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Pseudo-Einstein CR-structures on real hypersurfaces in a complex space form

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Abstract. We introduce the pseudo-Einstein structure on real hypersurfaces in a Kählerian manifold, namely, the Ricci curvature tensor for the generalized Tanaka-Webster connection (restricted) on the Levi subbundle D is proportional to the Levi form. In particular, we give a classification of pseudo-Einstein Hopf-hypersurfaces in a non-flat complex space form.

Key words: real hypersurfaces, complex space forms, the g.-Tanaka-Webster connection, pseudo-Einstein structures, g.-Tanaka-Webster flat structures.

1. Introduction

Let M be a $(2n-1)$-dimensional manifold and TM be its tangent bundle. A *CR-structure* on M is a complex rank $n-1$ subbundle $\mathcal{H} \subset \mathbb{C}TM = TM \otimes \mathbb{C}$ satisfying

(i) $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$,

(ii) $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$ (integrability),

where $\bar{\mathcal{H}}$ denotes the complex conjugation of $\mathcal{H}$.

Then there exists a unique subbundle $D = \mathrm{Re}\{\mathcal{H} \oplus \bar{\mathcal{H}}\}$, called the *Levi subbundle* (maximally holomorphic subbundle) of $(M, \mathcal{H})$, and a unique bundle map J such that $J^2 = -I$ and $\mathcal{H} = \{X - iJX \mid X \in D\}$. We call (D, J) the real representation of $\mathcal{H}$. Let $E \subset T^*M$ be the conormal bundle of D. If M is an oriented CR-manifold then E is a trivial bundle, hence admits globally defined a nowhere zero section η, i.e., a real one-form on M such that $\mathrm{Ker}(\eta) = D$. For (D, J) we define the Levi form by

$$L\colon D \times D \to \mathcal{F}(M), \quad L(X, Y) = d\eta(X, JY)$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on M. If the Levi form is Hermitian, then the CR-structure is called *pseudo-Hermitian*, in addition, if the Levi form is non-degenerate (positive or negative definite, resp.), then the CR-structure is called a *non-degenerate* (*strongly pseudo-*

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convex, resp.) *pseudo-Hermitian CR-structure*.

Tanaka-Webster connection ([20], [22]) is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. A real hypersurface in a Kählerian manifold has an (integrable) CR-structure $(D,\ J)$ which is associated with an almost contact metric structure $(\eta,\ \phi,\ \xi,\ g)$, but it is not guaranteed to be pseudo-Hermitian and strongly pseudo-convex, in general. In this context, the present author [7], [8] defined the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) $\hat{\nabla}^{(k)}$ for real number k for real hypersurfaces in Kählerian manifolds. In particular, if a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then its associated CR-structure is pseudo-Hermitian and strongly pseudo-convex, and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see Proposition 2 in Section 3). Very recently, the author and Kimura [9] proved a classification theorem of real hypersurfaces in a non-flat complex space form such that the holomorphic sectional curvatures for the g.-Tanaka-Webster connection are constant.

In this paper, we introduce a *pseudo-Einstein CR-structure* in a real hypersurface of a Kählerian manifold, says, the Ricci curvature tensor of type (0, 2) (restricted) on D for the g.-Tanaka-Webster connection is proportional to the Levi form. A real hypersurface M in a Kählerian manifold is called a *Hopf hypersurface* if its structure vector field ξ is a principal curvature vector field, that is $A\xi = \alpha_1\xi$. The main purpose of this paper is to prove

Main Theorem *Let M be a Hopf hypersurface of a non-flat complex space form $\widetilde{M}_n(c)$ $(c \neq 0)$ with constant holomorphic sectional curvature c. Suppose that M admits a pseudo-Einstein CR-structure* (*for the g.-Tanaka-Webster connection*). *Then M is locally congruent to one of the following*: (A_0) *a horosphere in $H_n\mathbb{C}$;* (A_1) *a geodesic hypersphere in $P_n\mathbb{C}$ or $H_n\mathbb{C}$, a homogeneous tube over $H_{n-1}\mathbb{C}$ in $H_n\mathbb{C}$; or* $\dim M = 3$ *and* (B) *a homogeneous tube over a complex quadric Q^{n-1} and $P_n\mathbb{R}$* (*resp. $H_n\mathbb{R}$*) *in $P_n\mathbb{C}$* (*resp. $H_n\mathbb{C}$*).

We note that a g.-Tanaka-Webster flat real hypersurface (whose curvature tensor $\hat{R}$ vanishes) is pseudo-Einstein. Before proving the Main Theorem, we show that a Hopf hypersurface in a non-flat complex space form admits a flat g.-Tanaka-Webster structure if and only if it is locally congruent to (A_0) a horosphere in $H_n\mathbb{C}$, or dim $M = 3$ and (B) in $P_n\mathbb{C}$ or

$H_n\mathbb{C}$.

2. Almost contact metric structures and the associated CR-structures

In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

First, we give a brief review of several fundamental notions and formulas which we will need later on. An odd-dimensional differentiable manifold M has an *almost contact structure* if it admits a (1, 1)-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1. \tag{1}$$

Then we can find always a compatible Riemannian metric, namely which satisfies

$$g(\phi X,\, \phi Y) = g(X,\, Y) - \eta(X)\eta(Y) \tag{2}$$

for all vector fields on M. We call $(\eta,\, \phi,\, \xi,\, g)$ an *almost contact metric structure* of M and $M = (M; \eta,\, \phi,\, \xi,\, g)$ an *almost contact metric manifold*. From (1) and (2) we easily get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X,\, \xi). \tag{3}$$

The tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D\colon p \to D_p$ defines a distribution orthogonal to ξ. For an almost contact metric manifold M, one may define naturally an almost complex structure on the product manifold $M \times \mathbb{R}$, where $\mathbb{R}$ denotes the real line. If the almost complex structure is integrable, M is said to be normal. The integrability condition for the almost complex structure is the vanishing of the tensor $[\phi,\, \phi] + 2d\eta \otimes \xi$, where $[\phi,\, \phi]$ denotes the Nijenhuis torsion of ϕ. For an almost contact metric manifold M, we define its fundamental 2-form Φ by $\Phi(X,\, Y) = g(\phi X,\, Y)$. If M satisfies in addition

$$\Phi = d\eta, \tag{4}$$

M is called a *contact metric manifold*. A normal contact metric manifold is called a Sasakian manifold. For more details about the general theory of almost contact metric manifolds, we refer to [5].

On the other hand, for an almost contact metric manifold $M = (M; \eta, \phi, \xi, g)$, the restriction $J = \phi \mid D$ of ϕ to D defines an almost complex structure in D. As soon as the following conditions are further satisfied:

$$[JX, JY] - [X, Y] \in D \text{ (or } [X, JY] + [JX, Y] \in D) \tag{5}$$

and

$$[J, J](X, Y) = 0 \tag{6}$$

for all $X, Y \perp \xi$, where $[J, J]$ is the Nijenhuis torsion of J, then the pair (η, J) is called an (integrable) CR-structure associated with the almost contact metric structure (η, ϕ, ξ, g). In addition that the associated Levi form L, defined by $L(X, Y) = d\eta(X, JY)$, $X, Y \perp \xi$, is Hermitian, then (η, J) is called a pseudo-Hermitian CR-structure. If its Levi form is non-degenerate (positive or negative definite, resp.), then (η, J) is called a non-degenerate (strongly pseudo-convex, resp.) pseudo-Hermitian CR-structure. In particular, for a contact metric manifold its associated Levi-form is Hermitian and positive definite, but its associated almost complex structure is not in general integrable. For further details about CR-structures, we refer for example to [3], [21].

3. The generalized Tanaka-Webster connection for real hypersurfaces

Let M be an (oriented) real hypersurface of a Kählerian manifold $\widetilde{M} = (\widetilde{M}; \widetilde{J}, \widetilde{g})$ and N a global unit normal vector on M. By $\widetilde{\nabla}$, A we denote the Levi-Civita connection in $\widetilde{M}$ and the shape operator with respect to N, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric of M induced from $\widetilde{g}$. An eigenvector (resp. eigenvalue) of the shape operator A is called a principal curvature vector (resp. principal curvature). For any vector field X tangent to M, we put

$$\widetilde{J}X = \phi X + \eta(X)N, \quad \widetilde{J}N = -\xi. \tag{7}$$

We easily see that the structure $(\eta,\, \phi,\, \xi,\, g)$ is an almost contact metric structure on M i.e., satisfies (1) and (2). From the condition $\widetilde{\nabla}\widetilde{J} = 0$, the relations (7) and by making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,\, Y)\xi, \tag{8}$$
$$\nabla_X \xi = \phi AX. \tag{9}$$

By using (8) and (9), we see that a real hypersurface in a Kählerian manifold always satisfies (5) and (6), the integrability condition of the associated CR-structure. From (4) and (9) we have

Proposition 1 *Let $M = (M; \eta,\, \phi,\, \xi,\, g)$ be a real hypersurface of a Kählerian manifold. The almost contact metric structure of M is contact metric if and only if $\phi A + A\phi = 2\phi$.*

The Tanaka-Webster connection ([20], [22]) is the canonical affine connection defined on non-degenerate pseudo-Hermitian CR-manifold. Tanno ([21]) defined the generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. We define the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) for real hypersurfaces of Kählerian manifolds by the naturally extended one of Tanno's generalized Tanaka-Webster connection. Now we recall the generalized Tanaka-Webster connection $\hat{\nabla}$ for contact metric manifolds:

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y$$

for all vector fields X and Y.

By taking account of (9), the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for real hypersurfaces of Kählerian manifolds, which is denoted by the same symbol for contact metric manifolds, is defined by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX,\, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{10}$$

for a non-zero real number k. We put

$$F_X Y = g(\phi AX,\, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y. \tag{11}$$

Then the torsion tensor $\hat{T}$ is given by $\hat{T}(X,\, Y) = F_X Y - F_Y X$. Also, by using (2), (3), (8), (9) and (10) we can see that

$$\hat{\nabla}^{(k)}\eta = 0,\ \hat{\nabla}^{(k)}\xi = 0,\ \hat{\nabla}^{(k)}g = 0,\ \hat{\nabla}^{(k)}\phi = 0, \tag{12}$$

and

$$\hat{T}(X,\, Y) = 2d\eta(X,\, Y)\xi, \quad X,\, Y \in D.$$

We note that the associated Levi form is $L(X,\, Y) = (1/2)g((J\bar{A} + \bar{A}J)X,\, JY)$, where we denote by $\bar{A}$ the restriction A to D. If M satisfies $\phi A + A\phi = 2k\phi$, then we see that the associated CR-structure is pseudo-Hermitian, strongly pseudo-convex and further satisfies $\hat{T}(\xi,\, \phi Y) = -\phi\hat{T}(\xi,\, Y)$. Hence the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see [7], [8]). Namely, we have

Proposition 2 *Let $M = (M; \eta,\, \phi,\, \xi,\, g)$ be a real hypersurface of a Kählerian manifold. If M satisfies $\phi A + A\phi = 2k\phi$, then the associated CR-structure is pseudo-Hermitian, strongly pseudo-convex, integrable, and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.*

Remark 1 The almost contact metric structure of M appearing in Proposition 2 is a contact metric structure only for the very special case $k = 1$. More precisely, a real hypersurface M in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ satisfies $\phi A + A\phi = 2k\phi$ if and only if M is locally congruent to one of real hypersurfaces of type (A_0) in $H_n\mathbb{C}$, (A_1) or (B) in $P_n\mathbb{C}$, $H_n\mathbb{C}$ among those ones in Theorems 5 and 6 in Section 4 (cf. [13] and [17]). With the help of the tables in [4] and [18], we see that the almost contact metric structures becomes contact metric only for a geodesic hypersphere of radius $\pi/4$ in $P_n\mathbb{C}$ and for a horosphere in $H_n\mathbb{C}$. Thus, we see that the real hypersurfaces of type (A_1) in $P_n\mathbb{C}$ except with the radius $r = \pi/4$ or (B) in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ are proper examples which has not contact structures but their associated CR structures are pseudo-Hermitian, strongly pseudo-convex, integrable.

We define the g.-Tanaka-Webster curvature tensor of $\hat{R}$ (with respect to $\hat{\nabla}^{(k)}$) by

$$\hat{R}(X,\, Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X,\,Y]}Z$$

for all vector fields X, Y, Z in M. Then we have

Proposition 3

$$\hat{R}(X, Y)Z = -\hat{R}(Y, X)Z,$$
$$g(\hat{R}(X, Y)Z, W) = -g(\hat{R}(X, Y)W, Z).$$

The first identity follows trivially from the definition of $\hat{R}$. Since the connection parallelizes the metric form, (i.e., $\hat{\nabla}g = 0$) we have also the second one by a similar way as the case of Riemanian curvature tensor. We remark that since the Tanaka-Webster connection is not torsion-free, the Jacobi- or Bianchi-type identities do not hold, in general.

The *g.-Tanaka-Webster Ricci* (*curvature*) *tensor* $\hat{\rho}$ (of $\hat{\nabla}^{(k)}$) is defined by

$$\hat{\rho}(X, Y) = \text{trace of } \{V \mapsto \hat{R}(V, X)Y\}, \quad V, X, Y \in D. \tag{13}$$

We define the pseudo-Einstein structure on real hypersurfaces in a Kählerian manifold.

Definition 4 Let M be a real hypersurface in a Kählerian manifold. Then the CR-structure (η, J) is said to be *pseudo-Einstein* if the g.-Tanaka-Webster Ricci tensor is proportional to the Levi form, namely,

$$\hat{\rho}(X, Y) = \lambda L(X, Y) \tag{14}$$

for $X, Y \perp \xi$, where λ is a real number.

Since $L(X, Y) = (1/2)g((\phi A + A\phi)X, \phi Y)$ for $X, Y \perp \xi$, λ in (14) is determined by $\hat{r} = \lambda(H - \alpha_1)$, where we have put $\alpha_1 = \eta(A\xi)$.

4. Pseudo-Einstein real hypersurfaces in a complex space form

Let $\widetilde{M} = \widetilde{M}_n(c)$ be a complex space form of constant holomorphic sectional curvature c and M a real hypersurface of $\widetilde{M}$. Then we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z =& \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{15}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{16}$$

for any tangent vector fields X, Y, Z on M. From (15) we get for the Ricci tensor S of type (1,1):

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + HAX - A^2X, \tag{17}$$

where $H(=$ trace of $A)$ denotes the mean curvature.

We now suppose that M is a Hopf hypersurface, that is ξ is a principal curvature vector field $A\xi = \alpha_1\xi$. Then we already know that α_1 is constant (cf. [12], [13]). Differentiating this covariantly, and then by using (9) we have

$$(\nabla_X A)\xi = \alpha_1\phi AX - A\phi AX,$$

and further by using (16) we obtain

$$(\nabla_\xi A)X = \frac{c}{4}\phi X + \alpha_1\phi AX - A\phi AX$$

for any vector field X on M. The symmetry of $\nabla_\xi A$ gives

$$2A\phi AX - \frac{c}{2}\phi X = \alpha_1(\phi A + A\phi)X.$$

If we assume that $AX = \mu X$ ($\|X\| = 1$) for X orthogonal to ξ, then we get

$$(2\mu - \alpha_1)A\phi X = \Big(\mu\alpha_1 + \frac{c}{2}\Big)\phi X. \tag{18}$$

If $2\mu - \alpha_1 = 0$, then the above equation gives $\mu^2 = -c/4$. This case determines the horosphere in $H_n\mathbb{C}$ (cf. [4]). We prepare some more which are needed soon to prove our results.

Theorem 5 ([10]) *Let M be a Hopf hypersurface of $P_n\mathbb{C}$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following*:

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$,*

(A_2) *a tube of radius r over a totally geodesic $P_l\mathbb{C}(1 \le l \le n-2)$, where $0 < r < \pi/2$,*

(B) *a tube of radius r over a complex quadric Q^{n-1} and $P_n\mathbb{R}$, where $0 < r < \pi/4$,*

(C) *a tube of radius r over $P_1\mathbb{C} \times P_{(n-1)/2}\mathbb{C}$, where $0 < r < \pi/4$ and n (≥ 5) is odd,*

(D) *a tube of radius r over a complex Grassmann $G_{2,5}\mathbb{C}$, where $0 < r < \pi/4$ and $n = 9$,*

(E) *a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$ and $n = 15$.*

Theorem 6 ([4]) *Let M be a Hopf hypersurface of $H_n\mathbb{C}$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following*:

(A_0) *a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,*

(A_2) *a tube over a totally geodesic $H_l\mathbb{C}(1 \le l \le n-2)$,*

(B) *a tube over a totally real hyperbolic space $H_n\mathbb{R}$.*

From the definition of $\hat{R}$, together with (10) and (11), we have

$$\hat{R}(X, Y)Z = R(X, Y)Z + (\nabla_X F)_Y Z + F_X F_Y Z - (\nabla_Y F)_X Z - F_Y F_X Z$$

for all vector fields X, Y, Z tangent to M. We put

$$E(X, Y)Z = (\nabla_X F)_Y Z + F_X F_Y Z - (\nabla_Y F)_X Z - F_Y F_X Z.$$

Use (9) to get

$$\begin{aligned} &E(X, Y)Z \\ &= (\nabla_X F)_Y Z - (\nabla_Y F)_X Z + F_X F_Y Z - F_Y F_X Z \\ &= g\big(\phi((\nabla_X A)Y - (\nabla_Y A)X),\, Z\big)\xi + 2g(\phi AY,\, Z)\phi AX \\ &\quad - 2g(\phi AX,\, Z)\phi AY + g\big((\nabla_X\phi)AY - (\nabla_Y\phi)AX,\, Z\big)\xi \\ &\quad - \eta(Z)\Big(\phi\big((\nabla_X A)Y - (\nabla_Y A)X\big) + (\nabla_X\phi)AY - (\nabla_Y\phi)AX\Big) \\ &\quad - k\Big(g\big((\phi A + A\phi)X, Y\big)\phi Z + \eta(Y)(\nabla_X\phi)Z - \eta(X)(\nabla_Y\phi)Z\Big) \\ &\quad + g(\phi AX,\, F_Y Z)\xi - \eta(F_Y Z)\phi AX - k\eta(X)\phi F_Y Z \\ &\quad - g(\phi AY,\, F_X Z)\xi + \eta(F_X Z)\phi AY + k\eta(Y)\phi F_X Z. \end{aligned} \tag{19}$$

Then E is a tensor field of type (1, 3), and

$$\hat{R}(X, Y)Z = R(X, Y)Z + E(X, Y)Z \tag{20}$$

for all vector fields X, Y, Z in M.

Here, we prove

Proposition 7 *Let M be a Hopf hypersurface of a non-flat complex space form $\widetilde{M}_n(c)$, $c \neq 0$. Then M admits a flat g.-Tanaka-Webster structure, namely, $\hat{R} = 0$ if and only if M is locally congruent to a horosphere in $H_n\mathbb{C}$, or $\dim M = 3$ and a homogeneous tube over a complex quadric Q^{n-1} and $P_n\mathbb{R}$ (resp. $H_n\mathbb{R}$) in $P_n\mathbb{C}$ (resp. $H_n\mathbb{C}$).*

Proof. Suppose that M is flat with respect to $\hat{\nabla}^{(k)}$, that is M satisfies $\hat{R} = 0$. Together with (11), (19) and (20), using (1), (2), (3), (8) and (16), then we have

$$\begin{aligned} &R(X, Y)Z \\ &= \frac{c}{4}\{\eta(X)g(Y, Z)\xi - \eta(Y)g(X, Z)\xi + \eta(Z)(\eta(Y)X - \eta(X)Y)\} \\ &\quad + \eta(AX)g(AY, Z)\xi - \eta(AY)g(AX, Z)\xi \\ &\quad + \eta(Z)(\eta(AY)AX - \eta(AX)AY) + kg((\phi A + A\phi)X, Y)\phi Z \\ &\quad + g(\phi AX, Z)\phi AY - g(\phi AY, Z)\phi AX. \end{aligned} \tag{21}$$

We assume that ξ is a principal curvature vector field, that is $A\xi = \alpha_1\xi$ on M. Then for $X \perp \xi$, $\|X\| = 1$, from (21) we get

$$\begin{aligned} &g(R(X, \phi X)\phi X, X) \\ &= -kg((\phi A + A\phi)X, \phi X) + g(\phi AX, \phi X)g(\phi A\phi X, X) \\ &\quad - g(\phi A\phi X, \phi X)g(\phi AX, X) \\ &= -k\big(g(AX, X) + g(A\phi X, \phi X)\big) \\ &\quad - g(AX, X)g(A\phi X, \phi X) + g(A\phi X, X)^2. \end{aligned}$$

But, from (15) we also get

$$g(R(X, \phi X)\phi X, X) = c + g(A\phi X, \phi X)g(AX, X) - g(AX, \phi X)^2$$

for any vector field $X \perp \xi$, $\|X\| = 1$. The above two equations give

$$\begin{aligned} &- k\big(g(AX, X) + g(A\phi X, \phi X)\big) \\ &\qquad - 2g(AX, X)g(A\phi X, \phi X) + 2g(AX, \phi X)^2 = c \end{aligned} \tag{22}$$

for any vector field $X \perp \xi$, $\|X\| = 1$.

Here, we divide our arguments into two cases: (i) $2\mu = \alpha_1$, (ii) $2\mu \neq \alpha_1$. We consider the case (i). Then we already knew that M is a horosphere in $H_n\mathbb{C}$. In fact, with its shape operator $A = I + \eta \otimes \xi$ in $H_n\mathbb{C}(-4)$ and (15) we can check that a horosphere satisfies the equation (21). This time we study the case (ii). If we assume that $AX = \mu X$, $X \perp \xi$, $\|X\| = 1$, then, from (22) by using (18) we have

$$(k + \alpha_1)\mu^2 + \frac{3}{2}c\mu - \frac{1}{2}c\alpha_1 + \frac{1}{4}ck = 0. \tag{23}$$

From (23), we see at once that $k \neq -\alpha_1$ (because $k = -\alpha_1$ implies that $2\mu = \alpha_1$). Further from (23), it follows that M has at most three distinct principal curvatures including α_1. So, in view of Takagi's list of homogeneous Hopf-hypersurfaces in $P_n\mathbb{C}$ or Berndt's list of Hopf-hypersurfaces of constant principal curvatures in $H_n\mathbb{C}$, we see that M is of type (A), (B) in $P_n\mathbb{C}$ or $H_n\mathbb{C}$.

First, we treat a real hypersurface of type (A). Then we know that those ones of type (A) are determined by the equation

$$\mu^2 - \alpha_1\mu - \frac{c}{4} = 0 \quad (A\phi = \phi A) \tag{24}$$

(cf. [16], [15]). From (23) and (24), we obtain $k^2 = -c/4$, $\alpha_1^2 = -c$, and $(\mu - \alpha_1/2)^2 = 0$, which can not occur. Thus, we see that among them of type (A) in $P_n\mathbb{C}$ or $H_n\mathbb{C}$, only a horosphere in $H_n\mathbb{C}$ admits a g.-Tanaka-Webster flat structure.

Next, we consider a real hypersurface of type (B). Its defining equation is

$$\alpha_1\mu^2 + c\mu - \frac{c}{4}\alpha_1 = 0 \quad \Big(A\phi + \phi A = -\frac{c}{\alpha_1}\phi\Big) \tag{25}$$

(cf. [13]). Together with (23), we get $\alpha_1 = 2k$. Thus, from (15) and (21), we have for any vector fields X, Z, $W \perp \xi$

$$\begin{aligned} &-\frac{c}{2}g(\phi Z, W)\phi X + g(\phi AZ, X)\phi AW - g(\phi AW, X)\phi AZ \\ &= \frac{c}{4}\big(g(W, X)Z - g(Z, X)W + g(\phi W, X)\phi Z \\ &\qquad - g(\phi Z, X)\phi W - 2g(\phi Z, W)\phi X\big) \\ &\quad + g(AW, X)AZ - g(AZ, X)AW. \end{aligned} \tag{26}$$

It arises naturally two subcases: (i) $\dim M \geq 5$, (ii) $\dim M = 3$.

In the case (i), if we put $X = Z$ (26) in and take an orthonormal pair $\{X, W\}$ belonging to an eigenspace $D(\mu)$ for an eigenvalue μ, then we get $c/4 + \mu^2 = 0$, which together with (25), yields a contradiction.

In case that (ii) $\dim M = 3$, we can check that (26) always holds for all the (possible) cases:

- $X = W \in D(\mu)$ and $Z = \phi X$;
- $X = Z \in D(\mu)$ and $W = \phi X$;
- $Z = W \in D(\mu)$ and $X = \phi Z$.

Conversely, we can also check that a 3-dimensional hypersurface M of type (B) in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ satisfies (21) with $k = \alpha_1/2$. In fact, we are aware that it holds always that $\hat{R}(X, Y)\xi = 0$ for any vector fields X and Y. Also, from (20) we can see that every Hopf hypersurface satisfies $\hat{R}(\xi, X)Y = 0$ for any vector fields X and Y. Together with Proposition 3 (the symmetry of $\hat{R}$) we can see that $\hat{R}$ vanishes for M. □

Now, we prove our Main Theorem.

Proof of the Main Theorem. Let M be a Hopf hypersurface in $P_n\mathbb{C}$ or $H_n\mathbb{C}$. First by (11), $F_X Y = g(\phi AX, Y)\xi$ for $X, Y \in D_p$ $(p \in M)$. Then (19) implies

$$\begin{aligned} g(E(X, Y)Z, W) = &g(\phi AY, Z)g(\phi AX, W) - g(\phi AX, Z)g(\phi AY, W) \\ &-kg((\phi A + A\phi)X, Y)g(\phi Z, W) \end{aligned}$$

for $X, Y, Z, W \in D_p$. Hence for an orthonormal basis $\{e_i\}$ on D_p $(p \in M)$, $i = 1, 2, \ldots, 2n-2$,

$$\sum_{i=1}^{2n-2} g(E(e_i, X)Y, e_i) = g(A\phi Y, \phi AX) + kg((\phi A + A\phi)X, \phi Y).$$

Moreover, from (20) we have

$$\begin{aligned} \hat{\rho}(X, Y) &= \sum_{i=1}^{2n-2} g(R(e_i, X)Y, e_i) + \sum_{i=1}^{2n-2} g(E(e_i, X)Y, e_i) \\ &= \rho(X, Y) - g(R(\xi, X)Y, \xi) + \sum_{i=1}^{2n-2} g(E(e_i, X)Y, e_i) \\ &= \rho(X, Y) - \frac{c}{4}g(X, Y) - \eta(A\xi)g(AX, Y) + \eta(AX)\eta(AY) \\ &\quad + g(A\phi Y, \phi AX) + kg((\phi A + A\phi)X, \phi Y) \end{aligned}$$

for $X, Y \perp \xi$, where we have put $\rho(X, Y) = g(SX, Y)$. Suppose that M is pseudo-Einstein, then by Definition 4 we have

$$\rho(X,\,Y)=\frac{c}{4}g(X,\,Y)+\Big(\frac{\lambda}{2}+\alpha_1-k\Big)g(AX,\,Y)\\ -\Big(\frac{\lambda}{2}-k\Big)g(\phi A\phi X,\,Y)+g(\phi A\phi AX,\,Y) \quad (27)$$

for $X,\,Y\perp\xi$. So, together with (17) we have

$$g(A^2X,\,Y)+\Big(\frac{\lambda}{2}+\alpha_1-H-k\Big)g(AX,\,Y)-\frac{c}{2}ng(X,\,Y)\\ +\Big(k-\frac{\lambda}{2}\Big)g(\phi A\phi X,\,Y)+g(\phi A\phi AX,\,Y)=0 \quad (28)$$

for any vector fields X and Y orthogonal to ξ.

As already seen in the proof of Proposition 7, a horosphere in $H_n\mathbb{C}$ (with $c=-4$) is a pseudo-Einstein space (with $\lambda=2k-2$). From now we consider the cases except a horosphere in $H_n\mathbb{C}$. Now we assume that $AX=\mu X$ ($\|X\|=1$) for X orthogonal to ξ, then from (28) we get

$$\mu^2+\Big(\frac{\lambda}{2}+\alpha_1-H-k\Big)\mu-\frac{c}{2}n+\Big(\frac{\lambda}{2}-k\Big)\bar{\mu}-\mu\bar{\mu}=0.$$

Here, we substitute $\bar{\mu}=(\alpha_1\mu+c/2)/(2\mu-\alpha_1)$. Then this is rewritten as

$$2\mu^3+(\lambda-2H-2k)\mu^2\\ +\Big(\alpha_1H-\alpha_1^2-cn-\frac{c}{2}\Big)\mu+\frac{c}{2}n\alpha_1+\frac{c}{4}\lambda-\frac{c}{2}k=0. \quad (29)$$

We denote its roots $\mu_1,\,\mu_2,\,\mu_3$ and we may assume that $\mu_3=\bar{\mu}_2$. Then from the roots and coefficients of (29), we have the following relations:

$$\begin{cases}\mu_1+\mu_2+\mu_3=-\dfrac{1}{2}(\lambda-2H-2k),\\ \mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3=\dfrac{1}{2}\Big(\alpha_1H-\alpha_1^2-cn-\dfrac{c}{2}\Big),\\ \mu_1\mu_2\mu_3=-\dfrac{1}{2}\Big(\dfrac{c}{2}n\alpha_1+\dfrac{c}{4}\lambda-\dfrac{c}{2}k\Big).\end{cases} \quad (30)$$

Let U be the maximal open and dense subset of M such that on each connected component of U the multiplicities m_i of the eigenvalue functions μ_i ($i=1,\,2$) of the shape operator A are constant. Let U_0 be a connected component of U, and we discuss our arguments on U_0. Then we may express $H=\alpha_1+m_1\mu_1+m_2\mu_2+m_2\bar{\mu}_2$, $m_1>1$. From the first equation of (30) we get

$$(1-m_2)\bar{\mu}_2=\alpha_1+(m_1-1)\mu_1+(m_2-1)\mu_2+k-\frac{1}{2}\lambda. \quad (31)$$

We may consider two cases divided: (i) $m_2 = 1$, (ii) $m_2 \neq 1$. First, we treat the case (i). Then (31) yields that μ_1 is constant. If $\mu_1 \neq 0$, then succeedingly the third equation of (30) gives $\mu_2\bar{\mu}_2 =$ constant. Since $\bar{\mu}_2 = (\alpha_1\mu_2 + c/2)/(2\mu_2 - \alpha_1)$ it follows that μ_2 is constant and then $\bar{\mu}_2$ is also constant. If $\mu_1 = 0$, from (31) we get $\lambda = 2(k + \alpha_1)$, and thus from the third one of (30) we get $\alpha_1 = 0$. Then with the second equation of (30) we get $\mu_2\bar{\mu}_2 = -cn - c/2$, which yields that μ_2 and $\bar{\mu}_2$ are constants. Next, we consider the case (ii). Then from (31) we obtain $\mu_1 = f_1(\mu_2)$, a function of μ_2. (If $\mu_1 = 0$, then from (31) we at once see that μ_2 is constant.) So, from the third equation of (30) we can see that μ_2 is constant, and hence μ_1 and $\bar{\mu}_2$ are also constants. Finally, since M is connected, we conclude that M has at most four distinct constant principal curvatures (including α_1) on M. Due to [4] and [10], we conclude that M is locally congruent to one of types (A_1), (A_2), (B) in $P_n\mathbb{C}$ or (A_0), (A_1), (A_2), (B) in $H_n\mathbb{C}$.

In a similar way as in the proof of Proposition 7, we first look at a real hypersurface of type (A). Then their characteristic property $\phi A = A\phi$ have the equation (28) be simpler:

$$(\lambda + \alpha_1 - H - 2k)g(AX, Y) - \frac{c}{2}ng(X, Y) = 0$$

for $X, Y \perp \xi$. This says that M is totally η-umbilical, that is $A = aI + b\eta \otimes \xi$ for constants a, b. As concerns of it, we already know that (A_1) in $P_n\mathbb{C}$ and (A_0), (A_1) in $H_n\mathbb{C}$ only have the property (cf. [15], [18]). Indeed, we compute the pseudo-Einstein constant $\lambda = (2n-2)a + 2k + c/2an$. (Here, $a \neq 0$ because $a = 0$ implies (rank of A_p) ≤ 1 at every point p, which is impossible (see, Theorem 2.3 in [13])).

Next, we deal with real hypersurfaces type (B). Use their determining relation $\phi A + A\phi = -(c/\alpha_1)\phi$ in (28) to obtain the quadratic equation for μ:

$$2\mu^2 + \Big(\frac{c}{\alpha_1} + \alpha_1 - H\Big)\mu + \Big(-\frac{c}{\alpha_1}\Big(\frac{\lambda}{2} - k\Big) - \frac{c}{2}n\Big) = 0.$$

Comparing the above equation with the defining equation (25) for (B), then we have

$$c = \alpha_1(\alpha_1 - H). \tag{32}$$

- For the case that M is of type (B) in $P_n\mathbb{C}(4)$, the principal curvatures and their eigenspaces are given as follows (cf. [2], [19]): $\mu_1 = (1 + x)/(1 -$

x), $\mu_2 = (x-1)/(x+1)$, $\alpha_1 = (x^2-1)/x$, where

$$x = \cot r,\ m(\mu_1) = n-1,\ m(\mu_2) = n-1,\ m(\alpha_1) = 1.$$
$$H = (n-1)\frac{(1+x)}{(1-x)} + (n-1)\frac{(x-1)}{(x+1)} + \frac{x^2-1}{x}.$$

Together with these data, (32) gives $n = 2$.

- In case that M is of type (B) in $H_n\mathbb{C}(-4)$, then the principal curvatures and their eigenspaces are given as follows (cf. [4]): $\mu_1 = x(= \coth r), \mu_2 = 1/x$, $\alpha_1 = 4x/(x^2+1)$, where $m(\mu_1) = n-1$, $m(\mu_2) = n-1$, $m(\alpha_1) = 1$. $H = (n-1)x + (n-1)(1/x) + 4x/(x^2+1)$. We also see that (32) only holds in $n = 2$.

In both cases the pseudo-Einstein constant $\lambda = 2k - \alpha_1$. After all, we have proved our Main Theorem. □

Remark 2 The name "pseudo-Einstein structure" in real hypersurfaces of a complex space form already used in [13]. Actually, the author adapt the notion by the same condition of "η-Einstein structure" in (almost) contact geometry (cf. [23]):

$$\rho = \alpha g + \beta\eta \otimes \eta, \tag{33}$$

for constants α, β. To avoid a confusion, we call an almost contact metric space satisfying (33) an η-Einstein space. In the same paper [13] he classified η-Einstein real hypersurfaces in $P_n\mathbb{C}$ for $n \geq 3$. Later, Cecil and Ryan [6], Montiel [14] developed this result for $P_n\mathbb{C}$, $H_n\mathbb{C}$, respectively. Indeed they classified (weakly) η-Einstein real hypersurfaces in $P_n\mathbb{C}$ or $H_n\mathbb{C}$, $n \geq 3$ for smooth functions α and β. They are realized as homogeneous real hypersurfaces of type (A): horospheres, tubes over $H_{n-1}\mathbb{C}$ in $H_n\mathbb{C}$, geodesic hyperspheres in $P_n\mathbb{C}$ or $H_n\mathbb{C}$, tubes of special radii r $(0 < r < \pi/2)$ over a totally geodesic $P_l\mathbb{C}$ $(1 \leq l \leq n-2)$, or homogeneous real hypersurfaces of type (B) in $P_n\mathbb{C}$: tubes of specific radii r $(0 < r < \pi/4)$ over a complex quadric Q^{n-1} and $P_n\mathbb{R}$. *There is no inclusion relation between the pseudo-Einstein real hypersurfaces and the η-Einstein real hypersurfaces.*

Remark 3 Ruled real hypersufaces in $P_n\mathbb{C}$ and $H_n\mathbb{C}$ given in [11] and [1], respectively. Let $\gamma\colon I \to \widetilde{M}_n(c)$ be a regular curve in $\widetilde{M}_n(c)$ ($P_n\mathbb{C}$ or $H_n\mathbb{C}$). Then for each $t \in I$, let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurfaces which is orthogonal to holomorphic plane Span$\{\dot\gamma,\ J\dot\gamma\}$. We have a ruled real hypersurface $M = \bigcup_{t\in I} M^{(t)}_{n-1}(c)$. These ruled ones are non-Hopf. The shape operator is written by the following form:

$$\begin{aligned} A\xi &= \alpha_1\xi + \nu V \ (\nu \neq 0), \\ AV &= \nu\xi, \\ AX &= 0 \quad \text{for any } X \perp \xi,\, V, \end{aligned}$$

where V is a unit vector orthogonal to ξ, and α_1, ν are differentiable functions on M. Moreover, we see that M *is Levi-flat*, that is, $L(X, Y) = (1/2)g((J\bar{A} + \bar{A}J)X, JY) = 0$ for any vector fields X, Y orthogonal to ξ. From (17), we have

$$\begin{aligned} S\xi &= f\xi, \\ SV &= gV, \\ SX &= \frac{c}{4}(2n+1)X \quad \text{for any } X \perp \xi,\, V, \end{aligned}$$

where $f = (c/2)(n-1) - \nu^2$ and $g = (c/4)(2n+1) - \nu^2$. Suppose that M admits the pseudo-Einstein structure. Then, together with (27), we get $cn = 0$, which is impossible. Thus, *a ruled real hypersurface M does not admit a pseudo-Einstein structure.*

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