*Hokkaido Mathematical Journal Vol. 37 (2008) p. 215–240*

# A generalized local limit theorem for mixing semi-flows

Yukiko Iwata

(Received April 4, 2007)

**Abstract.** Let $S_t\colon M \to M$ be a semi-flow preserving a probability measure $\mu$ on a compact manifold $M$ and $g\colon M \to \mathbb{R}$ a real-valued Borel measurable function. We show a generalized local limit theorem for the stationary process $\{g\circ S_t; t \geq 0\}$ on the probability space $(M, \mu)$ under certain conditions.

*Key words*: mixing semiflows, local limit theorem, semigroup of transfer operators.

## 1. Introduction

We consider a continuous dynamical system $S_t\colon M \to M$, $t \geq 0$, preserving a probability measure $\mu$ on a compact manifold $M$ and a real-valued Borel measurable function $g\colon M \to \mathbb{R}$. Then the family $\{g\circ S_t; t \geq 0\}$ gives a stationary process with $(M, \mu)$ as the underlying probability space. An important problem in ergodic theory is whether this process satisfies the limit theorems such as the strong law of large numbers, the central limit theorem and the law of iterated logarithm. In this paper, we study the local limit theorem for such a stationary process to measure the dependence between the values of $g$ at time 0 and time $t$ (see [B] for the local limit theorem for independent random variables in the case of discrete time.).

We say that the stationary process $\{g \circ S_t; t \geq 0\}$ with $\int_M g d\mu = 0$ satisfies the *local limit theorem* if

$$\lim_{t\to\infty} \sup_{z\in\mathbb{R}} \Big| \sqrt{t\sigma} \int_M u\Big(z + \int_0^t g\circ S_\tau d\tau\Big) d\mu - \frac{1}{\sqrt{2\pi}} e^{-z^2/2t\sigma} \int_{-\infty}^{\infty} u(\theta) d\theta \Big| = 0 \quad (1.1)$$

for all rapidly decreasing functions $u$, where we assume the limit

$$\sigma := \lim_{t\to\infty} \frac{1}{t} \int_M \Big( \int_0^t g\circ S_\tau d\tau\Big)^2 d\mu$$

exists and is not zero. This limit theorem implies that, for any finite interval

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*2000 Mathematics Subject Classification* : 37A10.

$I$, we have

$$\Big|\sqrt{t\sigma}\mu\Big(\int_0^t g\circ S_\tau d\tau \in I+z\Big) - m(I)\frac{1}{\sqrt{2\pi}}e^{-z^2/2t\sigma}\Big| \to 0 \quad \text{as } t\to\infty$$

uniformly for $z\in\mathbb{R}$, where $m$ is the Lebesgue measure on $\mathbb{R}$.

The local limit theorem are known to hold for some classes of discrete dynamical systems with appropriate choices of the measure $\mu$ and the function $g$. Rousseau-Egele [R-E] showed the local limit theorem (1.1) for Lasota-Yorke maps on the interval with an absolutely continuous invariant measure $\mu$ and a function of bounded variation $g$. Guivarc'h and Hardy [GH] gave the same result for Anosov diffeomorphisms with a Gibbs measure $\mu$ and a Hölder function $g$, by generalizing the argument of Rousseau-Egele, (acutually they showed the local limit theorem for mixing subshifts of finite type with a Gibbs measure $\mu$ and a Lipschitz function $g$). In these results, the function $g$ is assumed to satisfy a condition called aperiodic[1]. In [M], Morita generalized the result of Rousseau-Egele to the case where $g$ is not aperiodic (*i.e.* periodic). Afterwards, Aaronson and Denker [AD] proved the same result for more general class of discrete dynamical systems, called Gibbs-Markov maps. On the other hand, Gouëzel [G] showed the local limit theorem for non uniformly expanding maps by using methods for developed by Young, Aaronson and Denker.

However, for continuous dynamical systems, there are no results known for the local limit theorem at present. In fact, the local limit theorem is not known even for mixing Anosov flows, though some results on the central limit theorem are known ([HM], [R]). In this paper, we show the (generalized) local limit theorem for continuous dynamical systems under certain conditions.

We will show the generalized local limit theorem by considering spectral properties of the semigroups of transfer operators for a continuous dynamical system and their perturbations. This strategy is basically similar to the case of discrete dynamical systems in [R-E] and [M]. In order to modify these methods, we analyze the spectral property of continuous semigroups and their perturbations.

Our results are applied to discrete dynamical systems. For example, we can apply our methods to Anosov diffeomorphisms, not using Markov

[1] In Section 2, we give the definition of the aperiodic function.

partitions but depending on the result of [GL] (cf. [BT]).

As an application of the result in this paper, we show the local limit theorem for a certain class of susupension flows [T]. Unfortunately we do not know whether our conditions hold for Anosov flows, though we expect that this is the case.

I am grateful for the support of Professor M. Tsujii in my writing this paper. I would like to thank Professor T. Morita for introducing me to this topic and giving me some important comments.

## 2. Results

Let $C^0(M)$ be the space of complex-valued continuous functions on a compact topological space $M$, $C^0_R(M)$ the subspace of real-valued continuous functions on $M$, and $C^0_+(M)$ the space of non-negative valued continuous functions on $M$, equipped with the $C^0$ topology. We first take a Banach space $(B, \|\cdot\|_B)$ embedded in $C^0(M)$ and satisfying the conditions

(B1) $B$ contains the constant function $\mathbf{1}$.

(B2) $B \cap C^0_R(M)$ is dense in $C^0_R(M)$.

(B3) $B$ is a Banach algebra. i.e. $\|\psi\cdot\varphi\|_B \le \|\psi\|_B\|\varphi\|_B$ for all $\psi, \varphi \in B$.

Let $B^*$ be the dual space of $B$ equipped with the norm[2]

$$\|\nu\|_{B^*} = \sup_{\|\varphi\|_B \le 1} |\langle \nu, \varphi\rangle|.$$

By the condition (B3), we have the bounded operator $M_g\colon B^* \to B^*$ defined, for each $g \in B$, by

$$\langle M_g\nu, \varphi\rangle = \langle \nu, g\cdot\varphi\rangle.$$

Next we take a Banach space $(V, \|\cdot\|_V)$ that is embedded and dense in $B^*$. We assume that it satisfies the condition

(B4) The linear space $\langle V_+\rangle_{\mathbb{C}}$ spanned by the closed subset

$$V_+ := \{\nu \in V \mid \langle \nu, \varphi\rangle \ge 0 \quad \text{for all } \varphi \in B \cap C^0_+(M)\}$$

is dense in $V$.

**Remark 1** The conditions (B1)–(B4) hold, for instance, if $M$ is a closed manifold, $B$ the space of $C^r$ functions on $M$, $V$ a Banach space that is

[2] Let $x \in B$. It will be convenient to write $\langle x^*, x\rangle$ in place of $x^*(x)$ for every element $x^*$ of dual space $B^*$ of $B$.

embedded in the space of distributions of order $r$ and contains $C^\infty(M)$ as its dense subset.

**Remark 2** Each element $f$ of $V_+$ extends uniquely to a positive-valued continuous linear functional on $C^0(M)$, and thus it can be identified with a finite Borel measure on $M$ by Riesz's theorem [H, Theorem 2.1.7]. In the case where an element $f$ of $V$ extends (uniquely) to a finite signed measure on $M$, we will identify $f$ with that measure.

We consider a continuous semiflow $S_t\colon M \to M$, $t \geq 0$. This acts on $C^0(M)$ naturally and give the semigourp of operators

$$U_t\colon C^0(M) \to C^0(M), \quad \varphi \mapsto \varphi \circ S_t.$$

We assume that the Banach space $B$ is invariant with respect to this semigroup, that is, $U_t(B) \subset B$ for $t \geq 0$, and that $U_t\colon B \to B$ is bounded.

For each $g \in B \cap C^0_R(M)$, we introduce the one-parameter semigroup of operators

$$U_t(g)\colon C^0(M) \to C^0(M), \quad \varphi \mapsto \exp\Big(i\int_0^t g \circ S_s(x)ds\Big)\varphi \circ S_t(x),$$

which plays the central roll in our argument. Obviously we have $U_t(0) = U_t$. By the condition (B3), $U_t(g)$ is a bounded operator from $B$ to $B$. Let $P_t(g)\colon B^* \to B^*$ be the dual operator of $U_t(g)\colon B \to B$.

We fix $g \in B \cap C^0_R(M)$ and put the following assumptions on $P_t(\theta g)\colon B^* \to B^*$ for $\theta \in \mathbb{R}$: There exists $t_0 > 0$ such that, for all $\theta \in \mathbb{R}$,

(P1) The operator $M_h \circ P_t(\theta g)\colon B^* \to B^*$ is restricted to a bounded operator on $V$ for any $t \geq t_0$ and $h \in B$.
(P2) The essential spectral radius of the operator $P_{t_0}(\theta g)\colon V \to V$ is less than one.
(P3) $P_{t_0}(\theta g)\colon V \to V$ is of $C^3$ class with respect to $\theta \in \mathbb{R}$.
(P4) We have $\lim_{\varepsilon \to 0} \|P_{t+\varepsilon}(\theta g)\nu - P_t(\theta g)\nu\|_{V,V} = 0$ for all $\nu \in \langle V_+\rangle_{\mathbb{C}}$ and $t \geq t_0$. Moreover $\sup_{t_0 \leq t \leq 2t_0} \|P_t(\theta g)\|_{V,V} < \infty$.[3]

Under the assumptions (P1)–(P4), the following hold for any $t \geq t_0$ and $\theta \in \mathbb{R}$:

[3] We denote the operator norm of a linear operator from Banach space $(V, \|\cdot\|_V)$ to itself with respect to the norm $\|\cdot\|_V$ by $\|\cdot\|_{V,V}$

**Lemma 1** *The essential spectral radius of the operator $P_t(\theta g)\colon V \to V$ is less than one.*

**Lemma 2** *The spectral radius of the operator $P_t(\theta g)\colon V \to V$ is not more than one. If there exists an eigenvalue of $P_t(\theta g)\colon V \to V$ with modulus one, then the order of that eigenvalue is one.*

**Lemma 3** *$P_t(0)\colon V \to V$ has 1 as its eigenvalue and corresponding the eigenspace is spanned by finitely many $S_t$-invariant probability measures on $M$.* (*Recall* Remark 2.)

**Lemma 4** *If $P_t(\theta g)\colon V \to V$ has an eigenvalue $e^{i\eta}$ ($\eta \in \mathbb{R}$) on the unit circle, then all the corresponding eigenvectors are Borel finite signed measures, and absolutely continuous with respect to some probability measures that are eigenvectors for the eigenvalue 1 of $P_t(0)\colon V \to V$.*

We will prove Lemma 1–4 above (and Lemma 5 and 6 in the following) in Section 3. Finally we put the assumption

(P5) The eigenvalue 1 of $P_t(0)\colon V \to V$ is simple. The eigenvetor $\mu_0$ for the eigenvalue 1, which is identified with an $S_t$-invariant probability measure, is mixing with respect to $S_t$.

Under this additional assumption, we have

**Lemma 5** (1) *A complex number $e^{i\lambda t}$, $\lambda \in \mathbb{R}$ with modulus one, is an eigenvalue of $P_t(\theta g)\colon V \to V$ if and only if there exists a Borel measurable function $h\colon M \to \mathbb{C}$ with $|h(x)| \equiv 1$ $\mu_0$-a.e. $x$ such that*

$$h(S_t(x)) = \exp\Big(-i\lambda t + i\int_0^t \theta \cdot g \circ S_s(x)ds\Big)h(x) \quad \mu_0\text{-a.e. } x. \tag{2.1}$$

*The Borel measurable function $h$ above is unique up to multiplication by constants with modulus 1.*

(2) *$P_t(\theta g)\colon V \to V$ has at most one eigenvalue on the unit circle up to multiplicity.*

(3) *If there exists $s \geq t_0$ such that $P_s(\theta g)\colon V \to V$ has $e^{i\lambda s}$, $\lambda \in \mathbb{R}$, as its eigenvalue then, for any $t \geq t_0$, $P_t(\theta g)\colon V \to V$ has $e^{i\lambda t}$ as its eigenvalue.*

Put $\mathcal{U}_g := \{\theta \in \mathbb{R}\colon P_t(\theta g)$ has an eigenvalue with modulus one$\}$.[4]

**Lemma 6** *$\mathcal{U}_g$ is a closed additive subgroup of $\mathbb{R}$.*

[4]If $\theta \in \mathcal{U}_g$ then $P_t(\theta g)$ has an eigenvalue with modulus one for all $t \geq t_0$ by Lemma 5 (3).

We assume the following conditions on $g \in B \cap C_R^0(M)$.
(G1) $\int_M g d\mu_0 = 0$.
(G2) $\sigma_g^2 := \lim_{t\to\infty} \int_M (1/t)\big(\int_0^t g \circ S_\tau d\tau\big)^2 d\mu_0 > 0$.

**Remark 3** The existence of the limit $\sigma_g$ is proved in the proof of Lemma 11 in Section 4.

As we will see in Section 4, the condition (G1) and (G2) implies that $\mathcal{U}_g \neq \mathbb{R}$ and hence we have either $\mathcal{U}_g = a\mathbb{Z}$ for some $a > 0$ or $\mathcal{U}_g = \{0\}$. The function $g$ is called *periodic* in the former case, and *aperiodic* in the latter case.

It is not difficult to show the following central limit theorem under the assumptions we put above.

**Theorem 7** *For each $\beta \in \mathbb{R}$, we have*

$$\lim_{t\to\infty} \mu_0\Big\{x \in M; \frac{1}{\sigma_g\sqrt{t}} \int_0^t g \circ S_\tau(x) d\tau < \beta\Big\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-(1/2)u^2} du.$$

Our main result is

**Theorem 8** *If $g$ is aperiodic, then, for any rapidly decreasing function $u$ on $\mathbb{R}$, we have*

$$\lim_{t\to\infty} \sup_{z\in\mathbb{R}} \Big| \sigma_g\sqrt{t} \int_M u\Big(z + \int_0^t g \circ S_\tau d\tau\Big) d\mu_0 - \frac{e^{-z^2/2\sigma_g^2 t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(\theta) d\theta \Big| = 0.$$

*If $g$ is periodic (i.e. $\mathcal{U}_g = a\mathbb{Z}$ for some $a > 0$) then, for any rapidly decreasing function $u$ on $\mathbb{R}$, we have*

$$\lim_{t\to\infty} \sup_{z\in\mathbb{R}} \Big| \sigma_g\sqrt{t} \int_M u\Big(z + \int_0^t g \circ S_\tau d\tau\Big) d\mu_0 - \sum_{k=-\infty}^{\infty} \hat{u}(ak) e^{iakz} e^{\lambda(a)kt} \Big| \int_M h^k d\mu_0 \Big|^2 \frac{e^{-z^2/2\sigma_g^2 t}}{\sqrt{2\pi}} \Big| = 0,$$

*where $\hat{u}$ is the Fourier transform of $u$, $h$ is a Borel measurable function that satisfies the conditions in* Lemma 5 *for the eigenvalue $e^{\lambda(a)t_0}$ of $P_{t_0}(ag)$ with modulus one.*

As an application of Theorem 8, we introduce an example of a semiflow and two Banach spaces on a compact manifold satisfying the assumptions (B1)–(B3) and (P1)–(P4).

Fix integers $\ell \geq 2$ and $r \geq 3$. Let $T\colon \mathcal{S}^1 \to \mathcal{S}^1$ be the angle-multiplying map on the circle $\mathcal{S}^1 = \mathbb{R}/\mathbb{Z}$ defined by $Tx = \ell x$. Let $f$ be a positive-valued $C^r$ function on $\mathcal{S}^1$. The suspension semiflow $\{S_t\colon X_f \to X_f\}_{t\geq 0}$ of $T$ is defined on the subset

$$X_f := \{(x, y) \in \mathcal{S}^1 \times \mathbb{R}; 0 \leq y < f(x)\}$$

by

$$S_t(x, y) := \big(T^{n(x,y+t;f)}(x),\, y + t - f^{n(x,y+t;f)}(x)\big),$$

where $f^n(x) = \sum_{i=0}^{n-1} f(T^i x)$ and $n(x, t; f) = \max\{n \geq 0; f^n(x) \leq t\}$. Since $T$ is ergodic with respect to the Lebesgue measure $m$ on $\mathcal{S}^1$, the semiflow $S_t$ is also ergodic with respect to the invariant probability measure $d\nu := (\int_{\mathcal{S}^1} f(x)dm(x))^{-1}(dm \times dy)|_{X_f}$, where $dy$ is the Lebesugue measure on $\mathbb{R}$. Let $U_t$ be the operator on $L^2(X_f, \nu)$ defined by $U_t f = f \circ S_t$ for each $t \geq 0$ and $P_t\colon L^2(X_f, \nu) \to L^2(X_f, \nu)$ the dual operator of $U_t$, where we identify $L^2(X_f, \nu)$ with its dual space.

Tsujii showed the following [T, Theorem 1.2, Theorem1.5]: *If $S_t$ is weakly mixing, there exists a Hilbert space $C^1(X_f) \subset W_*(X_f) \subset L^2(X_f, \nu)$ such that the restriction of $P_t$ to $W_*(X_f)$ is the bounded operator whose essential spectral radius is less than* 1*, where $C^1(X_f)$ is the set of functions $\varphi$ on $X_f$ such that $U_t\varphi$ for any $t \geq 0$ is $C^1$ on the interior of $X_f$.*

Setting $B = C^1(X_f)$, $V = W_*(X_f)$ and $\mu_0 = \nu$, we can check immediately the assumptions (B1)–(B3) and (P1)–(P4). See [T] for more details. Therefore the conclusion of Theorem 8 holds for the stationary process $\{g \circ S_t; t \geq 0\}$ for $g \in C^1(X_f)$ satisfying (G1) and (G2). Besides we can show that $\sigma_g = 0$ if and only if there exists $f \in W_*(X_f)$ such that $(d/dt)(f \circ S_t)|_{t=0} = g$. See Appendix.

## 3. Proof of Lemma 1–6

*Proof of* Lemma 1. By the assumption (P2) and the definition of essential spectral radius (see [EN, p. 218]), we can find $n_0 \in \mathbb{N}$ and $0 < a < 1$ such that, for every $n \geq n_0$, there exists a compact operator $K_n\colon V \to V$ which satisfies $\|P_{nt_0}(\theta g) - K_n\|_{V,V} < a^n$. By the assumption (P4), we have

$$\begin{aligned}\|P_t(\theta g) - P_{t-nt_0}(\theta g)K_n\|_{V,V} &\leq \|P_{t-nt_0}(\theta g)\|_{V,V}\|P_{nt_0}(\theta g) - K_n\|_{V,V} \\ &\leq Ca^n\end{aligned}$$

for all $t \geq t_0$ where $n = [t/t_0] - 1$ and $C := \sup_{t_0 \leq t \leq 2t_0} \|P_t(\theta g)\|_{V,V}$. Since $P_{t-nt_0}(\theta g)K_n$ is a compact operator on $V$, this implies that the essential spectral radius of $P_t(\theta g)$ is not more than $a < 1$. □

*Proof of* Lemma 2. It is sufficient to consider the case where the spectral radius $\rho$ of $P_t(\theta g)\colon V \to V$ is not less than one. Let $\pi\colon V \to V$ be the spectral projector corresponding to the set of eigenvalues of $P_t(\theta g)$ with modulus $\rho$. Then we have the $P_t(\theta g)$-invariant decomposition $V = \pi(V) \oplus (I - \pi)(V)$, where the dimension of $\pi(V)$ is finite by Lemma 1. By the condition (B4), we can choose $\mu_i \in V_+$, $1 \leq i \leq k$, such that $v_i := \pi(\mu_i)$, $1 \leq i \leq k$ are bases on $\pi(V)$. Since $\mu_i - v_i \in (I - \pi)(V)$ for $1 \leq i \leq k$, we have

$$\lim_{n\to\infty} \big(\langle \rho^{-n} P_{nt}(\theta g)\mu_i, \varphi\rangle - \langle \rho^{-n} P_{nt}(\theta g)v_i, \varphi\rangle\big) = 0 \quad \forall \varphi \in B.$$

Since we can identify $\mu_i \in V_+$ with a measure, $\langle P_{nt}(\theta g)\mu_i, \varphi\rangle$ is bounded. Therefore, if $\rho > 1$, the first term on the left-hand side should converge to zero, and hence we should have $\lim_{n\to\infty}\langle \rho^{-n} P_{nt}(\theta g)v_i, \varphi\rangle = 0$. But this contradicts the definition of $v_i$. So we have $\rho = 1$. In the case $\rho = 1$, the latter term $\langle P_{nt}(\theta g)v_i, \varphi\rangle$ is bounded uniformly for $n \geq 0$ and $1 \leq i \leq k$. This implies that $P_t(\theta g)|_{\pi(V)}\colon V \to V$ is power bounded. Therefore orders of eigenvalues of $P_t(\theta g)$ with modulus one are one (cf. [K, p. 90]). □

*Proof of* Lemma 3. By the condition (B4), we can find an element $\mu \in V_+$ with $\langle \mu, \mathbf{1}\rangle > 0$. For such $\mu$, we have

$$\Big\langle \frac{1}{n}\sum_{k=0}^{n-1} P_{kt}(0)\mu, \mathbf{1}\Big\rangle = \Big\langle \mu, \frac{1}{n}\sum_{k=0}^{n-1} U_{kt}(0)(\mathbf{1})\Big\rangle = \langle \mu, \mathbf{1}\rangle > 0.$$

If one were not an eigenvalue of $P_t(0)\colon V \to V$, the left-hand side should converge to zero as $n \to \infty$ by Lemma 2. Therefore $P_t(0)\colon V \to V$ has one as its eigenvalue.

Let $\pi\colon V \to V$ be the spectral projector of $P_t(0)$ corresponding to the eigenvalue one. Then we have the $P_t(0)$-invariant decomposition $V = \pi(V) \oplus (I - \pi)(V)$. By the condition (B4), we can choose elements $\mu_i \in V_+$,

$1 \le i \le k$, such that $v_i := \pi(\mu_i)$, $1 \le i \le k$, are bases on $\pi(V)$. Take a subsequence $n_j \to \infty$ so that, for $1 \le i \le k$,

$$\frac{1}{n_j} \sum_{\ell=0}^{n_j-1} P_{\ell t}(0)(\mu_i)$$

converges weakly to a probability measure $\nu_i$. Since $v_i - \mu_i \in (I - \pi)(V)$, we have

$$\begin{aligned}\lim_{j\to\infty} \Big\langle \frac{1}{n_j} \sum_{\ell=0}^{n_j-1} P_{\ell t}(0)(\mu_i),\, \varphi \Big\rangle &= \lim_{j\to\infty} \Big\langle \frac{1}{n_j} \sum_{\ell=0}^{n_j-1} P_{\ell t}(0)(v_i),\, \varphi \Big\rangle \\ &= \langle v_i, \varphi\rangle \quad \forall \varphi \in B.\end{aligned}$$

This implies $\nu_i = v_i$. Hence the eigenspace $\pi(V)$ corresponding to the eigenvalue one is spanned by the probability measures $\nu_i$, $1 \le i \le k$. □

*Proof of* Lemma 4. Let $\pi \colon V \to V$ be the spectral projector corresponding to the eigenvalue $e^{i\eta t}$ of $P_t(\theta g) \colon V \to V$. Then we have the $P_t(\theta g)$-invariant decomposition $V = \pi(V) \oplus (I - \pi)(V)$. By the same argument as in the proof of Lemma 3, we can choose elements $\mu_i \in V_+$, $1 \le i \le k$, such that $v_i := \pi(\mu_i)$, $1 \le i \le k$ are bases on $\pi(V)$, and take some subsequence $n_j \to \infty$, so that, for $1 \le i \le k$, the two sequences of (signed) measures

$$\frac{1}{n_j} \sum_{\ell=0}^{n_j-1} e^{-i\ell\eta t} P_{\ell t}(\theta g)\mu_i, \quad \frac{1}{n_j} \sum_{\ell=0}^{n_j-1} P_{\ell t}(0)\mu_i$$

converge weakly to a finite signed measure $v_i$ and a finite measure $\omega_i \in V^+$ respectively. For all non-negative functions $\varphi \colon M \to \mathbb{R}$, we have $\langle \omega_i, \varphi\rangle \ge |\langle v_i, \varphi\rangle|$. This implies that $v_i$ is absolutely continuous with respect to $\omega_i$. From the proof of Lemma 3, $\omega_i$ is an element of the eigenspace corresponding to the eigenvalue 1 of $P_t(0) \colon V \to V$ □

*Proof of* Lemma 5. (1) Suppose that $P_t(\theta g)$ has an eigenvalue $e^{i\lambda t}$. Let $\nu$ be an element of the eigenspace corresponding to the eigenvalue $e^{i\lambda t}$. Since $\nu$ is absolutely continuous with respect to $\mu_0$ by Lemma 4, there exists a Borel measurable function $h$ such that $\nu = h\mu_0$. First we show that $|h(x)| \equiv 1$ for $\mu_0$-a.e. $x$. By the choice of $\nu$, we have

$$\langle P_t(\theta g)\nu,\, \varphi\rangle = \big\langle e^{i\lambda t}\nu,\, \varphi \big\rangle \quad \text{for all } \varphi \in B. \tag{3.1}$$

Since $B$ is dense in $C^0(M)$, (3.1) holds for $\varphi \in C^0(M)$ by the condition (B2). This implies that $P_t(\theta g)\nu$ is equal to $e^{i\lambda t}h\mu_0$ as a measure. Hence (3.1) holds for all $\varphi \in L^\infty(M, \mu_0)$. Put $A := \{x \in M\colon h(x) \neq 0\}$ and $\chi(x) := e^{-i\lambda t}\mathbf{1}_A|h|/h$. Replacing $\varphi$ by $\varphi \cdot \chi$ in (3.1), we have

$$\langle \mu_0,\, h \cdot U_t(\theta g)\chi \cdot U_t(0)\varphi\rangle = \langle \mu_0,\, |h| \cdot \varphi\rangle \tag{3.2}$$

for all $\varphi \in C^0(M)$. Since $|h \cdot U_t(\theta g)\chi| \le 1$ $\mu_0$-a.e., (3.2) with $\varphi \equiv \mathbf{1}$ implies

$$h \cdot U_t(\theta g)\chi = |h| \quad \text{for } \mu_0\text{-a.e. } x.$$

Using this equality in (3.2), we get

$$\langle \mu_0,\, |h| \cdot U_t(0)\varphi\rangle = \langle \mu_0,\, |h| \cdot \varphi\rangle \quad \text{for all } \varphi \in C^0(M).$$

This implies that $|\nu| = |h|\mu_0$ is an invariant measure. By ergodicity of $\mu_0$, $|h|$ should be a constant function. We may and do suppose $|h| = \mathbf{1}$ by multiplying $|h|$ by some constant. Letting $|h| = \mathbf{1}$ and $\varphi \equiv \mathbf{1}$ in (3.2), we obtain

$$\left\langle \mu_0,\, e^{-i\lambda t + i\theta \int_0^t g\circ S_\tau d\tau} \frac{h}{U_t(0)h} \right\rangle = \langle \mu_0,\, \mathbf{1}\rangle = \mu_0(M).$$

Since $|e^{-i\lambda t + i\theta \int_0^t g\circ S_\tau d\tau} h/U_t(0)h| = \mathbf{1}$ $\mu_0$-a.e., this implies (2.1).

Conversley, we assume that there exists a Borel measurable function $h$ with $|h(x)| = \mathbf{1}$ $\mu_0$-a.e. $x$ which satisfies (2.1). Suppose that $e^{i\lambda t}$ is not an eigenvalue of $P_t(\theta g)$. Then, for each $\varphi \in B$, the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1} e^{-ik\lambda t} P_{kt}(\theta g)(M_\varphi \mu_0)$$

should converge to zero in $V$. Further it converges weakly to zero as a sequence of measures. Take a positive number $\varepsilon > 0$ arbitrarily. By the condition (B2), we can find $\varphi \in B$ such that $\int |h - \varphi| d\mu_0 < \varepsilon$. Also, we have that $\langle M_h\mu_0,\, \psi\rangle = \langle e^{-i\lambda k t} P_{tk}(\theta g)(M_h\mu_0),\, \psi\rangle$ for all $\psi \in C^0(M)$ and $k \ge 0$ by (2.1). Therefore

$$\begin{aligned} |\langle M_h\mu_0,\, \psi\rangle| &= \left|\left\langle \frac{1}{n}\sum_{k=0}^{n-1} e^{-ik\lambda t} P_{kt}(\theta g)(M_h\mu_0),\, \psi \right\rangle\right| \\ &\le \left|\left\langle \frac{1}{n}\sum_{k=0}^{n-1} e^{-ik\lambda t} P_{kt}(\theta g)(M_\varphi\mu_0),\, \psi \right\rangle\right| + \varepsilon \|\psi\|_{C^0}. \end{aligned}$$

Since the first term on the right-hand side converges to 0 as $n \to \infty$, we obtain

$$|\langle M_h\mu_0,\, \psi\rangle| \le \varepsilon \|\psi\|_{C^0} \quad \text{for all } \psi \in C^0(M).$$

This implies $M_h\mu_0 = h\cdot\mu_0 = 0$ since $\varepsilon > 0$ is arbitrary. But this contradicts the fact $|h(x)| = \mathbf{1}$ $\mu_0$-a.e. $x$.

(2) Suppose that $P_t(\theta g)$ has eigenvalues $e^{i\lambda_1 t}$ and $e^{i\lambda_2 t}(\lambda_1,\, \lambda_2 \in \mathbb{R})$ on the unit circle. By (1) in Lemma 5, there exist Borel measurable functions $h_1$ and $h_2$ such that $|h_1| = |h_2| = \mathbf{1}$ $\mu_0$-a.e., and that (2.1) for $h = h_1$ and $h = h_2$. Then we have

$$\frac{h_1}{h_2}(S_t(x)) = e^{i(\lambda_1-\lambda_2)t}\frac{h_1}{h_2}(x) \quad \text{for } \mu_0\text{-a.e. } x. \tag{3.3}$$

Since $(S_t)_{t\ge0}$ is mixing with respect to $\mu_0$, this implies $e^{i(\lambda_1-\lambda_2)t} = 1$. Hence we have that $e^{i\lambda_1 t} = e^{i\lambda_2 t}$ and that $h_1(x)$ is a constant multiple of $h_2(x)$ for $\mu_0$-a.e. $x$. By the proof of (1) in Lemma 5, each eigenvector for the eigenvalue $e^{i\lambda_1 t}$ can be written in the form $h\mu_0$ where $h$ is a Borel measurable function which satisfies (2.1) with $\lambda = \lambda_1$. Therefore all the eigenvectors for the eigenvalue $e^{i\lambda_1 t}$ are scalar multiples of $h_1\mu_0$.

(3) We assume that there exists $s \ge t_0$ such that $P_s(\theta g)$ has an eigenvalue $e^{i\lambda s}$. Take an eigenvector $f_0 \ne 0$ for the eigenvalue $e^{i\lambda s}$. Suppose that there exists $t \ge t_0$ such that $P_t(\theta g)$ does not have eigenvalues with modulus 1. Then, by Lemma 1, the spectral radius of $P_t(\theta g)$ is smaller than 1, that is , there exist constants $K \ge 1$ and $b > 0$ such that

$$\|P_t^n(\theta g)\|_{V,V} \le Ke^{-bnt} \quad \text{for all } n \ge 0.$$

For each integer $m > 0$ and $s \ge t_0$, we have

$$\begin{aligned}\|f_0\|_V &= \|e^{-i\lambda ns}P_{ns}(\theta g)f_0\|_{V,V}\\ &= \|e^{-i\lambda ns}P_{ns-mt}(\theta g)P_{mt}(\theta g)f_0\|_{V,V}\\ &\le \|P_{ns-mt}(\theta g)P_{mt}(\theta g)f_0\|_{V,V}\\ &\le Ke^{-bmt}\sup_{t_0\le t\le 2t_0}\|P_t(\theta g)\|_{V,V}\cdot\|f_0\|_V,\end{aligned}$$

where $n \ge 0$ is the integer such that $nt - ms \in [t_0,\, 2t_0)$. The right-hand side converges to 0 as $m \to \infty$. But then this contradicts the fact $f_0 \ne 0$. Therefore $P_t(\theta g)$ has an eigenvalue on the unit circle for all $t \ge t_0$. □

*Proof of* Lemma 6. First we will show that $\mathcal{U}_g$ is an additive subgroup on $\mathbb{R}$. We fix $\theta_1$ and $\theta_2$ in $\mathcal{U}_g$. By Lemma 5, there exist Borel measurable functions $h_1$ and $h_2$ with $|h_1(x)| = |h_2(x)| = \mathbf{1}$ for $\mu_0$-a.e. $x$ and real numbers $\eta_1$ and $\eta_2$ satisfying, for all $t \geq t_0$,

$$h_j(S_t(x)) = e^{-i\eta_j t + \theta_j \int_0^t g(S_\tau(x))d\tau} h_j(x) \quad \text{for } \mu_0\text{-a.e. } x,$$

where $j = 1, 2$. Taking complex conjugation of the both sides above, we get $\overline{h_j(S_t(x))} = e^{i\eta_j t - i\theta_j \int_0^t g\circ S_\tau(x) d\tau}\overline{h_j}(x)$ $\mu_0$-a.e. This implies that $P_t(-\theta_j g)$ has an eigenvalue of modulus 1 by Lemma 5, that is, $-\theta_j \in \mathcal{U}_g$. One also has that

$$(h_1 \cdot h_2)(S_t(x)) = e^{-i(\eta_1+\eta_2)t + i(\theta_1+\theta_2)\int_0^t g(S_\tau(x))d\tau}(h_1 \cdot h_2)(x) \qquad \mu_0\text{-a.e. } x.$$

This implies that $P_t((\theta_1 + \theta_2)g)$ has an eigenvalue $e^{i\eta_1 t + i\eta_2 t}$ with modulus 1, that is, $\theta_1 + \theta_2 \in \mathcal{U}_g$.

Next we show that $\mathcal{U}_g$ is closed. By Lemma 1, for each $\theta_0 \in \mathbb{R} \setminus \mathcal{U}_g$, the spectral radius of $P_{t_0}(\theta_0 g)$ is less than 1. Since $P_{t_0}(\theta g)$ is continuous with respect to $\theta$, if $|\theta|$ is small enough then the spectral radius of $P_{t_0}((\theta_0 + \theta)g)$ is also less than 1. Hence $\mathbb{R} \setminus \mathcal{U}_g$ is open set. □

## 4. Spectral decomposition

We fix $\theta_0 \in \mathcal{U}_g$ in this section. For each $t \geq t_0$, $P_t(\theta_0 g)$ has one simple eigenvalue on the unit circle, and the rest of the spectrum is contained in the interior of the unit disc by Lemma 5. Therefore we can apply the perturbation theorem [AD, Lemma 4.2] to $P_t(\theta_0 g)$. In the following theorem, we add some claims which are required to deal with the semigroup $(P_t(\theta g))_{t\geq 0}$.

**Proposition 9** *There exist $c > 0$, $\epsilon > 0$ and a $C^3$ mapping $\lambda\colon [\theta_0 - c, \theta_0 + c] \to \{\mathrm{Re}(z) > -\epsilon\}$ such that*

(a) *For all $\theta$ with $|\theta| < c$ and $t \geq t_0$, $P_t((\theta_0+\theta)g)$ has $e^{\lambda(\theta_0+\theta)t}$ as its simple eigenvalue, and the rest of the spectrum of $P_t((\theta_0 + \theta)g)$ is contained in $\{|z| \leq e^{-\epsilon t}\}$.*

(b) *The spectral projector of $P_t((\theta_0 + \theta)g)$ corresponding to the eigenvalue $e^{\lambda(\theta_0+\theta)t}$ does not depend on $t$. We denote this spectral projector by $Q((\theta_0 + \theta)g)\colon V \to V$.*

(c) *Putting $R_t((\theta_0 + \theta)g) = P_t((\theta_0 + \theta)g)(I - Q((\theta_0 + \theta)g))$, we may write*

$P_t((\theta_0 + \theta)g)$ *as*

$$P_t((\theta_0 + \theta)g) = e^{\lambda(\theta_0+\theta)t} Q((\theta_0 + \theta)g) + R_t((\theta_0 + \theta)g) \quad \text{for all } t \geq t_0. \tag{4.1}$$

(d) *There exists a constant* $K \geq 1$ *such that, for all* $\theta$ *with* $|\theta| < c$ *, we have*

$$\|R_t((\theta_0 + \theta)g)\|_{V,V} \leq K e^{-\epsilon t} \quad \text{for all } t \geq t_0. \tag{4.2}$$

(e) *The correspondences* $\theta \mapsto Q((\theta_0 + \theta)g)$ *and* $\theta \mapsto R_t((\theta_0 + \theta)g)$ *are* $C^3$ *mappings from* $[\theta_0 - c, \theta_0 + c]$ *to the space of bounded linear operators on* $V$.

*Proof.* Applying the perturbation theorem [AD, Lemma 4.2] to the family of operators $P_{t_0}((\theta_0 + \theta)g)$ at the parameter $\theta = \theta_0$, we see that there exist $c > 0$, $\epsilon > 0$ and a $C^3$ function $\psi\colon [-c, c] \to \mathbb{C}$ satisfying $\psi(0) = 1$ and $\psi(\theta) \geq e^{-\epsilon t_0}$ such that $P_{t_0}((\theta_0 + \theta)g)$ has $\psi(\theta)$ as its simple eigenvalue and that the rest of its spectrum is contained in the disk $\{|z| \leq e^{-\epsilon t_0}\}$. For each $\theta \in [-c, c]$, let $Q((\theta_0 + \theta)g)$ be the spectral projector of $P_{t_0}((\theta_0 + \theta)g)$ corresponding the eigenvalue $\psi(\theta)$. Since $P_{t_0}((\theta_0 + \theta)g)$ commutes with $P_t((\theta_0 + \theta)g)$ for $t \geq t_0$, so does the spectral projector $Q((\theta_0 + \theta)g)$. Hence the image $V_0(\theta)$ and the kernel $V_1(\theta)$ of $Q((\theta_0 + \theta)g)$ are preserved by the operator $P_t((\theta_0 + \theta)g)$ for $t \geq t_0$, that is, $P_t((\theta_0 + \theta)g)(V_\sigma(\theta)) \subset V_\sigma(\theta)$ for $\sigma = 0, 1$. Since $V_0(\theta)$ is one dimensional, the restriction of $P_t((\theta_0 + \theta)g)$ to $V_0(\theta)$ is the multiplication by some scalar $\psi(\theta, t)$. We have $\psi(\theta, t_0) = \psi(\theta)$ and $\psi(\theta, t + s) = \psi(\theta, t)\psi(\theta, s)$ for $t, s \geq t_0$ and $\theta \in [-c, c]$. By the assumption (P3) and (P4), $\psi(\theta, t)$ is continuous in $(\theta, t)$ (and $C^3$ in $\theta$). Therefore, applying [EN, Theorem 1.4] to $\psi(\theta, \cdot)/\psi(\theta, t_0)$, we can find a $C^3$ function $\lambda\colon [\theta_0 - c, \theta_0 + c] \to \mathbb{C}$ such that $\psi(\theta, t) = e^{\lambda(\theta, t)}$ for $\theta \in [-c, c]$ and $t \geq t_0$. The spectrum of $R_{t_0}((\theta_0 + \theta)g)$ in the claim (c) is contained in the disk $\{|z| \leq e^{-\epsilon t_0}\}$. Hence, by using the assumption (P4), we see that

$$\lim_{t\to\infty} e^{\epsilon t} \|R_t((\theta_0 + \theta)g)\|_{V,V} = \lim_{n\to\infty} e^{\epsilon n t_0} \|R_{nt_0}((\theta_0 + \theta)g)\|_{V,V} = 0.$$

This implies the claim (d). The other claims are now obvious. □

We next consider the differentiation of $\lambda(\theta)$ in Proposition 9 at $\theta = \theta_0$.

**Lemma 10** $\lambda'(\theta_0) = 0 = \mu_0(g)$.

*Proof.* By Lemma 5, there exist a Borel measurable function $h$ and a real number $\eta \in \mathbb{R}$ such that

$$h(S_t(x)) = e^{-i\eta t + \theta_0 \int_0^t g(S_\tau(x))d\tau} h(x) \quad \text{and} \quad |h_j(x)| = 1 \quad \text{for } \mu_0\text{-a.e. } x.$$

for all $t \geq t_0$. By the assumption (B2), we take a sequence $\{h_j\}_{j=1}^\infty \subset B$ such that $\|h_j - h\|_{L^1(\mu_0)} \to 0$ as $j \to \infty$. Then we have

$$\begin{aligned}\big\langle \mu_0,\, e^{i\theta \int_0^t g\circ S_\tau d\tau/t}\big\rangle &= \big\langle \mu_0,\, e^{i((\theta/t)+\theta_0)\int_0^t g\circ S_\tau d\tau} e^{-\lambda(\theta_0)t} U_t(0)\bar{h}\cdot h\big\rangle \\ &= e^{-\lambda(\theta_0)t}\big\langle \mu_0,\, e^{i\theta\int_0^t g\circ S_\tau d\tau/t} U_t(\theta_0 g)(\bar{h}-\bar{h}_j)\cdot h\big\rangle \\ &\quad + e^{-\lambda(\theta_0)t}\Big\langle P_t\Big(\Big(\frac{\theta}{t}+\theta_0\Big)g\Big)(h\mu_0),\, \bar{h}_j\Big\rangle\end{aligned}$$

for every $\theta \in \mathbb{R}$ and $t \geq t_0$. Below we consider the limits of the both sides as $t \to \infty$.

First we consider the left-hand side. Since $\lim_{t\to\infty}(1/t)\int_0^t g\circ S_\tau d\tau = \int g d\mu_0 = 0$ $\mu_0$-a.e. by ergodicity of $\mu_0$, we have $\lim_{t\to\infty}\int e^{i\theta\int_0^t g\circ S_\tau d\tau/t}d\mu_0 = e^0 = 1$ by Lebesgue bounded convergence theorem. Hence we have

$$\lim_{t\to\infty}\big\langle \mu_0,\, e^{i\theta\int_0^t g\circ S_\tau d\tau/t}\big\rangle = 1 \quad \text{for all } \theta \in \mathbb{R}. \tag{4.3}$$

Next we consider the right-hand side. For the former term on the right-hand side, we have

$$\big|e^{-\lambda(\theta_0)t}\big\langle \mu_0,\, e^{i\theta\int_0^t g\circ S_\tau d\tau/t} U_t(\theta_0 g)(\bar{h}-\bar{h}_j)\cdot h\big\rangle\big| \leq \int |\bar{h}-\bar{h}_j| d\mu_0 \tag{4.4}$$

By (4.1) in Proposition 9, the latter term on the right-hand side is written as

$$e^{-\lambda(\theta_0)t}\Big\langle e^{\lambda((\theta/t)+\theta_0)t} Q\Big(\Big(\frac{\theta}{t}+\theta_0\Big)g\Big)(h\mu_0),\, \bar{h}_j\Big\rangle + e^{-\lambda(\theta_0)t}\Big\langle R_t\Big(\Big(\frac{\theta}{t}+\theta_0\Big)g\Big)(h\mu_0),\, \bar{h}_j\Big\rangle.$$

For each of these terms, we have

$$\lim_{t\to\infty} e^{-\lambda(\theta_0)t}\Big\langle e^{\lambda((\theta/t)+\theta_0)t} Q\Big(\Big(\frac{\theta}{t}+\theta_0\Big)g\Big)(h\mu_0),\, \bar{h}_j\Big\rangle$$

$$= e^{i\theta\lambda'(\theta_0)} \int h\bar{h}_j d\mu_0 \quad (4.5)$$

and

$$\lim_{t\to\infty} \Big| e^{-\lambda(\theta_0)t} \Big\langle R_t\Big(\Big(\frac{\theta}{t}+\theta_0\Big)g\Big)(h\mu_0),\, \bar{h}_j \Big\rangle \Big| \\ \leq \lim_{t\to\infty} \|\bar{h}_j\|_B \Big\| R_t\Big(\Big(\frac{\theta}{t}+\theta_0\Big)g\Big)(h\mu_0)\Big\|_{V,V} = 0 \quad (4.6)$$

by (4.2) and $\langle Q(\theta_0 g)(h\mu_0),\, \bar{h}_j\rangle = \int h\bar{h}_j d\mu_0$. By (4.3), (4.4), (4.5) and (4.6), we obtain

$$\Big|1 - e^{i\theta\lambda'(\theta_0)} \int h\bar{h}_j d\mu_0 \Big| \leq \int |\bar{h} - \bar{h}_j| d\mu_0 \quad \text{for all } \theta \in \mathbb{R}.$$

Since $\lim_{j\to\infty} \int h\bar{h}_j = 1$ and $\lim_{j\to\infty} \int |\bar{h} - \bar{h}_j| d\mu_0 = 0$, we conclude $e^{i\theta\lambda'(\theta_0)} = 1$ for all $\theta \in \mathbb{R}$. □

**Lemma 11** $\lambda''(\theta_0) = -\sigma_g^2$.

*Proof.* Take $h$ and $\{h_j\}_{j=1}^{\infty}$ in the same way as in the proof of Lemma 10 above. Then we have

$$\big\langle \mu_0,\, e^{i\theta\int_0^t g\circ S_\tau d\tau/\sqrt{t}} \big\rangle = e^{-\lambda(\theta_0)t} \big\langle \mu_0,\, e^{i\theta\int_0^t g\circ S_\tau d\tau/\sqrt{t}} U_t(\theta_0 g)(\bar{h} - \bar{h}_j)\cdot h \big\rangle \\ + e^{-\lambda(\theta_0)t} \Big\langle P_t\Big(\Big(\frac{\theta}{\sqrt{t}} + \theta_0\Big)g\Big)(h\mu_0),\, \bar{h}_j \Big\rangle. \quad (4.7)$$

We differentiate the both sides of (4.7) with respect to $\theta$ twice and consider the limit as $t \to \infty$. For the left-hand side, we obtain

$$\lim_{t\to\infty} \frac{\partial^2}{\partial\theta^2} \big\langle \mu_0,\, e^{i\theta\int_0^t g\circ S_\tau d\tau/\sqrt{t}} \big\rangle \Big|_{\theta=0} \\ = -\lim_{t\to\infty} \int \Big(\frac{1}{\sqrt{t}} \int_0^t g\circ S_\tau d\tau\Big)^2 d\mu_0 = -\sigma_g^2. \quad (4.8)$$

For the former term on the right-hand side of (4.7), we have

$$\Big| \frac{\partial^2}{\partial\theta^2} \big\langle \mu_0,\, e^{i\theta\int_0^t g\circ S_\tau d\tau/\sqrt{t}} U_t(\theta_0 g)(\bar{h} - \bar{h}_j)h \big\rangle \Big|_{\theta=0} \Big| \\ = \Big| \Big\langle \mu_0,\, \frac{-1}{t} \Big(\int_0^t g\circ S_\tau d\tau\Big)^2 U_t(\theta_0 g)(\bar{h} - \bar{h}_j) \Big\rangle \Big|$$

$$\leq \Big\langle \mu_0,\, \frac{1}{t}\Big(\int_0^t g\circ S_\tau d\tau\Big)^2 |U_t(\theta_0 g)(\bar h - \bar h_j)|\Big\rangle$$
$$\leq \Big\langle \mu_0,\, \frac{1}{t}\Big(\int_0^t g\circ S_\tau d\tau\Big)^2\Big\rangle \|\bar h - \bar h_j\|_{L^1(\mu_0)}.$$

This implies

$$\varlimsup_{t\to\infty}\Big|\frac{\partial^2}{\partial\theta^2}\big(e^{-\lambda(\theta_0)t}\langle \mu_0,\, e^{i\theta\int_0^t g\circ S_\tau d\tau/\sqrt{t}}U_t(\theta_0 g)(\bar h - \bar h_j)h\rangle\big)\Big|_{\theta=0}\Big| \leq \sigma_g^2 \|\bar h - \bar h_j\|_{L^1(\mu_0)}. \quad (4.9)$$

We decompose the latter term on the right-hand side of (4.7) as

$$\frac{\partial^2}{\partial\theta^2}\Big\langle P_t\Big(\Big(\frac{\theta}{\sqrt{t}}+\theta_0\Big)g\Big)(h\mu_0),\, \bar h_j\Big\rangle\Big|_{\theta=0}$$
$$= \frac{\partial^2}{\partial\theta^2}\Big\langle e^{\lambda((\theta/\sqrt{t})+\theta_0)t} Q\Big(\Big(\frac{\theta}{\sqrt{t}}+\theta_0\Big)g\Big)(h\mu_0),\, \bar h_j\Big\rangle\Big|_{\theta=0}$$
$$+ \frac{\partial^2}{\partial\theta^2}\Big\langle R_t\Big(\Big(\frac{\theta}{\sqrt{t}}+\theta_0\Big)g\Big)(h\mu_0),\, \bar h_j\Big\rangle\Big|_{\theta=0}.$$

As $\lambda'(\theta_0) = 0$, we see

$$\frac{\partial}{\partial\theta}(e^{\lambda(\theta/\sqrt{t}+\theta_0)t})\Big|_{\theta=0} = 0 \quad \text{and}$$
$$\frac{\partial^2}{\partial\theta^2}(e^{\lambda(\theta/\sqrt{t}+\theta_0)t})\Big|_{\theta=0} = e^{\lambda(\theta_0)t}\lambda''(\theta_0). \quad (4.10)$$

From (4.10), we have

$$\lim_{t\to\infty}\frac{\partial^2}{\partial\theta^2}\Big\langle e^{\lambda((\theta/\sqrt{t})+\theta_0)t-\lambda(\theta_0)t} Q\Big(\Big(\frac{\theta}{\sqrt{t}}+\theta_0\Big)g\Big)(h\mu_0),\, \bar h_j\Big\rangle\Big|_{\theta=0} = \lambda''(\theta_0)\int h\bar h_j d\mu_0 \quad (4.11)$$

for all $\theta \in \mathbb{R}$. Next we consider

$$\frac{\partial^2}{\partial\theta^2}\Big\langle e^{-\lambda(\theta_0)t} R_t\Big(\Big(\frac{\theta}{\sqrt{t}}+\theta_0\Big)g\Big)(h\mu_0),\, \bar h_j\Big\rangle\Big|_{\theta=0}.$$

Applying Leibniz formula for

$$R_t\Big(\Big(\frac{\theta}{\sqrt{t}}+\theta_0\Big)g\Big)$$

$$= \overbrace{R_{t_0}\Big(\Big(\frac{\theta}{\sqrt{t}}+\theta_0\Big)g\Big)\cdots R_{t_0}\Big(\Big(\frac{\theta}{\sqrt{t}}+\theta_0\Big)g\Big)}^{n\text{ times}} R_{t-nt_0}\Big(\Big(\frac{\theta}{\sqrt{t}}+\theta_0\Big)g\Big)$$

where $n = [t/t_0] - 1$ and using the estimate in (d) of Proposition 9, we can get

$$\lim_{t\to\infty}\frac{\partial^2}{\partial\theta^2}\Big\langle e^{-\lambda(\theta_0)t}R_t\Big(\Big(\frac{\theta}{\sqrt{t}}+\theta_0\Big)g\Big)(h\mu_0),\, \bar{h}_j\Big\rangle\Big|_{\theta=0} = 0. \tag{4.12}$$

By (4.8), (4.9), (4.11) and (4.12), we obtain

$$\left|-\sigma_g^2 - \lambda''(\theta_0)\int h\bar{h}_j d\mu_0\right| \le \sigma_g^2\|\bar{h} - \bar{h}_j\|_{L^1(\mu_0)}.$$

Since $\int h\bar{h}_j d\mu_0 \to 1$ and $\|\bar{h} - \bar{h}_j\|_{L^1(\mu_0)} \to 0$ as $j \to \infty$, we conclude $-\sigma_g^2 = \lambda''(\theta_0)$. □

## 5. Proof of Theorem 7 and Theorem 8

*Proof of* Theorem 7. Take $h$ and $\{h_j\}_{j=1}^{\infty}$ in the same way as in the proof of Lemma 10 in the case $\theta_0 = 0 \in \mathcal{U}_g$. By Lemma 10 and 11, we have

$$\begin{aligned}\lim_{t\to\infty}\int_M e^{i\theta\int_0^t g\circ S_\tau d\tau/\sqrt{t}}d\mu_0 &= \lim_{j\to\infty}\lim_{t\to\infty}\Big\langle P_t\Big(\frac{\theta}{\sqrt{t}}g\Big)(h\mu_0),\, \bar{h}_j\Big\rangle \\ &= e^{-\theta^2\sigma_g^2/2}.\end{aligned}$$

This shows the central limit theorem by Glivenko's theorem. □

Let $\mathcal{B}$, $\mathcal{S}$, $\mathcal{D}$, and $\mathcal{D}_K$ be the spaces of bounded smooth functions, rapidly decreasing functions, smooth functions with compact support, and smooth functions whose support is contained in the interval $[-K, K]$, respectively. The topology of $\mathcal{B}$ is defined as follows [S]: $\varphi_j \in \mathcal{B}$ converges to $\varphi \in \mathcal{B}$ in $\mathcal{B}$ if and only if $\sum_{|\alpha|\le m}\sup_{\theta\in\mathbb{R}}|\partial^\alpha\varphi_j(\theta) - \partial^\alpha\varphi(\theta)|$ converges to zero for all $m \ge 0$. (see [S] for the topologies of $\mathcal{S}$ and $\mathcal{D}$.)

*Proof of* Theorem 8. Notice that $\mathcal{U}_g \ne \mathbb{R}$ when $\sigma_g^2 > 0$ by Lemma 11. We will prove Theorem 8 in the case where $g$ is periodic. In the case where $g$ is aperiodic, the claim of Theorem 8 can be proved by the parallel argument.

Take $a > 0$ so that $\mathcal{U}_g = a\mathbb{Z}$. Let $e^{\lambda(a)t}$ be the eigenvalue of $P_t(ag)$ on the unit circle and let $h\mu_0$ be the corresponding eigenvector, where $h$ satisfies $|h| = 1$ $\mu_0$-a.e. and (2.1). Then, for all $k \in \mathbb{Z}$, $P_t(akg)$ has the eigenvalue $e^{\lambda(a)kt}$ and the corresponding eigenvector is $h^k\mu_0$. (See the proof of Lemma 6.) So we have, by (2.1),

$$\begin{aligned}\big\langle e^{-\lambda(a)kt}P_t(akg)\mu_0,\, \bar{h}^k\big\rangle &= e^{-\lambda(a)kt}\big\langle \mu_0,\, e^{iak\int_0^t g\circ S_\tau d\tau}U_t(0)\bar{h}^k\big\rangle \\ &= e^{-\lambda(a)kt}\big\langle \mu_0,\, e^{\lambda(a)kt}U_t(0)h^k\cdot \bar{h}^k\cdot U_t(0)\bar{h}^k\big\rangle \\ &= \langle \mu_0,\, \bar{h}^k\rangle \qquad (5.1)\end{aligned}$$

for all $t \geq t_0$. Let $Q(akg)$ be the spectral projector of $P_t(akg)$ for the eigenvalue $e^{\lambda(a)kt}$. Since the left-hand side of (5.1) converges to $\langle Q(akg)\mu_0,\, \bar{h}^k\rangle$ as $t \to \infty$ and since $Q(akg)\mu_0$ is a scalar multiple of $h^k\mu_0$, we obtain $Q(akg)\mu_0 = \langle \mu_0,\, \bar{h}^k\rangle h^k\mu_0$.

First we show Theorem 8 for all $u \in \mathcal{S}$ where the Fourier transform $\hat{u}$ belongs to $\mathcal{D}_K$. We see

$$\begin{aligned}&\sigma_g\sqrt{t}\int_M u\Big(z+\int_0^t g\circ S_\tau d\tau\Big)d\mu_0 \\ &\qquad = \frac{\sigma_g\sqrt{t}}{2\pi}\int_M\int_{-\infty}^{\infty} e^{i\theta z}\hat{u}(\theta)e^{i\theta\int_0^t g\circ S_\tau d\tau}d\theta d\mu_0 \\ &\qquad = \frac{\sigma_g\sqrt{t}}{2\pi}\int_{-\infty}^{\infty} e^{i\theta z}\hat{u}(\theta)\langle P_t(\theta g)\mu_0,\, \mathbf{1}\rangle d\theta.\end{aligned}$$

Let $\varepsilon\colon [t_0, \infty) \to (0, c)$ be a function satisfying

$$\lim_{t\to\infty}\varepsilon(t)^4 t^{3/2} = 0 \quad \text{and} \quad \lim_{t\to\infty}\varepsilon(t)t^{1/2} = \infty.$$

Using (5.1) and the standard formula

$$\frac{1}{\sqrt{2\pi}}e^{-z^2/2\sigma_g^2 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\Big(\frac{-\theta^2}{2}\Big)\exp\Big(\frac{-i\theta z}{\sigma_g\sqrt{t}}\Big)d\theta,$$

we get

$$\begin{aligned}&\sigma_g\sqrt{t}\int_M u\Big(z+\int_0^t g\circ S_\tau d\tau\Big)d\mu_0 \\ &\quad - \sum_{k=-\infty}^{\infty}\hat{u}(ak)e^{iakz}e^{\lambda(a)kt}\big|\langle \mu_0,\, h^k\rangle\big|^2\frac{1}{\sqrt{2\pi}}e^{-z^2/2\sigma_g^2 t} \\ &\qquad\qquad = \frac{1}{2\pi}\sum_{-\infty}^{\infty}[A_{1,k}+A_{2,k}+A_{3,k}+A_{4,k}+A_{5,k}],\end{aligned}$$

where

$$\begin{aligned}
A_{1,k} = & \int_{|\theta|<\varepsilon(t)\sigma_g\sqrt{t}} \exp\Big(\frac{-i\theta z}{\sigma_g\sqrt{t}} + iakz\Big)\hat{u}\Big(\frac{\theta}{\sigma_g\sqrt{t}} + ak\Big) \\
& \qquad\qquad \times \exp\Big(\lambda\Big(\frac{\theta}{\sigma_g\sqrt{t}} + ak\Big)t\Big)\big|\langle\mu_0,\, h^k\rangle\big|^2 d\theta \\
& - \int_{|\theta|<\varepsilon(t)\sigma_g\sqrt{t}} \hat{u}(ak)e^{iakz}e^{\lambda(a)kt}\exp\Big(\frac{-\theta^2}{2}\Big) \\
& \qquad\qquad \times \exp\Big(\frac{-i\theta z}{\sigma_g\sqrt{t}}\Big)\big|\langle\mu_0,\, h^k\rangle\big|^2 d\theta, \\
A_{2,k} = & \int_{|\theta|<\varepsilon(t)\sigma_g\sqrt{t}} \exp\Big(\frac{-i\theta z}{\sigma_g\sqrt{t}} + iakz\Big) \\
& \qquad\qquad \times \hat{u}\Big(\frac{\theta}{\sigma_g\sqrt{t}} + ak\Big)\exp\Big(\lambda\Big(\frac{\theta}{\sigma_g\sqrt{t}} + ak\Big)t\Big) \\
& \qquad\qquad \times \Big\langle Q\Big(\Big(\frac{\theta}{\sigma_g\sqrt{t}} + ak\Big)g\Big)\mu_0 - Q(ak)\mu_0,\, \mathbf{1}\Big\rangle d\theta, \\
A_{3,k} = & \int_{|\theta|<\varepsilon(t)\sigma_g\sqrt{t}} \exp\Big(\frac{-i\theta z}{\sigma_g\sqrt{t}} + iakz\Big)\hat{u}\Big(\frac{\theta}{\sigma_g\sqrt{t}} + ak\Big) \\
& \qquad\qquad \times \Big\langle R_t\Big(\Big(\frac{\theta}{\sigma_g\sqrt{t}} + ak\Big)g\Big)\mu_0,\, \mathbf{1}\Big\rangle d\theta, \\
A_{4,k} = & \int_{\varepsilon(t)\sigma_g\sqrt{t}\le|\theta|\le(a/2)\sigma_g\sqrt{t}} \exp\Big(\frac{-i\theta z}{\sigma_g\sqrt{t}} + iakz\Big)\hat{u}\Big(\frac{\theta}{\sigma_g\sqrt{t}} + ak\Big) \\
& \qquad\qquad\qquad \times \Big\langle P_t\Big(\Big(\frac{\theta}{\sigma_g\sqrt{t}} + ak\Big)g\Big)\mu_0,\, \mathbf{1}\Big\rangle d\theta, \\
A_{5,k} = & -\int_{\varepsilon(t)\sigma_g\sqrt{t}\le|\theta|} \hat{u}(ak)e^{iakz}e^{\lambda(a)kt}\exp\Big(\frac{-\theta^2}{2}\Big) \\
& \qquad\qquad \times \exp\Big(\frac{-i\theta z}{\sigma_g\sqrt{t}}\Big)\big|\langle\mu_0,\, h^k\rangle\big|^2 d\theta.
\end{aligned}$$

By Theorem 7 and Proposition 9, there exist positive constants $C_1(k), C_2(k)$, $K(k)$ and $\epsilon_k > 0$, which depend only on $k$, $g$ and $\mu_0$ such that

$$\begin{aligned}
&\Big|\exp\Big(\lambda\Big(\frac{i\theta}{\sigma_g\sqrt{t}} + ak\Big)t - \lambda(a)kt\Big) - \exp\Big(\frac{-\theta^2}{2}\Big)\Big| \le C_1(k)\frac{|\theta|^3}{\sqrt{t}}, \\
&\Big\|Q\Big(\Big(\frac{\theta}{\sigma_g\sqrt{t}} + ak\Big)g\Big) - Q(akg)\Big\|_{V,V} \le C_2(k)\frac{|\theta|}{\sqrt{t}},
\end{aligned}$$

$$\left\| R_t\Big(\Big(\frac{\theta}{\sigma_g\sqrt{t}}+ak\Big)g\Big)\right\|_{V,V} \le K(k)e^{-\epsilon_k t},$$

provided that $|\theta/(\sigma_g\sqrt{t})|$ is small enough. Hence we obtain the estimate

$$\begin{aligned}
&|A_{1,k}+A_{2,k}+A_{3,k}| \\
&\le C_1(k)(\|\hat{u}\|_{L^\infty}\cdot\varepsilon(t)^4t^{3/2}+\|\hat{u}'\|_{L^\infty}\cdot\varepsilon(t)^2t^{1/2}) \\
&\quad + C_2(k)\|\hat{u}\|_{L^\infty}\varepsilon(t)^2t^{1/2}\|\mu_0\|_V + K(k)\|\hat{u}\|_{L^\infty}e^{-\epsilon_k t}\varepsilon(t)t^{1/2}\|\mu_0\|_V \\
&\le B_1(k)(\varepsilon(t)^4t^{3/2}+\varepsilon(t)^2t^{1/2}+e^{-\epsilon_k t}\varepsilon(t)t^{1/2})(\|\hat{u}\|_{L^\infty}+\|\hat{u}'\|_{L^\infty}),
\end{aligned}$$

where $B_1(k)$ is a positive constant depending only on $g$ and $k$.

Next we estimate $|A_{4,k}|$. For all $\theta$ with $\varepsilon(t)\le|\theta/(\sigma_g\sqrt{t})|\le a/2$, the spectral radius of $P_t((\theta/(\sigma_g\sqrt{t})+ak)g)$ is less than one. By continuity of $P_t((\theta/(\sigma_g\sqrt{t})+ak)g)$ with respect to $\theta$, there exist constants $0<b<1$ and $C_3(k)>0$, which is a positive constant depending only on $k$ and $g$, such that

$$\max_{c\le|\theta|\le a/2}\left\| P_t\Big(\Big(\frac{\theta}{\sigma_g\sqrt{t}}+ak\Big)g\Big)\right\|_{V,V} \le C_3(k)e^{-bt}.$$

By Lemma 11, we have $\lim_{t\to\infty}(\lambda((\theta/(\sigma_g\sqrt{t}))+ak)t-\lambda(ak)t) = -\theta^2/2$ and thus

$$\begin{aligned}
\Big|\lambda\Big(\frac{\theta}{\sigma_g\sqrt{t}}+ak\Big)t\Big| &\le \Big|\lambda(ak)t-\frac{\theta^2}{2}\Big|+\frac{\theta^2}{4} \\
&= 1-\frac{\theta^2}{4} \quad \text{for sufficiently large } t\ge t_0.
\end{aligned}$$

Hence there exist positive constants $C_4(k)$ and $B_2(k)$, which depend only on $k$ and $g$, such that

$$\begin{aligned}
|A_{4,k}| &\le \|\hat{u}\|_{L^\infty}\Bigg(C_4(k)\int_{\varepsilon(t)\sigma_g\sqrt{t}\le|\theta|<c\sigma_g\sqrt{t}} e^{(1-\theta^2/4)}d\theta \\
&\qquad + \int_{c\sigma_g\sqrt{t}\le|\theta|\le(a/2)\sigma_g\sqrt{t}} C_3(k)e^{-bt}d\theta\Bigg) \\
&\le B_2(k)\sqrt{t}(e^{-\varepsilon(t)^2\sigma_g^2t/4}+2e^{-bt})\|\hat{u}\|_{L^\infty}.
\end{aligned}$$

It is easy to see $\lim_{t\to\infty}|A_{5,k}|=0$.

From the estimates on $A_{j,k}$ above, there exists a positive-valued function

$\{\gamma(t)\}_{t\geq 0}$ of real numbers with $\lim_{t\to\infty}\gamma(t)=0$ such that

$$|A_{1,k}+A_{2,k}+A_{3,k}+A_{4,k}+A_{5,k}|\leq B(K)\gamma(t)(\|\hat{u}\|_\infty+\|\hat{u}'\|_\infty) \tag{5.2}$$

for all $u\in\mathcal{S}$ with $\hat{u}\in\mathcal{D}_K$, where $B(K)$ is a positive constant depending only on $K$. Therefore the claim of Theorem 8 holds for all $u\in\mathcal{S}$ with $\hat{u}\in\mathcal{D}_K$.

In the following, we will show that the assumption $\hat{u}\in\mathcal{D}_K$ is actually not necessary. Let $\mu_{z,t}$ be a measure on $\mathbb{R}$ characterized by the condition

$$\int_{\mathbb{R}}\phi(\theta)\mu_{z,t}(d\theta)=\sigma_g\sqrt{t}\int\phi\Big(z+\int_0^t g\circ S_\tau d\tau\Big)d\mu_0 \qquad \text{for all } \phi\in C^0(\mathbb{R}).$$

We have that $\{\mu_{z,t}\}_{z,t}\subset\mathcal{B}^*\subset\mathcal{S}^*$. Let $\hat{\mu}_{z,t}$ be the Fourier transform of $\mu_{z,t}$. We first show that $\{\hat{\mu}_{z,t}\}_{t,z}$ is a bounded subset of $\mathcal{B}^*$. Obviously $\mu_{z,t}$ is a positive measure, i.e., $\mu_{z,t}(\phi)\geq 0$ for all non-negative functions $\phi\in\mathcal{S}$. This implies that

$$\langle\hat{\mu}_{z,t},\,\phi*\bar{\check{\phi}}\rangle=\frac{1}{\sqrt{2\pi}}\langle\mu_{z,t},\,\hat{\phi}\cdot\bar{\hat{\phi}}\rangle\geq 0\quad\text{for all }\phi\in\mathcal{D},$$

where $*$ is convolution and $\check{\phi}$ is defined by $\check{\phi}(\theta)=\phi(-\theta)$. That is, $\{\hat{\mu}_{z,t}\}_{z,t}$ is a set of distributions of positive type [S]. The set $\{\mu_{z,t}\}_{z,t}$ is bounded in $\mathcal{D}^*_K$. In fact, we have, by (5.2),

$$\begin{aligned}\Big|\int_{\mathbb{R}}\phi(\theta)\mu_{z,t}(d\theta)\Big|&=\Big|\sigma_g\sqrt{t}\int_M u\Big(z+\int_0^t g\circ S_\tau d\tau\Big)d\mu_0\Big|\\&\leq 2\Big|\sum_{k=0}^{[K/a]}\hat{u}(ak)e^{iakz}e^{\lambda(a)kt}|\langle\mu_0,\,h^k\rangle|^2\frac{1}{\sqrt{2\pi}}e^{-z^2/2\sigma_g^2t}\Big|\\&\quad+|A_{1,k}+A_{2,k}+A_{3,k}+A_{4,k}+A_{5,k}|\\&\leq\Big[\frac{K}{a}\Big]\frac{2}{\sqrt{2\pi}}\|\hat{u}\|_\infty+B(K)\gamma(t)(\|\hat{u}\|_\infty+\|\hat{u}'\|_\infty)\end{aligned}$$

for all $u\in\mathcal{S}$ with $\hat{u}\in\mathcal{D}_K$. We can now conclude that $\{\hat{\mu}_{z,t}\}_{t,z}$ is bounded in $\mathcal{B}^*$ using the following theorem [S, p. 276]: if the set of distributions of positive type is bounded in $\mathcal{D}^*_\Omega$, $\Omega$ is a neighbourhood of the origin in $\mathbb{R}$, then it is bounded in $\mathcal{B}^*$.

Take $u\in\mathcal{S}$ arbitrarily. Then take a sequence $(\rho_j)_{j=1}^\infty$ of probability measures on $\mathbb{R}$ such that $\hat{\rho}_j\in\mathcal{D}$ for each $i$ and converges weakly to Dirac measure $\delta_0$ as $j \to \infty$. For arbitary $\varepsilon > 0$, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} u(\theta)(\rho_j * \mu_{z,t})(d\theta) - \int_{\mathbb{R}} u(\theta)\mu_{z,t}(d\theta)\Big| \\
&\le \int_{|s|<\varepsilon}\Big|\int_{\mathbb{R}} u(\theta+s) - u(\theta)\mu_{z,t}(d\theta)\Big|\rho_j(ds) \\
&\quad + \int_{|s|\ge\varepsilon}\Big|\int_{\mathbb{R}} u(\theta+s) - u(\theta)\mu_{z,t}(d\theta)\Big|\rho_j(ds).
\end{aligned}$$

Since $\{\hat{\mu}_{z,t}\}_{t,z}$ is bounded in $\mathcal{B}^*$, and since $\hat{u} \in \mathcal{B}$, there exists constant $D_1(u)$ depending only on $u$ such that

$$\begin{aligned}
&\sup_{z\in\mathbb{R}}\sup_{s>0}\Big|\int_{\mathbb{R}} u(\theta+s)\mu_{z,t}(d\theta)\Big| \\
&\qquad = \sup_{z\in\mathbb{R}}\sup_{s>0}\frac{1}{2\pi}\Big|\int_{\mathbb{R}} \hat{u}(\theta)\sigma_g\sqrt{t}e^{i\theta(z+s)}\langle P_t(\theta g)\mu_0, \mathbf{1}\rangle d\theta\Big| \le D_1(u)
\end{aligned}$$

for all $t \ge t_0$. Moreover, since the set $\{s^{-1}(u(\theta+s) - u(\theta))\colon 0 < |s| \le 1\}$ is bouded in $\mathcal{B}$, we have

$$\sup_{z\in\mathbb{R}}\sup_{0<|s|\le1}\Big|\int_{\mathbb{R}} \frac{u(\theta+s)-u(\theta)}{s}\mu_{z,t}(d\theta)\Big| \le D_2(u) \quad \text{for all } t \ge t_0.$$

Thus we obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} u(\theta)(\rho_j * \mu_{z,t})(d\theta) - \int_{\mathbb{R}} u(\theta)\mu_{z,t}(d\theta)\Big| \\
&\qquad\qquad \le D_2(u)\varepsilon + \rho_j(\{s \in \mathbb{R}; |s| \ge \varepsilon\}) \cdot D_1(u).
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists an integer $j_0(\varepsilon)$ such that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} u(\theta)(\rho_j * \mu_{z,t})(d\theta) - \int_{\mathbb{R}} u(\theta)\mu_{z,t}(d\theta)\Big| \\
&\qquad \le D_3(u)\varepsilon \quad \text{for all } j \ge j_0(\varepsilon),\ z \in \mathbb{R},\ \text{and } t \ge t_0, \qquad (5.3)
\end{aligned}$$

where $D_3(u)$ is a constant depending only on $u$.

Take $j \ge j(\varepsilon)$. As $\widehat{\rho_j * \mu_{z,t}} \in \mathcal{D}$, we have

$$\lim_{t\to\infty}\Big|\int_{\mathbb{R}} u(\theta)(\rho_j * \mu_{z,t})(d\theta)$$

$$- \sum_{k=-\infty}^{\infty} \hat{u}(ak)\hat{\rho}_j(ak)e^{iakz}e^{\lambda(a)kt}\Big|\int h^k d\mu_0\Big|^2 \frac{e^{-z^2/2\sigma_g^2 t}}{\sqrt{2\pi}}\Bigg| = 0 \quad (5.4)$$

for all $z \in \mathbb{R}$. Obviously we have that $|\int h^k d\mu_0|^2 \le 1$ and $\sum_{k>k_0} |\hat{u}(ak)| \le \varepsilon$ for sufficiently large $k_0 \ge 0$. So we have

$$\Bigg|\sum_{k=-\infty}^{\infty} \hat{u}(ak)(\hat{\rho}_j(ak)-1)e^{iakz}e^{\lambda(iak)t}\Big|\int h^k d\mu_0\Big|^2\Bigg| \le \sum_{k\le k_0} |\hat{u}(ak)(\hat{\rho}_j(ak)-1)| + \varepsilon \le 3\varepsilon \quad (5.5)$$

for all $z \in \mathbb{R}$ and $t \ge t_0$. By (5.3), (5.4) and (5.5), we obtain

$$\begin{aligned}
&\overline{\lim_{t\to\infty}} \sup_{z\in\mathbb{R}}\Big|\sigma_g\sqrt{t}\int_M u\Big(z+\int_0^t g\circ S_\tau d\tau\Big)d\mu \\
&\qquad - \sum_{k=-\infty}^{\infty} \hat{u}(ak)e^{iakz}e^{\lambda(a)kt}|\langle\mu_0,\, h^k\rangle|^2 \frac{e^{-z^2/2\sigma_g^2 t}}{\sqrt{2\pi}}\Big| \\
&\le \overline{\lim_{t\to\infty}} \sup_{z\in\mathbb{R}}\Big|\int_{\mathbb{R}} u(\theta)\mu_{z,t}(d\theta) - \int_{\mathbb{R}} u(\theta)(\rho_j * \mu_{z,t})(d\theta)\Big| \\
&\quad + \overline{\lim_{t\to\infty}} \sup_{z\in\mathbb{R}}\Big|\int_{\mathbb{R}} u(\theta)(\rho_j * \mu_{z,t})(d\theta) \\
&\qquad - \sum_{k=-\infty}^{\infty} \hat{u}(ak)\hat{\rho}_j(ak)e^{iakz}e^{\lambda(a)kt}|\langle\mu_0,\, h^k\rangle|^2 \frac{e^{-z^2/2\sigma_g^2 t}}{\sqrt{2\pi}}\Big| \\
&\quad + \overline{\lim_{t\to\infty}} \sup_{z\in\mathbb{R}}\Big|\sum_{k=-\infty}^{\infty} (\hat{u}\hat{\rho}_j(ak) - \hat{u}(ak)) \\
&\qquad \times e^{iakz}e^{\lambda(iak)t}|\langle\mu_0,\, h^k\rangle|^2 \frac{e^{-z^2/2\sigma_g^2 t}}{\sqrt{2\pi}}\Big| \\
&\le (D_3(u)+3)\varepsilon \quad \text{for } j \ge j_0(\varepsilon).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this gives the conclusion of Theorem 8 for $u \in \mathcal{S}$, in the case where $g$ is periodic. □

## 6. Appendix

By the condition (B2), $B$ is dense in $L^2(M,\, \mu_0)$. Therefore the dual space $L^2(M,\, \mu_0)^*$ of $L^2(M,\, \mu_0)$ is identified with a subspace of $B^*$. Further, identifying $L^2(M,\mu_0)^*$ with $L^2(M,\mu_0)$ by Riesz's theorem, we can regard the operator $P_t$ as that on $L^2(M,\,\mu_0)$. Extending $U_t$ to the operator on $L^2(M,\,\mu_0)$, we have

$$\langle P_t\phi,\,\varphi\rangle_{L^2(M,\mu_0)} = \langle \phi,\,U_t\varphi\rangle_{L^2(M,\mu_0)} \quad \text{for all } \phi,\,\varphi \in L^2(M,\,\mu_0).$$

This implies that $P_t$ is the Perron-Frobenius operator on $L^2(M,\,\mu_0)$ in the usual sense (see [LM]). Note that we have $P_tU_t = Id$ on $L^2(M,\,\mu_0)$ since

$$\langle \phi,\,P_tU_t\varphi\rangle = \langle U_t\phi,\,U_t\varphi\rangle = \langle \phi,\,\varphi\rangle \quad \text{for all } \phi,\,\varphi \in L^2(M,\,\mu_0). \tag{6.1}$$

**Proposition 12** *We assume that the Banach space $V$ is embedded in $L^2(M,\,\mu_0)^*$ and that the real-valued function $g \in V \cap B$ satisfies the condition* (G1). *Then $\sigma_g = 0$ if and only if there exists a function $f \in L^2(M,\,\mu_0)$ such that $(d/dt)(f \circ S_t)|_{t=0} = g$.*

*Proof.* Suppose that there exists a function $f \in L^2(M,\,\mu_0)$ such that $(d/dt)(f \circ S_t)|_{t=0} = g$. Then we have

$$\frac{1}{t}\int_M \Big(\int_0^t g \circ S_\tau d\tau\Big)^2 d\mu_0 = \frac{1}{t}\int_M (f \circ S_t - f)^2 d\mu_0 \le \frac{4}{t}\int_M f^2 d\mu_0.$$

This implies $\sigma_g = 0$. Below we prove the converse.

Suppose that $\sigma_g = 0$. Since $V$ is identified with a subspace of $L^2(M,\,\mu_0)$ and since $g \in V \cap B$ by the assumption, $Lg := \int_0^\infty P_\tau g d\tau$ converges absolutely in $L^2(M,\,\mu_0)$. Let $g_t = \int_0^t g \circ S_\tau d\tau$ for $t > 0$. Then

$$\begin{aligned} 0 = \sigma_g^2 &= \sum_{k=-\infty}^{\infty} \int g_t \cdot g_t \circ S_{|k|t} d\mu_0 \\ &= \sum_{k=-\infty}^{\infty} \int P_{|k|t} g_t \cdot g_t d\mu_0 = \int \Big(2\sum_{k=0}^{\infty} P_{kt} g_t - g_t\Big) g_t d\mu_0. \end{aligned}$$

Putting $f_t = \sum_{k=0}^\infty P_{kt} g_t \in L^2(M,\mu_0)$, we have

$$0 = \sigma_g^2 = \int (2f_t - f_t + P_t f_t)(f_t - P_t f_t) d\mu_0 = \int f_t^2 - (P_t f_t)^2 d\mu_0.$$

Therefore, by (6.1), we obtain

$$\int (f_t - U_t P_t f_t)^2 d\mu_0 = \int f_t^2 - (P_t f_t)^2 d\mu_0 = 0.$$

That is, we have $f_t = U_t P_t f_t$. Using this, we obtain

$$\begin{aligned} f_t &= \sum_{k=0}^{\infty} P_{kt} \int_0^t U_\tau g d\tau \\ &= g_t + \sum_{k=1}^{\infty} P_{kt} \int_0^t U_\tau g d\tau = g_t + \sum_{k=1}^{\infty} P_{kt-t} P_t \int_0^t U_\tau g d\tau \\ &= g_t + \sum_{k=1}^{\infty} P_{kt-t} \int_0^t P_{t-\tau} g d\tau = g_t + \sum_{k=1}^{\infty} \int_{kt-t}^{kt} P_\tau g d\tau = g_t + Lg \end{aligned}$$

and

$$\begin{aligned} U_t P_t f_t &= \sum_{k=0}^{\infty} U_t P_{kt} \int_0^t P_{t-\tau} g d\tau = \sum_{k=0}^{\infty} U_t P_{kt} \int_0^t P_\tau g d\tau \\ &= \sum_{k=0}^{\infty} U_t \int_{kt}^{t+kt} P_\tau g d\tau = U_t \Big( \sum_{k=0}^{\infty} \int_{kt}^{t+kt} P_\tau g d\tau \Big) = U_t L g. \end{aligned}$$

Therefore we have $g_t = U_t Lg - Lg$. Mulutiplying the both sides of this equality by $1/t$ and considering the limit as $t \to 0$, we concluude $g = (d/dt)(Lg \circ S_t)|_{t=0}$. □

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Hokkaido University
School of Mathematics
Sapporo 060-0810, Japan
E-mail: iwata@math.sci.hokudai.ac.jp