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# Existence of horseshoe sets with nondegenerate one-sided homoclinic tangencies in $\mathbb{R}^3$

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**Abstract.** In this paper, we present some class of three dimensional $C^\infty$ diffeomorphisms with nondegenerate one-sided homoclinic tangencies $q$ associated with hyperbolic fixed points $p$ each of which exhibits a horseshoe set. A key point in the proof is the existence of a transverse homoclinic point arbitrarily close to $q$. This result together with Birkhoff-Smale Theorem implies the existence of a horseshoe set arbitrarily close to $q$.

*Key words*: Horseshoe sets, homoclinic tangencies, singular $\lambda$-Lemma, Birkhoff-Smale Theorem.

## 1. Introduction

Let $f$ be a two dimensional diffeomorphism with a nondegenerate one-sided homoclinic tangency $q$ associated with a hyperbolic fixed point $p$. The problem whether such a map has a horseshoe set is studied by some authors, e.g. Gavrilov-Silnikov [6, 7], Li [10], Homburg-Weiss [9], Gonchenko-Gonchenko-Tatjer [8] and so on. Gavrilov and Silnikov showed the existence of a horseshoe set arbitrarily close to the nondegenerate one-sided homoclinic tangency point as illustrated in Fig. 1.1. Li [10] presented existence theorems of a horseshoe set and a non-uniformly horseshoe set arbitrarily close to $q$ under the assumptions same as those of Gavrilov-Silnikov [6, 7]. Homburg and Weiss [9] studied the case which $q$ is a one-sided homoclinic tangency with finite order of contact as illustrated in Fig. 1.1. Moreover, they asked whether their results hold for diffeomorphisms of dimensions greater than two. Rayskin [15] studied the problem where $f$ is an $n(\geq 3)$-dimensional diffeomorphism which admits a two-sided homoclinic tangency associated with a hyperbolic point $p$ with the one-dimensional stable manifold and $(n-1)$-dimensional unstable manifold, as shown in Fig. 1.2. We note that the argument in [15] does not work if the homoclinic tangency is nondegenerate as well as one-sided. In this paper, we consider a three-

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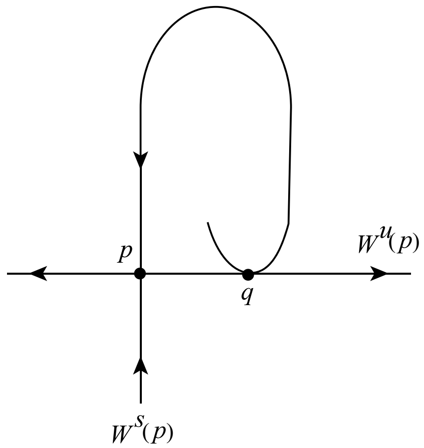


Fig. 1.1.

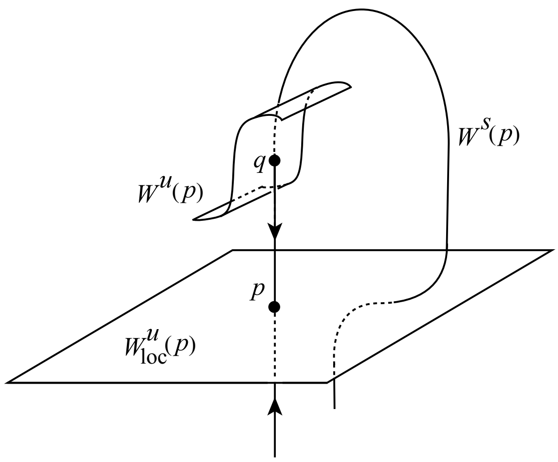


Fig. 1.2.

dimensional diffeomorphism $f$ with a hyperbolic fixed point $p$ with one-dimensional stable and two-dimensional unstable manifolds which admit a one-sided homoclinic quadratic tangency $q$, and show that, if $f$ satisfies some conditions given in Section 3, then $f$ has a horseshoe set. In our proof, a singular $\lambda$-lemma for one-sided homoclinic tangencies is crucial. The lemma corresponds to that for two-sided tangencies in [14, 15]. The one-sided singular $\lambda$-lemma in Section 4 is one of extension of the original $\lambda$-lemma

[12, 16], which can be applied to the case of non-transverse intersection. Once the lemma has been proved, the existences of our desired horseshoe sets is guaranteed by the Birkhoff-Smale Theorem. By the renormalization methods of Palis-Viana [11, 13] along one-parameter families through $f$ which has nondegenerate homoclinic tangency, one can get a sequence of diffeomorphisms arbitrarily $C^2$ close to $f$ which have horseshoe subsets created by Henon-like maps. In this paper, without such one-parameter families or $C^2$-perturbations, we will detect some horseshoe structures of $f$ arbitrary close to the homoclinic tangency.

## 2. Preliminaries

In this section, we will review some definitions and theorems needed to prove our main theorem.

**Definition 2.1** Let $f$ be a $C^r(r \geq 1)$ diffeomorphism on $\mathbb{R}^3$ having a hyperbolic fixed point $p$. The *stable* and *unstable manifolds* $W^s(p)$, $W^u(p)$ of $p$ are defined as

$$W^s(p) = \{x \in \mathbb{R}^3; \|f^n(p) - f^n(x)\| \to 0 \quad \text{for } n \to +\infty\},$$
$$W^u(p) = \{x \in \mathbb{R}^3; \|f^{-n}(p) - f^{-n}(x)\| \to 0 \quad \text{for } n \to +\infty\}.$$

The *local unstable* (resp. *local stable*) manifold, a small neighborhood of $p$ in $W^u(p)$ (resp. $W^s(p)$), is denoted by $W^u_{\text{loc}}(p)$ (resp. $W^s_{\text{loc}}(p)$). A point $q \in W^s(p) \cap W^u(p) \setminus \{p\}$ is called to be *homoclinic* for $p$ if $W^s(p) \cap W^u(p) \setminus \{p\} \neq \emptyset$. Also, if $T_qW^s(p) \oplus T_qW^u(p) = T_q\mathbb{R}^3$, the homoclinic point $q$ is called to be *transverse*. Otherwise, it is called a *homoclinic tangency*.

Let $U_\epsilon(A)$ denote an $\epsilon$-neighborhood of a given point $A \in \mathbb{R}^3$ and $\epsilon > 0$, and let $\text{dist}(x, S)$ be a value of metric function for given point $x$ and subset $S$ in $\mathbb{R}^3$.

**Definition 2.2** For an integer $l > 1$, $S_i$ $(i = 1, 2)$ be an $i$-dimensional $C^l$ immersed submanifold in $\mathbb{R}^3$ such that $S_1 \cap S_2$ has an isolated point $A$. We say that *the order of contact* of $S_1$ with $S_2$ at $A$ is $l$ if there exist positive real numbers $m$ and $M$ such that

$$m \leq \frac{\text{dist}(x, S_2)}{\|x - A\|^l} \leq M$$

for all $x \in S_{1,\epsilon} \setminus \{A\}$ where $S_{1,\epsilon}$ is a component containing $A$ of $S_1 \cap U_\epsilon(A)$ for a small $\epsilon > 0$. If the above integer $l > 1$ is even, the tangency point $A$ is called to be *one-sided*. Otherwise, it is called to be *two-sided*. The tangency is *nondegenerate* if the order of contact is two.

**Remark 2.3** ([15, Proposition 2.2]) The order of contact is preserved for any diffeomorphism of a neighborhood of tangency point.

**Remark 2.4** The order of contact in Definition 2.2 is a special case of the definition of order of contact given by Arnold, Zade and Varchenko [1]. They defined the order of contact $l$ for a $C^k$ ($k > l$) diffeomorphisms between $C^s$ ($s > k$) manifolds. While Rayskin [15, Definition 2.1 and 2.6] gives a definition of order of contact for pairs of two immersed $C^1$ manifolds in $\mathbb{R}^n$. In fact, the order of Rayskin's is greater than or equal to that of Arnold-Zade-Varchenko's [1].

The following theorem plays an important role in the proof of our main theorem.

**Lemma 2.5** (Birkhoff-Smale Theorem, see [2, 3, 5, 16]) *Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a $C^r$ ($r \geq 1$) diffeomorphism with a hyperbolic fixed point $p$ and a transverse homoclinic point $q$ of $p$. Then there exists a hyperbolic invariant set $\Lambda$ containing $p$, $q$ and an integer $m > 0$ such that $f^m|\Lambda$ is topologically conjugate to the two sided shift map $\sigma$ on the space $\Sigma(2)$ of two symbols.*

## 3. Assumptions in main theorem

Let $f$ be a $C^\infty$ diffeomorphism on $\mathbb{R}^3$ with a hyperbolic fixed point $p \in \mathbb{R}^3$ such that the eigenvalues of $Df(p)$ are real numbers $\mu$, $\lambda_1$, $\lambda_2$ with $0 < \mu < 1 < \lambda_2 < \lambda_1$. We suppose that $f$ satisfies the following conditions (i)–(v).

( i ) There exists a $C^\infty$ linearizing coordinate on a neighborhood $U$ of $p$ such that $p = (0, 0, 0)$ and

$$f(x_1, x_2, x_3) = (\lambda_1 x_1, \lambda_2 x_2, \mu x_3)$$

for any $(x_1, x_2, x_3) \in U$ with $f(x_1, x_2, x_3) \in U$. (It implies that $W^u_{\mathrm{loc}}(p) \cap U \subset \{x_3 = 0\}$ and $W^s_{\mathrm{loc}}(p) \cap U \subset \{x_1 = x_2 = 0\}$).

( ii ) $W^s(p) \cap W^u(p) \cap U$ contains a homoclinic tangency point $q = (0, 0, q_3)$, $q_3 \neq 0$ with the order of contact two. Moreover, the lines passing

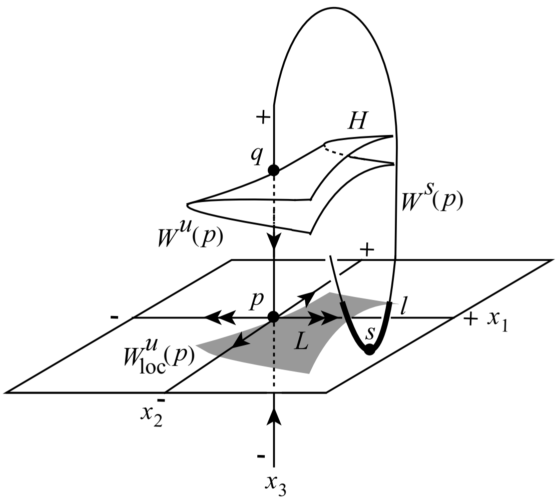


Fig. 3.1. The case of $q_3 > 0$ where $L$ meets the negative part of the $x_2$-axis and the second entry of the coordinate of $s$ is negative.

through $q$ and parallel respectively to the $x_1$ and $x_2$-axes meet $W^u(p)$ transversely at $q$.

(iii) For any sufficiently large $N \in \mathbb{N}$, the point $s = f^{-N}(q)$ is contained in $W^u_{\rm loc}(p) \cap U$ as illustrated in Fig. 3.1.

(iv) Let $l$ denote a small curve in $W^s(p)$ containing $s$. Assume that $l \setminus \{s\} \subset W^u_{\rm loc}(p) \times I_q$ where $I_q$ is the interval $(0,\, q_3]$ (resp. $[q_3,\, 0)$) if $q_3 > 0$ (resp. $q_3 < 0$).

( v ) Let $H$ be the intersection of $W^u(p)$ and a small neighborhood $V_q$ of $q$ in $\mathbb{R}^3$. The image $L = \mathrm{pr}(H)$ meets either the positive or negative parts of the $x_2$-axis non-trivially, but does not the opposite part, where $\mathrm{pr}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is the orthogonal projection defined by $\mathrm{pr}(x_1,\, x_2,\, x_3) = (x_1,\, x_2)$. Moreover, the second entry of the coordinate of $s$ is positive (resp. negative) if $L$ intersects the positive (resp. negative) part, as shown Fig. 3.1.

According to the Sternberg Linearizing Theorem [17], the linearizing condition (i) is generic in the space of $C^\infty$diffeomorphisms on $\mathbb{R}^3$. By the above (ii)–(iii) and Remark 2.3, $l$ is a curve tangent to $\mathbb{R}^2 \times \{0\}$ quadratically at $s$. Since the above assumptions do not depend on the global structure of the ambient space, the following result is true for diffeomorphisms defined on any three-dimensional Riemannian manifold.

**Theorem 3.1** *Under the assumption* (i)–(v) *for* $f \in \mathrm{Diff}^\infty(\mathbb{R}^3)$*, for any small* $\epsilon$*-neighborhood* $U_\epsilon(p)$ *and* $U_\epsilon(s)$ *of the saddle fixed point* $p$ *and the homoclinic tangency* $s$*, respectively, there exists an integer* $n_0 \geq \mathbb{N}$ *such that, for any* $n \geq n_0$*,* $f^n$ *has uniformly hyperbolic subset in* $U_\epsilon(p) \cap U_\epsilon(s)$ *topologically conjugate to the shift map of* $\Sigma(2)$*, that is,* $f$ *has a horseshoe set arbitrarily close to the nondegenerate one-sided homoclinic tangency.*

For the proof of the theorem, we need to show that $W^u(p)$ and $W^s(p)$ have a transverse intersection point contained in an arbitrarily small neighborhood of $s$ in $U$. In fact, the assertion is proved by using the *one-sided* singular $\lambda$-Lemma (Lemma 4.3). We remark that the similar results to Theorem 3.1 for *degenerate one-sided* homoclinic tangency are unproved yet which are not trivial from this paper as well as [15]. For example, the implicit function theorem is one of essential roles in the next Section 4, but can not be applied to such degenerate situations.

## 4. Proof of main theorem

From the condition (ii), we may assume that $H$ is represented as the graph of a $C^\infty$ function $x_1 = \varphi(x_2, x_3)$ if necessary replacing $V_q$ by a smaller neighborhood of $q$. The function $\varphi$ is satisfies with the following conditions.

$$\varphi(0, q_3) = 0, \quad \frac{\partial \varphi}{\partial x_3}(0, q_3) = 0, \quad \frac{\partial^2 \varphi}{\partial {x_3}^2}(0, q_3) \neq 0.$$

The former two conditions are derived immediately from the definition of $\varphi$. If $\partial^2\varphi(0, q_3)/\partial {x_3}^2 = 0$, then $q$ would be a tangency of $W^s(p)$ and $W^u(p)$ with order of contact greater than two. This contradicts the condition. By the implicit function theorem, there exists a $C^\infty$ function $x_3 = \eta(x_2)$ defined in a small neighborhood $V$ of 0 in the $x_2$-axis and such that $\eta(0) = q_3$ and $\partial\varphi(x_2, \eta(x_2))/\partial x_3 = 0$. We set

$$\tilde{h} = \{\big(\varphi(x_2, \eta(x_2)), x_2, \eta(x_2)\big); x_2 \in V\} \quad \text{and} \quad h = \mathrm{pr}(\tilde{h}) \subset L.$$

For two non-negative functions $a(u, v)$, $b(u, v)$, $a(u, v) \sim b(u, v)$ means that there exist constants $C_1$, $C_2 > 0$ independent of $u$, $v$ and satisfying

$$C_1 a(u,v) \leq b(u,v) \leq C_2 a(u,v)$$

for any $u$, $v$.

**Lemma 4.1**

$$\left|\frac{\partial\varphi}{\partial x_3}(x_2,\,x_3)\right| \sim \operatorname{dist}(h,\,(\varphi(x_2,\,x_3),\,x_2))^{1/2}.$$

*Proof.* By the Taylor expansion of $\varphi(x_2,\,x_3)$ at $x_3 = \eta(x_2)$ of order two, we have

$$\begin{aligned}\varphi(x_2,\,x_3) - \varphi(x_2,\,\eta(x_2)) &= \frac{\partial\varphi}{\partial x_3}(x_2,\,\eta(x_2))(x_3 - \eta(x_2)) \\ &\quad + \frac{1}{2}\frac{\partial^2\varphi}{\partial {x_3}^2}(x_2,\,\eta(x_2))(x_3 - \eta(x_2))^2 + \text{h.o.t.},\end{aligned} \tag{4.1}$$

where 'h.o.t.' represents a higher order term with respect to $x_3 = \eta(x_2)$. Since $\partial\varphi(x_2,\,\eta(x_2))/\partial x_3 = 0$ and $\partial^2\varphi(0,\,q_3)/\partial {x_3}^2 \neq 0$, if necessary replacing $V_q$ by a smaller neighborhood of $q$ in $\mathbb{R}^3$, one can suppose that

$$|\varphi(x_2, x_3) - \varphi(x_2, \eta(x_2))| \sim |x_3 - \eta(x_2)|^2$$

for any $(x_2,\,x_3)$ with $(\varphi(x_2,\,x_3),\,x_2,\,x_3) \in V_q$. By the condition (ii) in our assumption, the angle $\theta$ of $h$ and the $x_1$-axis satisfies $\theta$ with

$$0 < |\theta| < \frac{\pi}{2}.$$

Then, as illustrated in Fig. 4.1, we have

$$\begin{aligned}\operatorname{dist}(h,\,(\varphi(x_2,\,x_3),\,x_2)) &\sim \|(\varphi(x_2,\,\eta(x_2)),\,x_2) - (\varphi(x_2,\,x_3),\,x_2)\| \\ &= |\varphi(x_2,\,\eta(x_2)) - \varphi(x_2,\,x_3)| \\ &\sim |x_3 - \eta(x_2)|^2.\end{aligned} \tag{4.2}$$

Differentiating the both sides of (4.1) by $x_3$,

$$\left|\frac{\partial\varphi}{\partial x_3}(x_2, x_3)\right| \sim |x_3 - \eta(x_2)|.$$

Then the proof is completed by this approximation and (4.2). □

The curve $\tilde{h}$ divides $H$ into two components. Take a component $H_0$ of $H \setminus \tilde{h}$. The surface $H_0$ is represented as the graph of a $C^\infty$ function $\gamma(x_1,\,x_2)$ with domain $\operatorname{Int}(L) = L \setminus h$. Then any point $(x_1,\,x_2,\,x_3)$ of $H_0$ satisfies

$$\varphi(x_2,\,x_3) = x_1 \quad \text{and} \quad \gamma(x_1,\,x_2) = x_3. \tag{4.3}$$

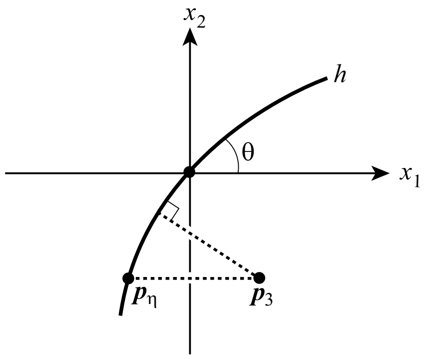


Fig. 4.1. The case of $\theta > 0$. $\boldsymbol{p}_\eta = (\varphi(x_2,\, \eta(x_2)),\, x_2)$, $\boldsymbol{p}_3 = (\varphi(x_2,\, x_3),\, x_2)$.

**Lemma 4.2** *For any* $(x_1,\, x_2) \in \mathrm{Int}(L)$,

$$\Big|\frac{\partial \gamma}{\partial x_1}(x_1,\, x_2)\Big| \sim \Big|\frac{\partial \gamma}{\partial x_2}(x_1,\, x_2)\Big| \sim \frac{1}{\mathrm{dist}\big(h,\, (x_1,\, x_2)\big)^{1/2}}.$$

*Proof.* By (4.3), $\varphi(x_2,\, \gamma(x_1,\, x_2)) = x_1$. Differentiating the both sides of the equation by $x_1$ and $x_2$, we have

$$\frac{\partial \varphi}{\partial x_3}(x_2,\, \gamma(x_1,\, x_2))\frac{\partial \gamma}{\partial x_1}(x_1,\, x_2) = 1,$$
$$\frac{\partial \varphi}{\partial x_2}(x_2,\, \gamma(x_1,\, x_2)) + \frac{\partial \varphi}{\partial x_3}(x_2,\, \gamma(x_1,\, x_2))\frac{\partial \gamma}{\partial x_2}(x_1,\, x_2) = 0.$$

These equations together with Lemma 4.1 show

$$\Big|\frac{\partial \gamma}{\partial x_1}(x_1,\, x_2)\Big| \sim \frac{1}{\mathrm{dist}\big(h,\, (x_1,\, x_2)\big)^{1/2}},$$
$$\Big|\frac{\partial \gamma}{\partial x_2}(x_1,\, x_2)\Big| \sim \frac{1}{\mathrm{dist}\big(h,\, (x_1,\, x_2)\big)^{1/2}}\Big|\frac{\partial \varphi}{\partial x_2}(x_2,\, \gamma(x_1,\, x_2))\Big|.$$

Since

$$\lim_{(x_1,x_2)\to(0,0)} \frac{\partial \varphi}{\partial x_2}(x_2,\, \gamma(x_1,\, x_2)) = \frac{\partial \varphi}{\partial x_2}(0,\, q_3) \neq 0$$

by the condition (ii), one can choose $H$ (and hence $L$) so that

$$\Big|\frac{\partial \gamma}{\partial x_2}(x_1,\, x_2)\Big| \sim \frac{1}{\mathrm{dist}\big(h,\, (x_1,\, x_2)\big)^{1/2}}$$

for any $(x_1, x_2) \in \mathrm{Int}(L)$. This completes the proof. $\square$

Set $W = U \cap \{x_3 = 0\}$ in $W^u(p)$ and let $\mathcal{T}$ be a small $\epsilon$-neighborhood of the $x_1x_3$-plane in $\mathbb{R}^3$ satisfying $\mathcal{T} \cap \{s\} = \emptyset$.

**Lemma 4.3** (One-sided singular $\lambda$-Lemma) *The sequence $\{f^n(H_0) \cap U \setminus \mathcal{T}\}$ $C^1$ converges to an open subsurface of $W^u(p) \setminus \mathcal{T}$ containing $s$ as $n \to +\infty$.*

*Proof.* We only consider the case when $L$ meets the negative parts of the $x_2$-axis non-trivially, as shown in Fig. 3.1. In the other case, the proof is done quite similarly. By the condition(i),

$$f^n(\lambda_1^{-n}x_1,\, \lambda_2^{-n}x_2,\, \mu^{-n}x_3) = (x_1,\, x_2,\, x_3)$$

for any $(x_1, x_2, x_3) \in U \setminus \mathcal{T}$ with $(\lambda_1^{-n}x_1, \lambda_2^{-n}x_2, \mu^{-n}x_3) \in V_q$. In particular, for all sufficiently large $n \in \mathbb{N}$, $f^n(\mathrm{Int}(L)) \cap W \setminus \mathcal{T}$ is equal to $W^- \setminus \mathcal{T}$ and hence it contains $s$, where $W^- = \{(x_1, x_2) \in W;\, x_2 < 0\}$.

For any $(x_1, x_2) \in W^- \setminus \mathcal{T}$, there exists an integer $n_0 > 0$ such that $\mathrm{Int}(L)$ contains $(\lambda_1^{-n}x_1, \lambda_2^{-n}x_2)$ if $n \geq n_0$. Then we set

$$\begin{aligned} g_n(x_1,\, x_2) &= f^n\big(\lambda_1^{-n}x_1,\, \lambda_2^{-n}x_2,\, \gamma(\lambda_1^{-n}x_1,\, \lambda_2^{-n}x_2)\big) \\ &= \big(x_1,\, x_2,\, \mu^n\gamma(\lambda_1^{-n}x_1,\, \lambda_2^{-n}x_2)\big). \end{aligned} \tag{4.4}$$

Note that $h$ is well $C^1$ approximated by the straight segment $\{(t,\, t\tan\theta);\ |t| < \delta\}$ for some $\delta > 0$ in a small neighborhood of $(0, 0)$ in the $x_1x_2$-plane. Moreover, since $0 < \lambda_1^{-1} < \lambda_2^{-1}$ and $|x_2| \geq \varepsilon$,

$$\mathrm{dist}(h,\, (\lambda_1^{-n}x_1,\, \lambda_2^{-n}x_2)) \sim |\lambda_1^{-n}x_1\tan\theta - \lambda_2^{-n}x_2| \sim \lambda_2^{-n}, \tag{4.5}$$

see Fig. 4.2.

By Lemma 4.2 together with (4.5),

$$\begin{aligned} \Big|\frac{\partial}{\partial x_1}\mu^n\gamma(\lambda_1^{-n}x_1,\, \lambda_2^{-n}x_2)\Big| &\sim \frac{\mu^n\lambda_1^{-n}}{\mathrm{dist}\big((\lambda_1^{-n}x_1,\, \lambda_2^{-n}x_2),\, h\big)^{1/2}} \\ &\sim \mu^n\lambda_1^{-n}\lambda_2^{n/2}, \\ \Big|\frac{\partial}{\partial x_2}\mu^n\gamma(\lambda_1^{-n}x_1,\, \lambda_2^{-n}x_2)\Big| &\sim \frac{\mu^n\lambda_2^{-n}}{\mathrm{dist}\big((\lambda_1^{-n}x_1,\, \lambda_2^{-n}x_2),\, h\big)^{1/2}} \\ &\sim \mu^n\lambda_2^{-n/2}. \end{aligned} \tag{4.6}$$

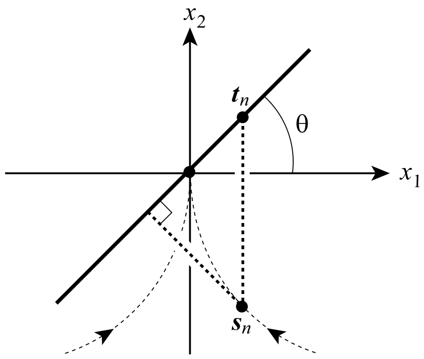


Fig. 4.2. $\boldsymbol{s}_n = (\lambda_1^{-n}x_1,\, \lambda_2^{-n}x_2)$, $\boldsymbol{t}_n = (\lambda_1^{-n}x_1,\, \lambda_1^{-n}x_1 \tan\theta)$.

Since $0 < \lambda_1^{-1}\lambda_2^{1/2} < \lambda_2^{-1/2} < 1$, these approximations imply that the map $g_n(x_1, x_2)$ on $W^- \setminus \mathcal{T}$ defined by (4.4) $C^1$ converges uniformly to $(x_1, x_2, 0)$ as $n \to \infty$. This completes the proof. □

**Lemma 4.4** *Let $l$ be the curve given in the condition* (iv)*. Then there exists an $n_0 \in \mathbb{N}$ such that $f^n(H_0)$ meets $l$ non-trivially and transversely for any integer $n \geq n_0$.*

*Proof.* Since $l$ is tangent to the $x_1x_2$-plane quadratically at $s$, the curve $l$ is parametrized as $\boldsymbol{a}(\tau) = \big(a_1(\tau),\, a_2(\tau),\, \tau^2 + O(|\tau|^3)\big)$ for any $\tau \in \mathbb{R}$ near 0 with $(a_1(0), a_2(0), 0) = s$ where each $a_i$ is a $C^\infty$ map on $\mathbb{R}$. By Lemma 4.3, $f^n(H_0) \cap l \neq \emptyset$ for all sufficiently large $n$, which shows the former part of this lemma, see Fig. 4.3.

Suppose that $\boldsymbol{a}(\tau_n) \in f^n(H_0) \cap l$. Since $\tau_n^2 + O(|\tau_n|^3) = \mu^n \gamma(\lambda_1^{-n}a_1(\tau_n), \lambda_2^{-n}a_2(\tau_n))$,

$$\lim_{n\to\infty} \frac{\tau_n^2}{\mu^n} = \lim_{(x_1,x_2)\to(0,0)} \gamma(x_1, x_2) = q_3. \tag{4.7}$$

By (4.4),

$$\begin{aligned}
\boldsymbol{b}(\tau_n) &:= \frac{\partial g_n}{\partial x_1}(a_1(\tau_n),\, a_2(\tau_n)) \\
&= \Big(1,\, 0,\, \mu^n \lambda_1^{-n} \frac{\partial \gamma}{\partial x_1}\big(\lambda_1^{-n}a_1(\tau_n),\, \lambda_2^{-n}a_2(\tau_n)\big)\Big), \\
\boldsymbol{c}(\tau_n) &:= \frac{\partial g_n}{\partial x_2}(a_1(\tau_n),\, a_2(\tau_n))
\end{aligned}$$

$$= \Big(0,\, 1,\, \mu^n \lambda_2^{-n} \frac{\partial \gamma}{\partial x_2}\big(\lambda_1^{-n} a_1(\tau_n),\, \lambda_2^{-n} a_2(\tau_n)\big)\Big).$$

Since $\boldsymbol{a}'(\tau_n) = \big(a_1'(\tau_n),\, a_2'(\tau_n),\, 2\tau_n + O(|\tau_n|^2)\big)$,

$$\begin{aligned}\boldsymbol{a}'(\tau_n) \cdot (\boldsymbol{b}(\tau_n) \times \boldsymbol{c}(\tau_n)) = 2\tau_n + O(|\tau_n|^2) \\ - a_1'(\tau_n)\mu^n \lambda_1^{-n} \frac{\partial \gamma}{\partial x_1}\big(\lambda_1^{-n} a_1(\tau_n),\, \lambda_2^{-n} a_2(\tau_n)\big) \\ - a_2'(\tau_n)\mu^n \lambda_2^{-n} \frac{\partial \gamma}{\partial x_2}\big(\lambda_1^{-n} a_1(\tau_n),\, \lambda_2^{-n} a_2(\tau_n)\big).\end{aligned}$$

By (4.6) and (4.7),

$$\lim_{n \to \infty} \mu^{-n/2} \boldsymbol{a}'(\tau_n) \cdot (\boldsymbol{b}(\tau_n) \times \boldsymbol{c}(\tau_n)) = 2\sqrt{q_3} \neq 0.$$

This means that $\boldsymbol{a}'(\tau_n)$ is not contained in the tangent space of $f^n(H_0)$ at $\boldsymbol{a}(\tau_n)$ for all sufficiently large $n \in \mathbb{N}$, see Fig. 4.4. Thus $l$ meets $f^n(H_0)$ transversely at $\boldsymbol{a}(\tau_n)$. This completes the proof. □

Lemma 4.4 implies that there exists a transversal homoclinic point associated with $p$ and arbitrarily close to the point $s$. Then, by Birkhoff-Smale Theorem (Lemma 2.5), we have a horseshoe set which is also arbitrarily close to $s$. This completes the proof of Theorem 3.1.

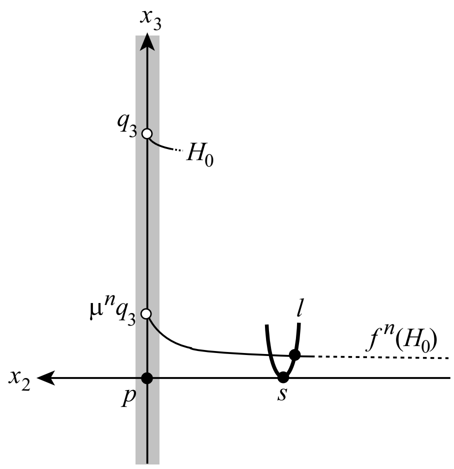


Fig. 4.3. The cross section along the $x_2x_3$-plane. The shaded region represents $\mathcal{T}$.

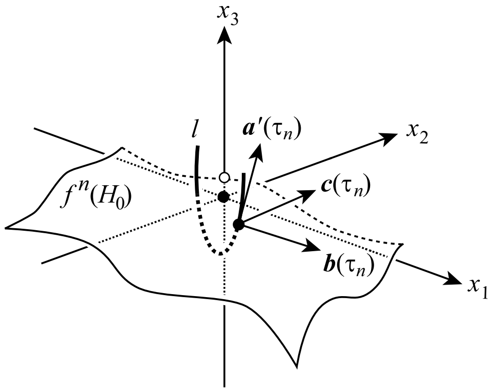


Fig. 4.4.

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