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Time regularity for aperiodic or irreducible random walks on groups

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Abstract. This paper studies time regularity for the random walk governed by a probability measure μ on a locally compact, compactly generated group G. If μ is eventually coset aperiodic on G and satisfies certain additional conditions, we establish that the associated Markov operator T_μ is analytic in $L^2(G)$, that is, one has an estimate $\|(I - T_\mu)T_\mu^n\| \le cn^{-1}$, $n \in \mathbb{N}$, in L^2 operator norm. Alternatively, if μ is irreducible with period d and satisfies certain conditions, we show that T_μ^d is analytic in $L^2(G)$. To obtain these results, we develop a number of interesting algebraic and spectral properties of coset aperiodic or irreducible measures on groups.

Key words: Locally compact group, probability measure, convolution operator, irreducible, random walk.

1. Introduction and statement of results

This paper continues the study of a fundamental question about time regularity for random walks on a locally compact group G. Namely, if T_μ denotes the Markov operator, acting in $L^2(G)$, associated with a probability measure μ on G, one asks: for which μ does an estimate

$$\|(I - T_\mu)T_\mu^n\|_{2\to 2} \le cn^{-1}$$

hold for all $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$? Here, $\| \cdot \|_{p\to p}$ denotes the operator norm for operators in $L^p(G)$, $1 \le p \le \infty$. Note that, more generally, a bounded linear operator $S \in \mathcal{L}(X)$ in a Banach space X is said to be *analytic* (see [6, 4]) if $\|(I - S)S^n\| \le cn^{-1}$ for some $c > 0$ and all $n \in \mathbb{N}$. For example, it is well known that if the Markov operator T_μ is symmetric then $(T_\mu)^2$ is analytic in $L^2(G)$ (this is a consequence of the spectral theorem for self-adjoint operators). Our aim, though, is to consider more general non-symmetric random walks.

The author recently studied analyticity of T_μ in [7] and [8], with the most comprehensive results being obtained in [8]. See also [1, 2] for es-

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timates implying analyticity in the particular case of groups which have polynomial volume growth. In the present paper, we extend the results of [8] in two new directions, involving *coset aperiodic* or *irreducible* probability measures. Our main theorems state that if μ is eventually coset aperiodic, (or irreducible of period $d \in \mathbb{N}$) and certain extra conditions hold, then T_μ (or respectively, T_μ^d) is analytic in L^2.

The hypotheses of these theorems are easy to check in particular cases, and the theorems apply in great generality on locally compact, compactly generated groups. We therefore feel that the current paper is a significant advance on [8].

Moreover, the algebraic and spectral properties of coset aperiodic or irreducible measures on groups, developed in the present paper, are of independent interest and might be useful in other studies.

To state our main results precisely we fix some notation (for further background material, see [14, 9, 8, 7]). Throughout the paper, G will be a compactly generated locally compact group and $\mathbb{P}(G)$ the set of regular Borel probability measures on G. For $\mu \in \mathbb{P}(G)$, the (right-invariant) Markov operator T_μ is defined by the convolution

$$(T_\mu f)(h) = (\mu * f)(h) = \int_G d\mu(g) f(g^{-1}h)$$

for $h \in G$ and $f \in L^p := L^p(G; dg)$, $1 \leq p \leq \infty$, where dg is a fixed left invariant Haar measure on G. Note that $\|T_\mu\|_{p\to p} \leq 1$ and

$$T_\mu^n f = \mu^n * f = T_{\mu^n} f$$

for $n \in \mathbb{N}$, where $\mu^n := \mu * \mu * \cdots * \mu$ denotes the n-th convolution power of μ.

For Borel measures ν_1, ν_2 on G, the notation $\nu_1 \geq \nu_2$ will mean that $\nu_1 - \nu_2$ is a positive measure. One says that $\mu \in \mathbb{P}(G)$ is *spread out* if there exist $n \in \mathbb{N}$, $c > 0$ and a non-empty open set $V \subseteq G$ such that $\mu^n \geq c\chi_V$ (here, χ_V denotes the characteristic function of V or, more precisely, the measure $\chi_V(g)dg$ on G).

Since G is compactly generated, we may fix an open, relatively compact neighborhood U of the identity $e \in G$ which is symmetric (that is, $U = U^{-1}$) and generates G. The modulus $\rho = \rho_U : G \to \mathbb{N}$ is defined by

$$\rho(g) = \inf\{n \in \mathbb{N} : g \in U^n\}, \quad g \in G,$$

where $U^n := \{g_1 g_2 \cdots g_n \colon g_1, \ldots, g_n \in U\}$.

One says that $\mu \in \mathbb{P}(G)$ is *centered* if

$$\int_G d\mu(g)\eta(g) = 0$$

for all $\eta \in \mathrm{Hom}(G, \mathbb{R})$, where $\mathrm{Hom}(G, \mathbb{R})$ is the set of all continuous group homomorphisms $\eta \colon G \to \mathbb{R}$, and the integral is assumed to converge absolutely.

Let δ_g denote the probability measure concentrated at a point $g \in G$, and for $\mu \in \mathbb{P}(G)$ define the involute $\mu^* \in \mathbb{P}(G)$ by $\mu^*(A) = \mu(A^{-1})$ for Borel sets $A \subseteq G$. One says that μ is *symmetric* if $\mu = \mu^*$; it is easy to see that any symmetric μ which satisfies $\int_G d\mu\rho < \infty$ is centered, but centered measures need not be symmetric in general.

We recall (compare [9, 13]) some standard concepts involving the support $\mathrm{supp}(\mu)$ of μ. One says that $\mu \in \mathbb{P}(G)$ is *adapted* (respectively, *irreducible*) if the smallest closed subgroup (respectively, the smallest closed sub-semigroup) of G containing $\mathrm{supp}(\mu)$ equals G. Clearly, an irreducible measure is adapted. Irreducibility is equivalent to the condition that

$$G = \overline{\bigcup_{n=1}^{\infty} (\mathrm{supp}(\mu))^n} \tag{1}$$

where the bar denotes topological closure.

Next, we say that $\mu \in \mathbb{P}(G)$ is *coset aperiodic* if there do not exist $g_0 \in G$ and a proper closed subgroup H of G such that $\mathrm{supp}(\mu)$ is contained in $g_0 H$ (proper means that $H \neq G$). Since $g_0 H = (g_0 H g_0^{-1}) g_0$, one can equivalently use right cosets $H g_0$ in place of left cosets $g_0 H$ in this definition. Lastly, say that $\mu \in \mathbb{P}(G)$ is *eventually coset aperiodic* if μ^{n_0} is coset aperiodic for some $n_0 \in \mathbb{N}$. A probability measure which is eventually coset aperiodic must be adapted (for if μ were not adapted, one would have $\bigcup_{n=1}^{\infty} \mathrm{supp}(\mu^n) \subseteq H$ for some proper closed subgroup $H \subseteq G$). However, simple examples show that (eventual) coset aperiodicity neither implies nor is implied by irreducibility: consider the measures $\mu_1 := 2^{-1}(\delta_0 + \delta_1)$, $\mu_2 := 2^{-1}(\delta_1 + \delta_{-1})$ on the group $\mathbb{Z}$ of integers.

The most important results of [8] can be summarized in the following theorem. Let $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$ be the open unit disc in $\mathbb{C}$, and denote by $\sigma_{L^2}(S)$ the L^2 spectrum of an operator $S \in \mathcal{L}(L^2)$.

Theorem 1.1 *Let $\mu \in \mathbb{P}(G)$ be centered, adapted, spread out, and such that $\int_G d\mu\rho^2 < \infty$. Then the semigroup $(e^{-t(I-T_\mu)})_{t\geq 0}$ is bounded analytic in L^2, in the sense that one has an estimate*

$$\|e^{-t(I-T_\mu)}\|_{2\to 2} + t\|(I-T_\mu)e^{-t(I-T_\mu)}\|_{2\to 2} \leq c$$

uniformly for all $t > 0$.

If, in addition, $\sigma_{L^2}(T_\mu) \subseteq \mathbb{D} \cup \{1\}$, then T_μ is analytic in L^2.

The second part of Theorem 1.1 follows immediately from the first part together with a general theorem of Nevanlinna (see [11, Theorem 4.5.4], [12, Theorem 2.1], and [4, 5]) about analytic operators. Indeed, Nevanlinna's theorem states that an operator $S \in \mathcal{L}(X)$ is analytic and power-bounded in the complex Banach space X (where power-bounded means that $\sup_{n\in\mathbb{N}} \|S^n\| < \infty$) if, and only if, the semigroup $(e^{-t(I-S)})_{t\geq 0}$ is bounded analytic and the spectrum of S is contained in $\mathbb{D} \cup \{1\}$.

Our main theorem for eventually coset aperiodic measures is the following.

Theorem 1.2 *Let $\mu \in \mathbb{P}(G)$ be spread out and eventually coset aperiodic. Then $\sigma_{L^2}(T_\mu) \subseteq \mathbb{D} \cup \{1\}$.*

If, in addition, μ is centered and $\int_G d\mu\,\rho^2 < \infty$, then T_μ is analytic in L^2.

Observe that the second statement of Theorem 1.2 follows immediately from the first statement together with Theorem 1.1.

When G is connected, we will see that every spread out probability measure on G is eventually coset aperiodic (see Remark (3) in Section 2 below). Hence Theorem 1.2 implies the following result, which seems to be new.

Corollary 1.3 *Let G be connected and let $\mu \in \mathbb{P}(G)$ be spread out. Then $\sigma_{L^2}(T_\mu) \subseteq \mathbb{D} \cup \{1\}$.*

If, in addition, μ is centered and $\int_G d\mu\rho^2 < \infty$, then T_μ is analytic in L^2.

We mention that our results may fail for non-spread out measures. For example, let $\mu \in \mathbb{P}(\mathbb{R})$ be a measure on $\mathbb{R}$ such that $\mathrm{supp}(\mu) = \{-1, 2^{1/2}, 3^{1/2}\}$. Then μ is coset aperiodic, but straightforward arguments using the Fourier transform (compare [8, Section 5]) show that $\{z \in \mathbb{C}\colon |z| = 1\} \subseteq \sigma_{L^2}(T_\mu)$.

Let us now consider irreducible measures. If $\mu \in \mathbb{P}(G)$ is irreducible and spread out, one may show (see Lemma 3.1 below) that $e \in \operatorname{supp}(\mu^n)$ for some $n \in \mathbb{N}$; hence the *period*

$$d = d(\mu) := \gcd\{n \in \mathbb{N} \colon e \in \operatorname{supp}(\mu^n)\} \tag{2}$$

of μ is a well-defined positive integer, where $\gcd(A)$ denotes the greatest common divisor of a non-empty set $A \subseteq \mathbb{N}$.

The basic properties of periods are well known in the situation of irreducible Markov chains on discrete spaces (for example, see [15]). In our setting, however, the group G need not be discrete and so some of the theory needs to be re-developed.

Our main analyticity result for irreducible measures on groups is the following.

Theorem 1.4 *Let $\mu \in \mathbb{P}(G)$ be centered, irreducible and spread out, such that $\int_G d\mu\rho^2 < \infty$. Then T_μ^d is analytic in L^2, where d is the period of μ. Hence one has an estimate*

$$\|(I - T_\mu^d)T_\mu^n\|_{2\to 2} \le cn^{-1}$$

for all $n \in \mathbb{N}$.

Theorem 1.4 is a fundamental result for the L^2 analysis of irreducible random walks, and appears to be new even in the well-studied case where G is a discrete group.

It turns out that irreducibility and coset aperiodicity are linked as follows (see Proposition 3.3 and Corollary 3.4 below). For μ irreducible and spread out of period d, μ^d is eventually coset aperiodic in the subgroup of G generated by $\operatorname{supp}(\mu^d)$. This fact will allow us to derive Theorem 1.4 using Theorem 1.2.

We remark that, whenever $\mu \in \mathbb{P}(G)$ is irreducible and spread out of period d, one has $\sigma_{L^2}(T_\mu) \subseteq \mathbb{D} \cup R_d$ where R_d is the set of complex d-th roots of unity. See the end of Section 4 for details.

Finally, we should mention that our results only provide useful information for groups which are *amenable*. In fact, if G is non-amenable and $\mu \in \mathbb{P}(G)$ is adapted, it is known (see [3]) that the L^2 spectral radius $r(T_\mu) := \lim_{n\to\infty}(\|T_\mu^n\|_{2\to 2})^{1/n}$ is strictly less than 1; then the norms $\|T_\mu^n\|_{2\to 2}$ decrease exponentially as $n \to \infty$, and it follows trivially that T_μ is analytic in L^2.

2. Coset aperiodicity and proof of Theorem 1.2

Before beginning the proof of Theorem 1.2, let us record some basic facts about coset aperiodicity and eventual coset aperiodicity which will be needed.

We shall often use the fact that $\mathrm{supp}(\nu_1 * \nu_2) = \overline{(\mathrm{supp}(\nu_1))(\mathrm{supp}(\nu_2))}$ for any probability measures $\nu_1, \nu_2 \in \mathbb{P}(G)$.

Lemma 2.1 *Let $\mu \in \mathbb{P}(G)$. The following conditions are all equivalent.*
(I) *μ is coset aperiodic.*
(II) *$\delta_g * \mu$ is adapted for all $g \in G$.*
(III) *$\mu * \delta_g$ is adapted for all $g \in G$.*
(IV) *$\mu^* * \mu$ is adapted.*

Moreover, if these conditions are satisfied, then $\nu * \mu$ and $\mu * \nu$ are coset aperiodic elements of $\mathbb{P}(G)$ for all $\nu \in \mathbb{P}(G)$.

Proof. Given $g \in G$, observe that $\delta_g * \mu$ is not adapted if and only if $\mathrm{supp}(\delta_g * \mu) \subseteq H$ for some proper closed subgroup H of G, or in other words, $\mathrm{supp}(\mu) \subseteq g^{-1}H$. From this one sees that Conditions (I) and (II) are equivalent.

An analogous argument shows that (I) and (III) are equivalent.

Next, given any $\nu \in \mathbb{P}(G)$, let $g_0 \in \mathrm{supp}(\nu)$ and note that

$$\mathrm{supp}(\delta_{gg_0} * \mu) = gg_0 \,\mathrm{supp}(\mu) \subseteq \mathrm{supp}(\delta_g * \nu * \mu)$$

for any $g \in G$. If Condition (II) holds for μ, then it follows that $\delta_g * \nu * \mu$ is adapted for all $g \in G$, and hence $\nu * \mu$ is coset aperiodic, by the equivalence of Conditions (I) and (II) applied to $\nu * \mu$.

In particular, by setting $\nu = \mu^*$ we see that (II) implies (IV).

By a similar argument, Condition (III) implies that $\mu * \nu$ is coset aperiodic for any $\nu \in \mathbb{P}(G)$.

Finally, if Condition (I) does not hold, then $\mathrm{supp}(\mu) \subseteq Hg_1$ for some $g_1 \in G$ and some proper closed subgroup H of G. Consequently

$$\mathrm{supp}(\mu^* * \mu) \subseteq (Hg_1)^{-1}(Hg_1) = g_1^{-1}Hg_1,$$

so that $\mu^* * \mu$ is not adapted. Thus (IV) implies (I), and the lemma is completely proved. □

Here are some further useful remarks about coset aperiodicity and eventual coset aperiodicity.

(1) A sufficient (though not necessary) condition for aperiodicity is the following. If $\mu \in \mathbb{P}(G)$ is adapted and $e \in \operatorname{supp}(\mu)$ then μ is coset aperiodic. (For, in this case

$$\operatorname{supp}(\mu) = e\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\mu^* * \mu)$$

which implies that $\mu^* * \mu$ is adapted. Hence μ is coset aperiodic by Lemma 2.1.)

(2) If $\mu \in \mathbb{P}(G)$ is eventually coset aperiodic, that is, μ^{n_0} is coset aperiodic for some $n_0 \in \mathbb{N}$, then μ^n is coset aperiodic for all $n \geq n_0$ (this remark follows by applying the last statement of Lemma 2.1 to μ^{n_0}).

(3) If G is connected and $\mu \in \mathbb{P}(G)$ is spread out, then μ is eventually coset aperiodic. (To show this fact, choose $n_0 \in \mathbb{N}$ such that $\mu^{n_0} \geq c\chi_V$ for some $c > 0$ and some non-empty open set $V \subseteq G$. One easily deduces that $(\mu^{n_0})^* * \mu^{n_0} \geq c'\chi_W$ for some open neighborhood W of e. Since G is connected one has $G = \bigcup_{j=1}^{\infty} W^j$, hence $(\mu^{n_0})^* * \mu^{n_0}$ is adapted and μ^{n_0} is coset aperiodic.)

Note that Corollary 1.3 is a consequence of Theorem 1.2 together with Remark (3) above.

We now begin the proof of Theorem 1.2. By Theorem 1.1, we only have to show the inclusion $\sigma_{L^2}(T_\mu) \subseteq \mathbb{D} \cup \{1\}$. Our strategy is to first show this inclusion under the extra assumptions that μ is coset aperiodic and $\int_G d\mu \rho^2 < \infty$, by a comparison of certain quadratic forms in L^2. Then we shall remove the extra assumptions.

Define the difference operators ∂_g by $(\partial_g f)(h) := f(g^{-1}h) - f(h)$ for $g, h \in G$ and functions $f\colon G \to \mathbb{C}$. The "Dirichlet norm" Γ_2 is defined by

$$\Gamma_2(f) = \Big(\int_U du \|\partial_u f\|_2^2\Big)^{1/2}$$

for all $f \in L^2$. The following quadratic form estimates are variations of estimates found in [8, 7] and [14, Chapter VII].

Proposition 2.2 *Let $\mu \in \mathbb{P}(G)$.*

(I) *If μ is adapted and spread out, then there exists $c > 0$ such that*

$$\operatorname{Re}((I - T_\mu)f, f) \geq c^{-1}\Gamma_2(f)^2$$

for all $f \in L^2$.

(II) *If* $\int_G d\mu\rho^2 < \infty$, *then there exists* $c' > 0$ *such that*

$$\mathrm{Re}((I - T_\mu)f,\, f) \leq c'\Gamma_2(f)^2$$

for all $f \in L^2$.

Proof. For part (I), see [8, Proposition 3.3].

To obtain part (II), one observes that

$$\mathrm{Re}((I - T_\mu)f,\, f) = ((I - T_{\overline{\mu}})f,\, f)$$

for $f \in L^2$, where $\overline{\mu} \in \mathbb{P}(G)$ is defined by $\overline{\mu} := 2^{-1}(\mu + \mu^*)$. A standard calculation using the symmetry $(\overline{\mu})^* = \overline{\mu}$ of $\overline{\mu}$ yields

$$((I - T_{\overline{\mu}})f,\, f) = 2^{-1} \int_G d\overline{\mu}(g) \|\partial_g f\|_2^2.$$

Then by the standard estimate (see [14, Proposition VII.3.2])

$$\|\partial_g f\|_2^2 \leq c\rho(g)^2\Gamma_2(f)^2$$

for all $g \in G$, $f \in L^2$, since $\int_G d\overline{\mu}\rho^2 = \int_G d\mu\rho^2 < \infty$ we obtain the estimate of part (II). □

The next lemma establishes a special case of Theorem 1.2. For a bounded linear operator $S \in \mathcal{L}(L^2)$, one defines the "numerical range"

$$\Theta(S) := \{(Sf,\, f) \colon f \in L^2,\, \|f\|_2 = 1\} \subseteq \mathbb{C}.$$

It is a standard fact (see [10, Corollary V.3.3]) that

$$\sigma_{L^2}(S) \subseteq \overline{\Theta(S)}$$

where the bar denotes closure in $\mathbb{C}$.

Lemma 2.3 *Let* $\mu \in \mathbb{P}(G)$ *be coset aperiodic. Suppose that* $\int_G d\mu\rho^2 < \infty$ *and that* $\mu \geq c\chi_V$ *for some* $c > 0$ *and some non-empty open set* $V \subseteq G$. *Then* $\sigma_{L^2}(T_\mu) \subseteq \overline{\Theta(T_\mu)} \subseteq \mathbb{D} \cup \{1\}$.

Proof. The hypotheses imply that the symmetric measure $\mu^* * \mu$ is spread out and, by Lemma 2.1, adapted. Thus Proposition 2.2 (I) yields an estimate

$$\begin{aligned}\mathrm{Re}((I - T_{\mu^**\mu})f,\, f) &= ((I - T_{\mu^**\mu})f,\, f) \\ &= \|f\|_2 - \|T_\mu f\|_2^2 \geq c^{-1}\Gamma_2(f)^2\end{aligned}$$

for all $f \in L^2$. Also, $\mathrm{Re}((I - T_\mu)f,\, f) \le c'\Gamma_2(f)^2$ by Proposition 2.2 (II). Hence for some $b > 0$ we have

$$\|f\|_2^2 - \|T_\mu f\|_2^2 \ge b\,\mathrm{Re}((I - T_\mu)f,\, f)$$

for all $f \in L^2$. Taking $\|f\|_2 = 1$, and noting that $\|T_\mu f\|_2^2 \ge |(T_\mu f,\, f)|^2$ by Cauchy-Schwartz, it follows that

$$\Theta(T_\mu) \subseteq \Gamma_b$$

where by definition $\Gamma_b := \{z \in \mathbb{C}\colon |z| \le 1 \text{ and } 1 - |z|^2 \ge b(1 - \mathrm{Re}(z))\}$. Since Γ_b is closed in $\mathbb{C}$, and $\Gamma_b \subseteq \mathbb{D} \cup \{1\}$, the lemma follows. □

Remark The last part of the proof of Lemma 2.3 is a quite general observation: for a bounded operator S in L^2 (or more generally in Hilbert space) with $\|S\| \le 1$, a quadratic form estimate of the type

$$\|f\|_2^2 - \|Sf\|_2^2 \ge b\,\mathrm{Re}((I - S)f,\, f), \quad f \in L^2,$$

implies that $\sigma_{L^2}(S) \subseteq \overline{\Theta(S)} \subseteq \mathbb{D} \cup \{1\}$.

The following comparison lemma (which is a generalization of [8, Theorem 1.7]) will allow us to extend the result of Lemma 2.3.

Lemma 2.4 *Let μ, $\nu \in \mathbb{P}(G)$ satisfy $\mu \ge \alpha\nu$ for some constant $\alpha > 0$. If $\overline{\Theta(T_\nu)} \subseteq \mathbb{D} \cup \{1\}$, then $\sigma_{L^2}(T_\mu) \subseteq \overline{\Theta(T_\mu)} \subseteq \mathbb{D} \cup \{1\}$.*

Proof. Consider the operator $S := T_\mu - \alpha T_\nu$, which acts by $Sf = (\mu - \alpha\nu) * f$ for all $f \in L^2$. Since $\mu - \alpha\nu$ is a positive measure by hypothesis, and $(\mu - \alpha\nu)(G) = 1 - \alpha$, it follows that $\alpha \in (0,\, 1]$ and that $\|S\|_{2\to2} \le 1 - \alpha$. For $f \in L^2$ with $\|f\|_2 = 1$, we have

$$(T_\mu f,\, f) = (Sf,\, f) + \alpha(T_\nu f,\, f) \in \{z \in \mathbb{C}\colon |z| \le 1 - \alpha\} + \alpha\overline{\Theta(T_\nu)}.$$

As a consequence,

$$\begin{aligned}\overline{\Theta(T_\mu)} &\subseteq \{z \in \mathbb{C}\colon |z| \le 1 - \alpha\} + \alpha\overline{\Theta(T_\nu)}\\ &\subseteq \{z \in \mathbb{C}\colon |z| \le 1 - \alpha\} + \alpha(\mathbb{D} \cup \{1\}) \subseteq \mathbb{D} \cup \{1\},\end{aligned}$$

where the last inclusion used that $\alpha \in (0,\, 1]$. □

The next lemma is elementary and not new, but we give a proof for the convenience of the reader.

Lemma 2.5 (See [8, Lemma 4.1]) *Let μ, $\nu \in \mathbb{P}(G)$ and g_0, $h_0 \in G$. Suppose that $g_0 \in \mathrm{supp}(\mu)$ and that $\nu \geq c\chi_V$ for some $c > 0$ and some relatively compact open set V with $h_0 \in V$. Then there exists $c' > 0$ and a relatively compact open set W such that $g_0 h_0 \in W$ and $\mu * \nu \geq c'\chi_W$. Moreover, W can be chosen to depend only on g_0, h_0 and V.*

Proof. Note that $(\mu * \chi_V)(g) = \mu(gV^{-1})$ for all $g \in G$. By continuity of the group multiplication, we can choose relatively compact open sets W, W' with $g_0 h_0 \in W$, $g_0 \in W'$ and $(W')^{-1}W \subseteq V$. It follows that $gV^{-1} \supseteq W'$ for all $g \in W$, and setting $\varepsilon := \mu(W') > 0$ we have

$$(\mu * \chi_V)(g) = \mu(gV^{-1}) \geq \varepsilon$$

for all $g \in W$. Then $\mu * \nu \geq c(\mu * \chi_V) \geq c\varepsilon\chi_W$. □

Proof of Theorem 1.2. Let $\mu \in \mathbb{P}(G)$ be spread out and eventually coset aperiodic. Since μ is spread out and by using Lemma 2.5, there exists an $n_1 \in \mathbb{N}$ such that

$$\mu^n \geq c_n \chi_{V_n}$$

for all $n \geq n_1$, where $c_n > 0$ are constants and $V_n \subseteq G$ are non-empty open sets. Also, by Remark (2) after Lemma 2.1 we may choose an $n_2 \in \mathbb{N}$ such that μ^n is coset aperiodic for all $n \geq n_2$.

Now set $N := \max\{n_1, n_2\}$, and take an arbitrary $n \in \mathbb{N}$ with $n \geq N$. Consider the measure

$$\nu := \gamma\rho^{-2}\mu^n,$$

where the constant $\gamma := (\int_G d\mu^n \rho^{-2})^{-1} > 0$ is such that $\nu \in \mathbb{P}(G)$. Then $\int_G d\nu\rho^2 < \infty$, and since the function $g \mapsto \rho(g)^{-2}$ is strictly positive on G, it is clear that the measure ν satisfies all the hypotheses of Lemma 2.3. Hence $\overline{\Theta(T_\nu)} \subseteq \mathbb{D} \cup \{1\}$ by Lemma 2.3. Because $\rho^{-2} \leq 1$ we have $\mu^n \geq \gamma^{-1}\nu$, so that applying Lemma 2.4 yields

$$\sigma_{L^2}(T_\mu^n) \subseteq \mathbb{D} \cup \{1\}.$$

But the spectral mapping theorem for bounded linear operators (see for example [16, Section VIII.7]) implies that $\sigma_{L^2}(T_\mu^n) = \{\lambda^n \colon \lambda \in \sigma_{L^2}(T_\mu)\}$. Therefore

$$\sigma_{L^2}(T_\mu) \subseteq \mathbb{D} \cup R_n$$

for any $n \in \mathbb{N}$ with $n \geq N$, where $R_n := \{\tau \in \mathbb{C} \colon \tau^n = 1\}$ denotes the set of n-th roots of unity. Since $R_N \cap R_{N+1} = \{1\}$, we conclude that $\sigma_{L^2}(T_\mu) \subseteq \mathbb{D} \cup \{1\}$. The proof of Theorem 1.2 is complete. □

We end this section with a simple example for which μ^2 is coset aperiodic although μ is not coset aperiodic. This demonstrates that, in general, coset aperiodicity is a stronger condition than eventual coset aperiodicity.

In what follows, for $m \in \mathbb{N}$ we use the notation $\mathbb{Z}_m := \{0, 1, \ldots, m-1\}$ for the finite cyclic group of integers modulo m, under the operation $+_m$ of addition modulo m.

Example Fix an odd integer $m \geq 3$, and let G be the semidirect product of $\mathbb{Z}_m$ with $\mathbb{Z}_2$ with respect to the action $\gamma \colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_m)$ defined by

$$\gamma(0)s = s, \quad \gamma(1)s = -s$$

for all $s \in \mathbb{Z}_m$, where $-s$ denotes the $\mathbb{Z}_m$-inverse of s. Thus $G = \{(s, z) \colon s \in \mathbb{Z}_m,\, z \in \mathbb{Z}_2\}$ is a finite solvable discrete group with product $(s, z)(s', z') = (s +_m \gamma(z)s',\, z +_2 z')$. In fact, G is the dihedral group of motions of a regular m-gon.

Consider the subgroup $H = \{(0, 0), (0, 1)\}$ of G, let $g_0 := (1, 0) \in G$, and let $\mu \in \mathbb{P}(G)$ be such that

$$\mathrm{supp}(\mu) = g_0 H = \{(1, 0), (1, 1)\}.$$

Thus μ is not coset aperiodic. But one computes that

$$\mathrm{supp}(\mu^2) = (g_0 H)(g_0 H) = \{(0, 0), (2, 0), (2, 1), (0, 1)\}.$$

Since m is odd, the element $(2, 0)$ generates the subgroup $\{(s, 0) \colon s \in \mathbb{Z}_m\}$ of G. One easily deduces that the set $\mathrm{supp}(\mu^2)$ generates G, and since $e = (0, 0) \in \mathrm{supp}(\mu^2)$ we conclude from Remark (1) above that μ^2 is coset aperiodic.

3. Irreducible measures

In this section, we develop the basic properties of irreducible, spread out measures on G. The key result is Proposition 3.3 below, which gives a finite partition of G into cosets such that, roughly speaking, the random walk on G governed by μ moves cyclically between the elements of the partition. The existence of such a partition is already well known for irreducible Markov chains on discrete spaces (see for example [15, p.3]).

Before giving Proposition 3.3, we need some lemmas. Adopt the convention that $\mu^0 := \delta_e$ for any $\mu \in \mathbb{P}(G)$.

Lemma 3.1 *Let $\mu \in \mathbb{P}(G)$ be irreducible and spread out. Then for any $g \in G$, there exist $n \in \mathbb{N}$, $c > 0$, and a non-empty open neighborhood V of g, such that*

$$\mu^n \geq c\chi_V.$$

As a consequence,

$$G = \bigcup_{n \in \mathbb{N}} \operatorname{supp}(\mu^n). \tag{3}$$

Lemma 3.1 implies that $\{n \in \mathbb{N} \colon e \in \operatorname{supp}(\mu^n)\}$ is non-empty, so that the period d of μ is well defined (see (2)).

Proof of Lemma 3.1. Since μ is spread out, one has $\mu^{n_1} \geq c\chi_W$ for some $n_1 \in \mathbb{N}$, $c > 0$ and a non-empty open set W. Because μ is irreducible, by (1) there exists an $n_2 \in \mathbb{N}$ such that $gW^{-1} \cap \operatorname{supp}(\mu^{n_2})$ is non-empty. By choosing a $w \in W$ such that $gw^{-1} \in \operatorname{supp}(\mu^{n_2})$, and writing $g = (gw^{-1})w$, we see from Lemma 2.5 that $\mu^{n_1+n_2} \geq c'\chi_V$ for some open set V containing g. □

Lemma 3.2 *Let $\mu \in \mathbb{P}(G)$ be irreducible and spread out with period d. Let $k, l \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and let $g, h \in G$ with $g \in \operatorname{supp}(\mu^k)$, $h \in \operatorname{supp}(\mu^l)$. Then $gh \in \operatorname{supp}(\mu^{k+l})$, and there exists an $n \in \mathbb{N}$ such that $nd > k$ and $g^{-1} \in \operatorname{supp}(\mu^{nd-k})$. If, in addition, $g \in \operatorname{supp}(\mu^{k'})$ for some $k' \in \mathbb{N}_0$, then $k' - k = md$ for some $m \in \mathbb{Z}$.*

Proof. That $gh \in \operatorname{supp}(\mu^{k+l})$ is clear. Next, by (3) we have $g^{-1} \in \operatorname{supp}(\mu^j)$ for some $j \in \mathbb{N}$. Then

$$e = gg^{-1} \in \operatorname{supp}(\mu^{k+j}),$$

and it follows from the definition of the period that d divides $k + j$. Thus $j = nd - k$ for some $n \in \mathbb{N}$, as desired.

Finally, suppose also that $g \in \operatorname{supp}(\mu^{k'})$. By what was just proved, we have $g^{-1} \in \operatorname{supp}(\mu^{sd-k'})$ for some $s \in \mathbb{N}$ with $sd > k'$. Then $e = gg^{-1} \in \operatorname{supp}(\mu^{k+sd-k'})$ and consequently d divides $k - k'$, as required. □

Proposition 3.3 *Let $\mu \in \mathbb{P}(G)$ be irreducible and spread out with period d. Define $G' := \bigcup_{n\in\mathbb{N}_0} \operatorname{supp}(\mu^{nd})$. Then G' is an open normal subgroup of G, $\operatorname{supp}(\mu^d) \subseteq G'$, and $G/G' \cong \mathbb{Z}_d$.*

Let $y \in \operatorname{supp}(\mu)$. Then yG' generates the group G/G', and $G/G' = \{y^j G' \colon j = 0, 1, \ldots, d-1\}$. Moreover, $\operatorname{supp}(\mu) \subseteq yG'$.

If $f \in L^p$ and $\operatorname{supp}(f) \subseteq y^j G'$ for some $j \in \{0, 1, \ldots, d-1\}$, then $\operatorname{supp}(\mu^n * f) \subseteq y^{n+_d j} G'$ for all $n \in \mathbb{N}_0$ (where $+_d$ denotes addition modulo d).

When regarded as an element of $\mathbb{P}(G')$, the measure μ^d is irreducible and spread out with period 1, and eventually coset aperiodic.

Proof of Proposition 3.3. For each $j \in \{0, 1, \ldots, d-1\}$ define a subset $Y_j \subseteq G$ by

$$Y_j := \bigcup_{n\in\mathbb{N}_0} \operatorname{supp}(\mu^{nd+j}),$$

and put $G' := Y_0$. Then $G = \bigcup_{j=0}^{d-1} Y_j$ by Lemma 3.1, and the Y_j's are pairwise disjoint by the last statement of Lemma 3.2. Lemma 3.2 also implies that

$$Y_0^{-1} = Y_0, \quad Y_j^{-1} = Y_{d-j}$$

for $j \in \{1, \ldots, d-1\}$, and that

$$Y_k Y_{k'} \subseteq Y_{k+_d k'}$$

for all $k, k' \in \{0, 1, \ldots, d-1\}$, where $+_d$ denotes addition modulo d. In particular, it follows that $G' = Y_0$ is a subgroup of G. Since μ^d is spread out, G' has non-empty interior and is therefore an open subgroup of G.

Fix an element $y \in \operatorname{supp}(\mu)$. For $j \in \{1, \ldots, d-1\}$, since $y^j \in Y_j$ and $y^{-j} \in Y_{d-j}$, we find that

$$y^j Y_0 \subseteq Y_j Y_0 \subseteq Y_j = y^j(y^{-j} Y_j) \subseteq y^j Y_0,$$

proving that $Y_j = y^j Y_0$. A similar argument yields that $Y_j = Y_0 y^j$. We conclude that $G' = Y_0$ is a normal subgroup of G, and that the group G/G' consists of the d distinct elements $y^j G'$, $j \in \{0, 1, \ldots, d-1\}$. Thus $yG' \in G/G'$ generates G/G', and $G/G' \cong \mathbb{Z}_d$.

Next, it is obvious from the definitions that $\operatorname{supp}(\mu^d) \subseteq Y_0 = G'$ and $\operatorname{supp}(\mu) \subseteq Y_1 = yG'$. The statement about $\operatorname{supp}(\mu^n * f)$ follows easily from what we have already proved.

It is clear that μ^d is spread out and irreducible as an element of $\mathbb{P}(G')$. Moreover, the definition of d implies that $gcd(S') = 1$, where by definition

$$S' := \{n \in \mathbb{N}\colon e \in \operatorname{supp}(\mu^{nd})\}.$$

Thus μ^d has period 1 as an element of $\mathbb{P}(G')$.

Finally, we show that μ^d is eventually coset aperiodic in G'. Now S' has the semigroup property that n, $n' \in S'$ implies $n + n' \in S'$; since $\gcd(S') = 1$, it follows that there exist n_1, $n_2 \in S'$ with $n_2 - n_1 = 1$. Then

$$\operatorname{supp}(\mu^d) = e \operatorname{supp}(\mu^d) \subseteq \operatorname{supp}(\mu^{n_1 d}) \operatorname{supp}(\mu^d) \subseteq \operatorname{supp}(\mu^{n_2 d}),$$

and it follows that $\mu^{n_2 d}$ is adapted in G'. But $e \in \operatorname{supp}(\mu^{n_2 d})$, so by Remark (1) in Section 2, $\mu^{n_2 d}$ is coset aperiodic in G'. □

Corollary 3.4 *Let $\mu \in \mathbb{P}(G)$ be irreducible and spread out with period d. Then μ is eventually coset aperiodic in G if and only if $d = 1$.*

Proof of Corollary 3.4. If $d = 1$, the statement of the corollary is contained in Proposition 3.3. If $d \geq 2$, then it is clear from Proposition 3.3 that for every $n \in \mathbb{N}$, μ^{nd} is not adapted in G, hence μ cannot be eventually coset aperiodic in G (see Remark (2) in Section 2). □

We remark that in the example at the end of Section 2, $\mu \in \mathbb{P}(G)$ is irreducible and spread out with period 1, eventually coset aperiodic but not coset aperiodic.

Corollary 3.5 *Let $\mu \in \mathbb{P}(G)$ be irreducible and spread out with period d. If G is connected, then $d = 1$.*

Proof. By Remark (3) in Section 2, μ is eventually coset aperiodic, and hence $d = 1$ by Corollary 3.4. □

The following positivity result (which is not used in the rest of the paper) gives further information on the convolution powers of an irreducible measure.

Proposition 3.6 *Adopt the hypotheses and notation of* Proposition 3.3. *If K is any compact subset of G', then there exists an $N \in \mathbb{N}$ and numbers $c_n > 0$ such that*

$$\mu^{nd} \geq c_n \chi_K \tag{4}$$

for all $n \in \mathbb{N}$ with $n \geq N$.

Proof. Since μ^d is irreducible in the group G' with period 1, we will assume without loss of generality that $d = 1$ and $G' = G$.

Let n_1 be as in the proof of Proposition 3.3, so that $e \in \mathrm{supp}(\mu^{n_1})$ and $e \in \mathrm{supp}(\mu^{n_1+1})$. For $n \in \mathbb{N}$ with $n \geq n_1^2$, by division

$$n = qn_1 + r = (q - r)n_1 + r(n_1 + 1)$$

for some integers q, r with $q \geq n_1$ and $0 \leq r < n_1$. It follows that

$$e \in \mathrm{supp}(\mu^n)$$

for all $n \geq n_1^2$. Combining this result with Lemmas 3.1 and 2.5, it is easy to argue that for any $g \in G$ there exist an open neighborhood V_g of g and $N(g) \in \mathbb{N}$ such that

$$\mu^n \geq c(n, g)\chi_{V_g}$$

for all $n \geq N(g)$ and some numbers $c(n, g) > 0$. The Proposition then follows by compactness of K. □

4. Proof of Theorem 1.4

Essentially, Theorem 1.4 will be derived by applying Theorem 1.2 to the measure μ^d where d is the period. To see that μ^d is centered in the subgroup of G generated by $\mathrm{supp}(\mu^d)$, we will need the following non-trivial result, whose proof is deferred until Section 5.

Theorem 4.1 *Let G' be an open, normal subgroup of G such that $G/G' \cong \mathbb{Z}_d$ for some $d \in \mathbb{N}$. Fix $y \in G$ such that the element $yG' \in G/G'$ generates the group G/G', and let $\mu \in \mathbb{P}(G)$ with $\mathrm{supp}(\mu) \subseteq yG'$. Then $\mathrm{supp}(\mu^d) \subseteq G'$, so that we may regard μ^d as an element of $\mathbb{P}(G')$.*

If, in addition, $\int_G d\mu\rho < \infty$ and μ is centered in G, then μ^d is centered in G'.

Note that in the statement of Theorem 4.1, μ is not assumed to be irreducible and the integer d need not be the period.

Let us prove Theorem 1.4. Take $\mu \in \mathbb{P}(G)$ as in the hypothesis of the Theorem, with period d, and let G' be defined as in Proposition 3.3. Fix an element $y \in \mathrm{supp}(\mu)$. By Proposition 3.3, $\mu^d \in \mathbb{P}(G')$ is irreducible, spread out, and eventually coset aperiodic on G'. Theorem 4.1 shows that μ^d is centered in G'.

Fix a modulus function $\rho'\colon G' \to \mathbb{N}$ for the compactly generated group G'. Since G/G' is finite, it is easy to see that for some $c > 0$ one has

$$c^{-1}\rho(g') \le \rho'(g') \le c\rho(g') \tag{5}$$

for all $g' \in G'$. From the assumption $\int_G d\mu\rho^2 < \infty$ it is then easy to deduce that $\int_{G'} d\mu^d(\rho')^2 < \infty$. Thus μ^d satisfies all of the hypotheses of Theorem 1.2 on the group G', so by Theorem 1.2 T_{μ^d} is analytic in $L^2(G')$, that is,

$$\|(I - T_\mu^d)T_\mu^{nd} f\|_{L^2(G')} \le cn^{-1}\|f\|_{L^2(G')} \tag{6}$$

for all $n \in \mathbb{N}$ and $f \in L^2(G')$.

Since G' is an open subgroup of G, we may identify $L^2(G')$ with the subspace of $L^2(G)$ consisting of functions with support contained in G' (to justify this, note that left Haar measure on G' is just the restriction to G' of left Haar measure dg on G). Define the right translation operators $R(g) \in \mathcal{L}(L^2(G))$, $g \in G$, by $(R(g)f)(h) := f(hg)$, $f \in L^2(G)$, $h \in G$. For any $f \in L^2(G)$, since G is the disjoint union of the sets $y^jG' = G'y^j$ for $j \in \{0, 1, \ldots, d-1\}$, we can write

$$f = \sum_{j=0}^{d-1} R(y^j)f_j$$

where

$$f_j := R(y^{-j})(\chi_{y^jG'}f) \in L^2(G') \subseteq L^2(G).$$

Because $\|R(g)F\|_2 = \Delta(g)^{-1/2}\|F\|_2$ for any $F \in L^2(G)$, $g \in G$, where $\Delta\colon G \to (0, \infty)$ is the modular function of G, one has

$$\|f_j\|_2 \le c\|f\|_2$$

where $c > 0$ is a constant independent of f. By these observations, together with (6) and the fact that $R(y^j)$ commutes with T_μ, we obtain an estimate

$$\|(I - T_\mu^d)T_\mu^{nd} f\|_2 = \Big\|\sum_{j=0}^{d-1} R(y^j)(I - T_\mu^d)T_\mu^{nd} f_j\Big\|_2$$

$$\leq c' \sum_{j=0}^{d-1} \|(I - T_\mu^d) T_\mu^{nd} f_j\|_2$$

$$\leq cn^{-1} \sum_{j=0}^{d-1} \|f_j\|_2 \leq c' n^{-1} \|f\|_2$$

for all $f \in L^2(G)$ and $n \in \mathbb{N}$. This proves Theorem 1.4. □

Remark For any $\mu \in \mathbb{P}(G)$ which is irreducible and spread out with period d, one has

$$\sigma_{L^2}(T_\mu) \subseteq \mathbb{D} \cup R_d$$

where $R_d = \{\tau \in \mathbb{C}\colon \tau^d = 1\}$. To see this, first apply Theorem 1.2 to the measure $\mu^d \in \mathbb{P}(G')$ to obtain that

$$\sigma_{L^2(G')}(T_{\mu^d}) \subseteq \mathbb{D} \cup \{1\}.$$

But by reasoning similar to the last part of the proof of Theorem 1.4, one can show that $\sigma_{L^2(G')}(T_{\mu^d}) = \sigma_{L^2(G)}(T_{\mu^d})$. Then the claimed result follows by the spectral mapping theorem.

Note that for irreducible random walks on discrete spaces, related spectral results are given in, for example, [15, p. 94–96].

5. Proof of Theorem 4.1

In this section, we shall assume the hypotheses of Theorem 4.1. In particular, G' is an open normal subgroup of G with $G/G' \cong \mathbb{Z}_d$ for some $d \in \mathbb{N}$, and $y \in G$ is a fixed element such that yG' generates G/G'.

The first statement of Theorem 4.1 is easy to prove: for $\mu \in \mathbb{P}(G)$ with $\text{supp}(\mu) \subseteq yG'$, one has

$$\text{supp}(\mu^d) \subseteq (yG')^d = y^d G' = G'$$

as claimed, using the fact that $y^d \in G'$.

It is not trivial, however, to show that μ^d is centered in G'. The reason is that a probability measure on G' need not be centered in G' even if it is centered in G (it is easy to find examples of this where G is a finite extension of $G' \cong \mathbb{R}^s$ for some s).

Recall that $\text{Hom}(H, \mathbb{R})$ denotes the vector space of continuous homomorphisms from a locally compact group H into $\mathbb{R}$. To compare centeredness on G and on G', the following concept is useful. An element $\eta \in \mathrm{Hom}(G', \mathbb{R})$ is said to be G-*invariant* if

$$\eta(gxg^{-1}) = \eta(x)$$

for all $x \in G'$ and $g \in G$. The G-invariant elements form a vector subspace, denoted $\mathrm{Hom}_G(G', \mathbb{R})$, of $\mathrm{Hom}(G', \mathbb{R})$.

Lemma 5.1 *Given $\eta \in \mathrm{Hom}(G', \mathbb{R})$, the following two conditions are equivalent.*

(I) *There exists an $\overline{\eta} \in \mathrm{Hom}(G, \mathbb{R})$ which is an extension of η, that is, $\overline{\eta}(x) = \eta(x)$ for all $x \in G'$.*

(II) *η is G-invariant.*

Moreover, if these conditions are satisfied, then the extension $\overline{\eta} \in \mathrm{Hom}(G, \mathbb{R})$ of η is unique.

Remark Lemma 5.1 implies that $\mathrm{Hom}(G, \mathbb{R})$ is isomorphic as a vector space with $\mathrm{Hom}_G(G', \mathbb{R})$.

Proof of Lemma 5.1. If Condition (I) holds then $\overline{\eta}(gxg^{-1}) = \overline{\eta}(x)$ for all $x \in G'$ and $g \in G$, because $\overline{\eta} \in \mathrm{Hom}(G, \mathbb{R})$. Thus (II) holds.

Conversely, to prove that (II) implies (I), let $\eta \in \mathrm{Hom}(G', \mathbb{R})$ be G-invariant. Since G is the disjoint union of the sets $y^j G'$ for $j \in \{0, 1, \ldots, d-1\}$, we may define $\overline{\eta} \colon G \to \mathbb{R}$ by

$$\overline{\eta}(y^j x) := j d^{-1} \eta(y^d) + \eta(x) \tag{7}$$

for $j \in \{0, 1, \ldots, d-1\}$ and $x \in G'$. By taking $j = 0$, it is clear that $\overline{\eta}$ is an extension of η. To show that $\overline{\eta} \in \mathrm{Hom}(G, \mathbb{R})$, we shall check that

$$\overline{\eta}((y^j z)(y^k w)) = \overline{\eta}(y^j z) + \overline{\eta}(y^k w) \tag{8}$$

for all $j, k \in \{0, 1, \ldots, d-1\}$ and $z, w \in G'$. One writes

$$(y^j z)(y^k w) = y^{j+k}((y^{-k} z y^k) w)$$

in case $j + k \le d - 1$, or

$$(y^j z)(y^k w) = y^{j+k-d}(y^d (y^{-k} z y^k) w)$$

in case $j + k \ge d$, and uses the G-invariance of η to see that both sides of (8) are equal to

$$(j + k) d^{-1} \eta(y^d) + \eta(z) + \eta(w)$$

(we leave the details to the reader). Thus $\overline{\eta} \in \mathrm{Hom}(G, \mathbb{R})$, proving that (II) implies (I).

For the final statement of the lemma, observe that if $\eta_1 \in \mathrm{Hom}(G, \mathbb{R})$ is an extension of $\eta \in \mathrm{Hom}(G', \mathbb{R})$ then η_1 is uniquely determined by the formula

$$\eta_1(g) = d^{-1}\eta_1(g^d) = d^{-1}\eta(g^d)$$

for all $g \in G$, because $g^d \in G'$. □

The idea of the next result is that by averaging with respect to the action of the finite group G/G', we can associate a G-invariant element with any element of $\mathrm{Hom}(G', \mathbb{R})$.

Lemma 5.2 *Given* $\eta \in \mathrm{Hom}(G', \mathbb{R})$, *there exists a unique element* $\widehat{\eta} \in \mathrm{Hom}(G, \mathbb{R})$ *such that*

$$\widehat{\eta}(x) = \frac{1}{d}\sum_{j=0}^{d-1} \eta(y^j x y^{-j}) \tag{9}$$

for all $x \in G'$.

Proof. Define a map $\widehat{\eta}\colon G' \to \mathbb{R}$ by equation (9), and note that $\widehat{\eta} \in \mathrm{Hom}(G', \mathbb{R})$. Because $y^d \in G'$, one has $\eta(y^d x y^{-d}) = \eta(x)$ for all $x \in G'$. From this and (9), it is easy to see that

$$\widehat{\eta}(y^k x y^{-k}) = \widehat{\eta}(x)$$

for all $x \in G'$ and $k \in \{0, 1, \ldots, d-1\}$. Since any $g \in G$ can be written as $g = y^k z$ with $k \in \{0, 1, \ldots, d-1\}$ and $z \in G'$, we deduce that $\widehat{\eta}$ is G-invariant. Hence by Lemma 5.1, $\widehat{\eta}$ extends uniquely to an element $\widehat{\eta} \in \mathrm{Hom}(G, \mathbb{R})$. □

Now the key result to obtain Theorem 4.1 is the following.

Lemma 5.3 *Let* $\mu \in \mathbb{P}(G)$ *satisfy* $\int_G d\mu\rho < \infty$ *and* $\mathrm{supp}(\mu) \subseteq yG'$. *Then for any* $\eta \in \mathrm{Hom}(G', \mathbb{R})$ *one has*

$$\int_{G'} d\mu^d(x)\eta(x) = d\Big(\int_G d\mu(g)\widehat{\eta}(g)\Big), \tag{10}$$

where $\widehat{\eta} \in \mathrm{Hom}(G, \mathbb{R})$ *is as in* Lemma 5.2.

Proof. Because $\operatorname{supp}(\mu) \subseteq yG' = G'y$, the measure

$$\nu_0 := \mu * \delta_{y^{-1}} \in \mathbb{P}(G)$$

has support contained in G', and can therefore be regarded as an element of $\mathbb{P}(G')$. Similarly, for $j \in \{1, \ldots, d-1\}$ the measures $\nu_j := \delta_{y^j} * \nu_0 * \delta_{y^{-j}}$ are elements of $\mathbb{P}(G')$. Now the crucial observation is that

$$\mu^d = (\nu_0 * \delta_y)^d = \nu_0 * \nu_1 * \cdots * \nu_{d-1} * \delta_{y^d}, \tag{11}$$

where the convolutions on the right side can be regarded as taking place on the group G'. Also note that $\int_{G'} d\nu_j \rho' < \infty$ where ρ' is a modulus on G' (see (5)).

For any $\tau_1, \tau_2 \in \mathbb{P}(G')$ satisfying the moment condition $\int_{G'} d\tau_i \rho' < \infty$, $i = 1, 2$, it is easy to see that

$$\int_{G'} d(\tau_1 * \tau_2)\eta = \int_{G'} d\tau_1 \eta + \int_{G'} d\tau_2 \eta$$

for all $\eta \in \operatorname{Hom}(G', \mathbb{R})$ (the moment condition ensures that the integrals are all finite). Combining this fact with (11) and (9), one obtains

$$\begin{aligned} \int_{G'} d\mu^d(x)\eta(x) &= \sum_{j=0}^{d-1} \int_{G'} d\nu_j(x)\eta(x) + \int_{G'} d\delta_{y^d}(x)\eta(x) \\ &= \sum_{j=0}^{d-1} \Big(\int_{G'} d\nu_0(x)\eta(y^j x y^{-j}) \Big) + \eta(y^d) \\ &= d\Big(\int_G d\nu_0(g)\widehat{\eta}(g) \Big) + \eta(y^d); \end{aligned}$$

in the last line, we used the fact that $\operatorname{supp}(\nu_0) \subseteq G'$ to change the integration space from G' to G. But since $\nu_0 = \mu * \delta_{y^{-1}}$ and $\widehat{\eta} \in \operatorname{Hom}(G, \mathbb{R})$, we have

$$\int_G d\nu_0(g)\widehat{\eta}(g) = \int_G d\mu(g)\widehat{\eta}(g) + \widehat{\eta}(y^{-1})$$

and $\widehat{\eta}(y^{-1}) = -d^{-1}\widehat{\eta}(y^d) = -d^{-1}\eta(y^d)$, with the last equality a consequence of (9). Combining these equalities yields (10). □

The second statement of Theorem 4.1 follows immediately from Lemma 5.3, since for μ centered in G the right side of (10) equals zero. This finishes the proof of Theorem 4.1 and Theorem 1.4.

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