Hokkaido Mathematical Journal Vol. 37 (2008) p. 349–362

Comparison results for a class of weakly coupled systems of eikonal equations

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(Received November 24, 2006; Revised June 20, 2007)

Abstract. We present two comparison theorems for a class of weakly coupled systems of Hamilton-Jacobi equations with convex, coercive Hamiltonians. These results apply in particular to systems arising in large deviations theory for random evolution processes.

Key words: systems of Hamilton-Jacobi equations, viscosity solutions, large deviations theory.

1. Introduction

We study comparison results for the weakly coupled system of Hamilton-Jacobi equations

$$H_i(x, Du^i) + \sum_{j=1}^{M} c_{ij}(x)(u^i - u^j) = 0 \quad x \in D. \tag{1.1}$$

Systems of this type arise in the optimal control of a random evolution process (see [7], [12]).

Another motivation for (1.1) comes from large deviations theory for random evolution processes (see [4], [5]). Large deviations functionals defined on the sample paths of this type of stochastic process typically solve weakly coupled systems of second order linear equations [5]. A standard way to prove large deviations results via PDE methods is to take the logarithmic transform of the path functional and to pass to the limit in the equation so obtained (see [1], [2], [8], [11]). In the case of random evolution processes, passing to the limit in the system satisfied by the logarithmic transform of the path functional we *formally* get the system (1.1) where the Hamiltonians are the large deviations ones

$$H_i(x, p) = \frac{|p|^2}{2} - b_i(x) \cdot p. \tag{1.2}$$

2000 Mathematics Subject Classification : Primary 49L25; Secondary 35F30, 35B25.

To make rigorous the previous method, a key tool is a comparison result for the limit problem (1.1). But the system does not satisfy monotonicity assumptions with respect to the variable u which are usually assumed for this type of results. In fact, rewriting the system (1.1) as

$$H_i(x,\, Du_i) + d_{ii}(x)u_i + \sum_{j\neq i} d_{ij}(x)u_j = 0$$

(we prefer the former notation, instead of more common latter one, in view of the application to singular perturbations) where $d_{ii} = \sum_j c_{ij}$ is the coefficient of u_i and $d_{ij} = -c_{ij}$ are the coefficients of u_j, $j \neq i$, in the i-th equation, then we have

$$d_{ii} + \sum_{j\neq i} d_{ij} = 0$$

for $x \in D$, $i = 1, \ldots, M$, while in general it is required that $d_{ii} + \sum_{j\neq i} d_{ij} \geq c_0 > 0$. In other words, in the terminology of [9], the system (1.1) is *quasi-monotone* but not *monotone* (see also [7], [12]).

If the system (1.1) has only one component, i.e. $M = 1$, it reduces to

$$H(x,\, Du) = 0, \quad x \in D. \tag{1.3}$$

It is well known that the existence of a strict subsolution is a sufficient, and also necessary, condition to get comparison result and uniqueness for Hamilton-Jacobi equations without zero order terms (i.e. quasi-monotone). More recently, by the Aubry-Mather theory for critical Hamilton-Jacobi equations [6], it can been shown that to a convex, coercive Hamiltonian $H(x,\, p)$, it is possible to associate a closed, possible empty set $\mathcal{A}$, said the Aubry set of H (for the definition of Aubry set and its properties we refer to [6]). The main property of the Aubry set is the following

There exists a C^1-function ψ and a nonnegative continuous function f such that

$$H(x,\, D\psi) \leq -f(x), \quad x \in D$$

with $f > 0$ out of $\mathcal{A}$.
In particular, if $\mathcal{A}$ is empty, then there exists a strict subsolution in all D and it is possible to have a comparison result for (1.3) ([9]). Otherwise, the Aubry set $\mathcal{A}$ behaves as a sort of interior boundary where, to have uniqueness, the value of the solution has to be prescribed ([6]).

The idea we follows for (1.1) shares several analogy with the case of a single equation. We consider a function $\psi \in C^1$ which is a subsolution of all the equations $H_i(x, Du) = 0$, $i = 1, \ldots, M$, the crucial point being that ψ is the same for all the Hamiltonians H_i. We introduce some assumptions to control the sets where ψ fails to be a strict subsolution of the equation corresponding to the Hamiltonian H_i. Assuming the existence of such a strict subsolution, the proof of the comparison result for (1.1) is similar to the one in [9] for (1.3). Also in this case the difficulty is given by the absence of a zero order term in the equation. For this reason, given a subsolution u and a supersolution v to (1.1) we compare v with $u_\lambda = (\lambda u_1 + (1-\lambda)\psi, \ldots, \lambda u_M + (1-\lambda)\psi)$, where $\lambda \in (0, 1)$ and ψ is the strict subsolution. Then we get the result sending $\lambda \to 1$.

In the last section, we discuss our assumptions, in particular we prove that an assumption introduced in [5], named *strong Levinson's condition*, implies the existence of a strict subsolution to (1.1) when H_i are the large deviations Hamiltonians (1.2).

2. Definitions and assumptions

Consider the weakly-coupled system of Hamilton-Jacobi equations

$$H_i(x, Du_i) + \sum_{j=1}^{M} c_{ij}(x)(u_i - u_j) = 0 \quad x \in D,\ i = 1, \ldots, M \tag{2.1}$$

where D is a bounded set with Lipschitz-continuous boundary. In all the paper we will assume that

(2.2) $c_{ij} \colon D \to \mathbb{R}$ are continuous for $x \in D$, $i, j = 1, \ldots, M$,

(2.3) $H_i(x, p)$ is continuous in (x, p), convex and coercive in p for $i = 1, \ldots, M$

(2.4) there exist a C^1 function ψ and continuous functions $f_i \geq 0$ such that $H_i(x, D\psi) \leq -f_i(x)$ in D, $i = 1, \ldots, M$.

For the uniqueness results, we consider two different sets of assumptions. We define

$$\mathcal{A}_i = \{x \in D \colon f_i(x) = 0\}$$

and we will assume either

(2.5) $c_{ij}(x) \geq 0$ for $x \in D$, $i, j = 1, \ldots, M$, $i \neq j$, and $\mathcal{A}_i$ is empty, for any $i = 1, \ldots, M$, or

(2.6) $c_{ij}(x) > 0$ for $x \in D$, $i, j = 1, \ldots, M$, $i \neq j$, and $\bigcap_{i=1}^{M} \mathcal{A}_i$ is empty.

Note that assumption (2.6) implies that the intersection of Aubry sets, see [6], of the Hamiltonians H_i is empty. But it is stronger in the sense that it also require that there is a same subsolution for all the Hamiltonians which is a strict subsolution at any point of D for at least one Hamiltonian.

For a function $u\colon E \to \mathbb{R}^M$, we say that $u = (u_1, \ldots, u_M)$ is u.s.c. and we write $u \in \mathrm{USC}(E)$ if all the components u_i, $i = 1, \ldots, M$, are u.s.c. in E. Similarly $v \in \mathrm{LSC}(E)$ if all the components v_i, $i = 1, \ldots, M$, are l.s.c. in E.

If $u = (u_1, \ldots, u_M)$, $v = (v_1, \ldots, v_M)$, are two functions defined in a set E we write $u \leq v$ in E if $u_i \leq v_i$ in E, $i \in \{1, \ldots, M\}$.

We recall the definition of viscosity solution for weakly coupled systems (see [7], [9] for more details)

Definition 2.1

i) An u.s.c. function $u\colon D \to \mathbb{R}^M$ is said a viscosity subsolution of (2.1) if whenever $\phi \in C^1(D)$, $i \in \{1, \ldots, M\}$ and $u_i - \phi$ attains a local maximum at $x \in D$, then

$$H_i(x, D\phi(x)) + \sum_{j=1}^{M} c_{ij}(x)(u_i - u_j) \leq 0.$$

ii) A l.s.c. $v\colon D \to \mathbb{R}^M$ is said a viscosity supersolution of (2.1) if whenever $\phi \in C^1(D)$, $i \in \{1, \ldots, M\}$ and $v_i - \phi$ attains a local minimum at $x \in D$, then

$$H_i(x, D\phi(x)) + \sum_{j=1}^{M} c_{ij}(x)(v_i - v_j) \geq 0.$$

iii) A continuous function u is said a viscosity solution of (2.1) if it is both a viscosity sub- and supersolution of (2.1).

Proposition 2.2 *Let $u \in \mathrm{USC}(\overline{D})$ be a viscosity subsolution to* (2.1)*. Then u is Lipschitz continuous in D.*

Proof. It is sufficient to observe that u_i, $i = 1, \ldots, M$, is a viscosity subsolution of

$$H_i(x, Du) \leq C \quad x \in D$$

where C is a sufficiently large constant such that $\|\sum_{j=1}^{M} c_{ij}(x)(u_i - u_j)\|_\infty \leq C$. Then the Lipschitz continuity of u_i is consequence of the coercitivity of the Hamiltonian H_i (see [1, Lemma 2.5]). □

3. Two maximum principles for weakly coupled systems of eikonal type

Aim of this section is to show a comparison theorem for (2.1) under either assumption (2.5) or assumption (2.6).

Theorem 3.1 *Assume* (2.2)–(2.4) *and* (2.5). *Let* $u \in \mathrm{USC}(\overline{D})$ *and* $v \in \mathrm{LSC}(\overline{D})$ *be respectively a subsolution and a supersolution of* (2.1) *such that* $u \leq v$ *on* ∂D. *Then*

$$u \leq v \quad \text{in } \overline{D}.$$

The proof of the theorem is based on the following lemma.

Lemma 3.2 *Let* $g_i \in C^0(\overline{D})$, $i = 1, \ldots, M$, *and assume that* (2.2)–(2.4) *and* (2.5) *hold with* g_i *in place of* f_i. *Let* $u \in \mathrm{USC}(\overline{D})$ *be a subsolution of*

$$H_i(x, Du_i) + \sum_{j=1}^{M} c_{ij}(x)(u_i - u_j) \leq -g_i(x), \quad x \in D, \tag{3.1}$$

$v \in \mathrm{LSC}(\overline{D})$ *a supersolution of* (2.1) *and assume that* $u \leq v$ *on* ∂D. *Then*

$$u \leq v \quad \text{in } \overline{D}.$$

Proof. We set $A = \{1, \ldots, M\}$ and we assume by contradiction that

$$M = \max_{\substack{i \in A \\ x \in \overline{D}}} \{u_i - v_i\} > 0.$$

We define

$$\Psi_\varepsilon(x, y, i) = u_i(x) - v_i(y) - \frac{|x - y|^2}{2\varepsilon^2}.$$

Let $(x_\varepsilon, y_\varepsilon, i_\varepsilon)$ be such that $\Psi_\varepsilon(x_\varepsilon, y_\varepsilon, i_\varepsilon) = \max_{\overline{D} \times \overline{D} \times A} \Psi(x, y, j)$. Since

$$M \leq \Psi_\varepsilon(x_\varepsilon, y_\varepsilon, i_\varepsilon)$$

we have

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon^2} \leq u_{i_\varepsilon}(x_\varepsilon) - v_{i_\varepsilon}(y_\varepsilon) - M \tag{3.2}$$

and therefore $\lim_{\varepsilon\to 0}|x_\varepsilon - y_\varepsilon| = 0$. Let $(x_{\varepsilon_m}, y_{\varepsilon_m}, i_{\varepsilon_m})$ be a converging sequence for $\varepsilon_m \to 0$ and $(x, y, i) = \lim_{m\to\infty}(x_{\varepsilon_m}, y_{\varepsilon_m}, i_{\varepsilon_m})$. Then $x = y$ and by (3.2)

$$\limsup_{m\to\infty} \frac{|x_{\varepsilon_m} - y_{\varepsilon_m}|^2}{2\varepsilon_m^2} \le u_i(x) - v_i(x) - M \le 0.$$

It follows that $\lim_{\varepsilon\to 0}|x_\varepsilon - y_\varepsilon|^2/(2\varepsilon^2) = 0$ and, by (3.2), $x_\varepsilon, y_\varepsilon \in D$ for ε sufficiently small. By (3.2) we also have

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon^2} \le u_{i_\varepsilon}(x_\varepsilon) - u_{i_\varepsilon}(y_\varepsilon) + u_{i_\varepsilon}(y_\varepsilon) - v_{i_\varepsilon}(y_\varepsilon) - M \le L|x_\varepsilon - y_\varepsilon|$$

where L, see Proposition 2.2, is the maximum of the Lipschitz constants for u_i, $i = 1, \ldots, M$. Therefore

$$\frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon^2} \le 2L.$$

Since $\Psi(x, y_\varepsilon, i_\varepsilon)$ has a maximum point at x_ε, we have

$$H_{i_\varepsilon}\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}\Big) + \sum_{j\in A} c_{i_\varepsilon j}(x_\varepsilon)(u_{i_\varepsilon}(x_\varepsilon) - u_j(x_\varepsilon)) \le -g_{i_\varepsilon}(x_\varepsilon) \quad (3.3)$$

Since $-\Psi(x_\varepsilon, y, i_\varepsilon)$ has a minimum point at y_ε, we have

$$H_{i_\varepsilon}\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}\Big) + \sum_{j\in A} c_{i_\varepsilon j}(y_\varepsilon)(v_{i_\varepsilon}(y_\varepsilon) - v_j(y_\varepsilon)) \ge 0. \quad (3.4)$$

Subtracting (3.4) by (3.3), we get

$$\begin{aligned} &H_{i_\varepsilon}\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}\Big) - H_{i_\varepsilon}\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}\Big) \\ &+ \sum_{j\in A}(c_{i_\varepsilon j}(x_\varepsilon) - c_{i_\varepsilon j}(y_\varepsilon))(u_{i_\varepsilon}(x_\varepsilon) - u_j(x_\varepsilon)) \\ &+ \sum_{j\in A} c_{i_\varepsilon j}(y_\varepsilon)\big((u_{i_\varepsilon}(x_\varepsilon) - v_{i_\varepsilon}(y_\varepsilon)) - (u_j(x_\varepsilon) - v_j(y_\varepsilon))\big) \le -g_{i_\varepsilon}(x_\varepsilon) \end{aligned}$$

Recalling (2.5) and observing that $\Psi(x_\varepsilon, y_\varepsilon, j) \le \Psi(x_\varepsilon, y_\varepsilon, i_\varepsilon)$ implies that $u_j(x_\varepsilon) - v_j(y_\varepsilon) \le u_{i_\varepsilon}(x_\varepsilon) - v_{i_\varepsilon}(y_\varepsilon)$, $j = 1, \ldots, M$, we get a contradiction for ε sufficiently small. $\square$

Proof of Theorem 3.1. For $\lambda \in (0, 1)$, set $u_\lambda = (\lambda u_1 + (1-\lambda)\psi, \dots, \lambda u_M + (1-\lambda)\psi)$, where ψ is as in (2.4). Then, by convexity of H_i, it is straightforward to verify that u_λ is a subsolution of

$$H_i(x, Du_i) + \sum_{j \in A} c_{ij}(x)(u_i - u_j) = -(1-\lambda) f_i(x) \qquad x \in D,\ i = 1, \dots, M \quad (3.5)$$

Since ψ is defined up to a constant, we can assume that $\psi \le \min_{j \in A}\{u_j\}$ in $\overline{D}$, hence $u_\lambda \le v$ on ∂D. By Lemma 3.2, we have for any $\lambda \in (0, 1)$

$$u_\lambda \le v \quad \text{in } D$$

and we get the statement for $\lambda \to 1$. □

For the maximum principle under assumption (2.6), we need to assume the continuity of the supersolution.

Theorem 3.3 *Assume* (2.2)–(2.4) *and* (2.6). *Let* $u \in \mathrm{USC}(\overline{D})$ *and* $v \in \mathrm{LSC}(\overline{D}) \cap C^0(D)$ *be respectively a subsolution and a supersolution of* (2.1) *such that* $u \le v$ *on* ∂D*, then*

$$u \le v \quad \text{in } \overline{D}.$$

Given the following lemma, the proof of Theorem 3.3 is exactly the same of that of Theorem 3.1.

Lemma 3.4 *Let* $g_i \in C^0(\overline{D})$, $i = 1, \dots, M$, *and assume that* (2.2)–(2.4) *and* (2.6) *hold with* g_i *in place of* f_i. *Let* $u \in \mathrm{USC}(\overline{D})$ *be a subsolution of* (3.1), $v \in \mathrm{LSC}(\overline{D}) \cap C^0(D)$ *a supersolution of* (2.1) *and assume that* $u \le v$ *on* ∂D. *Then*

$$u \le v \quad \text{in } \overline{D}.$$

Proof. We set $A = \{1, \dots, M\}$ and we assume by contradiction that

$$M = \max_{\substack{i \in A \\ x \in \overline{D}}} \{u_i - v_i\} > 0.$$

Let $z \in \overline{D}$ be a point where the maximum is achieved. Then $z \in D$. We distinguish two cases:
A) There exists $k \in A$ such that $u_k(z) - v_k(z) \le M - \delta$ for some $\delta > 0$;
B) $u_i(z) - v_i(z) = M$ for any $i \in A$.

We start proving A). Let $i \in A$ be an index such that $u_i(z) - v_i(z) = M$. Define

$$\Psi_\varepsilon(x, y) = u_i(x) - v_i(y) - \frac{|x-y|^2}{2\varepsilon^2} - |y-z|^2 \tag{3.6}$$

and let $(x_\varepsilon, y_\varepsilon) \in \overline{D} \times \overline{D}$ be such that $\Psi_\varepsilon(x_\varepsilon, y_\varepsilon) = \max_{\overline{D}\times\overline{D}} \Psi$. Let us show that

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon^2} + |y_\varepsilon - z|^2 \longrightarrow 0 \quad \text{for } \varepsilon \to 0^+. \tag{3.7}$$

By $\Psi_\varepsilon(z, z) \le \Psi_\varepsilon(x_\varepsilon, y_\varepsilon)$ and recalling that z is a maximum point for $u - v$, we get

$$\begin{aligned}&\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon^2} + |y_\varepsilon - z|^2 \le u_i(y_\varepsilon) - v_i(y_\varepsilon)\\ &-(u_i(z) - v_i(z)) + u_i(x_\varepsilon) - u_i(y_\varepsilon) \le L|x_\varepsilon - y_\varepsilon|\end{aligned}$$

where L, see Proposition 2.2, is the maximum of the Lipschitz constants for u_i, $i = 1, \dots, M$. Therefore

$$\frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon^2} \le 2L$$

and

$$\lim_{\varepsilon\to 0} x_\varepsilon = \lim_{\varepsilon\to 0} y_\varepsilon = z. \tag{3.8}$$

Since $\Psi(x, y_\varepsilon)$ has a maximum at x_ε, we have

$$H_i\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}\Big) + \sum_{j\in A} c_{ij}(x_\varepsilon)(u_i(x_\varepsilon) - u_j(x_\varepsilon)) \le -g_i(x_\varepsilon). \tag{3.9}$$

Since $-\Psi(x_\varepsilon, y)$ has a minimum at y_ε we get

$$H_i\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} - 2(y_\varepsilon - z)\Big) + \sum_{j\in A} c_{ij}\big(y_\varepsilon)(v_i(y_\varepsilon) - v_j(y_\varepsilon)\big) \ge 0. \tag{3.10}$$

Subtracting (3.10) by (3.9), we get

$$H_i\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}\Big) - H_i\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} - 2(y_\varepsilon - z)\Big)$$

$$
\begin{aligned}
&+\sum_{j\in A}(c_{ij}(x_\varepsilon)-c_{ij}(y_\varepsilon))(u_i(x_\varepsilon)-u_j(x_\varepsilon))\\
&+\sum_{j\neq k}c_{ij}(y_\varepsilon)\big((u_i(x_\varepsilon)-v_i(y_\varepsilon))-(u_j(x_\varepsilon)-v_j(y_\varepsilon))\big)\\
&+c_{ik}(y_\varepsilon)\big((u_i(x_\varepsilon)-v_i(y_\varepsilon))-(u_k(x_\varepsilon)-v_k(y_\varepsilon))\big)\le 0
\end{aligned}
$$

Recalling (2.2), (2.6), (3.8) and that $u_i(z)-v_i(z)\ge u_j(z)-v_j(z)$, $j\in A$ and $u_i(z)-v_i(z)\ge u_k(z)-v_k(z)+\delta$, we get a contradiction for ε sufficiently small since u, $v\in C^0(D)$.

To prove B), let i be such that $g_i(z)>0$ and define Ψ_ε as in (3.6). Repeating the same argument of the case A), we get

$$
\begin{aligned}
&H_i\Big(x_\varepsilon,\,\frac{x_\varepsilon-y_\varepsilon}{\varepsilon^2}\Big)-H_i\Big(y_\varepsilon,\,\frac{x_\varepsilon-y_\varepsilon}{\varepsilon^2}-2(y_\varepsilon-z)\Big)\\
&+\sum_{j\in A}(c_{ij}(x_\varepsilon)-c_{ij}(y_\varepsilon))(u_i(x_\varepsilon)-u_j(x_\varepsilon))\\
&+\sum_{j\in A}c_{ij}(x_\varepsilon)\big((u_i(x_\varepsilon)-v_i(y_\varepsilon))-(u_j(x_\varepsilon)-v_j(y_\varepsilon)\big)\le -g_i(x_\varepsilon)
\end{aligned}
$$

which gives immediately a contradiction for ε sufficiently small since $g_i(z)<0$. ☐

Remark 3.5 Concerning the existence of a solution to (2.1), Ishii in [9] extended the Perron's method for viscosity solutions to weakly coupled systems of nonlinear equations. We remark that for the Perron's method, besides standard continuity assumptions, it is sufficient to assume that the system is quasi-monotone. It follows that if the system (2.1) satisfies assumptions (2.2)–(2.4) and there exist a subsolution u and a supersolution v of (2.1) such that $u=v$ on ∂D, then there exists a solution to (2.1) which assume the same boundary datum.

4. Examples

In this section we discuss two examples of systems satisfying the hypotheses (2.5) or (2.6).

4.1. Systems with Eikonal Hamiltonians

Consider the system (2.1) where $H_i(x,p)=|p|-F_i(x)$, with F_i non-negative, continuous functions such that either $\{x\in D\colon F_i(x)=0\}$ is empty for any $i=1,\dots,M$ or $\bigcap_{i=1}^M\{x\in D\colon F_i(x)=0\}$ is empty. Then $\psi\equiv 0$

and $f_i(x) = F_i(x)$ satisfy (2.3)–(2.4) and either (2.5) or (2.6).

We also give an example of system having infinite many solutions. Consider the one-dimensional system

$$\begin{cases} |Du_1| - F(x) + u_1(x) - u_2(x) = 0 & x \in (-1,\,1) \\ |Du_2| - F(x) + u_2(x) - u_1(x) = 0 & x \in (-1,\,1) \\ u_i(\pm 1) = 0 & i = 1,\,2 \end{cases} \tag{4.1}$$

where $F(x) = 2|x|$. Then $u_1(x) = u_2(x) = 1 - x^2$ and $u_1(x) = u_2(x) = \min\{1 - x^2,\, x^2 + C\}$, $C \in (0,\,1)$, are viscosity solutions to (4.1). Note that the set $\mathcal{A}_i$ for the Hamiltonian $H_i(x,\,p) = |p| - 2|x|$ coincide with $\{0\}$ and therefore the assumptions (2.5) and (2.6) are not satisfied.

4.2. Systems with large deviations Hamiltonians

Consider a right continuous strong Markov process $(X_t^\varepsilon,\, \nu_t^\varepsilon)$ with phase space $\mathbb{R}^N \times \{1,\,\dots,\,M\}$. The first component of process satisfies

$$dX_t^\varepsilon = b_{\nu_t^\varepsilon}(X_t^\varepsilon)dt + \varepsilon^{1/2} dW_t \tag{4.2}$$

where $X_0^\varepsilon = x \in D$, while the second component ν_t^ε is a random process with states $\{1,\,\dots,\,M\}$ for which

$$\mathbb{P}\{\nu_{t+\Delta}^\varepsilon = j \mid \nu_t^\varepsilon = i,\, X_t^\varepsilon = x\} = c_{ij}(x) + O(\Delta) \tag{4.3}$$

for $\Delta \to 0$, $i,\, j = 1,\,\dots,\,M$, $i \neq j$. For $\varepsilon = 0$ the process $(X_t^\varepsilon,\, \nu_t^\varepsilon)$ degenerates into the random process $(X_t^0,\, \nu_t^0)$ defined by

$$\frac{dX^0}{dt} = b_{\nu_t^0}(X_t^0) \tag{4.4}$$

with ν_t^0 satisfying (4.3) for $\varepsilon = 0$. The process $(X_t^0,\, \nu_t^0)$ is said a *Random Evolution process*. Large deviations functionals defined on the paths of the process (4.2) satisfy weakly coupled systems of second order linear equations. Passing to the limit in the system satisfied by the log-transform of the large-deviations functional (see [5]) we get the system (2.1) where the Hamiltonians H_i are the large-deviations ones

$$H_i(x,\,p) = \frac{|p|^2}{2} - b_i(x) \cdot p. \tag{4.5}$$

When $\nu_\varepsilon(t)$ has only one state for any ε, then (4.2) corresponds to a small random perturbation of a dynamical system. In this setting, a classical assumption which gives large deviations principles is the regularity of the vector field b. This corresponds to require that

(4.6) $\exists T > 0$ such that for any integral curve of $\dot{x} = b(x(t))$ with $x(0) \in D$, there exists $s < T$ for which $x(s) \notin \overline{D}$

Condition (4.6) is known in the probabilistic literature as the Levinson's condition for the dynamical system $\dot{x} = b(x(t))$. It has also a significative consequence in the PDE approach to large deviations. In fact (see [1, Lemma 6.1] for the proof)

Proposition 4.1 *The following conditions are equivalent*

i) *Condition* (4.6)

ii) *There exists a C^1 function ψ such that*

$$\frac{|D\psi|^2}{2} - b(x) \cdot D\psi < 0 \quad \textit{for } x \in D$$

As discussed in the introduction the existence of a strict subsolution is fundamental to get comparison results for Hamilton-Jacobi equations without zero order terms. For this reason, condition (4.6) has been used by several authors to study singular perturbation results in the framework of viscosity solution theory (see [1, Chapter VI], [2], [8], [11]).

An analogous of the Levinson's condition for random evolution processes is that the process (4.4) exits out of D a.s. in a uniformly bounded time, i.e. there exists $T < \infty$ such that

$$\mathbb{P}_{x,i}(\tau^0 \leq T) = 1, \quad \text{for any } x \in D \text{ and } i = 1, \dots, M \tag{4.7}$$

where τ^0 is the exit-time of the process $X^0(t)$ from D and $\mathbb{P}_{x,i}$ is the conditional probability with respect to the initial condition $X^0(0) = x$, $\nu^0(0) = i$. A sufficient condition for (4.7) (see [5]) is the following condition

(4.8) For any smooth vector field $\lambda(x) = (\lambda_1(x), \dots, \lambda_M(x))$ satisfying $\lambda_i(x) \geq 0$, $\sum_{i=1}^M \lambda_i(x) = 1$ for $x \in D$ the vector field $\bar{b}(x) = \sum_{i=1}^M \lambda_i(x) b_i(x)$ is regular, i.e. it satisfies (4.6).

In the next proposition, we show that (4.8) is equivalent to (2.5) for the Hamiltonians (4.5) and we also give a simple geometric condition which guarantees the property

Proposition 4.2 *The following three conditions are equivalent*:

i) *Condition* (4.8).

ii) *For any $x \in D$, the null vector does not belong to the convex hull*

$\text{co}\{b_1(x), \ldots, b_M(x)\}$.

iii) *There exists a C^1 function ψ such that $H_i(x, D\psi) < 0$ in D, for any $i = 1, \ldots, M$.*

We need a preliminary lemma. Let $B(a, r) = \{x \in \mathbb{R}^M : |x - a| \le r\}$ for $a \in \mathbb{R}^M$ and $r > 0$.

Lemma 4.3 *Given M vectors $v_1, \ldots, v_M$ in $\mathbb{R}^N$, then $0 \notin \text{co}\{v_1, \ldots, v_M\}$ if and only if $B = \bigcap_{i=1}^M B(v_i, |v_i|)$ has non empty interior* (co *stands for the convex hull*).

Proof. Assume that $0 \notin C := \text{co}\{v_1, \ldots, v_M\}$. Let P be a point which realizes the minimum of the distance between $\text{cl}(C)$ and 0 and let ℓ be the line through 0 and P. Denote by ℓ_i, $i = 1, \ldots, M$, the intersection between ℓ and $B(v_i, |v_i|)$. ℓ_i is a segment of positive length centered at w_i, the projection of v_i on ℓ. Because of the choice of P, all the segments ℓ_i are on the same side of ℓ with respect to the origin. Hence their intersection is a segment of positive length and any point of its interior is in the interior of B.

Now assume that B has nonempty interior. Hence there exists a non-null vector p such that $p \in \text{int}(B(v_i(x), |v_i|))$ or, equivalently,

$$\frac{|p|^2}{2} - v_i \cdot p < 0, \quad i = 1, \ldots, M. \tag{4.9}$$

If, by contradiction, $0 \in \text{co}\{v_1, \ldots, v_M\}$, then there exists $\lambda_i \in [0, 1]$, $\sum_{i=1}^M \lambda_i = 1$ such that $\sum_{i=1}^M \lambda_i v_i = 0$. By (4.9),

$$0 > \sum_{i=1}^M \lambda_i \Big(\frac{|p|^2}{2} - v_i \cdot p \Big) = \frac{|p|^2}{2} - \Big(\sum_{i=1}^M \lambda_i v_i \Big) \cdot p = \frac{|p|^2}{2},$$

hence a contradiction since $|p| \neq 0$. □

Proof of Proposition 4.2. ii) $\Rightarrow$ iii): By Lemma 4.3, for any $x_0 \in D$, there exists $p \in \text{int}\big(\bigcap_{i=1}^M B(b_i(x_0), |b_i(x_0)|)\big)$. Then $H_i(x_0, p) < 0$ for any $i = 1, \ldots, M$ and, by continuity, the inequalities $H_i(x, p) \le 0$ hold for $x \in B(x_0, \delta)$, for δ sufficiently small. Hence the function $\psi_{x_0}(x) = p \cdot (x - x_0)$ is a C^1-strict subsolution of $H_i(x, Du) = 0$ in $B(x_0, \delta)$ for any $i = 1, \ldots, M$. Now applying the argument in [6, Theorem 3.3], based on a partition of the unity of D, to each Hamiltonian H_i we can construct a function ψ which satisfies iii).

iii) $\Rightarrow$ i): Let ψ be a C^1 function satisfying iii) and take a vector field λ as in i). Set $H(x, p) = |p|^2/2 - \overline{b}(x) \cdot p$, where $\overline{b}(x) = \sum_{i=1}^M \lambda_i(x) b_i(x)$. We have for $x \in D$

$$H(x, D\psi) = \frac{|D\psi|^2}{2} - \overline{b}(x) \cdot D\psi = \frac{|D\psi|^2}{2} - \sum_{i=1}^M \lambda_i(x) b_i(x) \cdot D\psi$$
$$= \sum_{i=1}^M \lambda_i(x) \Big(\frac{|D\psi|^2}{2} - b_i(x) \cdot D\psi \Big) = \sum_{i=1}^M \lambda_i(x) H_i(x, D\psi) < 0$$

Since ψ is a strict subsolution of the Hamilton-Jacobi equation corresponding to $\overline{b}(x)$, then, by Proposition 4.1, the vector field $\overline{b}$ is regular and therefore i) holds.
i) $\Rightarrow$ ii): Assume by contradiction that $0 \in \mathrm{co}\{b_1(x_0), \ldots, b_M(x_0)\}$, for some $x \in D$, hence there exists μ_i, $i = 1, \ldots, M$ such that $\mu_i \in [0, 1]$, $\sum_{i=1}^M \mu_i = 1$ and $\sum_{i=1}^M \mu_i b_i(x_0)$ is the null vector. By iii) and Proposition 4.1, we know that for any $\lambda(x)$ as in (4.8), there exists $\psi \in C^1(D)$ such that

$$H(x, D\psi) = \frac{|D\psi|^2}{2} - \sum_{i=1}^M \lambda_i(x) b_i(x) \cdot D\psi < 0 \tag{4.10}$$

Now choosing $\lambda(x)$ in such a way that

$$\lambda_i(x_0) = \mu_i, \quad i = 1, \ldots, M$$

and substituting in (4.10), we get for $x = x_0$

$$\frac{|D\psi(x_0)|^2}{2} < 0$$

and therefore a contradiction. $\square$

Remark 4.4 Taking $\lambda(x)$ such that for any $x \in D$ $\lambda_i(x) = 1$ and $\lambda_j(x) = 0$ for $j \neq i$ in (4.8), we get that the vector field $b_i(x)$, for any $i = 1, \ldots, M$, is regular in the sense of definition (4.6).

But, even if all the vector fields b_i are regular, not necessarily (4.8) is satisfied. In fact, taking $b_1(x) = (1, \ldots, 1)$ and $b_2(x) = -b_1(x)$, then b_1 and b_2 are regular vector fields in $D = [0, 1]^N$. But, since $0 \in \mathrm{co}\{b_1(x), b_2(x)\}$ for any $x \in D$ by Proposition 4.2, (4.8) is not satisfied. Another interesting example of the same phenomenon is given in [5].

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