_e be a

$$\varphi_e : (x, y, z, p, q, r, s, t, a, B, c, e) \mapsto \left(x, y, z - \frac{e_0}{2}x^2y, p - e_0xy, q - \frac{e_0}{2}x^2, r - e_0y, s - e_0x, t, a, B, c - e_0a, e - e_0\right)$$

Then

$$\begin{cases} \varphi_e^*\varpi_0 = \varpi_0 & \varphi_e^*\varpi_y = \varpi_y \\ \varphi_e^*\varpi_1 = \varpi_1 & \varphi_e^*\varpi_r = \varpi_r - e_0\varpi_y \\ \varphi_e^*\varpi_2 = \varpi_2 & \varphi_e^*\varpi_s = \varpi_s. \end{cases}$$

Therefore, φ_e leaves D invariant and sends a germ $(0, \ldots, 0, a_0, 0, c_0, e_0)$ to a germ $(0, \ldots, 0, a_0, 0, c_0', 0)$ where $c_0' = c_0 - e_0a_0$.

Similarly, Let φ_a, φ_c be

$$\varphi_a : (x, y, z, p, q, r, s, t, a, B, c, e) \mapsto \left(x, y, z, p + a_0q, q, r + 2a_0s + a_0^2t, s + a_0t, t, a - a_0, B, c + 2ea_0, e\right)$$

$$\varphi_c : (x, y, z, p, q, r, s, t, a, B, c, e) \mapsto$$
$$\left(x, y, z - \frac{c_0'}{6}x^3, p - \frac{c_0'}{2}x^2, q, r - c_0'x, s, t, a, B, c - c_0', e\right)$$

Then these maps preserve D and the composition $\varphi_c \circ \varphi_a$ sends a germ $(0, \dots, 0, a_0, 0, c_0', 0)$ to a germ $(0, \dots, 0, 0, 0, 0, 0)$, where above isomorphisms are obtained by focussing on the form $\varpi_c = dc + 2ade - eda$ in the proof of Proposition 5.5 and leaving the form invariant to keep the symbol algebras. □

5.2. Classification of Σ_2

Finally, we will classify points in Σ_2. From the Proposition 5.2, $w \in \Sigma_2$, we can not define the symbol algebra at w. But $\partial^{(1)}D$ and $\partial^{(2)}D$ are subbundle, so we can define graded Lie algebra at w as follows;

For $w \in \Sigma_2$, we put $\mathfrak{g}_{-1}(w) := D^{-1}(w) = D(w)$, $\mathfrak{g}_{-2}(w) := D^{-2}(w)/D^{-1}(w)$, $\mathfrak{g}_{-3}(w) := D^{-3}(w)/D^{-2}(w)$, $\mathfrak{g}_{-4}(w) := T_w(\Sigma(J^2))/D^{-3}(w)$.

$$\mathfrak{m}(w) = \mathfrak{g}_{-1}(w) \oplus \mathfrak{g}_{-2}(w) \oplus \mathfrak{g}_{-3}(w) \oplus \mathfrak{g}_{-4}(w).$$

We define Lie bracket by the same way of the usual symbol algebra except for $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}]$. For $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}]$, we define $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}] = 0$.

Note that this graded Lie algebra does not satisfy the generating condition $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}] = \mathfrak{g}_{-4}$.

Remark 5.8 Note that the above graded Lie algebra at $w \in \Sigma_2$ is an example of the associated Lie algebra of filtered manifold $(\Sigma(J^2), F)$ by setting;

$$F^{-4}(w) = T_w(J^2), \quad F^{-3}(w) = \partial^{(2)}D(w),$$
$$F^{-2}(w) = \partial^{(1)}D(w), \quad F^{-1}(w) = D(w).$$

Lemma 5.9 *For $w_0 \in \Sigma_2$, there exists $w \in U_{rs}$ such that w is locally isomorphic to w_0.*

Proof. Note that Σ_2 is coverd by $U_{rs} \cup U_{rt} \cup U_{st}$. From the symmetry x and y, U_{st} is isomorphic to U_{rs}. So it is enough to consider the points in $U_{rt} \backslash (U_{rs} \cup U_{st})$. $U_{rt} \backslash (U_{rs} \cup U_{st})$ is a set consisting of a point w_0. w_0 is the origin in the coordinate U_{rt}, i.e., w_0 is the integral element given by;

$$w_0 = \{dx = dy = dz = dp = dq = ds = 0\} = \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial t} \right\rangle.$$

Then we consider the isomorphism $\varphi : (J^2, C^2) \to (J^2, C^2)$;

$$\begin{aligned} \varphi : (x, y, z, p, q, r, s, t) &\mapsto (\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t}) \\ &= (x - y, y, z, p, q + p, r, s + r, t + 2s + r). \end{aligned}$$

This isomorphism φ sends the integral element w_0 to $\bar{w}_0$ where the $\bar{w}_0$ is expressed by

$$\bar{w}_0 = \{d\bar{x} = d\bar{y} = d\bar{z} = d\bar{p} = d\bar{q} = d(\bar{s} - \bar{r}) = 0\} = \left\langle \frac{\partial}{\partial \bar{t}}, \frac{\partial}{\partial \bar{r}} + \frac{\partial}{\partial \bar{s}} \right\rangle,$$

in the new coordinate system. Thus $\bar{w}_0 \in U_{\bar{r}\bar{s}}$ in this new coordinate system. $\square$

From above lemma, it is enough to classify the points in U_{rs}.

Proposition 5.10 *For $w \in \Sigma_2$, graded Lie algebra $\mathfrak{m}(w)$ is isomorphic to $\mathfrak{m}(E, F)$, $\mathfrak{m}(E, F) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}$ and $[,]$ is given by;*

$$[X_B, X_r] = X_y - FX_x,\ [X_D, X_s] = X_y,\ [X_B, X_s] = X_x,\ [X_D, X_r] = EX_x$$
$$[X_E, X_r] = X_t,\ [X_F, X_s] = X_t$$
$$[X_r, X_x] = X_p,\ [X_s, X_y] = X_p + FX_q,\ [X_s, X_x] = X_q,\ [X_r, X_y] = EX_q$$

the other is trivial,

where $\{X_z, X_p, X_q, X_x, X_y, X_t, X_r, X_s, X_B, X_D, X_E, X_F\}$ are basis which satisfy

$$\begin{aligned} \mathfrak{g}_{-1} &= \langle \{X_r, X_s, X_B, X_D, X_E, X_F\} \rangle \\ \mathfrak{g}_{-2} &= \langle \{X_x, X_y, X_t\} \rangle \\ \mathfrak{g}_{-3} &= \langle \{X_p, X_q\} \rangle \\ \mathfrak{g}_{-4} &= \langle \{X_z\} \rangle, \end{aligned}$$

and $E, F \in \mathbb{R}$ are parameters.

Proof. We may assume $w \in U_{rs}$ by the above lemma. From the proof of Theorem 3.3, in U_{rs}, D is expressed by $D = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = \varpi_t = 0\}$ where $(x, y, z, p, q, r, s, t, B, D, E, F)$ is the coordinate and

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE - BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

Recall that $w \in \Sigma_2$ if and only if $B = D = 0$ in this coordinate. Let $\{X_z, X_p, X_q, X_x, X_y, X_t, X_r, X_s, X_B, X_D, X_E, X_F\}$ be the dual frame of the coframe $\{\varpi^0, \varpi^1, \varpi^2, \varpi_x, \varpi_y, \varpi_t, dr, ds, dB, dD, dE, dF\}$. From the proof of Proposition 5.2, the structure equations are;

$$\begin{cases} d\varpi_i \equiv 0 & (i = 0, 1, 2) \\ d\varpi_x = -(EdD - FdB) \wedge dr - dB \wedge ds \not\equiv 0 & \\ d\varpi_y = -dB \wedge dr - dD \wedge ds \not\equiv 0 & \\ d\varpi_t = -dE \wedge dr - dF \wedge ds \not\equiv 0 & (\mathrm{mod}\ D). \end{cases}$$

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv -dr \wedge \varpi_x - ds \wedge \varpi_y \not\equiv 0 \\ d\varpi_2 \equiv -ds \wedge \varpi_x - (Edr + Fds) \wedge \varpi_y \not\equiv 0 \\ \qquad (\mathrm{mod}\ \partial D, \varpi_\alpha \wedge \varpi_\beta\ (\alpha, \beta = x, y, t)). \end{cases}$$

$$\begin{cases} d\varpi_0 \equiv -\varpi_1 \wedge \varpi_x - \varpi_2 \wedge \varpi_y \not\equiv 0 \quad (\mathrm{mod}\ \partial^2 D) \\ d\varpi_0 \equiv 0 \quad (\mathrm{mod}\ \partial^2 D, \varpi_\alpha \wedge \varpi_\beta\ (\alpha, \beta \in \{x, y, t, 1, 2\})). \end{cases}$$

Thus we obtain the result from the argument of the proof of the Proposition 5.5. □

For the graded Lie algebra $\mathfrak{m}(E, F)$, the followings are intrinsic;

$$\mathfrak{g}^V_{-1} = \{X \in \mathfrak{g}_{-1} \mid ad(X)|_{\mathfrak{g}_{-2}} = 0\} = \langle X_B, X_D, X_E, X_F \rangle$$

$$\mathfrak{g}^V_{-2} = \{X \in \mathfrak{g}_{-2} \mid ad(X)|_{\mathfrak{g}_{-1}} = 0\} = \langle X_t \rangle$$

$$\tilde{\mathfrak{g}}_{-1} = \{X \in \mathfrak{g}_{-1} \mid \operatorname{Im} ad(X)|_{\mathfrak{g}_{-1}} \in \mathfrak{g}^V_{-2}\} = \langle X_E, X_F \rangle,$$

i.e. the above subalgebras are preserved by Lie algebra isomorphisms induced by isomorphisms of differential systems.

Lemma 5.11 *For the graded Lie algebra $\mathfrak{m}(E,F)$, let $Ch(\mathfrak{m}(E,F))$ be a set of the characteristic directions, that is,*

$$Ch(\mathfrak{m}(E,F)) = \big\{V \subset \mathfrak{g}_{-1} : 1\text{-dimensional subspace} \mid X \in V, X \neq 0, \operatorname{rank} ad(X)|_{\mathfrak{g}_{-2}} = 1\big\}.$$

Then

$$\#Ch(\mathfrak{m}(E,F)) = \begin{cases} 2 & (F^2 + 4E > 0) \\ 1 & (F^2 + 4E = 0) \\ 0 & (F^2 + 4E < 0). \end{cases}$$

Remark 5.12 For $w \in \Sigma_2$, w is said to be *hyperbolic, elliptic* or *parabolic* according to whether $F^2 + 4E$ is positive, negative or zero, respectively.

Proof. For $X \in \mathfrak{g}_{-1}$,

$$X = \xi X_r + \eta X_s + X^V \quad (\xi, \eta \in \mathbb{R},\ X^V \in \mathfrak{g}^V_{-1})$$

Then

$$\begin{cases} ad(X)(X_x) = \xi X_p + \eta X_q \\ ad(X)(X_y) = \xi(EX_q) + \eta(X_p + FX_q) = \eta X_p + (\xi E + \eta F) X_q \\ ad(X)(X_t) = 0. \end{cases}$$

Hence X is a characteristic direction if and only if X is a null direction for the quadratic form

$$\xi(\xi E + \eta F) - \eta^2 = E\xi^2 + F\xi\eta - \eta^2.$$

Therefore the determinant of this quadratic form classifies the number of the characteristic directions. □

From above lemma, Σ_2 has at least 3 components. We put

$$\Sigma_h = \{w \in \Sigma_2 \mid w \text{ is a hyperbolic point}\}$$
$$\Sigma_e = \{w \in \Sigma_2 \mid w \text{ is a elliptic point}\}$$
$$\Sigma_p = \{w \in \Sigma_2 \mid w \text{ is a parabolic point}\}.$$

Then this classification is sufficient by the following theorem.

Theorem 5.13 *There exists a decomposition of* Σ_2

$$\Sigma_2 = \Sigma_h \cup \Sigma_e \cup \Sigma_p$$

into disjoint three subsets such that, if $w \in \Sigma_h$*, then* $(\Sigma(J^2(M^{1+2}, 2), D)$ *around* w *is locally isomorphic to the germ of* $(\mathbb{R}^{12}, \bar{D})$ *at* $(0, \ldots, 0, 1, 0)$*, if* $w \in \Sigma_e$*, then,* $(0, \ldots, 0, -1, 0)$*, and if* $w \in \Sigma_p$*, then,* $(0, \ldots, 0, 0, 0)$*. Here* $(\mathbb{R}^{12}, \bar{D})$ *is given as in the introduction.*

Proof. First, we introduce the isomorphisms $\varphi_a (a \in \mathbb{R}) : U_{rs} \to U_{rs}$ and $\psi : U_{rs} \to U_{rs}$. these isomorphisms will preserve the determinant of the quadratic form;

$$F^2 + 4E = \bar{F}^2 + 4\bar{E}.$$

For nonzero $a \in \mathbb{R}$, we define φ_a by;

$$\begin{aligned} \varphi_a : (x, y, z, p, q, r, s, t, B, D, E, F) &\mapsto (\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t}, \bar{B}, \bar{D}, \bar{E}, \bar{F}) \\ &= \left(\frac{x}{a^2}, \frac{y}{a}, \frac{z}{a^4}, \frac{p}{a^2}, \frac{q}{a^3}, r, \frac{s}{a}, \frac{t}{a^2}, \frac{B}{a}, D, \frac{E}{a^2}, \frac{F}{a} \right). \end{aligned}$$

ψ is defined by;

$$\begin{aligned} \psi : (x, y, z, p, q, r, s, t, B, D, E, F) &\mapsto (\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t}, \bar{B}, \bar{D}, \bar{E}, \bar{F}) \\ &= \Big(x - \frac{y}{2}, y, z, p, q + \frac{p}{2}, r, s + \frac{r}{2}, t + \frac{r}{4} + s, \\ &\qquad B - \frac{D}{2}, D, E - \frac{F}{2} - \frac{1}{4}, F + 1 \Big). \end{aligned}$$

Then

$$\begin{cases} \psi^*\bar{\varpi}_0 = \varpi_0 & \psi^*\bar{\varpi}_x = \varpi_x - \frac{1}{2}\varpi_y \\ \psi^*\bar{\varpi}_1 = \varpi_1 & \psi^*\bar{\varpi}_y = \varpi_y \\ \psi^*\bar{\varpi}_2 = \varpi_2 + \frac{1}{2}\varpi_1 & \psi^*\bar{\varpi}_t = \varpi_t. \end{cases}$$

(1) For $w \in \Sigma_h$, we may assume $w = (0,0,0,0,0,0,0,0,0,0,E_0,F_0) \in U_{rs}$. Then $F_0^2 + 4E_0 > 0$. If $F_0 \neq 0$, φ_{-F_0} sends w to $w' = \left(0,\dots,0,\frac{E_0}{F_0^2},-1\right)$. The isomorphism ψ sends w' to $w'' = \left(0,\dots,0,\frac{E_0}{F_0^2}+\frac{1}{2}-\frac{1}{4},0\right)$. Furthermore $\varphi_{\sqrt{E_0'}}$ sends w'' to $(0,\dots,0,1,0)$, where $E_0' = \frac{E_0}{F_0^2}+\frac{1}{2}-\frac{1}{4}$.

If $F_0 = 0$, $\varphi_{\sqrt{E_0}}$ sends w to $(0,\dots,0,1,0)$.

(2) For $w \in \Sigma_e$, we may assume $w = (0,0,0,0,0,0,0,0,0,0,E_0,F_0) \in U_{rs}$. Then $F_0^2 + 4E_0 < 0$. If $F_0 \neq 0$, φ_{-F_0} sends w to $w' = \left(0,\dots,0,\frac{E_0}{F_0^2},-1\right)$. The isomorphism ψ sends w' to $w'' = \left(0,\dots,0,\frac{E_0}{F_0^2}+\frac{1}{2}-\frac{1}{4},0\right)$. Furthermore $\varphi_{\sqrt{-E_0'}}$ sends w'' to $(0,\dots,0,-1,0)$, where $E_0' = \frac{E_0}{F_0^2}+\frac{1}{2}-\frac{1}{4}$.

If $F_0 = 0$, $\varphi_{\sqrt{-E_0}}$ sends w to $(0,\dots,0,-1,0)$.

(3) For $w \in \Sigma_p$, we may assume $w = (0,0,0,0,0,0,0,0,0,0,E_0,F_0) \in U_{rs}$. Then $F_0^2 + 4E_0 = 0$. If $F_0 = 0$, then $E_0 = 0$.

If $F_0 \neq 0$, φ_{-F_0} sends w to $w' = \left(0,\dots,0,-\frac{1}{4},-1\right)$. The isomorphism ψ sends w' to $(0,\dots,0,0,0)$. □

Remark 5.14 Note that the normal forms of the graded Lie algebras are obtained by the above local normal forms. Namely, for $w \in \Sigma_2$, $\mathfrak{m}(w)$ is isomorphic to $\mathfrak{m}(1,0), \mathfrak{m}(-1,0), \mathfrak{m}(0,0)$ (given as in Proposition 5.10) according to whether $w \in \Sigma_h$, $w \in \Sigma_e$ or $w \in \Sigma_p$, respectively.

We summarize

Corollary 5.15

$$\Sigma(J^2) = \Sigma_0 \cup \Sigma_1 \cup (\Sigma_h \cup \Sigma_e \cup \Sigma_p)$$

where

$$\Sigma_0 = \{w \in \Sigma(J^2) \mid \dim(w \cap fiber) = 0\} = J^3$$

$$\Sigma_1 = \{w \in \Sigma(J^2) \mid \dim(w \cap fiber) = 1\}$$

$$\Sigma_2 = \{w \in \Sigma(J^2) \mid \dim(w \cap fiber) = 2\}$$
$$\Sigma_2 = \Sigma_h \cup \Sigma_p \cup \Sigma_e$$
$$\Sigma_h = \Sigma_2 \cap \{w : \textit{hyperbolic point}\}$$
$$\Sigma_e = \Sigma_2 \cap \{w : \textit{elliptic point}\}$$
$$\Sigma_p = \Sigma_2 \cap \{w : \textit{parabolic point}\}$$

Σ_0 *is an open set in* $\Sigma(J^2)$. Σ_1 *is an codimemsion* 1 *submanifold in* $\Sigma(J^2)$ Σ_2 *is an codimemsion* 2 *submanifold in* $\Sigma(J^2)$ *and* P^2*-bundle over* J^2. Σ_h, Σ_e *are also codimemsion* 2 *submanifolds in* $\Sigma(J^2)$. Σ_p *is an codimemsion* 3 *submanifold in* $\Sigma(J^2)$.

Moreover, the each component have the following normal forms;

(0) Σ_0 *has jet type normal form.*

(1) $w \in \Sigma_1$ *is locally isomorphic to a germ at the origin in* $(\mathbb{R}^{12}, \hat{D})$ *where* $(\mathbb{R}^{12}; x, y, z, p, q, r, s, t, a, B, c, e)$ *is coordinate and* $\hat{D}$ *is expressed by* $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = \varpi_s = 0\}$, *where*

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt. \end{cases}$$

(2) $w \in \Sigma_h$ *is locally isomorphic to a germ at* $(0, \dots, 0, 1, 0)$ *in* $(\mathbb{R}^{12}, \bar{D})$.
$w \in \Sigma_e$ *is locally isomorphic to a germ at* $(0, \dots, 0, -1, 0)$ *in* $(\mathbb{R}^{12}, \bar{D})$,
$w \in \Sigma_p$ *is locally isomorphic to a germ at* $(0, \dots, 0, 0, 0)$ *in* $(\mathbb{R}^{12}, \bar{D})$.

where $(\mathbb{R}^{12}; x, y, z, p, q, r, s, t, B, D, E, F)$ *is coordinate and* $\bar{D}$ *is expressed by* $\bar{D} = \{\varpi^0 = \varpi^1 = \varpi^2 = \varpi_x = \varpi_y = \varpi_t = 0\}$ *where*

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE - BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

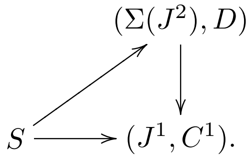
Appendix

A description of integral manifolds of $(\Sigma(J^2), D)$

In this appendix, we will consider integral manifolds of $(\Sigma(J^2), D)$. If S is a 2-dimensional submanifold of $\Sigma(J^2)$ and satisfies $TS \subset D$, then S is called a 2-dimensional integral manifold of $(\Sigma(J^2), D)$. If S is a 2-dimensional integral manifold of $(\Sigma(J^2), D)$ with $\Omega|_S \neq 0$, then S is called an integral manifold of $(\Sigma(J^2), D)$ with independence condition Ω, where Ω is a 2-form on $(\Sigma(J^2), D)$ independent modulo D.

We describe the relation between the integral manifolds of $(\Sigma(J^2), D)$ and singular solutions of partial differential equations of second order.

Here 2-dimensional integral manifold S of $(\Sigma(J^2), D)$ is a *singular solution*, if the projection of S to (J^1, C^1) has singularity,



For $(J^k(M^{m+n}, n), C^k)$, the integral manifolds S with independence condition $dx_1 \wedge \cdots \wedge dx_n$ correspond to the graphs of the k-jet extensions of m functions of n variables. Hence, the integral manifolds with independence condition $dx_1 \wedge \cdots \wedge dx_n$ depend on m functions of n variables.

Example 5.16 Let $(x, y, z, p, q, r, s, t, p_{111}, p_{112}, p_{122}, p_{222})$ be the canonical coordinate on $J^3(M^{1+2}, 2)$. The solution S with independence condition $dx \wedge dy$ is expressed by;

$$S = \big(x, y, z(x,y), z_x(x,y), z_y(x,y), z_{xx}(x,y), z_{xy}(x,y), z_{yy}(x,y),$$
$$z_{xxx}(x,y), z_{xxy}(x,y), z_{xyy}(x,y), z_{yyy}(x,y)\big).$$

Therefore the integral manifolds depend on 1 function of 2 variables $z(x, y)$.

Now, we consider the integral manifolds of $(\Sigma(J^2), D)$ passing through $\Sigma_1 = \{\dim(w \cap \text{fiber}) = 1\}$ with some independence condition. Note that the integral manifolds $S \subset \{\dim(w \cap \text{fiber}) = 0\}$ are treated in the above example.

Proposition 5.17 *Let $(x, y, z, p, q, r, s, t, a, B, c, e)$ be the canonical coordinate around Σ_1 on $(\Sigma(J^2), D)$ (Theorem 5.7).*

If S is an integral manifold of $(\Sigma(J^2), D)$ passing through Σ_1 with independence condition $dx \wedge dt$, then S is written by;

$$S = \left(x, y(x,t), \int q y_t dt + z_0(x), z_x - q y_x, \int t y_t dt + q_0(x), \right.$$
$$\left. p_x - s y_x, q_x - t y_x, t, y_x, y_t, r_x, s_x \right). \quad (1)$$

In other wards, the integral manifolds depend on 1 function of 2 variables $y(x,t)$ such that $y_t(0,0)$ and 2 functions of 1 variable $q_0(x), z_0(x)$.

Conversely, for any $y(x,t)$ with $y_t(0,0) = 0$ and $q_0(x), z_0(x)$. We can construct the integral manifold by (1).

Proof. Let S be an integral manifold, then $S = (x, y(x,t), z(x,t), \ldots, t, \ldots, e(x,t))$ from independence condition. Moreover S satisfies $S^* \varpi_i = 0$ $(i = 0, 1, 2, y, r, s)$;

$$S^* \varpi_0 = (z_x - q y_x - p) dx + (z_t - q y_t) dt = 0 \quad (2)$$

$$S^* \varpi_1 = (p_x - r - s y_x) dx + (p_t - s y_t) dt = 0 \quad (3)$$

$$S^* \varpi_2 = (q_x - s - t y_x) dx + (q_t - t y_t) dt = 0 \quad (4)$$

$$S^* \varpi_y = (y_x - a) dx + (y_t - B) dt = 0 \quad (5)$$

$$S^* \varpi_r = (r_x - c) dx + (r_t - (a^2 + eB)) dt = 0 \quad (6)$$

$$S^* \varpi_s = (s_x - e) dx + (s_t + a) dt = 0. \quad (7)$$

$a = y_x$ and $B = y_t$ is determined by (5), and note that the condition passing through Σ_1 is $B = y_t = 0$. From (4), $q = \int t y_t dt + q_0(x)$ where $q_0(x)$ is a function on S depending only on x, and $s = q_x - t y_x$. From (7), $e = s_x$. From (2), $z = \int q y_t dt + z_0(x)$ where $z_0(x)$ is a function on S depending only on x, and $p = z_x - q y_x$. From (3), $r = p_x - s y_x$. From (6) $c = r_x$. Therefore

$$S = \left(x, y(x,t), \int q y_t dt + z_0(x), z_x - q y_x, \int t y_t dt + q_0(x), \right.$$
$$\left. p_x - s y_x, q_x - t y_x, t, y_x, y_t, r_x, s_x \right). \qquad (8)$$

Conversely, for any $y(x,t)$ with $y_t(0,0) = 0$ and $q_0(x), z_0(x)$. We define the 2-dim submanifold S by (1), then $y_t(0,0) = 0$ ensure that passing through Σ_1 and the rest 3 conditions in $(2), \ldots, (7)$ are satisfied by definition, automatically. □

Corollary 5.18 *The projection of the integral manifolds passing through Σ_1 with independence condition $dx \wedge dt$ have singularities at the origin.*

Corollary 5.19 *Let S be an integral manifolds with independence condition $dx \wedge dt$. Assume $S \subset \Sigma_1$, then the projection of S is a regular curve.*

Proof. The condition is $B = y_t \equiv 0$. Hence $y(x,t) = y(x)$ depends only on x. From the above theorem, we have

$$\begin{aligned}
x &= x \\
y &= y(x) \\
z &= z_0(x) \\
p &= z_0' - q_0' y' \\
q &= q_0(x) \\
r &= (q_0'' + 3)t^3 - 3q_0'' xt + z_0'' \\
s &= q_0' - ty' \\
t &= t \\
a &= y' \\
B &= 0 \\
c &= z_0''' - (q_0''' y' + q_0'' y'') - (q_0'' y'' + q_0' y''') - (q_0'' - ty'')y' - (q_0' - ty')y'' \\
e &= q_0'' - ty''.
\end{aligned}$$
□

Example 5.20 (cuspidal edge) Let $y(x,t) = t^3 - 3xt, z_0(x)$ and $q_0(x)$. Then the integral manifold $S(x,t)$ is

$$
\begin{aligned}
x &= x\\
y &= t^3 - 3xt\\
z &= \frac{9}{28}t^7 - \frac{27}{20}xt^5 + \left(q_0 + \frac{3}{2}x^2\right)t^3 - 3q_0xt + z_0\\
p &= \frac{9}{10}t^5 + \left(q_0' - \frac{3}{2}x\right)t^3 - 3q_0'xt + z_0'\\
q &= \frac{3}{4}t^4 - \frac{3}{2}xt^2 + q_0\\
r &= (q_0'' + 3)t^3 - 3q_0''xt + z_0''\\
s &= \frac{3}{2}t^2 + q_0'\\
t &= t\\
a &= -3t\\
B &= 3t^2 - 3x\\
c &= q_0'''t^3 - 3(q_0'' + xq_0''')t + z_0'''\\
e &= q_0''
\end{aligned}
$$

from direct calculation.

Example 5.21 (Cartan's overdeteremined system) We consider the Cartan's overdeteremined system

$$r = \frac{1}{3}t^3, \quad s = \frac{1}{2}t^2.$$

The Lie algebra of infinitesimal contact transformations of the system is isomorphic to the 14-dim exceptional simple Lie algebra G_2.

Let $y(x,t) = -xt$, $z_0(x) = 0$ and $q_0(x) = 0$. Then the integral manifold $S(x,t)$ is

$$
\begin{aligned}
x &= x & s &= \frac{1}{2}t^2\\
y &= -xt & t &= t
\end{aligned}
$$

$$z = \frac{1}{6}x^2t^3 \qquad a = -t$$

$$p = -\frac{1}{6}xt^3 \qquad B = -x$$

$$q = -\frac{1}{2}xt^2 \qquad c = 0$$

$$r = \frac{1}{3}t^3 \qquad e = 0.$$

Therefore the projection of the integral manifold $S(x,t)$ is a singular solution of the Cartan's overdeteremined system, where the projection is $\Sigma(J^2) \to J^2$.

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