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The Cauchy integral operator on Hardy space

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Abstract. We show that the Cauchy integral operator is bounded from $H^p(\mathbb{R}^1)$ to $h^p(\mathbb{R}^1)$ (local Hardy space). To prove our theorem we shall introduce generalized atom and consider a variant of "Tb theorem".

Key words: the Cauchy integral, Calderón-Zygmund operator, Hardy space, local Hardy space.

1. Introduction

Consider the Cauchy integral operator

$$C_A f(x) = \text{p.v.} \int_{\mathbb{R}^1} \frac{1}{x - y + i(A(x) - A(y))} f(y)dy.$$

This operator is very important in real and complex analysis, and many studies have been carried out (see, for example, [11]). L^p boundedness of C_A is well-known, but H^p boundedness of C_A has not been studied yet.

In this paper we shall show that C_A is a bounded operator from $H^p(\mathbb{R}^1)$ to $h^p(\mathbb{R}^1)$ (local Hardy space). In [8] (see Theorem A in Section 2), the author proved the $H^p(\mathbb{R}^1) \to h^p(\mathbb{R}^1)$ boundedness of Calderon's commutator

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^1} \frac{A(x) - A(y)}{(x - y)^2} f(y)dy.$$

The relation between C_A and T_A is written in [11]. Compared with Calderon's commutator, the Cauchy integral operator is difficult to study. Because we can calculate $T_A 1$ and apply "$T1$ theorem" by David and Journé [5], but we can not calculate $C_A 1$ directly. To prove our theorem we shall introduce generalized atom and consider a variant of "Tb theorem".

2. Definitions and notation

Throughout this paper we assume that, unless otherwise stated, all given functions are complex valued. Strictly speaking, only functions A

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and φ are real valued.

The following notation is used: For a set $E \subset \mathbb{R}^n$ and a locally integrable function b, we denote the Lebesgue measure of E by $|E|$ and $b(E) = \int_E b(x)dx$. We indicate the characteristic function of E by χ_E. We write a ball of radius r centered at x_0 by $B(x_0, r) = \{x \in \mathbb{R}^n; |x - x_0| < r\}$.

First we shall define two maximal functions and two Hardy spaces. Let φ be a fixed real valued Schwartz function in $\mathcal{S}(\mathbb{R}^n)$ such that $\mathrm{supp}(\varphi) \subset B(0, 1)$ and $\int \varphi(x)dx \neq 0$, then we define

$$M_\varphi f(x) = \sup_{t>0}\left|\int f(y)\varphi_t(x-y)dy\right|,$$
$$m_\varphi f(x) = \sup_{1>t>0}\left|\int f(y)\varphi_t(x-y)dy\right|,$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$.

Definition 1 (Fefferman–Stein's Hardy space [6])

$$H^p(\mathbb{R}^n) = \{f \in \mathcal{S}'; \|f\|_{H^p} = \|M_\varphi f\|_{L^p} < \infty\}, \quad \text{where } 0 < p < \infty.$$

Definition 2 (local Hardy space [7])

$$h^p(\mathbb{R}^n) = \{f \in \mathcal{S}'; \|f\|_{h^p} = \|m_\varphi f\|_{L^p} < \infty\}, \quad \text{where} \quad 0 < p < \infty.$$

Remark $\|f\|_{h^p} \leq \|f\|_{H^p}$.

Definition 3 (Lipschitz space)

$$\mathrm{Lip}_\alpha(\mathbb{R}^n) = \left\{f; \|f\|_{\mathrm{Lip}_\alpha} = \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty\right\} \quad \text{for } 0 < \alpha < 1.$$

Remark $(H^p)^* = \mathrm{Lip}_{n(1/p-1)}$ where $n/(n+1) < p < 1$ (For the duality, see [6] or [10], p. 54).

Next we shall define Calderón–Zygmund operator.

Definition 4 Let $0 < \delta \leq 1$. We say a function $K(x, y)$ defined on $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n; x \neq y\}$ is a δ-Calderón-Zygmund kernel if K satisfies the following conditions.

$$|K(x, y)| \leq \frac{C_1}{|x-y|^n}, \tag{i}$$

$$|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq C_1\frac{|y-z|^\delta}{|x-z|^{n+\delta}}, \tag{ii}$$

if $2|y-z|<|x-z|$.

Definition 5 We say an operator T is a δ-Calderón-Zygmund operator (associated with δ-Calderón-Zygmund kernel K) if

$$Tf(x)=\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}K(x,\,y)f(y)dy$$

exists for almost all x where $f\in L^2(\mathbb{R}^n)$ and T is bounded on $L^2(\mathbb{R}^n)$;

$$\|Tf\|_{L^2}\le C_2\|f\|_{L^2}.$$

Definition 6 The transpose of a operator T is denoted by

$${}^tTf(x)=\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}K(y,\,x)f(y)dy.$$

Definition 7 For a bounded function b, we define

$$\widetilde{{}^tT}b(x)=\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\Big\{K(y,\,x)-K(y,\,0)\chi_{\{|y|\ge 1\}}(y)\Big\}b(y)dy.$$

Note that if $b\in L^2\cap L^\infty$ then $\widetilde{{}^tT}b(x)={}^tTb(x)+C_b$ a.e. where C_b is a constant.

Remark There are more elaborate definitions of Calderón-Zygmund operator and $\widetilde{{}^tT}b$ (see, for example, [3] and [4]). But we are interested in the Cauchy integral operator, so our definitions will do.

Following [3] and [4], we define accretivity condition on functions. We shall use this condition in Section 3.

Definition 8 Let $\beta>0$. An bounded function b is said to be β-accretive if $\operatorname{Re} b(x)\ge\beta$ for almost all x.

Next we define the Cauchy integral operator and Calderón's commutator.

Definition 9 (the Cauchy integral operator) Let A be a real valued function on $\mathbb{R}^1$. We define

$$C_Af(x)=\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{1}{x-y+i(A(x)-A(y))}f(y)dy.$$

Definition 10 (Calderón's commutator) Let A be a real valued function on $\mathbb{R}^1$. We define

$$T_A f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{A(x)-A(y)}{(x-y)^2} f(y)dy.$$

The following propositions are most essential (see [3] and [4]).

Proposition 1 *If $A' \in L^\infty(\mathbb{R}^1)$ then the Cauchy integral operator C_A is a 1-Calderón-Zygmund operator, that is, bounded on $L^2(\mathbb{R}^1)$.*

Proposition 2 *If $A' \in L^\infty(\mathbb{R}^1)$ then Calderón's commutator T_A is a 1-Calderón-Zygmund operator, that is, bounded on $L^2(\mathbb{R}^1)$.*

The author [8] proved the next theorem.

Theorem A *If $A' \in L^\infty(\mathbb{R}^1) \cap \mathrm{Lip}_\alpha(\mathbb{R}^1)$, then T_A is a bounded operator from $H^p(\mathbb{R}^1)$ to $h^p(\mathbb{R}^1)$ where $1/(1+\alpha) \le p \le 1$.*

Remark In [9], the author proved the $H^p \to h^p$ boundedness of higher order commutators.

3. Theorem

Our result is the following:

Theorem *Let $n/(n+\delta) < p \le 1$ and $n/(n+\alpha) \le p$. We assume that T is a δ-Calderón-Zygmund operator. If there exists a β-accretive function b such that b, $\widetilde{{}^tTb} \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ then T is a bounded operator from $H^p(\mathbb{R}^n)$ to $h^p(\mathbb{R}^n)$ and*

$$\|Tf\|_{h^p} \le C_{p,n,\delta,\beta,b} \|f\|_{H^p},$$

where $C_{p,n,\delta,\beta,b}$ is a positive constant depending only on $p, n, C_1, C_2, \delta, \beta$, $\|b\|_\infty$, $\|b\|_{\mathrm{Lip}_\alpha}$ and $\|\widetilde{{}^tTb}\|_{\mathrm{Lip}_\alpha}$.

As a corollary of this theorem we obtain the boundedness of the Cauchy integral.

Corollary *If $A' \in L^\infty(\mathbb{R}^1) \cap \mathrm{Lip}_\alpha(\mathbb{R}^1)$, then C_A is a bounded operator from $H^p(\mathbb{R}^1)$ to $h^p(\mathbb{R}^1)$ where $1/(1+\alpha) \le p \le 1$.*

Proof. Note that C_A is a 1-Calderón-Zygmund operator (Proposition 1). Let $b(x) = 1 + iA'(x)$. Then b is 1-accretive and $b \in \mathrm{Lip}_\alpha(\mathbb{R}^1)$. By an elementary calculus of complex analysis (refer the calculation in [12], p. 407),

$$\begin{aligned}
&\widetilde{{}^tC_A}b(x)\\
&= \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\Big\{\frac{1+iA'(y)}{y-x+i(A(y)-A(x))}\\
&\qquad\qquad -\frac{1+iA'(y)}{y+i(A(y)-A(0))}\chi_{\{|y|\geq 1\}}(y)\Big\}dy\\
&= \text{ constant.}
\end{aligned}$$

Therefore we can apply theorem. □

4. Lemmas

In this section we shall define atoms and molecules on $H^p(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$ and show some properties of Hardy spaces. Let $n/(n+1) < p < 1$.

First we define atoms on $H^p(\mathbb{R}^n)$.

Definition 11 A function $a(x)$ is a H^p-atom centered at x_0 if there exists a ball $B(x_0, r)$ such that the following conditions are satisfied

$$\operatorname{supp}(a) \subset B(x_0, r), \tag{1}$$

$$\|a\|_{L^\infty} \leq r^{-n/p}, \tag{2}$$

$$\int a(x)dx = 0. \tag{3}$$

Proposition 3 (atomic decomposition of H^p) *If $f \in H^p(\mathbb{R}^n)$ then f can be written as $f = \sum_{j=1}^\infty \lambda_j a_j$ where a_j's are H^p-atoms and $\sum_{j=1}^\infty |\lambda_j|^p \approx \|f\|_{H^p}^p$.*

For the proof of this proposition, see [10], p. 20. The following Lemma 1 is trivial.

Lemma 1 *If a function $a(x)$ is a H^p-atom supported in $B(x_0, r)$, then $\|a\|_{H^q} \leq C_{n,q} r^{n(1/q-1/p)}$ where $n/(n+1) < q \leq 1$.*

Next we define atoms on $h^p(\mathbb{R}^n)$.

Definition 12 A function $a(x)$ is a large h^p-atom centered at x_0 if there exists a ball $B(x_0, r)$ of radius $r > 1$ which satisfies the conditions (1) and

$$\|a\|_{L^1} \leq r^{n(1-1/p)}. \tag{2'}$$

Lemma 2 ([7]) *If a function $a(x)$ is a large h^p-atom then $\|a\|_{h^p} \le C_{p,n}$.*

Remark The condition $p < 1$ is essential.

Definition 13 A function $a(x)$ is a small h^p-atom centered at x_0 if there exists a ball $B(x_0, r)$ of radius $r \le 1$ which satisfies the conditions (1), (2′) and

$$\Big|\int a(x)dx\Big| \le 1. \tag{3′}$$

Lemma 3 ([8]) *If a function $a(x)$ is a small h^p-atom then $\|a\|_{h^p} \le C_{p,n}$.*

In [8], the author introduced small h^p-atom. In this paper we need to consider generalization of this atom.

Definition 14 Let b be β-accretive. We say a function $a(x)$ is a small (h^p, b)-atom centered at x_0 if there exists a ball $B(x_0, r)$ of radius $r \le 1$ which satisfies the conditions (1), (2′) and

$$\Big|\int a(x)b(x)dx\Big| \le 1. \tag{3″}$$

Lemma 4 *We assume that b is β-accretive and $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$. If a function a is a small (h^p, b)-atom centered at x_0 then*

$$\|a\|_{h^p} \le C_{p,n,\beta,b},$$

where $\alpha \ge n(1/p - 1)$.

Proof. We assume that $\mathrm{supp}(a) \subset B(x_0, r)$, then

$$\begin{aligned}\Big|\int a(x)dx\Big| &\le \Big|\frac{1}{b(x_0)}\int_{B(x_0,r)} a(x)(b(x)-b(x_0))dx\Big| \\ &\quad + \Big|\frac{1}{b(x_0)}\int a(x)b(x)dx\Big| \\ &\le \frac{C_n}{\beta}\|b\|_{\mathrm{Lip}_\alpha} r^\alpha \cdot r^{n(1-1/p)} + \frac{1}{\beta} \le \frac{C_n\|b\|_{\mathrm{Lip}_\alpha}+1}{\beta}.\end{aligned}$$

Therefore by Lemma 3 we obtain the desired result. □

Next we define molecules on $h^p(\mathbb{R}^n)$.

Definition 15 Let $\delta > n(1/p - 1)$. A function $M(x)$ is a large (h^p, δ)-molecule centered at x_0 if there exists $r > 1$ such that the following conditions are satisfied

$$\int_{|x-x_0|<2r} |M(x)|dx \le r^{n(1-1/p)}, \tag{M$_1$}$$

$$\int_{|x-x_0|\ge 2r} |M(x)||x-x_0|^{\delta} dx \le r^{\delta+n(1-1/p)}. \tag{M$_2$}$$

Definition 16 We assume that b is β-accretive and $\delta > n(1/p-1)$. We say a function $M(x)$ is a small (h^p, δ, b)-molecule centered at x_0 if there exists $r \le 1$ which satisfies (M_1), (M_2) and

$$\left| \int M(x)b(x)dx \right| \le 1. \tag{M$_3$}$$

The following two lemmas are most essential to prove our theorem.

Lemma 5 *If a function $M(x)$ is a large (h^p, δ)-molecule centered at x_0 then $\|M\|_{h^p} \le C_{p,n,\delta}$.*

Lemma 6 *Let $\delta > n(1/p-1)$ and $\alpha \ge n(1/p-1)$. We assume that b is β-accretive and $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$. If a function $M(x)$ is a small (h^p, δ, b)-molecule centered at x_0 then $\|M\|_{h^p} \le C_{p,n,\alpha,\beta,\delta,b}$.*

Remark For the definition of H^p-molecule, see [10]. The author [8] introduced h^p-molecule when $b \equiv 1$.

The proofs of Lemmas 5 and 6 are similar, so we shall prove Lemma 6 only. The proof of Lemma 5 is easier.

Proof of Lemma 6. We use the same argument as in [8]. Let $E_0 = \{x; |x-x_0| < 2r\}$ and $E_i = \{x; 2^i r \le |x-x_0| < 2^{i+1} r\}$, $i = 1, 2, 3, \ldots$, and let $\chi_i(x) = \chi_{E_i}(x)$, $\tilde{\chi}_i(x) = (1/b(E_i))\chi_{E_i}(x)$, $m_i = (1/b(E_i)) \int_{E_i} M(y)b(y)dy$, $\tilde{m}_i = \int_{E_i} M(y)b(y)dy$ and $M_i(x) = (M(x) - m_i)\chi_i(x)$. Note that $b(E_i) \ne 0$.

We write

$$M(x) = \sum_{i=0}^{\infty} M_i(x) + \sum_{i=0}^{\infty} m_i \chi_i(x) = \sum_{i=0}^{\infty} M_i(x) + \sum_{i=0}^{\infty} \tilde{m}_i \tilde{\chi}_i(x).$$

Let $N_j = \sum_{k=j}^{\infty} \tilde{m}_k$ and we write

$$\begin{aligned} M(x) &= \sum_{i=0}^{\infty} M_i(x) + \sum_{i=1}^{\infty} N_i(\tilde{\chi}_i(x) - \tilde{\chi}_{i-1}(x)) + N_0 \tilde{\chi}_0(x) \\ &= I + II + III. \end{aligned}$$

First we estimate I.

It is clear that $\operatorname{supp}(M_i) \subset B(x_0,\, 2^{i+1}r)$ and $\int M_i(x)b(x)dx = 0$.

By the condition (M_1) and the accretivity condition of b, we have

$$\int |M_0(x)|dx \le \Big(1 + \frac{\|b\|_\infty}{\beta}\Big) r^{n(1-1/p)}.$$

So by Lemma 2 or 4 we have $\|M_0\|_{h^p} \le C_{p,n,\beta,b}$.

When $i \ge 1$, by using the condition (M_2), we have

$$\begin{aligned}\int |M_i(x)|dx &\le C_{n,\beta,b}(2^i r)^{-\delta} \int_{E_i} |M(x)||x - x_0|^\delta dx \\ &\le C_{n,\beta,b}(2^i r)^{-\delta} r^{\delta+n(1-1/p)} \le C_{n,\beta,b} 2^{-\delta i} r^{n(1-1/p)}.\end{aligned}$$

By Lemma 2 or 4 we have

$$\begin{aligned}\|M_i\|_{h^p} &\le C_{p,n,\beta,b} 2^{-\delta i} r^{n(1-1/p)} (2^{i+1}r)^{n(1/p-1)} \\ &= C_{p,n,\beta,b} 2^{(-\delta+n(1/p-1))i}.\end{aligned}$$

Since $\delta > n(1/p - 1)$, we obtain $\sum_{i=1}^\infty \|M_i\|_{H^p}^p \le C_{p,n,\delta,\beta,b}$ and $\|I\|_{H^p} \le C_{p,n,\delta,\beta,b}$.

Next we estimate II.

Let $A_i(x) = N_i(\tilde\chi_i(x) - \tilde\chi_{i-1}(x))$. It is clear that $\operatorname{supp}(A_i) \subset B(x_0,\, 2^{i+1}r)$ and $\int A_i(x)b(x)dx = 0$. Using the condition (M_2), we have

$$\begin{aligned}\|A_i\|_{L^\infty} &\le C_{n,\beta,b}(2^i r)^{-n} \int_{|x-x_0|\ge 2^i r} |M(x)|dx \\ &\le C_{n,\beta,b}(2^i r)^{-n}(2^i r)^{-\delta} \int_{|x-x_0|\ge 2^i r} |M(x)||x - x_0|^\delta dx \\ &\le C_{n,\beta,b} 2^{i(-n-\delta)} r^{-n-\delta} r^{\delta+n(1-1/p)} = C_{n,\beta,b} 2^{i(-n-\delta)} r^{-n/p}.\end{aligned}$$

By Lemma 2 or 4 we have

$$\|A_i\|_{H^p} \le C_{p,n,\beta,b} 2^{i(-n-\delta)} r^{-n/p} (2^{i+1}r)^{n/p} \le C_{p,n,\beta,b} 2^{i(-\delta+n(1/p-1))}.$$

Since $\delta > n(1/p - 1)$, we obtain $\sum_{i=1}^\infty \|A_i\|_{H^p}^p \le C_{p,n,\delta,\beta,b}$ and $\|II\|_{H^p} \le C_{p,n,\delta,\beta,b}$.

Finally we estimate III.

It is clear that $\operatorname{supp}(N_0\tilde\chi_0) \subset B(x_0,\, 2r)$. Using the conditions ($\mathrm{M}_1$) and

(M_2), we have

$$\begin{aligned}\|N_0\tilde{\chi}_0\|_{L^1} &\le C_{\beta,b}\int |M(x)|dx \\ &\le C_{\beta,b}\Big(\int_{|x-x_0|<2r}|M(x)|dx + (2r)^{-\delta}\int_{|x-x_0|\ge 2r}|M(x)||x-x_0|^{\delta}dx\Big) \\ &\le C_{\beta,b}\big(r^{n(1-1/p)} + (2r)^{-\delta}r^{\delta+n(1-1/p)}\big) \le C_{\beta,b}r^{n(1-1/p)}.\end{aligned}$$

Using the condition (M_3), we have

$$\Big|\int N_0\tilde{\chi}_0(x)b(x)dx\Big| = \Big|\int M(x)b(x)dx\Big| \le 1.$$

By Lemma 2 or 4 we have $\|N_0\tilde{\chi}_0\|_{h^p} \le C_{p,n,\beta,b}$. □

5. Proof of Theorem

Applying the interpolation theorem between L^2 and H^p or h^p, we may assume $p<1$. By the atomic decomposition of H^p, it suffices to show that $\|Ta\|_{h^p}\le C$ for every H^p-atom a, where C is a positive constant depending only on p, n, C_1, C_2, δ, β, $\|b\|_\infty$, $\|b\|_{\mathrm{Lip}_\alpha}$ and $\|\widetilde{{}^tTb}\|_{\mathrm{Lip}_\alpha}$.

We assume H^p-atom a is supported in $B(x_0, r)$. We shall show that if $r\ge 1$ then $Ta(x)$ is a constant multiple of a large (h^p, δ)-molecule, and if $r<1$ then $Ta(x)$ is a constant multiple of a small (h^p, δ, b)-molecule.

We have to check that if $r\ge 1$ then Ta satisfies (M_1) and (M_2), and if $r<1$ then Ta satisfies three conditions of Definition 16.

Since T is bounded on L^2, we have

$$\int_{|x-x_0|\le 2r}|Ta(x)|dx \le C\|Ta\|_{L^2}r^{n/2} \le Cr^{n(1-1/p)}. \tag{4}$$

If $|x-x_0|\ge 2r$, we have

$$|Ta(x)| = \Big|\int (K(x, y)-K(x-x_0))a(y)dy\Big| \le C\frac{r^{n(1-1/p)+\delta}}{|x-x_0|^{n+\delta}}. \tag{5}$$

If $r\ge 1$, by (4), (5) and Lemma 5, we have $\|Ta\|_{h^p}\le C$. If $r<1$, by the duality of $H^{n/(n+\alpha)}$ and Lip_α and Lemma 1, we have

$$\begin{aligned}\Big|\int Ta(x)b(x)dx\Big| &= |(a, \widetilde{{}^tTb})| \le C\|a\|_{H^{n/(n+\alpha)}}\|\widetilde{{}^tTb}\|_{\mathrm{Lip}_\alpha} \\ &\le C\|\widetilde{{}^tTb}\|_{\mathrm{Lip}_\alpha}r^{\alpha+n(1-1/p)} \le C,\end{aligned} \tag{6}$$

because $\alpha \geq n(1/p - 1)$.

By (4), (5), (6) and Lemma 6, we obtain $\|Ta\|_{h^p} \leq C$. □

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