Hokkaido Mathematical Journal Vol. 37 (2008) p. 309–329

$C^l - \mathcal{G}_V$-determinacy of weighted homogeneous function germs on weighted homogeneous analytic varieties

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(Received October 10, 2006; Revised June 15, 2007)

Abstract. We provide estimates on the degree of $C^l - \mathcal{G}_V$-determinacy ($\mathcal{G}$ is one of Mather's groups $\mathcal{R}$ or $\mathcal{K}$) of weighted homogeneous function germs which are defined on weighted homogeneous analytic variety V and satisfies a convenient Lojasiewicz condition. The result gives an explicit order such that the C^l-geometrical structure of a weighted homogeneous polynomial function germ is preserved after higher order perturbations, which generalize the result on $C^l - \mathcal{K}$-determinacy of weighted homogeneous functions germs given by M.A.S. Ruas.

Key words: $C^l - \mathcal{R}_V$-determinacy, $C^l - \mathcal{K}_V$-determinacy, weighted homogeneous polynomial function germs, controlled vector field, weighted homogeneous control functions.

1. Introduction

In singularity theory of smooth functions and maps, a fundamental question is raised for a given equivalence relation: when is a map germ equivalent to a finite part of its Tayor expansion?

It is concerned with determinacy of map-germs and trivialization for families of map-germs.

There is an extensive literature related to trivialization and determinacy for families of map-germs. In C^l-determinacy theory ($l \geq 0$), the question of determining the degree of $C^0 - \mathcal{G}$-determinacy of weighted homogeneous map germs has been considered by several authors (e.g. ref. [4], [6], [8], [5]). Estimates for the degree of $C^l - \mathcal{G}$-determinacy of weighted homogeneous map-germs have been studied by M.A.S. Ruas and M.J. Saia (ref. [10]). Recently, the sufficient conditions for the topological triviality of families of germs of functions defined on an analytic variety V are provided by M.A.S. Ruas and J.N. Tomazella (ref. [11]). But these results do not include estimates for the degree of $C^l - \mathcal{G}_V$-determinacy of function germs defined

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2000 Mathematics Subject Classification : 58A35.

This work was supported by the National Nature Science Foundation of China under grant, No. 10671009, No. 60534080.

on real analytic varieties.

In this paper, when $(V, 0)$ is a weighted homogeneous analytic variety, we construct a $C^l - \mathcal{G}_V$- ($\mathcal{G}$ is one of Mather's groups $\mathcal{R}$ or $\mathcal{K}$) trivialization for a one parameter family $f_t = f + t\theta$ of a weighted homogeneous polynomial function germ f, defined on a class of weighted homogeneous real analytic varieties V satisfying a convenient Lojasiewicz condition, and give an explicit order for the filtration of a map-germ $\theta\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ such that the C^l-geometrical structure of f (consistent with V) is preserved after higher order perturbations. So a weighted homogeneous polynomial function germ f (consistent with V) after higher order perturbations is finite determined. Our method is concretely offering a controlled vector field whose integration gives a $C^l - \mathcal{G}_V$-trivialization.

An application of our result to free arrangement is also presented.

2. Preliminaries

In this paper, we assume that germs are in real analytic category.

Let $\mathcal{O}_n$ be the ring of germs of analytic functions $h\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$. This is a local ring with maximal ideal $\mathcal{M}_n$, consisting of these $h \in \mathcal{O}_n$ such that $h(0) = 0$. We denote by $J^k(n, p)$ the set of $k-$jets of elements of $\mathcal{O}_n$.

A germ of a subset $(V, 0) \subset (\mathbf{R}^n, 0)$ is the germ of a real analytic variety if there exist germs of real analytic functions $f_1, \ldots, f_r$ such that $V = \{x\colon f_1(x) = \cdots = f_r(x) = 0\}$.

Definition 2.1 Let $\mathcal{R}$ be the group of germs of diffeomorphisms of $(\mathbf{R}^n, 0)$, and $\mathcal{R}_V = \{\phi \in \mathcal{R}\colon \phi(V) = V\}$, i.e. $\mathcal{R}_V$ be the group of germs of diffeomorphisms preserving $(V, 0)$. Also, let $C^l - \mathcal{R}_V$ $(l > 0)$ denote the group of germs of C^l-diffeomorphisms preserving $(V, 0)$.

$\mathcal{R}_V$ $(C^l - \mathcal{R}_V)$ acting on germ $h_0\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ is given by composition on the right.

Two germs h_1 and $h_0\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ are $C^l - \mathcal{R}_V$-equivalent iff there exists a germ of C^l-diffeomorphism $\phi\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ with $\phi(V) = V$ and $h_1 \circ \phi = h_2$.

A one-parameter deformation $h\colon (\mathbf{R}^n \times \mathbf{R}, 0) \longrightarrow (\mathbf{R}, 0)$ of $h_0\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ is $C^l - \mathcal{R}_V$-trivial if there exists C^l-diffeomorphism

$$\varphi\colon (\mathbf{R}^n \times \mathbf{R}, 0) \longrightarrow (\mathbf{R}^n \times \mathbf{R}, 0), \quad \varphi(x, t) = (\overline{\varphi}(x, t), t)$$

such that $h \circ \varphi(x, t) = h_0(x)$ and $\varphi(V \times \mathbf{R}) = V \times \mathbf{R}$.

We denote by θ_n the set of germs of tangent vector fields in $(\mathbf{R}^n, 0)$. Then θ_n is a free $\mathcal{O}_n$-module of rank n.

Let $I(V)$ be the ideal in $\mathcal{O}_n$ consisting of germs of real analytic function vanishing on V. We denote by

$$\Theta_V = \{\eta \in \theta_n; \eta(I(V)) \subset I(V)\},$$

the submodule of germs of vector fields tangent to V. Θ_V is a finitely generated $\mathcal{O}_n$-module for it is a submodule of θ_n which is Noetherian since $\mathcal{O}_n$ is.

Since $I(V) \cdot \{\partial/\partial x_i\} \subset \Theta_V$, it follows in the real analytic case that

$$\Theta_V = \mathcal{O}_n\{\zeta_j\}_{1\le j\le p} + I(V)\mathcal{O}_n\Big\{\frac{\partial}{\partial x_i}\Big\}$$

where $\{\zeta_j\}_{1\le j\le p}$ together with $I(V) \cdot \{\partial/\partial x_i\}$ generate the $\mathcal{O}_n$-module of vector fields tangent to V.

In the real analytic category, let

$$\begin{aligned} T\mathcal{R}_{V,e} = \Big\{ \frac{\partial\overline{\varphi}(x,\,t)}{\partial t}\Big|_{t=0} &: \varphi(x,\,t) = (\overline{\varphi}(x,\,t),\,t) \\ &: (\mathbf{R}^n \times \mathbf{R},\, 0 \longrightarrow \mathbf{R}^n \times \mathbf{R},\, 0), \\ &\quad \text{with } \overline{\varphi}(x,\,0) = \text{id} \text{ and } \varphi(V \times \mathbf{R}) = V \times \mathbf{R} \Big\}. \end{aligned}$$

Then

$$T\mathcal{R}_{V,e} = \Theta_V = \mathcal{O}_n\{\zeta_j\}_{1\le j\le p} + I(V)\mathcal{O}_n\Big\{\frac{\partial}{\partial x_i}\Big\}, \quad \text{(p. 59 of ref. [1])}$$

Moreover the tangent space of $\mathcal{R}_V$

$$T\mathcal{R}_V = \mathcal{M}_n\{\zeta_j\}_{1\le j\le p} + I(V)\mathcal{O}_n\Big\{\frac{\partial}{\partial x_i}\Big\} = \Theta_V^{\circ}$$

is computed by taking families of germs in $\mathcal{R}_V$.
The tangent space of the orbit of h under the action of the group is $T\mathcal{R}_V(h) = dh(\Theta_V^{\circ})$. ([11])

Definition 2.2 ([11])

(a) We assign weights $w_1, w_2, \ldots, w_n$, $w_i \in \mathbf{Q}^+$, $i = 1, \ldots, n$ to a given coordinate system $x_1, \ldots, x_n$ in $\mathbf{R}^n$. The filtration of a monomial $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ with respect to this set of weights is defined by $\text{fil}(x^\beta) = \sum_{i=1}^n \beta_i w_i$.

(b) We define a filtration in the ring $\mathcal{O}_n$ via the function defined by

$$\mathrm{fil}(f) = \inf_{|\beta|}\Big\{\mathrm{fil}(x^\beta)\colon \frac{\partial^{|\beta|} f}{\partial x^\beta}(0) \neq 0\Big\}, \quad |\beta| = \beta_1 + \cdots + \beta_n,$$

for any germ f in $\mathcal{O}_n$. This definition can be extended to $\mathcal{O}_{n+r}$, the ring of $r-$parameter families of germs in n-variables, by defining $\mathrm{fil}(x^\alpha t^\beta) = \mathrm{fil}(x^\alpha)$. For any map germ $f = (f_1, \ldots, f_p)\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ we call $\mathrm{fil}(f) = (d_1, \ldots, d_p)$, where $d_i = \mathrm{fil}(f_i)$ for each $i = 1, \ldots, p$.

(c) We extend the filtration to Θ_V, defining $\mathrm{fil}(\partial/\partial x_i) = -w_i$ for all $i = 1, \ldots, n$, so that given $\xi = \sum_{j=1}^n \xi_j(\partial/\partial x_j) \in \Theta_V$, then $\mathrm{fil}(\xi) = \inf_j\{\mathrm{fil}(\xi_j) - w_j\}$.

(d) For given $(w_1, \ldots, w_p\colon d_1, \ldots, d_p)$, $w_i, d_j \in \mathbf{Q}^+$, a map germ $f\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ is weighted homogeneous of type $(w_1, \ldots, w_n\colon d_1, \ldots, d_p)$ if for all $\lambda \in \mathbf{R} - \{0\}$,

$$f(\lambda^{w_1}x_1, \lambda^{w_2}x_2, \ldots, \lambda^{w_n}x_n) = (\lambda^{d_1}f_1(x), \lambda^{d_2}f_2(x), \ldots, \lambda^{d_p}f_p(x)).$$

Definition 2.3 ([10]) Let $(w_1, \ldots, w_n\colon 2k)$ be fixed. We define the standard control function $\rho_k(x)$ by $\rho_k(x) = x_1^{2\alpha_1} + x_2^{2\alpha_2} + \cdots + x_n^{2\alpha_n}$, where the α_i are chosen in such way that the function ρ_k is weighted homogeneous of type $(w_1, \ldots, w_n\colon 2k)$.

Remark ([10]) We observe that ρ_k satisfies a Lojasiewicz condition $\rho_k \geq c|x|^{2\alpha}$ for some constants c and α.

Lemma 2.4 (Lemma 1 of [10]) *Let $h(x)$ be a weighted homogeneous polynomial of type $(w_1, \ldots, w_n\colon 2k)$ and $h_t(x)$, $t \in [0, 1]$ a deformation of h, which is weighted homogeneous of the same type as h. Then*:

(a) *There exists a constant c_1 such that $|h_t(x)| \leq c_1\rho_k(x)$.*

(b) *If there exist constants c and α such that $|h_t(x)| \geq c|x|^\alpha$, then $|h_t(x)| \geq c_2\rho_k(x)$ for some constant c_2.*

Lemma 2.5 (Lemma 2 of [10]) *Let $h(x)$ be a weighted homogeneous polynomial of type $(w_1, \ldots, w_n\colon 2k)$, with $w_1 \leq w_2 \leq \cdots \leq w_n$, $\rho(x)$ the standard control function of same type as h and $h_t(x)$ a deformation of h such that*

$$\mathrm{fil}(h_t) \geq 2k + lw_n + 1, \quad t \in [0, 1], \, l \geq 1.$$

Then the function $\nu(x) = h_t(x)/\rho(x)$ is differentiable of class C^l.

Definition 2.6 ([11]) The germ of an analytic variety $(V, 0) \subseteq (\mathbf{R}^n, 0)$ is weighted homogeneous if it is defined by a weighted homogeneous map germ $f\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$.

A set of generators $\{\gamma_1 \cdots \gamma_m\}$ of Θ_V is called weighted homogeneous of type $(w_1, \ldots, w_n\colon d_1, \ldots, d_m)$ if $\gamma_i = \sum_{j=1}^n \gamma_{ij}(\partial/\partial x_j)$ and γ_{ij} $(i = 1, \ldots, m; j = 1, \ldots, n)$ are weighted homogeneous polynomials of type $(w_1, \ldots, w_n\colon d_i + w_j)$ whenever $\gamma_{ij} \neq 0$.

When V is a weighted homogeneous variety, we can always choose weighted homogeneous generators for Θ_V. (see [2] or [11])

Definition 2.7 ([11]) Let V be defined by weighted homogeneous polynomials. We say that h is weighted homogeneous consistent with V if h is weighted homogeneous with respect to the same set of weights assigned to V.

3. Estimates for the degree of $C^l - \mathcal{R}_V$-determinacy of weighted homogeneous function germs on a class of weighted homogeneous real analytic varieties

Theorem 3.1 *Let V be a weighted homogeneous subvariety of $(\mathbf{R}^n, 0)$, and let there exist a system $\{\gamma_1 \cdots \gamma_m\}$ of weighted homogeneous generators of type $(w_1, \ldots, w_n;\ d_1, \ldots, d_m)$ for Θ_V^0, where $w_1 \leq w_2 \leq \cdots \leq w_n$ and $w_i \in \mathbf{Z}^+$ and $\gamma_j = \sum_{i=1}^n \gamma_{ji}(\partial/\partial x_i)$;*
If
(a) $f\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ *is a weighted homogeneous function-germ of type $(w_1, \ldots, w_n; d)$, which is consistent with V;*
(b) f *satisfies a Lojasiewicz condition*

$$N_{\mathcal{R}_V} f(x) = (df(\gamma_j))^2 = \sum_{j=1}^m \Big(\sum_{i=1}^n \frac{\partial f}{\partial x_i}\gamma_{ji}\Big)^2 \geq c|x|^\alpha$$

for some constants c and α.
Then deformations of f defined by

$$f_t(x) = f(x) + t\theta(x), \quad t \in [0, 1],$$

with $\mathrm{fil}(\theta) \geq d + lw_n - w_1$*, for all $t \in [0, 1]$ and $l > 1$ are $C^l - \mathcal{R}_V$-trivial.*

Remark We discuss the case $l = 1$ in another paper. Actually, when $l = 1$, f_t is V-bilipschitz triviality.

Firstly we observe $df(\gamma_j)$. Because

$$df(\gamma_j) = df\Big(\sum_{i=1}^{n} \gamma_{ji} \frac{\partial}{\partial x_i}\Big) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \gamma_{ji},$$

it follow that

$$\begin{aligned} \mathrm{fil}\big(df(\gamma_j)\big) &= \inf_{1 \le i \le n} \Big\{ \mathrm{fil}\Big(\frac{\partial f}{\partial x_i} \gamma_{ji}\Big)\Big\} \\ &= \inf_{1 \le i \le n} \{\mathrm{fil}(f) - w_i + (d_j + w_i)\} \\ &= \mathrm{fil}(f) + d_j. \end{aligned}$$

Let $s_j = \mathrm{fil}(f) + d_j$ and $N^*_{\mathcal{R}_V} f$ be defined by $N^*_{\mathcal{R}_V} f = \sum_{j=1}^{m} (df(\gamma_j))^{2\alpha_j}$, where $\alpha_j = k/s_j$, $k = \mathrm{l.c.m.}(s_j)$. Then $N^*_{\mathcal{R}_V} f$ is a weighted homogeneous control function of type $(w_1, \ldots, w_n; 2k)$. By Remark of Definition 2.3, $N^*_{\mathcal{R}_V} f \ge c_1 (N_{\mathcal{R}_V} f)^{\beta}$ for some constants c_1 and β.

For a deformation f_t of f, let $N^*_{\mathcal{R}_V} f_t$ by $N^*_{\mathcal{R}_V} f_t = \sum_{j=1}^{m} (df_t)_x(\gamma_j))^{2\alpha_j}$, where $(df_t)_x = \sum_{i=1}^{n} (\partial f_t / \partial x_i) dx_i$, and α_j are same as above. If f_t is weighted homogeneous of same type as f, then $N^*_{\mathcal{R}_V} f_t$ is weighted homogeneous of type $(w_1, \ldots, w_n; 2k)$ for all t. If $f_t(x) = f(x) + t\theta(x)$ and $fil(\theta) \ge d$, it follows that $\mathrm{fil}(N^*_{\mathcal{R}_V} f_t) \ge \mathrm{fil}(N^*_{\mathcal{R}_V} f)$.

Lemma 3.2 *Let f and f_t satisfy the condition of the above theorem. Then, there exist positive constants a_1 and a_2 such that*

$$a_2 \rho_k(x) \le N^*_{\mathcal{R}_V} f_t \le a_1 \rho_k(x)$$

Proof. When f_t is weighted homogeneous of the same type as f, the result follows from Lojasiewicz condition and Lemma 2.4.

If $\mathrm{fil}(f_t) > \mathrm{fil}(f)$, we write $N^*_{\mathcal{R}_V} f_t = N^*_{\mathcal{R}_V} f + tR(x, t)$ where $R(x, t)$ is a polynomial with $\mathrm{fil}(R(x, t)) > \mathrm{fil}(N^*_{\mathcal{R}_V} f)$.

Then $N^*_{\mathcal{R}_V} f \le N^*_{\mathcal{R}_V} f_t + |R_t(x)|$, for $0 \le t \le 1$. Because $N^*_{\mathcal{R}_V} f \ge c_1 (N_{\mathcal{R}_V} f)^{\beta}$ for some constants c_1 and β, we have $N^*_{\mathcal{R}_V} f \ge cc_1 |x|^{\alpha\beta}$ by condition (b). So by Lemma 2.4, there exists a constant a_2 such that

$$a_2 \rho_k(x) \le N^*_{\mathcal{R}_V} f \le N^*_{\mathcal{R}_V} f_t + |R_t(x)|.$$

Again since $\mathrm{fil}(R_t(x)) > \mathrm{fil}(N^*_{\mathcal{R}_V} f)$, it follows that $\lim_{x \to 0} |R_t(x)| / \rho_k(x) =$

0. Thus $a_2\rho_k(x) \le N^*_{\mathcal{R}_\mathcal{V}} f_t$.

It is easy to see that there exists a constant a_1 such that $N^*_{\mathcal{R}_V} f_t \le a_1\rho_k(x)$ for small t. □

Lemma 3.3 *Let $\rho(x)$ be the standard control function of same type $(w_1, \ldots, w_n\colon 2k)$, with $w_1 \le w_2 \le \cdots \le w_n$, $h_t(x)$ an analytic deformation of a analytic function $h\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ such that*

$$\mathrm{fil}(h_t) \ge 2k + lw_n + 1, \quad t \in [0, 1],\, l \ge 1.$$

And $Z(x)$ is differentiable and satisfies

$$a_2\rho_k(x) \le Z(x) \le a_1\rho_k(x).$$

Then the function $\lambda(x) = h_t(x)/Z(x)$ is differentiable of class C^l.

Proof. We will proceed by induction on the class of differentiability (similar to the proof of Lemma 2 of ref. [10]).

In fact, $\lambda(x) = h_t(x)/Z(x)$ is C^l if $x \ne 0$. It is sufficient to prove that $\lambda(x) = h_t(x)/Z(x)$ is C^l when $x = 0$.

Firstly we consider $l = 1$. The gradient of $\lambda(x) = h_t(x)/Z(x)$ is

$$\nabla\lambda = \frac{\nabla h_t(x)}{Z(x)} - \frac{\nabla Z(x) \cdot h(x)}{(Z(x))^2}$$

because

$$a_2\rho_k(x) \le Z(x) \le a_1\rho_k(x),$$

So

$$\begin{aligned}\frac{\nabla h_t(x)}{a_1\rho(x)} - \frac{\nabla(Z(x)) \cdot h_t(x)}{(a_2\rho(x))^2} &\le \frac{\nabla h_t(x)}{Z(x)} - \frac{\nabla Z(x) \cdot h_t(x)}{(N^*_{\mathcal{R}_V} f_t)^2}\\ &\le \frac{\nabla h_t(x)}{a_2\rho(x)} - \frac{\nabla(Z(x)) \cdot h_t(x)}{(a_1\rho(x))^2},\end{aligned}$$

with

$$\inf_i\Big\{\mathrm{fil}\Big(\frac{\partial Z(x)}{\partial x_i}(x)\Big)\Big\} \ge 2k - w_n$$

and

$$\mathrm{fil}(h_t(x)) \ge 2k + w_n + 1,$$

then

$$\operatorname{fil}(|\nabla(Z(x)) \cdot h_t(x)|) \geq 4k + 1.$$

Each term of

$$\frac{\nabla h_t(x)}{a_1\rho(x)} - \frac{\nabla(Z(x)) \cdot h_t(x)}{(a_2\rho(x))^2}$$

and

$$\frac{\nabla h_t(x)}{a_2\rho(x)} - \frac{\nabla(Z(x)) \cdot h_t(x)}{(a_1\rho(x))^2}$$

is of form $g(x) \cdot m(x)/\rho(x)$, where $m(x)$ is weighted homogeneous of type $(w_1, \ldots, w_n; 2k)$ and $\lim_{x\to 0} g(x) = 0$. It follows from Lemma 2.4 that $m(x)/\rho(x)$ is bounded, hence $\nabla\lambda$ is continuous.

Let us assume by induction that for all function $\lambda(x) = h_t(x)/Z(x)$ with $\operatorname{fil}(h_t) \geq 2k + (l-1)w_n + 1$, λ is of class C^{l-1}.

Let $\lambda(x) = h_t(x)/Z(x)$ with $\operatorname{fil}(h_t) \geq 2k + lw_n + 1$. Then $\nabla\lambda(x) = H(x)/Z(x)$ with $\operatorname{fil}(H) \geq 2k + (l-1)w_n + 1$ is of class C^{l-1}, and $\lambda(x)$ is of class C^l. □

Proof of Theorem 3.1. Because

$$\frac{\partial f_t}{\partial t}\big([(df_t)_x(\gamma_j)]^{2\alpha_j}\big) = (df_t)_x\Big(\frac{\partial f_t}{\partial t}\big(\big((df_t)_x(\gamma_j)\big)^{2\alpha_j-1}\gamma_j\big)\Big)$$

and

$$N^*_{\mathcal{R}_V} f_t = \sum_{j=1}^{m}\big([(df_t)_x(\gamma_j)]^{2\alpha_j}\big),$$

then

$$\frac{\partial f_t}{\partial t} N^*_{\mathcal{R}_V} f_t = (df_t)_x(W_{\mathcal{R}_V}),$$

where

$$W_{\mathcal{R}_V} = \sum_{j=1}^{m} W_j\gamma_j \quad \text{and} \quad W_j = \frac{\partial f_t}{\partial t}\big((df_t)_x(\gamma_j)\big)^{2\alpha_j-1}.$$

Now we compute $\operatorname{fil}(W_j)$.

Because $\gamma_j = \sum_{i=1}^{n}\gamma_{ji}(\partial/\partial x_i)$, where γ_{ji} are weighted homogeneous polynomials of type $(w_1, \ldots, w_n\colon d_j + w_i)$,

$$\begin{aligned}\operatorname{fil}\big((df_t)_x(\gamma_j)\big) &= \operatorname{fil}\Big(\sum_{i=1}^{n} \gamma_{ji} \frac{\partial f_t}{\partial x_i}\Big)\\ &= \inf_{i=1,\ldots,n} \{\operatorname{fil}(f_t) - w_i + (d_j + w_i)\}\\ &= \operatorname{fil}(f_t) + d_j\\ &= \operatorname{fil}(f) + d_j\\ &= d + d_j\end{aligned}$$

$$\begin{aligned}\operatorname{fil}(W_j) &= \operatorname{fil}\Big(\frac{\partial f_t}{\partial t}\Big) + (2\alpha_j - 1)\operatorname{fil}\big(df_t(\gamma_j)\big)\\ &\geq (d + lw_n - w_1) + (2\alpha_j - 1)(d + d_j)\\ &= (d + lw_n - w_1) + (2\alpha_j - 1)s_j\\ &= d + lw_n - w_1 + 2k - d - d_j\\ &= 2k + lw_n - w_1 - d_j\end{aligned}$$

Again because $\gamma_j W_j = \sum_{1\leq i\leq n} \gamma_{ji} W_j(\partial/\partial x_i)$, then

$$\begin{aligned}\operatorname{fil}(\gamma_{ji} W_j) &\geq d_j + w_i + 2k + lw_n - w_1 - d_j\\ &= 2k + lw_n + w_i - w_1\\ &\geq 2k + (l-1)w_n + 1.\end{aligned} \tag{$*$}$$

Now from equation $(\partial f_t/\partial t)(x, t) = (df_t)_x(W_{\mathcal{R}_V}/N^*_{\mathcal{R}_V} f_t)$, we obtain

$$\frac{\partial f_t}{\partial t}(x, t) - (df_t)_x\Big(\frac{W_{\mathcal{R}_V}}{N^*_{\mathcal{R}_V} f_t}\Big) = 0.$$

Let $\nu\colon (\mathbf{R}^n \times \mathbf{R}, 0) \to (\mathbf{R}^n \times \mathbf{R}, 0)$ be the stratified vector field

$$\nu(x) = \begin{cases} \dfrac{W_{\mathcal{R}_V}}{N^*_{\mathcal{R}_V} f_t} + \dfrac{\partial}{\partial t}, & x \neq 0\\ \dfrac{\partial}{\partial t}, & x = 0\end{cases}$$

where $W_{\mathcal{R}_V}/N^*_{\mathcal{R}_V} f_t \in T\mathcal{R}_V = \Theta^0_V$.

Again let $W_{\mathcal{R}_V}/N^*_{\mathcal{R}_V} f_t = \sum_{i=1}^{n} \nu_i(x,t)(\partial/\partial x_i)$, where

$$\nu_i(x, t) = \sum_{j=1}^{m} \frac{\gamma_{ji} W_j}{N^*_{\mathcal{R}} f_t}, \quad i = 1, \ldots, n$$

and it denotes the $i-$th component of ν. Owning to $(*)$ and Lemma 3.2, it follows from Lemma 3.3 that $\gamma_{ji}W_j/N^*_{\mathcal{R}_V}f_t$ $(1 \le i \le n; 1 \le j \le m)$ are differentiable of class C^{l-1}. So ν is of class C^{l-1}, where $l > 1$.

Moreover the orbits of a vector field ν on $\mathbf{R} \times \mathbf{R^n}$ are the integral curves (i.e., the graphs of solutions) of the first order system of differential equations:

$$\begin{cases} \dfrac{dx_0}{dt} = 1 \\ \dfrac{dx_i}{dt} = \nu_i(x, t), \quad i = 1, \ldots, n. \end{cases}$$

By hypothesis, $\mathbf{R} \times 0$ is an orbit of ν. Designate by $t \to (t, \phi(t, x))$ the solutions of the above system of differential equations corresponding to the initial condition $\phi(0, x) = x$. By the fundamental theorems of first order differential equations and Lemma 3.5 of [11], the mapping $\phi_t \colon x \to \phi(t, x)$ is a diffeomorphism. Again because $W_{\mathcal{R}_V}/N^*_{\mathcal{R}_V}f_t = \sum_{i=1}^n \nu_i(x, t)(\partial/\partial x_i) \in T\mathcal{R}_V$, it follows that $\phi_t \in \mathcal{R}_V$. Now equation

$$\frac{\partial f_t}{\partial t}(x, t) - (df_t)_x\Big(\sum_{i=1}^n \nu_i(x, t)\frac{\partial}{\partial x_i}\Big) = 0$$

implies $d(f_t \circ \phi) = 0$ and this equation implies the $C^l - \mathcal{R}_V$-triviality of family $f_t(x)$ in a neighborhood of $t = 0$. Since the same argument is true in a neighborhood of $t = \bar{t}$, for all $\bar{t} \in [0, 1]$, the proof is complete. □

Example (Example 3.12 of [11]) Let $V = \phi^{-1}(0)$ where $\phi(x, y, z) = 2x^2y^2 + y^3 - z^2 + x^4y$. We have ϕ is weighted homogeneous with respect to the weights $w_1 = 1$, $w_2 = 2$, $w_3 = 3$. Let $f(x, y, z) = x^{12} + y^6 + z^4$. Then f is a weighted homogeneous function-germ of type $(1, 2, 3; 12)$ and consistent with V.

The module Θ_V° is generated by

$$\begin{aligned} \alpha_1 &= (2x, 4y, 6z), \\ \alpha_2 &= (0, 2z, x^4 + 4x^2y + 3y^2), \\ \alpha_3 &= (x^2 + 3y, -4xy, 0), \\ \alpha_4 &= (z, 0, 2x^3y + 2xy^2). \end{aligned}$$

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is weighted homogeneous of type $(1, 2, 3; 0, 1, 1, 2)$. Moreover

$$df = (12x^{11},\, 6y^5,\, 4z^3), \quad df(\alpha_1) = 24x^{12} + 24y^6 + 24z^4.$$

then

$$\sum_{i=1}^{4} \big(df(\alpha_i)\big)^2 \geq 24x^{12} + 24y^6 + 24z^4.$$

But $24x^{12} + 24y^6 + 24z^4$ is weighted homogeneous function of type (1, 2, 3). So that there are some constants c and α such that

$$24x^{12} + 24y^6 + 24z^4 \geq c(x^2 + y^2 + z^2)^{\alpha}.$$

Therefore the example satisfies condition of Theorem 3.1. If we let

$$f_t(x,\, y,\, z) = f(x,\, y,\, z) + t(ax^{20} + by^{10} + cx^2z^6),$$

where

$$\theta(x) = ax^{20} + by^{10} + cx^2z^6.$$

Moreover

$$\text{fil}(\theta) = 20 \geq 12 + 3 \times 3 - 1 = 12 + 9 - 1 = 20,$$

where $l = 3$. Then deformation f_t of f is $C^3 - \mathcal{R}_V$-trivial by Theorem 3.1.

Now we present a application of Theorem 3.1 to free arrangement.

Let $\mathbf{R}$ be the real field and $V_{\mathbf{R}}$ a vector space of dimension l. A hyperplane H in $V_{\mathbf{R}}$ is an affine subspace of dimension $l - 1$.

A hyperplane arrangement $\mathcal{A}_{\mathbf{R}}$ is a finite set of hyperplanes in $V_{\mathbf{R}}$. Each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial α_H of degree 1. The product $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$ is called a defining polynomial of $\mathcal{A}$. Let

$$\Theta_{\mathcal{A}} = \{\eta \in \theta_l \colon \eta(Q(\mathcal{A})) \in (Q(\mathcal{A})) \\ : (Q(\mathcal{A})) \text{ is ideal generated by } Q(\mathcal{A})\},$$

$\Theta^{\circ}_{\mathcal{A}}$ is the submodule of $\Theta_{\mathcal{A}}$ given by the vector fields that are zero at zero.

If each $H \in \mathcal{A}$ contains the origin, we call $\mathcal{A}$ a central arrangement. Then $V_{\mathcal{A}} = \bigcup_{H \in \mathcal{A}} H$ is defined by homogeneous polynomial $Q(\mathcal{A})$. $\Theta_{V_{\mathcal{A}}} = \Theta_{\mathcal{A}}$ and $\Theta^{\circ}_{V_{\mathcal{A}}} = \Theta^{\circ}_{\mathcal{A}}$. Moreover we always choose homogeneous generators for $\Theta_{\mathcal{A}}$ and $\Theta^{\circ}_{\mathcal{A}}$ by Lemma 3.2 of [2].

Corollary 3.4 *Let*

(a) $\mathcal{A}$ *be a central arrangement,* $\Theta^{\circ}_{\mathcal{A}}$ *has a system of generators consisting of* l *homogeneous elements* $\{\zeta_1, \zeta_2, \ldots, \zeta_l\}$;

(b) $f\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ *be a homogeneous function*;

(c)

$$N_{\mathcal{R}_{V_{\mathcal{A}}}} f(x) = \sum_{i=1}^{l} \big(df(\zeta_i)\big)^2 \geq c|x|^{\alpha}$$

for some constants c *and* α.

Then deformations of f *defined by*

$$f_t(x) = f(x) + t\Theta(x),$$

with $\text{degree}(\Theta) \geq \text{degree}(f) + p$, *for all* $t \in [0, 1]$ *and* $p > 1$, *are* $C^p - \mathcal{R}_{V_{\mathcal{A}}}$*-trivial.*

This proof is obvious.

4. Estimates for the degree of $C^l - \mathcal{K}_V$-determinacy of weighted homogeneous function germs on a class of weighted homogeneous real analytic varieties

First we give some basic notations. (see [3]) The contact group $\mathcal{K}$ consists of pair of germs of diffeomorphisms (H, h) with $H\colon (\mathbf{R}^{n+p}, 0) \to (\mathbf{R}^{n+p}, 0)$ and $h\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that

(1) $H \circ i = i \circ h$ for $i(x) = (x, 0)$ and

(2) $\pi \circ H = h \circ \pi$ for $\pi(x, y) = x$.

such an H acts on $\mathcal{O}_{n,p}$ by $(h(x), H \cdot f(x)) = H(x, f(x))$ (i.e. $\text{grap}(H \cdot f) = H(\text{grap}(f))$)

Let $(V, 0)$ be the germ of a real subvariety of $\mathbf{R}^p$ defined by a finitely generated ideal I of $\mathcal{O}_p$. The group $\mathcal{K}_V$ is the subgroup of $\mathcal{K}$ consisting of elements $(H, h) \in \mathcal{K}$ such that $H(\mathbf{R}^n \times V) = \mathbf{R}^n \times V$. It is a geometric subgroup of $\mathcal{K}$ in the sense of ref. [11]. In particular, if $V = \{0\}$ then this is just contact equivalence.

We say that f and g are $\mathcal{K}_V$-equivalent if there is an element $(H, h) \in \mathcal{K}_V$ such that $(H, h) \cdot f = g$, where the action is that of contact equivalence.

The function $h\colon (\mathbf{R}^n \times \mathbf{R}, 0) \longrightarrow (\mathbf{R}, 0)$ is $k - C^l - \mathcal{K}_V$-determined iff for all $g\colon (\mathbf{R}^n \times \mathbf{R}, 0) \longrightarrow (\mathbf{R}, 0)$ with the same k-jet as h the germs h and g are $C^l - \mathcal{K}_V$-equivalent.

In the real analytic case,

$$T\mathcal{K}_{V,e} = \theta_n \oplus \mathcal{O}_{x,y}\{\zeta_i\} \qquad \text{(see [3])}$$

where $\{\zeta_i\}$ a set of generators for Θ_V.

$$T\mathcal{K}_V = \mathcal{M}_x \cdot \theta_n \oplus \mathcal{O}_{x,y}\{\zeta_i\}) \qquad \text{(see [3])}$$

Moreover

$$T\mathcal{K}_{V,e} \cdot f = \mathcal{O}_x\Big\{\frac{\partial f}{\partial x_i}\Big\} + \mathcal{O}_x\{\zeta_i \circ f\} \qquad \text{(see [9])}$$

$$T\mathcal{K}_V \cdot f = \mathcal{M}_x\Big\{\frac{\partial f}{\partial x_i}\Big\} + \mathcal{O}_x\{\zeta_i \circ f\} \qquad \text{(see [3])}$$

Definition 4.1 Θ_V is called free if Θ_V is a free module over $\mathcal{O}_y$.

Lemma 4.2 *Let $(V, 0)$ be the germ of an real subvariety of $\mathbf{R}^p$ defined by a weighted homogeneous polynomial g. If Θ_V be free, then it has a basis consisting of p weighted homogeneous elements.*

Proof. We can always choose weighted homogeneous generators $\{\zeta_i\}$ for Θ_V (see [2] or [11]).

Let r be the rank of the free $\mathcal{O}_y$-module Θ_V. Note that

$$g\theta_p \subset \Theta_V \subset \theta_p.$$

Since θ_p contains the p linearly independent elements $\partial/\partial y_1, \ldots, \partial/\partial y_p$, and $g\theta_p$ contains the p linearly independent elements $g(\partial/\partial y_1), \ldots, g(\partial/\partial y_p)$, it follows from Proposition A, 3(1) of ref. [7] that $p \le r \le p$. □

Definition 4.3 If $\zeta \in \theta_p$, then $\zeta = \sum_{j=1}^p \zeta_j(y)(\partial/\partial y_j)$. Given vector fields $\zeta_1, \ldots, \zeta_p \in \theta_p$, define the coefficient matrix $M(\zeta_1, \ldots, \zeta_p) = (\zeta_{ij}(y))$.

If $f\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ is a weighted homogeneous polynomial map germ of type $(r_1, \ldots, r_n; w_1, \ldots, w_p)$, we define

$$N_{\mathcal{C},V} f = \sum_{i=1}^{p}\Big(\sum_{j=1}^{p}(\zeta_{ij} \circ f)^2\Big)$$

and define $N_{\mathcal{R}} f = \sum_I M_I^2$, where each M_I is a $p \times p$ minor of the Jacobean matrix of f, $I = (i_1, \ldots, i_p) \subset (1, \ldots, n)$.

We observe that for each $p \times p$ minor M_I, there is an s_I such that M_I is weighted homogeneous of type $(r_1, \ldots, r_n; s_I)$.

Let $N^*_{\mathcal{R}}f$ be defined by $N^*_{\mathcal{R}}f = \sum_I M_I^{2\beta_I}$, where $\beta_I = k/s_I$, $k =$ l.c.m.(s_I). Then $N^*_{\mathcal{R}}f$ is a weighted homogeneous control function of type $(r_1, \ldots, r_n; 2k)$.

For deformations f_t of f defined by $f_t(x) = f(x) + t\Theta(x)$, $\Theta = (\Theta_1, \ldots, \Theta_p)$, we define the control $N^*_{\mathcal{R}}f_t$ by $N^*_{\mathcal{R}}f_t = \sum_I M_{t_I}^{2\beta_I}$, where M_{t_I} are the $p \times p$ minors of Jacobean matrix J_{f_t} of f_t and β_I are same as above. If f_t is weighted homogeneous of same type as f, then $N^*_{\mathcal{R}}f_t$ is weighted homogeneous of type $(r_1, \ldots, r_n; 2k)$ for all t. If $f_t(x) = f(x) + t\Theta(x)$ and $fil(\Theta_i) \geq d_i$, it follows that $\text{fil}(N^*_{\mathcal{R}}f_t) \geq \text{fil}(N^*_{\mathcal{R}}f)$.

Theorem 4.4 *Let*

(a) *V be a weighted homogeneous subvariety of $(\mathbf{R}^p, 0)$, which is defined by a weighted homogeneous polynomial g;*

(b) *Θ_V a free $\mathcal{O}_y$-module, $\zeta_1, \ldots, \zeta_p$ be a basis of weighted homogeneous of type $(w_1, \ldots, w_p; d_1, \ldots, d_p)$, with $d_1 \leq d_2 \leq \cdots \leq d_p$, for Θ_V, where*

$$\zeta_i = \sum_{j=1}^{p} \zeta_{ij}(y)\frac{\partial}{\partial y_j} = \sum_{j=1}^{p} \zeta_{ij}\frac{\partial}{\partial y_j}$$

and ζ_{ij} are weighted homogeneous polynomials of type $(w_1, \cdots, w_p; d_i + w_j)$;

(c) *$f\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ a weighted homogeneous polynomial map germ of type $(r_1, \ldots, r_n; w_1, \ldots, w_p)$ with $r_1 \leq r_2 \leq \cdots \leq r_n$, $w_1 \leq w_2 \leq \cdots \leq w_p$;*

(d) *$N_{\mathcal{K},V}f = N_{\mathcal{C},V}f + N_{\mathcal{R}}f \geq c \mid x \mid^\alpha$, for some constants c and α.*
Then deformations of f defined by

$$f_t(x) = f(x) + t\Theta(x), \quad \Theta = (\Theta_1, \ldots, \Theta_p)$$

with $\text{fil}(\Theta_i) \geq d_i + w_p + lr_n + 1$, *for all i, $t \in [0, 1]$ and $l > 1$ are $C^l - \mathcal{K}_V$-trivial.*

Remark Condition (b) is satisfied by Lemma 4.2.

Proof. We firstly define vector fields ν_1 and W_R in following (1) and (2).
(1) Let

$$N^*_{\mathcal{C},V}f = \sum_{i=1}^{p}\Big(\sum_{j=1}^{p}(\zeta_{ij} \circ f)^{2\beta_{ij}}\Big),$$

where $\zeta_{ij} \circ f$ is a weighted homogeneous polynomial of type $(r_1, \ldots, r_n; d_i +$

$w_j)$, $\beta_{ij} = k_1/(d_i + w_j)$ and $k_1 = \text{l.c.m.}\{d_i + w_j \mid 1 \le i \le p,\, 1 \le i \le p\}$.

Let

$$N^*_{\mathcal{C},V} f_t = \sum_{i=1}^{p}\Big(\sum_{j=1}^{p}(\zeta_{ij}\circ f_t)^{2\beta_{ij}}\Big)$$

where each β_{ij} is the same as above. Now

$$\begin{aligned} N^*_{\mathcal{C},V} f_t \cdot \frac{\partial f_t}{\partial t} &= \sum_{i=1}^{p}\Big(\sum_{j=1}^{p}(\zeta_{ij}\circ f_t)^{2\beta_{ij}-1}\cdot\frac{\partial f_t}{\partial t}\cdot\zeta_{ij}\circ f_t\Big)\\ &= \sum_{i=1}^{p}\Big(\sum_{j=1}^{p}(\zeta_{ij}\circ f_t)^{2\beta_{ij}-1}\cdot\frac{\partial f_t}{\partial t}\cdot f_t^*(\zeta_{ij})\Big) \end{aligned}$$

We define

$$W_{ij}(x,\, t) = (\zeta_{ij}\circ f_t)^{2\beta_{ij}-1}\cdot\frac{\partial f_t}{\partial t},$$

then we have

$$N^*_{\mathcal{C},V} f_t \frac{\partial f_t}{\partial t} = \sum_{i=1}^{p}\Big(\sum_{j=1}^{p} W_{ij}(x,\, t)\cdot f_t^*(\zeta_{ij})\Big) \qquad (\diamond)$$

and

$$\begin{aligned} \text{fil}(W_{ij}(x,\, t)) &= \text{fil}\Big((\zeta_{ij}\circ f_t)^{2\beta_{ij}-1}\cdot\frac{\partial f_t}{\partial t}\Big)\\ &\ge (d_i + w_j)(2\beta_{ij} - 1) + d_i + w_p + lr_n + 1\\ &= 2k_1 - d_i - w_j + d_i + w_p + lr_n + 1\\ &\ge 2k_1 + lr_n + 1 \end{aligned}$$

Let

$$\nu_1\colon (\mathbf{R}^n\times\mathbf{R}^p\times\mathbf{R},\, 0) \longrightarrow (\mathbf{R}^n\times\mathbf{R}^p\times\mathbf{R},\, 0)$$

be the vector field defined by $(0,\, V_p,\, 0)$, where $V_p = \sum_{i=1}^{p}\sum_{j=1}^{p} W_{ij}\zeta_{ij}$.

We can show ν_1 belong to $T\mathcal{K}_V$ when $y \ne 0$. Since

$$\begin{aligned} &(\zeta_{ij}\circ f_t)^{2\beta_{ij}-1}\cdot\frac{\partial f_t}{\partial t}\cdot\zeta_{ij}\\ &= \Big((\zeta_{ij}\circ f_t)^{2\beta_{ij}-1}\cdot\Big(\frac{\partial f_t}{\partial t}\Big)_1\cdot\zeta_{ij},\, \ldots,\, (\zeta_{ij}\circ f_t)^{2\beta_{ij}-1}\cdot\Big(\frac{\partial f_t}{\partial t}\Big)_p\cdot\zeta_{ij}\Big) \end{aligned}$$

Let

$$\begin{cases} (\zeta_{ij}\circ f_t)^{2\beta_{ij}-1}\cdot\Big(\dfrac{\partial f_t}{\partial t}\Big)_1\cdot\zeta_{ij}=b_1\zeta_{11}(y)+b_2\zeta_{21}(y)+\cdots+b_p\zeta_{p1}(y)\\ \qquad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\ (\zeta_{ij}\circ f_t)^{2\beta_{ij}-1}\cdot\Big(\dfrac{\partial f_t}{\partial t}\Big)_p\cdot\zeta_{ij}=b_1\zeta_{1p}(y)+b_1\zeta_{2p}(y)+\cdots+b_1\zeta_{pp}(y)\end{cases}$$

Because Θ_V is free, it implies $\det M(\zeta_1,\ldots,\zeta_p)\neq 0$ over the ring $\mathcal{O}_y$, where

$$M(\zeta_1,\ldots,\zeta_p)=\begin{pmatrix}\zeta_{11}(y) & \cdots & \cdots & \zeta_{p1}(y)\\ \vdots & & & \vdots\\ \zeta_{1p}(y) & \cdots & \cdots & \zeta_{pp}(y)\end{pmatrix}.$$

By Crammer's rule, that

$$(\mathrm{Det}\,M)b_k\in\Big(\mathcal{O}_y\Big(\frac{\partial f_t}{\partial t}\Big)_1+\cdots+\mathcal{O}_y\Big(\frac{\partial f_t}{\partial t}\Big)_p\Big)(\zeta_{ij}\circ f_t)^{2\beta_{ij}-1}\cdot\zeta_{ij}$$

Therefore if $y\neq 0$, then $b_k\in\mathcal{O}_{x,y}$, $(k=1,\ldots,p)$ so that $\nu_1\in T\mathcal{K}_V$.

(2) We construct the vector field $W_R\colon(\mathbf{R}^n,0)\longrightarrow(\mathbf{R}^n,0)$ as in the proof of the Proposition 2.2. of [10] ([9] for more details)

We observe that for each $p\times p$ minor M_I of the Jacobean matrix J_f of f, $I=(i_1,i_2,\ldots,i_p)\subset(1,2,\ldots,n)$, there is an s_I such that M_I is weighted homogeneous of type $(r_1,\ldots,r_n;s_I)$.

Let $N^*_{\mathcal{R}}f$ be defined by $N^*_{\mathcal{R}}f=\sum_I M_I^{2\alpha_I}$, where $2\alpha_I=k/s_I$ and $k=$ l.c.m.(s_I). Then, $N^*_{\mathcal{R}}f$ is a weighted homogeneous control function of type $(r_1,r_2,\ldots,r_n;2k)$.

For deformations f_t of f, we define the control $N^*_{\mathcal{R}}f_t$ by $N^*_{\mathcal{R}}f=\sum_I M_{t_I}^{2\alpha_I}$, where M_{t_I} are the $p\times p$ minor of J_{f_t}, and the α_I are the same as above. □

Now there exists a vector field W_I associated to M_{t_I}, such that $(\partial f_t/\partial t)M_{t_I}=df_t(W_I)$, where

$$W_I=\sum_{i=1}^n u_i\frac{\partial}{\partial x_i},\ \text{with:}\begin{cases}u_i=0, & i\bar{\in}I\\ u_{i_m}=\displaystyle\sum_{j=1}^p N_{ji_m}\Big(\frac{\partial f_t}{\partial t}\Big)_j, & i_m\in I\end{cases}$$

and N_{ji_m} is the $(p-1)\times(p-1)$ minor cofactor of $\partial f_{tj}/\partial x_{i_m}$ in $(df_t)_x$.

Then

$$\frac{\partial f_t}{\partial t} N^*_{\mathcal{R}} f_t = (df_t)_x(W_R), \tag{$\star$}$$

where $W_R = (\sum_I M_{t_I}^{2\alpha_I - 1}) u_i(\partial/\partial x_i)$.
(3) To find a $C^l - \mathcal{K}_V$-equivalence between f and f_t, we consider the following unfolding of the graph of f

$$\begin{aligned} F\colon (\mathbf{R}^n \times \mathbf{R},\, 0) &\longrightarrow (\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R},\, 0) \\ (x,\, t) &\longmapsto (x,\, f_t(x),\, t), \qquad t \in [0,\, 1]. \end{aligned}$$

We aim to find C^l retractions h and k of $\mathrm{id}_{\mathbf{R}^n \times 0}$ and $\mathrm{id}_{\mathbf{R}^n \times \mathbf{R}^p \times 0}$ respectively, such that the following diagram commutes:

$$\begin{array}{ccccc} (\mathbf{R}^n \times 0) & \xrightarrow{(\mathrm{id},\, f)} & (\mathbf{R}^n \times \mathbf{R}^p \times 0) & \xrightarrow{\pi_{\mathbf{R}^n}} & (\mathbf{R}^n \times 0) \\ h\uparrow & & k\uparrow & & h\uparrow \\ (\mathbf{R}^n \times \mathbf{R}, 0 \times I) & \xrightarrow{F} & (\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R},\ 0 \times 0 \times I) & \xrightarrow{\pi_{\mathbf{R}^n \times \mathbf{R}}} & (\mathbf{R}^n \times \mathbf{R}, 0 \times I) \end{array}$$

where

$$k(\mathbf{R}^n \times V \times \mathbf{R}) = \mathbf{R}^n \times V$$

; $\pi_{\mathbf{R}^n \times \mathbf{R}}$ and $\pi_{\mathbf{R}^n}$ are the canonical projections.

If we can do so, then

$h_1\colon (\mathbf{R}^n,\, 0) \longrightarrow (\mathbf{R}^n,\, 0)$ defined by $h_1 = h(x,\, 1)$ and
$k_1\colon (\mathbf{R}^n \times \mathbf{R}^p \times 0) \longrightarrow (\mathbf{R}^n \times \mathbf{R}^p \times 0)$
defined by $k_1(x,\, y) = k(x,\, y,\, 1)$,

will give a $C^l - \mathcal{K}_V$-equivalence between f and f_t.

We shall construct h and k in neighborhood of $t = 0$ as follows:

Firstly we need to define the control function $N^*_{\mathcal{K},V} f$ by $N^*_{\mathcal{K},V} f = (N^*_{\mathcal{R}} f)^\lambda + (N^*_{\mathcal{C},V} f)^\mu$ where λ and μ are constants such that $N^*_{\mathcal{K},V} f$ is weighted homogeneous.

For deformations f_t of f, we define the control $N^*_{\mathcal{K},V} f_t$ by $N^*_{\mathcal{K},V} f_t = N^*_{\mathcal{R}} f_t^\lambda + N^*_{\mathcal{C},V} f_t^\mu$ where λ and μ are the same as above. By condition (d), there exist some constants a, c and β such that

$$N^*_{\mathcal{K},V} f \geq a(N_{\mathcal{K},V} f)^\beta \geq ac|x|^{\alpha\beta}.$$

From Lemma 2.4, similarly to the proof of Lemma 3.2, we obtain that there exist constants c_3 and c_4 such that:

$$c_3\rho_k(x,\, y) \le N^*_{\mathcal{K},V} f_t \le c_4\rho_k(x,\, y) \tag{$\star\star$}$$

where k is the weight of $N^*_{\mathcal{K},V} f_t$.
Now

$$N^*_{\mathcal{K},V} f_t \frac{\partial f_t}{\partial t} = ((N^*_{\mathcal{R}} f_t)^\lambda + (N^*_{\mathcal{C},V} f_t)^\mu)\frac{\partial f_t}{\partial t}.$$

By $(\star)$,

$$(N^*_{\mathcal{R}} f_t)^\lambda \frac{\partial f_t}{\partial t} = df_t((N^*_{\mathcal{R}} f_t)^{\lambda-1} W_R).$$

By $(\diamond)$

$$(N^*_{\mathcal{C},V} f_t)^\mu \frac{\partial f_t}{\partial t} = \sum_{i=1}^{p}\Big(\sum_{j=1}^{p}(N^*_{\mathcal{C},V} f_t)^{\mu-1} W_{ij}(x,\, t)\cdot f_t^*(\zeta_{ij})\Big).$$

So we obtain

$$\begin{aligned}\frac{\partial f_t}{\partial t} &= df_t\Big[\frac{(N_{\mathcal{R}})^* f_t^{\lambda-1} W_R}{N^*_{\mathcal{K},V} f_t}\Big]\\ &\quad + \Big[\frac{\sum_{i=1}^p(\sum_{j=1}^p (N^*_{\mathcal{C},V} f_t)^{\mu-1} W_{ij}(x,\, t)\cdot f_t^*(\zeta_{ij}))}{N^*_{\mathcal{K},V} f_t}\Big]\end{aligned} \tag{$\star\star\star$}$$

To complete the proof, it remains to find germs of C^l vector fields

$$\xi\colon (\mathbf{R}^n \times \mathbf{R},\, 0) \longrightarrow (\mathbf{R}^n \times \mathbf{R},\, 0) \quad \pi_{\mathbf{R}}\circ\xi = \frac{\partial}{\partial t},\ \pi_{\mathbf{R}^n}\circ\xi(0,\, t) = 0,$$

and

$$\eta\colon (\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R},\, 0) \longrightarrow (\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R},\, 0),$$

such that ξ is a lift for η over F, that is $dF(\xi) = \eta\circ F$.

So let

$$\xi(x,\, t) = -\frac{(N^*_{\mathcal{R}} f_t)^{\lambda-1} W_R}{N^*_{\mathcal{K},V} f_t} + \frac{\partial}{\partial t}$$

and

$$\eta(x,\, y,\, t) = \frac{(N^*_{\mathcal{R}} f_t)^{\lambda-1} W_R}{N^*_{\mathcal{K},V} f_t} + \frac{(N^*_{\mathcal{C},V} f_t)^{\mu-1}\nu_1}{N^*_{\mathcal{K},V} f_t} + \frac{\partial}{\partial t}$$

Then

$$dF(\xi) = \Big(-\frac{(N^*_{\mathcal{R}} f_t)^{\lambda-1} W_R}{N^*_{\mathcal{K},V} f_t},\, df_t\Big(-\frac{(N^*_{\mathcal{R}} f_t)^{\lambda-1} W_R}{N^*_{\mathcal{K},V} f_t}\Big) + \frac{\partial f_t}{\partial t},\, \frac{\partial}{\partial t}\Big).$$

From equation $(\star\star\star)$, it follows that $dF(\xi) = \eta \circ F$.

By $(\star\star)$ and Lemma 2.5, ξ and η are class C^l.

Moreover

$$-\frac{(N^*_{\mathcal{R}} f_t)^{\lambda-1} W_R}{N^*_{\mathcal{K},V} f_t} \in \theta_n \subset T\mathcal{K}_V$$

and

$$\frac{(N^*_{\mathcal{R}} f_t)^{\lambda-1} W_R}{N^*_{\mathcal{K},V} f_t} + \frac{(N^*_{\mathcal{C},V} f_t)^{\mu-1} \nu_1}{N^*_{\mathcal{K},V} f_t} \in T\mathcal{K}_V$$

The vector fields $\xi(x,\, t)$ and $\eta(x,\, y,\, t)$ are clearly integrable, hence they determine C^l-diffeomorphisms H and K in $(\mathbf{R}^n \times \mathbf{R},\, 0)$ and $(\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R},\, 0)$ respectively.

The properties of $\xi(x,\, t)$ and $\eta(x,\, y,\, t)$ imply that $\pi_{\mathbf{R}^n} \circ H = h$ and $\pi_{\mathbf{R}^n \times \mathbf{R}^p} \circ K = k$ are the desired retractions. It implies the $C^l - \mathcal{K}$-triviality of the family f_t in a neighborhood of $t = 0$. Since the same argument in a neighborhood of $t = \bar{t}$, for $t \in [0,\, 1]$, the proof is complete.

Remark This Theorem generalizes a result of M.A.S. Ruas and M.J. Saia (Proposition 2.5 of [10]).

Corollary 4.5 (Proposition 2.5 of [10]) *Let*

(a) $f\colon (\mathbf{R}^n,\, 0) \longrightarrow (\mathbf{R}^p,\, 0)$ *be a weighted homogeneous polynomial map germ of type* $(r_1,\, \dots,\, r_n; w_1,\, \dots,\, w_p)$ *with* $r_1 \le r_2 \le \cdots \le r_n$, $w_1 \le w_2 \le \cdots \le w_p$;

(b) $V = \{0\}$ *be a weighted homogeneous subvariety which is defined by the ideal* $\mathcal{M}_p$;

(c) Θ_V *be free* $\mathcal{O}_y$*-module,* $(y_1 \cdot (\partial/\partial y_1),\, \dots,\, y_p \cdot (\partial/\partial y_p))$ *be a basis of weighted homogeneous of type* $(w_1,\, \dots,\, w_p; 0,\, \dots,\, 0)$ *for* Θ_V;

(d)

$$N_{\mathcal{K},\{0\}} f = N_{\mathcal{C},\{0\}} f + N_{\mathcal{R}} f = N_{\mathcal{C}} f + N_{\mathcal{R}} f \ge c|x|^{\alpha},$$

for constants c *and* α.

Then deformations of f defined by

$$f_t(x) = f(x) + t\Theta(x), \quad \Theta = (\Theta_1, \ldots, \Theta_p)$$

with $\mathrm{fil}(\Theta_i) \geq w_p + lr_n + 1$, *for all* i, $t \in [0, 1]$ *and* $l > 1$ *are* C^l-*trivial.*

Proof. Since $V = \{0\}$, $\Theta_V = \mathcal{O}_y\{y_i(\partial/\partial y_i)\}$ and it is free. $\{y_i(\partial/\partial y_i)\}$ is a basis for Θ_V so that

$$N^*_{\mathcal{C},\{0\}} f = \sum_{i=1}^{p} f_i^{2\beta_i},$$

where $\beta_i = k/w_i$ and $k = \mathrm{l.c.m.}(w_i)$. By Theorem 4.4, the proof is complete. □

Acknowledgment The authors thanks the referee for his/her careful reading and very useful comments which improved the final version of this paper.

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