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The role of compactification theory in the type problem

(To the memory of my esteemed teacher the late Professor Kiyoshi Noshiro on the occasion of the centennial anniversary of his birth)

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Abstract. We are concerned with the type problem for the covering surface S_Γ or more precisely $(S_\Gamma, S, \pi_\Gamma)$ of the base parabolic Riemann surface S with its projection π_Γ, where S_Γ is the infinitely sheeted covering Riemann surface constructed from the sequence of replicas S_n of S and the family $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$ of pasting arcs $\gamma_n \subset S$ with $\gamma_{n-1} \cap \gamma_n = \emptyset$ $(n \in \mathbb{N})$ in such a fashion that $S_n \setminus (\gamma_{n-1} \cup \gamma_n)$ is joined to $S_{n+1} \setminus (\gamma_n \cup \gamma_{n+1})$ crosswise along γ_n for each $n \in \mathbb{N}$. Here $\mathbb{N}$ is the set of positive integers and the parabolicity of a surface is characterized by the nonexistence of the Green function on the surface. The central object of this paper is to show by using the theory of Wiener and Royden compactifications that the type of the covering surface S_Γ is parabolic if the sequence of capacities $\mathrm{cap}(\gamma_n, S \setminus \gamma_0)$ of $\gamma_n \in \Gamma$ with respect to the surface S less the arc γ_0 fixed in S disjoint from all other arcs $\gamma_n \in \Gamma$ for all sufficiently large $n \in \mathbb{N}$ converges to zero so rapidly as to satisfy the condition $\sum_{n\in\mathbb{N}} \mathrm{cap}(\gamma_n, S \setminus \gamma_0) < +\infty$.

Key words: capacity, covering surface, Evans-Selberg potential, Green function, hyperbolic, parabolic, pasting arc, Riemann surface, Royden compactification, type problem, Wiener compactification.

1. Introduction

Consider a covering Riemann surface (R, S, π), where R and S are Riemann surfaces and π is an analytic mapping (i.e. a projection mapping) of R onto S (cf. e.g. [1]). The (generalized) *type problem* (cf. [15], [11], [12], etc.) is to determine whether $R \in O_G$ or not, where O_G is the class of parabolic Riemann surfaces characterized by the nonexistence of Green functions on them. Surfaces not in O_G are referred to as being hyperbolic. In this problem the surface S is always supposed to be parabolic in advance due to the following reason (cf. e.g. Tsuji [15]):

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Effect of base hyperbolicity *The hyperbolicity of the base surface S implies that of the covering surface R.*

The proof is straightforward. In fact, the hyperbolicity of S is characterized by the existence of a potential p on S, i.e. a positive superharmonic function p which does not admit any positive harmonic function h with $0 < h \le p$ on S, or equivalently, by the existence of a positive superharmonic function p which is not harmonic on S. Then $p \circ \pi$ is a positive nonharmonic superharmonic function on R along with p on S and we must conclude that R is also hyperbolic. Actually this is in essence nothing but the Liouville property. Another point to be observed in this problem is the following:

Effect of finite sheetedness *If S is parabolic and the covering surface (R, S, π) is finitely n sheeted, then R is parabolic.*

We understand by being finitely n sheeted for a covering surface (R, S, π) that the number of points in the fiber $\pi^{-1}(z)$ is a constant $n \in \mathbb{N}$, the set of positive integers, for every $z \in S$, where branch points are counted according to their multiplicities. Contrariwise suppose that R is hyperbolic. Then a potential p on R exists but we take a special one that is the Green function on R with its pole at $\alpha \in R$, i.e. the potential $p \in H(R \setminus \{\alpha\})$, where $H(X)$ is the class of harmonic functions on an open subset X of a Riemann surface, such that $p(\zeta) - \log(1/|\zeta|)$ is harmonic at α with ζ a local parameter at α, i.e. p has a logarithmic pole at α. Then clearly

$$q(z) := \sum_{t \in \pi^{-1}(z)} p(t) \quad (z \in R)$$

is a positive superharmonic function on S with logarithmic pole at $\pi(\alpha)$ so that q is not harmonic on the whole S. The existence of such a q contradicts the parabolicity of S and we are done. For these two reasons mentioned above, we always suppose in our (generalized) type problem for covering surfaces (R, S, π) that they are infinitely sheeted and the base surfaces S are parabolic. If we restrict ourselves to the special base surface $S = \widehat{\mathbb{C}}$, the complex sphere, or $S = \mathbb{C}$, the complex plane, and to simply connected covering surfaces R, we come back to the genuine classical type problem.

In the present article we are concerned with the following type of special covering surfaces $(S_\Gamma, S, \pi_\Gamma)$ described in detail in the sequel. As mentioned above we take arbitrarily and then fix a parabolic open Riemann surface S as the base surface of our covering surfaces. We do not exclude closed surfaces S_0 for our base surfaces but it is treated in the following fashion: we fix a point $s_0 \in S_0$ and then consider $S := S_0 \setminus \{s_0\}$ viewing s_0 as the point at infinity of S_0. Now take a family $\Gamma := \{\gamma_n \colon n \in \mathbb{N}\}$ of simple arcs γ_n in S, referred to as a family of *pasting arcs* for S_Γ, with initial points a_n and terminal points b_n for all $n \in \mathbb{N}$ and in this paper we assume that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty, \tag{1.1}$$

where ∞ is the Alexandroff point of S. By the scissoring and pasting method using S and Γ we now construct a Riemann surface S_Γ. The above condition (1.1) is to require that the projections $\{a_n, b_n \colon n \in \mathbb{N}\}$ of branch points of S_Γ only accumulate at the point at infinity of S. On the disposal of arcs in Γ we suppose

$$\gamma_{n-1} \cap \gamma_n = \emptyset \quad (n \in \mathbb{N}), \tag{1.2}$$

where γ_0 is a simple arc in S disjoint from all other γ_n for sufficiently large $n \in \mathbb{N}$.

In general we denote by

$$(X \setminus \gamma) \biguplus_\gamma (Y \setminus \gamma) \tag{1.3}$$

the Riemann surface obtained from two Riemann surfaces X and Y by joining $X \setminus \gamma$ and $Y \setminus \gamma$ crosswise along a common arc γ in X and Y. More precisely, let U_X and U_Y be parametric discs at some points in X and Y and $\gamma_X \subset U_X$ and $\gamma_Y \subset U_Y$ be some arcs. If we identify U_X and U_Y as the unit disc $\mathbb{D}$ and assume that $\gamma_X = \gamma_Y = \gamma$ in this identification, then we say that X and Y contain the common arc γ. Considering the boundaries γ in $X \setminus \gamma$ and $Y \setminus \gamma$ in their Carathéodory compactifications we denote them by $\gamma^+ \cup \gamma^-$, i.e. we denote by γ^+ (γ^-, resp.) the upper (lower, resp.) shore of the slit γ considered in $\mathbb{D}$. Identification γ^+ in $X \setminus \gamma$ ($Y \setminus \gamma$, resp.) with γ^- in $Y \setminus \gamma$ ($X \setminus \gamma$, resp.) produces the surface (1.3) and then introduce the local conformal structure by the local parameters z in X and Y at each point of $(X \setminus \gamma) \biguplus_\gamma (Y \setminus \gamma)$ except at end points of γ and finally give the local conformal structure at the end points of γ by $\sqrt{z}$ with z the original local parameters in X and Y. This gives rise to a Riemann surface (1.3).

We take the sequence $(S_n : n \in \mathbb{N})$ of replicas S_n of S, $S_n = S$ $(n \in \mathbb{N})$. If it is preferable even in the expense of cumbersomeness to make things precise, then we denote by γ_{nj} the arc γ_n if it is considered (i.e. embedded) in S_j, i.e. $\gamma_{nj} = \gamma_n$ and $\gamma_{nj} \subset S_j$. We place γ_0 in S_1, i.e. $\gamma_0 = \gamma_{01}$. We construct the sequence $(V_n)_{n\in\mathbb{N}}$ of Riemann surfaces V_n $(n \in \mathbb{N})$ inductively as follows. Let

$$V_1 := S_1. \tag{1.4}$$

If $V_1, \ldots, V_n$ have been constructed, then we set

$$V_{n+1} := (V_n \setminus \gamma_{nn}) \textstyle\bigcup\!\!\!\!\times_{\gamma_n} (S_{n+1} \setminus \gamma_{n,n+1}) \quad (\gamma_{nn} = \gamma_{n,n+1} = \gamma_n). \tag{1.5}$$

From the sequence $(V_n)_{n\in\mathbb{N}}$ we construct the sequence $(W_n)_{n\in\mathbb{N}}$ of Riemann surfaces W_n given by

$$W_n := V_n \setminus \gamma_{nn} \qquad (n \in \mathbb{N}), \tag{1.6}$$

which is a subsurface of V_{n+1} and therefore of W_{n+1}. The relative boundary ∂W_n of W_n relative to e.g. W_{n+1} is a Jordan curve $\gamma_n^+ \cup \gamma_n^-$. Since the sequence $(W_n)_{n\in\mathbb{N}}$ satisfies the inclusion relations

$$W_1 \subset \overline{W}_1 \subset W_2 \subset \overline{W}_2 \subset W_3 \subset \cdots \subset W_n \subset \overline{W}_n \subset W_{n+1} \subset \cdots \qquad (n \in \mathbb{N}),$$

where inclusions above are understood to indicate Riemann subsurfaces, and now we can define a Riemann surface S_Γ by

$$S_\Gamma := \bigcup_{n\in\mathbb{N}} W_n. \tag{1.7}$$

Although W_n is not relatively compact since it has the ideal boundary δW_n consisting of the Alexandroff points ∞_j of S_j $(1 \leq j \leq n)$ which is however of parabolic character, we can give the role to W_n played by the relatively compact subregion as far as the O_G character is concerned. Then $(W_n)_{n\in\mathbb{N}}$ is an exhaustion of S_Γ in the wider sense. Take an arbitrary $\zeta \in S_\Gamma$. There is an $n \in \mathbb{N}$ such that either $\zeta \in S_n \setminus (\gamma_{n-1,n} \cup \gamma_{nn})$ or $\zeta \in (\gamma_{n-1,n} \cup \gamma_{nn})$ and in any case ζ may be viewed as a point ζ_n in $S_n = S$. We then set $\pi_\Gamma(\zeta) = \zeta_n$. Then $\pi = \pi_\Gamma \colon S_\Gamma \to S$ is a natural projection and we obtain a covering Riemann surface

$$(S_\Gamma, S, \pi_\Gamma). \tag{1.8}$$

Our type problem is to determine whether $S_\Gamma \in O_G$ or $S_\Gamma \notin O_G$. There is a unique harmonic function w_n on $W_n \setminus \overline{W_1}$ with boundary values 0 on ∂W_1 and 1 on ∂W_n and $0 < w_n < 1$ on $W_n \setminus \overline{W_1}$ $(n \geq 2)$. Then $S_\Gamma \in O_G$, for example, if and only if $(w_n)_{n\geq 2}$ converges to zero.

For a closed subset F in a Riemann surface R and an open subset U of R such that $F \subset U$, the *capacity* $\mathrm{cap}(F, U)$ of F relative to U, or more precisely the *variational* 2-*capacity*, is given by

$$\mathrm{cap}(F,\, U) := \inf_{\varphi \in \mathcal{F}(F,\, U)} \int_R |\nabla\varphi(z)|^2 dxdy \quad (z = x + iy), \tag{1.9}$$

where $\mathcal{F}(F, U)$ is the class of functions $\varphi \in C^\infty(R)$ such that $\varphi|F \geq 1$ and $\varphi|(R \setminus U) \leq 0$. In terms of the capacity we are apt to expect the condition (1.10) stated below is the necessary condition for S_Γ to be parabolic. Needless to say the expectation is in general in vain as is seen by simple examples of $S_\Gamma \in O_G$ with Γ such as containing two subsequences $(\gamma_{n'})$ and $(\gamma_{n''})$ such that $n' \,\mathrm{cap}(\gamma_{n'},\, S \setminus \gamma_0) \to 0$ $(n' \to \infty)$ and $(\mathrm{cap}(\gamma_{n''},\, S \setminus \gamma_0))_{n''}$ does not converge to zero. Thus we have to have some condition to realize the above expectation and at present we are able to prove this under a certain mild, which we hope, restriction on Γ. We say that Γ is *monotonically disposed* toward ∞ if there are simple curves α and β on S starting from a_0 and b_0 and tending to ∞ such that $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ are on α in this order and similarly $b_0, b_1, b_2, b_3, \ldots, b_n, \ldots$ are on β in this order, and the curve $-\alpha + \gamma_0 + \beta$ surrounds a simply connected subregion of S, a strip type region, for which each γ_n is a cross cut $(n \in \mathbb{N})$ and if we denote by $\alpha(m, n)$ ($\beta(m, n)$, resp.) the subarc of α (β, resp.) starting from a_m (b_m, resp.) and terminating at a_n (b_n, resp.) for $m < n$, then $\alpha(m, n) + \gamma_n - \beta(m, n) - \gamma_m$ forms a Jordan curve. Observe that the monotonically disposedness of Γ implies the pasting arcs in Γ are mutually disjoint. Then we have the following:

Proposition 1.1 *Suppose that Γ is monotonically disposed toward ∞. Then the parabolicity of S_Γ implies that*

$$\lim_{n\to\infty} \mathrm{cap}(\gamma_n,\, S \setminus \gamma_0) = 0 \tag{1.10}$$

for one and hence for every choice of admissible arc γ_0 in S.

In the above condition (1.10) we may replace γ_0 by any compact continuum with connected complement in S as will be seen in §2. Based on the above observation our object of this paper more concretely stated is to study whether the condition (1.10) shown above to be necessary for the parabolicity is also sufficient for S_Γ to be parabolic. The monotonical disposedness of Γ is only assumed in Proposition 1.1 above and in the above mentioned sufficiency problem no condition whatsoever except (1.1) is imposed upon Γ beforehand. We have shown (see [8], [9], [10]) that in the case $S = \mathbb{C}$ (the complex plane) and initial points a_n and terminal points b_n of pasting arcs $\gamma_n \in \Gamma$ satisfy the following symmetry conditions

$$-\overline{a}_n = b_n \quad (n \in \mathbb{N}),$$

then (1.10) certainly implies the parabolicity of S_Γ. The purpose of this paper is to prove that if, instead of the above geometric condition, we require the quantitative condition that the speed of the convergence of the sequence $(\mathrm{cap}(\gamma_n,\, S \setminus \gamma_0))_{n\in\mathbb{N}}$ to zero is sufficiently fast, then the desired conclusion can be derived.

The Main Theorem *If the sequence* $(\mathrm{cap}(\gamma_n,\, S \setminus \gamma_0))_{n\in\mathbb{N}}$ *of capacities of* γ_n *in* Γ *converges to zero so rapidly as to satisfy*

$$\sum_{n\in\mathbb{N}} \mathrm{cap}(\gamma_n,\, S \setminus \gamma_0) < +\infty \tag{1.11}$$

for one and hence for every choice of admissible arc γ_0 *in* S*, then* S_Γ *is parabolic.*

In the above condition (1.11) we may replace γ_0 by any compact continuum with connected complement in S as will be seen in §2. The parabolicity of the base surface S can also be characterized by the existence of an Evans-Selberg potential $E(z,\, z_0)$ on S with its pole z_0 in S chosen arbitrarily in advance (see [6], [13], [12]) like the hyperbolicity of S is characterized by the existence of the Green function $G(z,\, z_0)$ on S with its pole z_0 in S. Given an arbitrary point $z_0 \in S$. An Evans-Selberg potential $E(z,\, z_0)$ on S with its negative pole z_0 in S is first of all harmonic on $S \setminus \{z_0\}$ with its negative pole at z_0, i.e. $E(z,\, z_0) + \log(1/|z|)$ is harmonic at z_0 for any local parameter z centered at z_0, such that

$$\lim_{z\to\infty} E(z,\, z_0) = +\infty, \tag{1.12}$$

where ∞ is the point at infinity of S. If $S = \mathbb{C}$, then its Evans-Selberg potential $E(z,\, z_0)$ is unique and given by $-\log(1/|z - z_0|)$. However, in general, it is only unique if and only if the harmonic dimension of S is 1. In any case we can take an arbitrary Evans-Selberg potential $E(z,\, z_0)$ on S if it is parabolic such that $E(\,\cdot\,,\,\cdot\,)$ is $[-\infty,\, +\infty)$-valued continuous on $S \times S$ and finitely continuous on $S \times S$ less its diagonal with $E = -\infty$ on the diagonal of $S \times S$ and $E(\,\cdot\,, z_0) - E(\,\cdot\,, z_0')$ is bounded near the point ∞ at infinity of S for any pair of points z_0 and z_0' in S (see [12]). We fix such an E for S and we consider the following metrical quantities: for any subset X in S we set $\delta(X) = 0$ if $z_0 \in X$ and

$$\delta(X) := \exp(\sup\{t \in [-\infty, \infty) \colon X \subset \{z \in S \colon E(z, z_0) > t\}\}) \quad (1.13)$$

if $z_0 \notin X$ so that $0 \leq \delta(X) \leq +\infty$. In view of (1.12) we can understand that the bigness of $\delta(X)$ is equivalent to the closeness of X to the point at infinity. In the case $S = \mathbb{C}$ and $X \subset \mathbb{C}$,

$$\delta(X) = \mathrm{dis}(z_0, X) := \inf_{z \in X} |z - z_0|.$$

We now consider the following condition for Γ:

$$\lim_{n \to \infty} \frac{1}{\delta(\gamma_n)} = 0 \quad (1.14)$$

for one and hence for every choice of the reference point z_0 in S and the following condition for Γ stronger than the above condition:

$$\sum_{n \in \mathbb{N}} \frac{1}{\delta(\gamma_n)} < +\infty. \quad (1.15)$$

for one and hence for every choice of the reference point z_0 in S. Concerning conditions (1.10) and (1.11) for Γ in terms of capacities and those (1.14) and (1.15) stated above for Γ in terms of the metric induced by Evans-Selberg potential we have the following relations:

Proposition 1.2 *The capacity condition* (1.10) *and the metric condition* (1.14) *are mutually equivalent. The stronger metric condition* (1.15) *implies the stronger capacity condition* (1.11).

Therefore, as a corollary of the main theorem, we deduce the following metric version of the sufficiency criterion for S_Γ to be parabolic. It seems that this metric criterion is much more convenient in practical application than the capacity criterion although the former is theoretically less general than the latter.

Corollary to the Main Theorem *If $\gamma_n \in \Gamma$ converges to the point at infinity of S so rapidly as to satisfy the condition* (1.15)*, then S_Γ is parabolic.*

We will prove the main theorem by using the theory of Wiener and

Royden compactifications (cf. e.g. [3], [12]). It is really tailor made for the application to the type problem. For example, the parabolicity of S_Γ can be characterized by the nonexistence of the Wiener harmonic boundary and also by the nonexistence of the Royden harmonic boundary of S_Γ, which are the sets of regular points for the Dirichlet problem on S_Γ with respect to the Wiener boundary and also the Royden boundary of S_Γ. Therefore, for example, we have to show that the Wiener or Royden harmonic boundary of S_Γ is empty if the condition (1.11) is postulated. Actually it is easier to show that the condition (1.11) implies the emptiness of the Royden harmonic boundary of S_Γ although it is equivalent to the emptiness of the Wiener harmonic boundary of S_Γ

The contents of this paper is as follows. After the present §1, Introduction, reversing the order due to the convenience sake, we first prove Proposition 1.2 in §2 and then Proposition 1.1 in §3. The important remark made in §2 is that the arc γ_0 in (1.11) may be replaced by any admissible continuum K (i.e. a compact continuum K with connected $S \setminus K$ (cf. §2 below)) so that what the main theorem really asserts is that, if

$$\sum_{n\in\mathbb{N}} \mathrm{cap}(\gamma_n, S \setminus K) < +\infty$$

for one and hence for every admissible continuum K, then S_Γ is parabolic. The proof of the main theorem will be given in §§4–7. In §4, that the projection $\pi = \pi_\Gamma\colon S_\Gamma \to S$ is not a Fatou mapping will be shown based on an erroneous assumption that $S_\Gamma \notin O_G$ as a result of the general result to judge whether a given analytic mapping is Fatou or not, giving whose proof is the main achievement in §5. Briefly reviewing the necessary tools in the Royden compactification theory in §6, we will prove that the projection mapping $\pi\colon S_\Gamma \to S$ is a Fatou mapping again based on the erroneous assumption that $S_\Gamma \notin O_G$. This contradicts the result in §4 and thus the proof of the main theorem will be completed in this final section §7.

2. Proof of Proposition 1.2

Since $E(\,\cdot\,, z_0) - E(\,\cdot\,, z_0')$ is bounded on S near ∞ and both of $E(\,\cdot\,, z_0)$ and $E(\,\cdot\,, z_0')$ tend to $+\infty$ as $z \to \infty$, we can find a relatively compact subregion Ω of S such that $K \cup \{z_0, z_0'\} \subset \Omega$, where K is an admissible continuum in S (see the definition right after Claim 2.1 below) and both of $E(\,\cdot\,, z_0)$ and $E(\,\cdot\,, z_0')$ are strictly positive on $S \setminus \Omega$ so that there is a constant $\kappa > 1$ with

$$E(\,\cdot\,,\,z_0) - \log\kappa \le E(\,\cdot\,,\,z_0') \le E(\,\cdot\,,\,z_0) + \log\kappa \tag{2.1}$$

on $S \setminus \Omega$. Let δ' be the δ in (1.13) obtained from $E(\,\cdot\,,\,z_0')$. Then we deduce from the above inequality that

$$\kappa^{-1}\delta(X) \le \delta'(X) \le \kappa\delta(X) \tag{2.2}$$

for any $X \subset S \setminus \Omega$. If $\delta(\gamma_n) \to +\infty$ $(n \to \infty)$, then $\gamma_n \subset S \setminus \Omega$ for every sufficiently large n and hence $\delta'(\gamma_n) \to +\infty$ $(n \to \infty)$. We can thus conclude the following.

Claim 2.1 The conditions (1.14) and (1.15) do not depend on the choice of the reference point z_0.

We will call a subset $K \subset S$ an *admissible continuum* if, first of all, it is a nonempty compact continuum and $S \setminus K$ is connected. The reference arc γ_0 in (1.10) and (1.11) is an example. Generalizing γ_0 to any admissible continuum $K \subset S$ we consider the condition

$$\lim_{n\to\infty} \operatorname{cap}(\gamma_n,\, S \setminus K) = 0 \tag{2.3}$$

for any admissible continuum K more general than (1.10) and also the condition

$$\sum_{n\in\mathbb{N}} \operatorname{cap}(\gamma_n,\, S \setminus K) < +\infty \tag{2.4}$$

for any admissible K more general than (1.11). Now fix an arbitrary admissible continuum $K \subset S$ and any reference point $z_0 \in S$ for δ. For any real number c we set $\Omega_c := \{z \in S \colon E(z,\,z_0) < c\}$, which is a relatively compact subregion of S. Choose arbitrarily but then fix a real number a such that $\Omega_a \supset K$. Now suppose that (1.14) ((1.15), resp.) is valid. Then there is an $n_0 \in \mathbb{N}$ such that $\gamma_n \cap \overline{\Omega}_a = \emptyset$ $(n \ge n_0)$. Observe that $\gamma_n \subset S \setminus \Omega_{\delta(\gamma_n)}$ and moreover $\delta(\gamma_n) > a$ for every $n \ge n_0$. Let ω be the extremal (i.e. capacitary) function for $\operatorname{cap}(\gamma_n,\, S\setminus K)$ so that $\omega \in H(S\setminus(K\cup\gamma_n))\cap C(S)$, $\omega|K = 0$, $\omega|\gamma_n = 1$, $0 \le \omega \le 1$ on S, and

$$D\big(\omega,\, S \setminus (K \cup \gamma_n)\big) := \int_{S\setminus(K\cup\gamma_n)} |\nabla\omega(z)|^2 dxdy \quad (z = x + iy).$$

Similarly let ω' be the extremal function for $\operatorname{cap}(S \setminus \Omega_{\delta(\gamma_n)},\, S \setminus \Omega_a)$. Then by the Dirichlet principle

$$\begin{aligned}\operatorname{cap}(\gamma_n,\, S\setminus K) &= D(\omega,\, S\setminus(K\cup\gamma_n))\\ &\le D(\omega',\, S\setminus(\Omega_{\delta(\gamma_n)}\cup\Omega_a)) = \operatorname{cap}(S\setminus\Omega_{\delta(\gamma_n)},\, S\setminus\Omega_a).\end{aligned}$$

On the other hand $\omega' = (E(\,\cdot\,,\, z_0)-a)/(\delta(\gamma_n)-a)$ on $\Omega_{\delta(\gamma_n)}\setminus\overline{\Omega}_a$ and thus, by the Green formula, we deduce that

$$\begin{aligned}D(\omega',\, S\setminus(\Omega_{\delta(\gamma_n)}\cup\Omega_a)) &= (\delta(\gamma_n)-a)^{-1}\int_{\partial\Omega_{\delta(\gamma_n)}} *dE(\,\cdot\,,\, z_0)\\ &= \frac{2\pi}{\delta(\gamma_n)-a}.\end{aligned}$$

Since $a<\delta(\gamma_n)\to+\infty$ $(n\to\infty)$, we can find an $n_1\ge n_0$ in $\mathbb{N}$ such that $\delta(\gamma_n)-a>\delta(\gamma_n)/2$ $(n\ge n_1)$ so that

$$\operatorname{cap}(\gamma_n,\, S\setminus K)\le\frac{4\pi}{\delta(\gamma_n)}\quad(n\ge n_1). \tag{2.5}$$

Therefore we can conclude the following.

Claim 2.2 The condition (1.14) ((1.15), resp.) for any reference point z_0 in S implies the condition (2.3) ((2.4), resp.) for every admissible continuum K and in particular the condition (1.10) ((1.11), resp.) for every admissible arc γ_0.

Take and then fix an arbitrary admissible continuum K in S. In addition to the variational 2-capacity $\operatorname{cap}(\gamma) := \operatorname{cap}(\gamma,\, S\setminus K)$ it is convenient to consider the kernel capacities: the Green capacity $c_g(\gamma)$ and the logarithmic capacity $c_l(\gamma)$ for arcs $\gamma\subset S$. Let $g(z,\,\zeta)$ be the Green kernel on $S\setminus K$ so that $g(\,\cdot\,,\,\zeta)$ is the Green function on $S\setminus K$ with its pole at ζ. Let $\mathcal{F}_\gamma$ be the family of unit Borel measures μ on γ. Then consider

$$V_g(\gamma) := \inf_{\mu\in\mathcal{F}_\gamma} I_g(\mu),$$

where $I_g(\mu)$ is referred to as the Green energy of μ and given by

$$I_g(\mu) := \iint g(z,\,\zeta)\,d\mu(z)\,d\mu(\zeta).$$

The quantity $V_g(\gamma)$ has the unique extremal measure $\mu_{g,\gamma}\in\mathcal{F}_\gamma$ referred to as the equilibrium measure, i.e. $V_g(\gamma)=I(\mu_{g,\gamma})$, and the Green potential of the measure $\mu_{g,\gamma}$

$$g^{\mu_{g,\gamma}}(\,\cdot\,) := \int g(\,\cdot\,,\,\zeta)d\mu_{g,\gamma}(\zeta),$$

referred to as the equilibrium potential of the set γ, satisfies that $g^{\mu_{g,\gamma}}/V_g(\gamma)$ is the extremal function for the capacity $\mathrm{cap}(\gamma) = \mathrm{cap}(\gamma,\, S \setminus K)$. We define the Green capacity $c_g(\gamma)$ of γ by

$$c_g(\gamma) := \frac{1}{V_g(\gamma)}. \tag{2.6}$$

In view of the energy identity

$$D(g^\nu,\, S \setminus K) := \int_{S\setminus K} |\nabla g^\nu(z)|^2 dxdy = 2\pi I_g(\nu)$$

for any $\nu \in \mathcal{F}_g$, we can conclude that

$$\mathrm{cap}(\gamma) = \frac{2\pi}{V_g(\gamma)} = 2\pi c_g(\gamma). \tag{2.7}$$

We set $l(z,\,\zeta) := \log(1/|z-\zeta|)$ for every $(z,\,\zeta) \in \mathbb{C} \times \mathbb{C}$. The function $l(z,\,\zeta)$ on $\mathbb{C} \times \mathbb{C}$ is referred to as the logarithmic kernel on $\mathbb{C}$. For any point $p \in S$ fix a $\rho \in (0,\,1/4)$ and consider the parametric disc $(U,\,z)$ at p such that $U := \{|z| < \rho\}$. We denote by $U_j := \{z \in U \colon |z| < j\rho/3\}$ $(j = 1,\,2)$. We can define $I_l(\mu)$ and $V_l(\gamma)$ by exactly the same fashion as $I_g(\mu)$ and $V_g(\gamma)$ are given. However the logarithmic capacity $c_l(\gamma)$ of γ is defined differently from (2.6) by

$$c_l(\gamma) := \exp(-V_l(\gamma)). \tag{2.8}$$

We denote by M_l (M_g, resp.) the maximum of $l(z,\,\zeta)$ ($g(z,\,\zeta)$, resp.) on $(\partial U) \times \overline{U}_2$ and by m_l (m_g, resp.) the minimum of $l(z,\,\zeta)$ ($g(z,\,\zeta)$, resp.) on $(\partial U) \times \overline{U}_2$. By the choice of ρ we see that $M_l \geq m_l > 0$ and if we set $\kappa := \max(M_g/m_l,\, M_l/m_g)$, then

$$\kappa^{-1} l(z,\,\zeta) \leq g(z,\,\zeta) \leq \kappa l(z,\,\zeta) \quad ((z,\,\zeta) \in \overline{U} \times \overline{U}_2)$$

so that

$$\kappa^{-1} V_l(\gamma) \leq V_g(\gamma) \leq \kappa V_l(\gamma)$$

for any arc $\gamma \subset \overline{U}_2 \setminus U_1$. Hence by (2.6) and (2.8) we see that

$$\frac{\kappa^{-1}}{\log(1/c_l(\gamma))} \leq c_g(\gamma) \leq \frac{\kappa}{\log(1/c_l(\gamma))} \tag{2.9}$$

for any arc $\gamma \subset \overline{U}_2$. By considering the transfinite diameter of any arc $\gamma \subset \overline{U}_2$ for the logarithmic kernel it is readily seen that $c_l(\gamma) \geq d(\gamma)/4$, where $d(\gamma)$ is the diameter of γ (cf. [15]). If $\overline{\gamma}$ is a cross cut of the annulus $U_2 \setminus \overline{U}_1$, then, since $d(\gamma) \geq \rho/3$, we finally conclude that $c_l(\overline{\gamma}) \geq \rho/12$ for any cross cut $\overline{\gamma}$ of $U_2 \setminus \overline{U}_1$. Hence with the first inequality of (2.9) we deduce

$$c_g(\overline{\gamma}) \geq \frac{1}{\kappa \log(12/\rho)} \tag{2.10}$$

for any cross cut $\overline{\gamma}$ of the annulus $U_2 \setminus \overline{U}_1 = \{\rho/3 < |z| < 2\rho/3\}$ with $0 < \rho < 1/4$, which will be used in a moment below.

As for the proof of Proposition 1.2, only the implication from (1.10) to (1.14) is left to be proved. We will prove the stronger version of this: the implication from (2.3) to (1.14) for an arbitrarily chosen and then fixed admissible continuum K in S. The condition (2.3) of course contains implicitly the requirement that $K \cap \gamma_n = \emptyset$ for every sufficiently large $n \in \mathbb{N}$ and thus, without loss of generality, we can assume that $K \cap \gamma_n = \emptyset$ for every $n \in \mathbb{N}$. Suppose contrariwise that (1.14) is not the case, i.e. there is a bounded subsequence $(\delta(\gamma_n))_{n\in\mathbb{N}'}$ of $(\delta(\gamma_n))_{n\in\mathbb{N}}$, where $\mathbb{N}'$ is a cofinal subset of $\mathbb{N}$. This means that there is a point $p \in S$ and $c_n \in \gamma_n$ $(n \in \mathbb{N}'')$ such that $c_n \to p$ $(n \in \mathbb{N}'', n \to +\infty)$, where $\mathbb{N}''$ is a cofinal subset of $\mathbb{N}'$ and thus of $\mathbb{N}$. Take a parametric disc $U := \{|z| < \rho\}$ $(0 < \rho < 1/4)$ at p and set $U_j := \{|z| < j\rho/3\}$ $(j = 1, 2)$ as above. Once more choose a cofinal subset $\mathbb{N}'''$ of $\mathbb{N}''$ such that $c_n \in U_1$ and one of a_n and b_n is in the outside of U and hence of U_2 $(n \in \mathbb{N}''')$. Then we can choose a subarc $\overline{\gamma}_n$ of γ_n for each $n \in \mathbb{N}'''$ such that $\overline{\gamma}_n$ is a cross cut of the annulus $U_2 \setminus \overline{U}_1$. Then by (2.7) and (2.10) we conclude that

$$\operatorname{cap}(\gamma_n, S \setminus K) \geq \operatorname{cap}(\overline{\gamma}_n, S \setminus K) \geq 2\pi c_g(\overline{\gamma}_n) \geq \frac{1}{\kappa \log(12/\rho)} > 0 \qquad (n \in \mathbb{N}'''),$$

which contradicts (2.3). Hence we have the following result.

Claim 2.3 The condition (2.3) implies the condition (1.14). In particular, the condition (1.10) implies the condition (1.14).

Putting Claims 2.1, 2.2, and 2.3 together assures the validity of Proposition 1.2. This also proves that the condition (2.3) does not depend upon the choice of admissible continuum K but cannot deduce the same for (2.4). We append here a direct proof for that conditions (2.3) and (2.4) do not depend on the choice of an admissible continuum K in S. In fact, take arbitrary admissible continua K and K' in S. We will show under the condition (2.3) for K there is a finite constant $M = M_{K,K'} > 0$ such that

$$\operatorname{cap}(\gamma_n,\, S \setminus K') \leq M \operatorname{cap}(\gamma_n,\, S \setminus K) \tag{2.11}$$

for every large $n \in \mathbb{N}$. Then the validity of (2.3) ((2.4), resp.) for K implies that of (2.3) ((2.4), resp.) for K'. Fix a reference point z_0 in K for δ in (1.14) and suppose (2.3) is valid for K, which also follows from assuming (2.4) for K. By Claim 2.3 we have (1.14) and $K' \cap \gamma_n = \emptyset$ for $n \in \mathbb{N}' := \{n \in \mathbb{N} \colon n \geq n'\}$ with n' a sufficiently large integer. Let Ω be a regular subregion of S such that $K \cup K' \subset \Omega$ and $\gamma_n \cap \overline{\Omega} = \emptyset$ $(n \in \mathbb{N}')$. Let w_n (w'_n, resp.) be the extremal (i.e. capacitary) function for $\operatorname{cap}(\gamma_n,\, S \setminus K)$ ($\operatorname{cap}(\gamma_n,\, S \setminus K')$, resp.) for $n \in \mathbb{N}'$ and w be the extremal function for $\operatorname{cap}(S \setminus \Omega,\, S \setminus K)$. Fix an arbitrary $\lambda \in (0,\, 1)$ such that $\{w \leq \lambda\} \supset K \cup K'$. Since $w_n < w$ on Ω, $\{w_n \leq \lambda\} \supset K'$. Consider

$$u_n(z) = \begin{cases} \dfrac{w_n(z) - \lambda}{1 - \lambda} & (z \in \{w_n \geq \lambda\}), \\ 0 & (z \in \{w_n \leq \lambda\}). \end{cases}$$

Then u_n ($1 - u_n$, resp.) is the extremal function for $\operatorname{cap}(\gamma_n,\, S \setminus \{w_n \leq \lambda\})$ ($\operatorname{cap}(\{w_n \leq \lambda\},\, S \setminus \gamma_n)$, resp.). Since $\operatorname{cap}(\gamma_n,\, S \setminus \{w_n \leq \lambda\}) = \operatorname{cap}(\{w_n \leq \lambda\},\, S \setminus \gamma_n)$ and $\{w_n \leq \lambda\} \supset K'$, the monotoneity of the capacity (or the Dirichlet principle) implies that

$$\begin{aligned} &\operatorname{cap}(\gamma_n,\, S \setminus K') \leq \operatorname{cap}(\gamma_n,\, S \setminus \{w_n \leq \lambda\}) \\ &= D(u_n,\, S \setminus (\gamma_n \cup \{w_n \leq \lambda\}) = (1 - \lambda)^{-2} D(w_n,\, \{\lambda < w_n < 1\}) \\ &< (1 - \lambda)^{-2} D(w_n,\, S \setminus (K \cup \gamma_n)) = (1 - \lambda)^{-2} \operatorname{cap}(\gamma_n,\, S \setminus K) \end{aligned}$$

for every $n \in \mathbb{N}'$, i.e. we have shown the validity of (2.11) with $M = (1 - \lambda)^{-2}$.

Claim 2.4 The conditions (2.3) ((2.4), resp.) is valid for any admissible continuum $K \subset S$ if and only if (2.3) ((2.4), resp.) is valid for one admissible continuum $K \subset S$. □

3. Proof of Proposition 1.1

Under the assumption that Γ is monotonically disposed toward ∞ we are to prove the validity of (1.10) can be deduced from the postulation that $S \in O_G$, i.e. S_Γ is parabolic. We use freely without further explanation notations used to introduce the notion of monotonical disposedness toward ∞ given just in the preceding paragraph before Proposition 1.1 is stated. We will deduce (1.10) by contradiction. We thus assume erroneously that (1.10) is invalid. By Proposition 1.2 we know that (1.10) is equivalent to (1.14) and therefore it amounts to the same that we are assuming the invalidity of (1.14) so that we can find a cofinal subnet $\mathbb{N}'$ of $\mathbb{N}$ such that

$$\lim_{n' \in \mathbb{N}',\, n' \to \infty} \delta(\gamma_{n'}) < +\infty \tag{3.1}$$

exists. Based on the fact that Γ is monotonically disposed toward ∞, the simply connected subregion G of S bounded by the simple curve $-\alpha + \gamma_0 + \beta$ is not relatively compact in S and the relative boundary $\partial_S G$ of G with respect to S is

$$\partial_S G = -\alpha + \gamma_0 + \beta.$$

We denote by G_n the Jordan subregion of S bounded by the Jordan curve $-\alpha(0, n) + \gamma_0 + \beta(0, n) - \gamma_n$ so that $\partial_S G_n = -\alpha(0, n) + \gamma_0 + \beta(0, n) - \gamma_n$ and

$$G_1 \subset G_2 \subset \cdots \subset G_n \subset G_{n+1} \subset \cdots .$$

We denote by F_n the complement of $\overline{G_n} \setminus \gamma_n$ with respect to $\overline{G}$:

$$F_n := (\overline{G} \setminus \overline{G_n}) \cup \gamma_n \quad (n \in \mathbb{N}).$$

Then F_n is a continuum in S and

$$F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$$

so that we see that the set

$$F := \bigcap_{n \in \mathbb{N}} F_n$$

is again a continuum in S unless it is empty. It is obvious that $F \subset \overline{G}$ but in fact we have

$$F \subset G. \tag{3.2}$$

If this is not the case, since $\overline{G} = G \cup (\alpha \cup \beta \cup \gamma_0)$, we can find a point $a \in F \cap (\alpha \cup \beta \cup \gamma_0)$. Then $a \in \alpha(0, m) \cup \beta(0, m) \cup \gamma_0$ for a sufficiently large $m \in \mathbb{N}$ and hence $a \in F \cap (\alpha(0, m) \cup \beta(0, m) \cup \gamma_0)$. On the other hand, $F_n \cap (\alpha(0, m) \cup \beta(0, m) \cup \gamma_0) = \emptyset$ for every $n > m$ and therefore $F \cap (\alpha(0, m) \cup \beta(0, m) \cup \gamma_0) = \emptyset$, which contradicts the existence of an $a \in F \cap (\alpha(0, m) \cup \beta(0, m) \cup \gamma_0) = \emptyset$, and thus (3.2) is here established. For any arbitrarily chosen $n \in \mathbb{N}$ we have

$$\gamma_{n'} \subset F_{n'} \subset F_n$$

for every $n' \in \mathbb{N}'$ with $n' > n$ and a fortiori we have

$$\delta(F_n) \leq \delta(F_{n'}) \leq \delta(\gamma_{n'}) \quad (n' \in \mathbb{N}',\, n' > n)$$

and from this relation we deduce

$$\limsup_{n\to\infty} \delta(F_n) \leq \limsup_{n'\in\mathbb{N}', n'\to\infty} \delta(F_{n'}) \leq \lim_{n'\in\mathbb{N}', n'\to\infty} \delta(\gamma_{n'}) < +\infty.$$

Hence we can find a point z_n in F_n for each $n \in \mathbb{N}$ such that the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded in the sense of δ: there is a finite positive number B such that $\delta(z_n) \leq B$ for every $n \in \mathbb{N}$. Thus the sequence $(z_n)_{n\in\mathbb{N}}$ is contained in the compact subset $\{z \in S\colon E(z, z_0) \leq \log B\}$ of S and therefore we can find a cofinal subnet $\mathbb{N}''$ of N such that the sequence $(z_{n''})_{n''\in\mathbb{N}''}$ converges to a point $b \in S$. For any F_m we have $z_{n''} \in F_{n''} \subset F_m$ for every $n'' > m$ with $n'' \in \mathbb{N}''$ and then $(z_{n''})_{n''\in\mathbb{N}'', n''>m} \subset F_m$, which implies that $b \in F_m$ for every $n'' > m$ so that $b \in F = \bigcap_{m\in\mathbb{N}} F_m$. By (3.2), we see that $b \in F \subset G$. Next we assert that F accumulates at ∞, the point at infinity of S. If this is not the case, then there is a regular subregion Ω of S such that $\Omega \supset F = \bigcap_{n\in\mathbb{N}} F_n$. Viewing this as $\Omega \supset F = F \cap \Omega = \bigcap_{n\in\mathbb{N}}(F_n \cap \Omega)$ and observing that $b \in F \subset F_n \cap \Omega$ for every $n \in \mathbb{N}$, we see that

$$F_n \cap \Omega \neq \emptyset \quad (n \in \mathbb{N}).$$

Viewing $\Omega \supset F = \bigcap_{n\in\mathbb{N}} F_n$ as $\Omega \cap \overline{\Omega} \supset \bigcap_{n\in\mathbb{N}}(F_n \cap \overline{\Omega})$, we see that

$$\partial\Omega = \overline{\Omega} \setminus (\Omega \cap \overline{\Omega}) \subset \overline{\Omega} \setminus \bigcap_{n\in\mathbb{N}} (F_n \cap \overline{\Omega}) = \bigcup_{n\in\mathbb{N}} (\overline{\Omega} \setminus (F_n \cap \overline{\Omega})).$$

Here $\partial\Omega$ is compact and $\overline{\Omega} \setminus (F_n \cap \overline{\Omega})$ is open in the space $\overline{\Omega}$ and therefore there must exist an $m \in \mathbb{N}$ such that

$$\partial\Omega \subset \bigcup_{n \leq m} \left(\overline{\Omega} \setminus (F_n \cap \overline{\Omega})\right) = \overline{\Omega} \setminus (F_m \cap \overline{\Omega}).$$

Hence we obtain that

$$\Omega = \overline{\Omega} \setminus \partial\Omega \supset \overline{\Omega} \setminus \left(\overline{\Omega} \setminus (F_m \cap \overline{\Omega})\right) = F_m \cap \overline{\Omega} \supset F_m \cap \partial\Omega,$$

which shows that $F_m \cap \partial\Omega = \emptyset$. Clearly F_m accumulates at ∞, the point at infinity of S, and hence $F_m \cap (S \setminus \Omega) \neq \emptyset$. Since F_m is connected, $F_m \cap \Omega \neq \emptyset$ and $F_m \cap (S \setminus \Omega) \neq \emptyset$ have to imply that $F_m \cap \partial\Omega \neq \emptyset$, contradicting the above conclusion $F_m \cap \partial\Omega = \emptyset$. Thus F accumulates at ∞, which with $b \in F$ assures that F is a nondegenerate continuum contained in G. Thus

$$G \setminus F \subset S \tag{3.3}$$

is a simply connected subregion of S with

$$\partial_S(G \setminus F) = (-\alpha + \gamma_0 + \beta) \cup \partial_S F,$$

where $-\alpha + \gamma_0 + \beta$ is a simple arc in S disjoint from $\partial_S F$ and these two sets are nondegenerate continua so that every point of the relative boundary $\partial_S(G \setminus F)$ of $G \setminus F$ relative to S is regular with respect to the Dirichlet problem on $G \setminus F$. We take the relative boundary values φ on $\partial_S(G \setminus F)$ given by

$$\varphi(\zeta) = \begin{cases} 0 & (\zeta \in \alpha \cup \beta \cup \gamma_0), \\ 1 & (\zeta \in \partial_S F). \end{cases}$$

The solution $\omega := H_\varphi^{G \setminus F}$ of the Dirichlet problem on $G \setminus F$ with the above boundary function φ satisfies that $0 < \omega < 1$ on $G \setminus F$ and takes the boundary values 0 on $-\alpha + \gamma_0 + \beta$ and 1 on $\partial_S F$.

Thus far we have been considering $G \setminus F$ as a subregion of S (cf. (3.3)). We next show that the same surface $G \setminus F$ may be embedded naturally in S_Γ so that it is considered this time as a subregion of S_Γ. We consider the subregion $G_{n-1,n}$ of S bounded by the Jordan curve $-\alpha(n-1, n) + \gamma_{n-1} + \beta(n-1, n) - \gamma_n$. We view that $G_{n-1,n}$ is a subregion of S_n and any one of arcs $\alpha(n-1, n)$, $\beta(n-1, n)$, γ_{n-1}, and γ_n are embedded in S_n so that

$$\partial_{S_\Gamma} G_{n-1,n} = \partial_{S_n} G_{n-1,n} = -\alpha(n-1, n) + \gamma_{n-1} + \beta(n-1, n) - \gamma_n.$$

We also view that

$$G_n = G_{0,1} \cup \gamma_1 \cup G_{1,2} \cup \gamma_2 \cup \cdots \cup \gamma_{n-1} \cup G_{n-1,n} \subset W_n \quad (n \in \mathbb{N})$$

and as a result of this we see that

$$G \setminus F = G \setminus \bigcap_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} (G \setminus F_n) = \bigcup_{n\in\mathbb{N}} G_n \subset \bigcup_{n\in\mathbb{N}} W_n = S_\Gamma$$

so that $G \setminus F$ can also be considered as a relatively noncompact subregion of S_Γ (cf. (3.3)):

$$G \setminus F \subset S_\Gamma. \tag{3.4}$$

Moreover, by regarding $\alpha = \sum_{n\in\mathbb{N}} \alpha(n-1,\, n) \subset S_\Gamma$ and $\beta = \sum_{n\in\mathbb{N}} \beta(n-1,\, n) \subset S_\Gamma$, we see that

$$\partial_{S_\Gamma}(G \setminus F) = -\alpha + \gamma_0 + \beta.$$

Therefore $\omega \in HB(G \setminus F; \partial_{S_\Gamma}(G \setminus F))$, the class of bounded functions harmonic on $G \setminus F$ with vanishing relative boundary values on $\partial_{S_\Gamma}(G \setminus F)$, and $\omega > 0$ on $G \setminus F$. Hence we can conclude that $G \setminus F \notin SO_{HB}$, where SO_{HB} is the family of subregions Y of some Riemann surfaces X with $HB(Y; \partial_X Y) = \{0\}$. The fact that S_Γ contains a subregion $G \setminus F$ not contained in SO_{HB} is known to be equivalent to that $S_\Gamma \notin O_G$, i.e. S_Γ is hyperbolic (cf. [12]). This contradicts the assumption $S_\Gamma \in O_G$ of our Proposition 1.1 so that we are done. □

4. Fatou mappings

From now on we start the proof of our main theorem in this paper. A part of the proof contains some general recognition, which, we belive, is worth discussing independently and led to a result of a certain independent interest in its own right. Hence we present here and also in the next section the above part of the proof in a slightly wider frame than really needed.

An analytic mapping f of a Riemann surface X to a Riemann surface Y is said to be a *Fatou mapping* (cf. [3]) of X to Y if f can be extended to a continuous mapping of the Wiener compactification of X to the Wiener compactification of Y (cf. [3], [12]). If both of X and Y are simultaneously hyperbolic or parabolic, then any analytic mapping of X to Y is a Fatou mapping. The proof for the case both of X and Y are parabolic is straightforward but that for the case both of X and Y are hyperbolic requires a considerable elaboration. Far from considering the question whether a given analytic mapping being Fatou or not, there is even no nontrivial analytic mappings themselves f of X to Y if X is parabolic and Y is hyperbolic. In fact, assuming the nontriviality of f, $f(X)$ is a hyperbolic Riemann surface along with Y. Viewing $f\colon X \to f(X)$ as a covering surface $(X, f(X), f)$, by the effect of the base hyperbolicity remarked in Introduction, we see that the hyperbolicity of the base surface $f(X)$ should imply that of X, contradicting the parabolicity assumption of X. Thus the essential case concerning Fatou mappings in which some careful thoughts are required is when X is hyperbolic and Y is parabolic. In this case we have the following characterization of Fatou mappings due to Constantinescu and Cornea (see [3]):

Fatou map criterion *An analytic mapping f of a hyperbolic Riemann surface X to a parabolic Riemann surface Y is a Fatou mapping if and only if there exists a nonpolar closed subset E of Y such that the balayage $\widehat{\mathbf{R}}_1^{f^{-1}(E),X}$ of the constant superharmonic function 1 in X with respect to the closed subset $f^{-1}(E)$ of X is a potential on X.*

Here the balayage $\widehat{\mathbf{R}}_s^{K,X}$ of a positive superharmonic function s on a hyperbolic Riemann surface X in X with respect to a closed subset K of X is the infimum of the class of positive superharmonic functions which are not less than s quasi everywhere on K. A potential on X is, by definition, a positive superharmonic function whose greatest harmonic minorant is zero.

In this section we are concerned with analytic mappings π which are projections of covering surfaces (X, Y, π). We will show that if X is hyperbolic and Y is parabolic and X covers Y rather regularly in some sense specified below, then π can never be a Fatou mapping. Suppose the given covering surface (X, Y, π) satisfies that $X \notin O_G$, $Y \in O_G$, and it is infinitely sheeted. For each $a \in Y$ and $0 < r < 1$ we say that $\{y \in Y\colon |z(y)| < r\}$ ($\{y \in Y\colon |z(y)| \le r\}$, resp.) is a disc (closed disc, resp.) with radius r centered at a and denoted by $\Delta(a, r)$ ($\overline{\Delta}(a, r)$, resp.), where z is a local parameter at a so that $z(a) = 0$ and $\{y \in Y\colon |z(y)| < 1\}$ is its parametric disc at a. Strictly speaking, $\Delta(a, r)$ and $\overline{\Delta}(a, r)$ depend upon the choice of local parameter z and thus we assume such a z is fixed in advance when we consider $\Delta(a, r)$ and $\overline{\Delta}(a, r)$. We say that a subset $K \subset X$ is a *ν-sheeted disc* (*ν-sheeted closed disc,* resp.) over $\Delta(a, r)$ ($\overline{\Delta}(a, r)$, resp.) centered at the point $b \in X$ if K is a component of $\pi^{-1}(\Delta(a, r))$ ($\pi^{-1}(\overline{\Delta}(a, r))$, resp.)

and there exists a point $b \in X$ and a local parameter ζ at b with $\pi(b) = a$, $\zeta(b) = 0$, and the parametric disc of ζ at b is $\{x \in X : |\zeta(x)| < 1\}$ such that

$$K = \{x \in X : |\zeta(x)| \leq r^{1/\nu}\} \quad (\nu \in \mathbb{N})$$

and the local expression of $\pi\colon K \to \Delta(a, r)$ ($\pi\colon K \to \overline{\Delta}(a, r)$, resp.) in terms of ζ and z is given by

$$z = \pi(\zeta) = \zeta^\nu.$$

Here if and only if $\nu > 1$, then the point $b \in X$ is a branch point of the covering surface (X, Y, π) with multiplicity $\nu > 1$.

Let B be the set of projections of branch points of the covering surface (X, Y, π) and $\overline{B}$ be, as usual, the closure of B in Y. The covering surface is referred to as being *sparsely branched* if $\overline{B}$ is polar in Y. By the assumption (1.1), in our covering surfaces (S_Γ, S, π) the set B is an isolated set in S so that $\overline{B} = B$ is still isolated and a fortiori polar. Hence our covering surfaces (S_Γ, S, π) are sparsely branched.

We say that the covering surface (X, Y, π) is *complete* (cf. [1]) if for every $a \in Y$ there is a closed disc $\overline{\Delta}(a, r)$ around a such that each component of $\pi^{-1}(\overline{\Delta}(a, r))$ is compact in X. When (X, Y, π) is smooth, i.e. there is no branch point in X so that $B = \emptyset$, the completeness of (X, Y, π) is equivalent to the regularity of (X, Y, π), where (X, Y, π) is regular if, for any arc γ in Y and any point $\tilde{a} \in X$ lying over the initial point a of γ, there always exists a continuation $\tilde{\gamma}$ on X along γ starting from $\tilde{a}$, i.e. there is an arc $\tilde{\gamma}$ on X with initial point $\tilde{a}$ such that $\pi(\tilde{\gamma}) = \gamma$ (cf. [1]). The covering surfaces (S_Γ, S, π) with (1.10) are clearly complete. For a role played by the completeness in the type problem, we refer to [9] and [7].

Suppose the covering surface (X, Y, π) is complete and $Y \setminus \overline{B} \neq \emptyset$, which is the case e.g. if the complete (X, Y, π) is moreover sparsely branched. Then to each $a \in Y \setminus \overline{B}$ there is a closed disc $\overline{\Delta(a, r)} = \{y \in Y : |z(y)| \leq r\}$ ($z(a) = 0$) contained in $Y \setminus \overline{B}$ such that each connected component K_n of $\pi^{-1}(\overline{\Delta}(a, r))$ in its decomposition into connected components

$$\pi^{-1}(\overline{\Delta}(a, r)) = \bigcup_{n \in \mathbb{N}} K_n \tag{4.1}$$

is a 1-sheeted closed disc over $\overline{\Delta}(a, r)$ centered at $b_n \in X$ due to the monodromy theorem so that $K_n = \{x \in X : |\zeta(x)| \leq r\}$ ($\zeta(b_n) = 0$) and the local expression of $\pi\colon K_n \to \overline{\Delta}(a, r)$ is given by $z = \pi(\zeta) = \zeta$. Hence, by

$\pi(K_n) = \overline{\Delta}(a, r)$, we can identify K_n with $\overline{\Delta}(a, r)$:

$$K_n = \overline{\Delta}(a, r) \quad (n \in \mathbb{N}) \tag{4.2}$$

in the sense of conformal equivalence. Thus considering a function h on the set $\pi^{-1}(\overline{\Delta}(a, r))$ and considering a family $\{h_n : n \in \mathbb{N}\}$ of functions h_n on $\overline{\Delta}(a, r)$ related by $h_n = h|K_n$ or more precisely by $h_n = (h|K_n) \circ \pi^{-1}$ for every $n \in \mathbb{N}$ amount to the same. The closed discs $\overline{\Delta}(a, r)$ such as those chosen above will be referred to as *distinguished* closed discs at a. If $\overline{\Delta}(a, r)$ is distinguished at a, then clearly so is every $\overline{\Delta}(a, s)$ $(0 < s \le r)$.

The following result not only plays a decisive role in our proof of the main theorem but also may have a certain independent and general interest:

Theorem 4.1 *Suppose that the covering surface X is hyperbolic and the base surface Y is parabolic in the infinitely sheeted covering surface (X, Y, π) with its projection π. If (X, Y, π) is complete and sparsely branched, then the projection π as the analytic mapping $\pi\colon X \to Y$ can never be a Fatou mapping.*

We will use the above result in the following restricted situation for the proof of our main theorem:

Corollary to Theorem 4.1 *In the covering surface (S_Γ, S, π) with (1.10) (for which with (1.11) suffices), the hyperbolicity of S_Γ implies that the projection π as the analytic mapping $\pi\colon S_\Gamma \to S$ is not a Fatou mapping.*

5. Proof of Theorem 4.1

Our proof of Theorem 4.1 is by contradiction so that we assume the projection $\pi\colon X \to Y$ is a Fatou mapping. By the Fatou map criterion mentioned in §4, making the above assumption and assuming the existence of a nonpolar closed subset E in Y such that the balayage $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),X}$ is a potential on X amount to the same. Here we remark that we can moreover assume that the existence of a distinguished closed disc $\overline{\Delta}(a, r(a))$ at some point $a \in Y \setminus \overline{B}$ such that $E \subset \Delta(a, r(a))$ by choosing a suitable subset of E as new E if necessary. To each $c \in Y \setminus \overline{B}$ fix a distinguished closed disc $\overline{\Delta}(c, t(c))$ at c contained in $Y \setminus \overline{B}$ and observe that

$$E \setminus \overline{B} \subset \bigcup_{c \in E \setminus \overline{B}} \Delta\Big(c, \frac{t(c)}{2}\Big).$$

By the Lindelöf covering theorem there is a countable subset $\{c_n\}_{n\in\mathbb{N}}$ of the set $E \setminus \overline{B}$ such that

$$E \setminus \overline{B} \subset \bigcup_{n\in\mathbb{N}} \Delta\Big(c_n,\, \frac{t(c_n)}{2}\Big).$$

On setting $E_n := E \cap \overline{\Delta}(c_n,\, t(c_n)/2)$ $(n \in \mathbb{N})$, which are all closed (and in fact compact), we have

$$E = \overline{B} \cup \Big(\bigcup_{n\in\mathbb{N}} E_n\Big).$$

Since E is nonpolar and $\overline{B}$ is polar, at least one of the members in $\{E_n \colon n \in \mathbb{N}\}$, say E_n, must be nonpolar. We put $E' := E_n$, $a := c_n$, and $r(a) := t(a) = t(c_n)$. Then $E' \subset \Delta(a,\, r(a))$ and E' is a nonpolar closed subset of E in Y. We denote by D_m the interior of K_m $(m \in \mathbb{N})$, where

$$\pi^{-1}(\overline{\Delta}(a, r(a))) = \bigcup_{m\in\mathbb{N}} K_m$$

is the decomposition of $\pi^{-1}(\overline{\Delta}(a,\, r(a)))$ into connected components K_m, closed discs in X. Since $E'_m := \pi^{-1}(E') \cap K_m$ is a nonpolar compact subset of D_m, there is a regular point in E'_m with respect to the Dirichlet problem on $X \setminus \pi^{-1}(E')$. Then $\widehat{\mathbf{R}}_1^{\pi^{-1}(E'),X} | (X \setminus \pi^{-1}(E'))$ is the Dirichlet solution on $X \setminus \pi^{-1}(E')$ with boundary values 1 on those boundary points in $\pi^{-1}(E')$ and 0 at the ideal boundary of X. Therefore we see that

$$0 < \widehat{\mathbf{R}}_1^{\pi^{-1}(E'),X} \le \widehat{\mathbf{R}}_1^{\pi^{-1}(E),X}$$

on X. That $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),X}$ is a potential implies that $\widehat{\mathbf{R}}_1^{\pi^{-1}(E'),X}$ is a potential. Thus we may start our proof of Theorem 4.1 from the following erroneous assumption and we are to derive a contradiction.

Assumption 5.1 There exists a distinguished closed disc $\overline{\Delta}(a,\, r(a)) \subset Y \setminus \overline{B}$ and a nonpolar compact set $E \subset \Delta(a,\, r(a))$ such that

$$u := \widehat{\mathbf{R}}_1^{\pi^{-1}(E),X} \tag{5.1}$$

is a potential on X, i.e. a positive superharmonic function whose greatest harmonic minorant is zero on X.

The mother function u on X in (5.1) above will give birth to a child function v on Y given below which will be raised by the assistance of a nurse function u_0 on Y also given below. First of all note that $0 < u < 1$ on $X \setminus \pi^{-1}(E)$ and $0 < u \leq 1$ on X. Consider a function v on X given by

$$v(z) := \inf_{\zeta \in \pi^{-1}(z)} u(\zeta) \quad (z \in Y).. \tag{5.2}$$

In addition to the mother function (5.1) we consider the nurse function u_0 on Y given by

$$u_0 := \begin{cases} \widehat{\mathbf{R}}_1^{E, \Delta(a, r(a))} & \text{on } \overline{\Delta}(a,\, r(a)), \\ 0 & \text{on } Y \setminus \overline{\Delta}(a,\, r(a)). \end{cases} \tag{5.3}$$

In order to study v above it is convenient to prepare the following simple lemma. Let W be a plane region and

$$\mathcal{H} := \{h_n \in H(W) \colon |h_n| \leq M \ (n \in \mathbb{N})\}$$

be a countably infinite family, where $H(W)$ is a class of harmonic functions on W and M a positive constant. We do not require any standard usual conditions on $\mathcal{H}$ such as directedness and the like other than its uniform boundedness. We define a function h on W by

$$h(z) := \inf_{n \in \mathbb{N}} h_n(z) \quad (z \in W). \tag{5.4}$$

Lemma 5.1 *The function h is a continuous superharmonic function on W.*

The uniform boundedness of $\mathcal{H}$ implies its equicontinuity (cf. e.g. [2], [14]) and hence, for any point $w \in W$ and any positive number $\varepsilon > 0$ there is an open neighborhood V of w such that the oscillations of functions in $\mathcal{H}$ on V are less than $\varepsilon/2$:

$$\sup_{j \in \mathbb{N}} \Big(\sup_{w_1, w_2 \in V} |h_j(w_1) - h_j(w_2)| \Big) < \frac{\varepsilon}{2}. \tag{5.5}$$

First we show the *continuity* of h on V. Fix an arbitrary $w' \in V$. By the definition (5.4) of h, there is an $n \in \mathbb{N}$ and an $m \in \mathbb{N}$ such that

$$h(w) + \frac{\varepsilon}{2} > h_n(w), \qquad h(w') + \frac{\varepsilon}{2} > h_m(w').$$

Hence by (5.5)

$$h(w) + \frac{\varepsilon}{2} > h_n(w) > h_n(w') - \frac{\varepsilon}{2} \geq h(w') - \frac{\varepsilon}{2}$$

so that we deduce that $h(w) - h(w') > -\varepsilon$. Similarly

$$h(w') + \frac{\varepsilon}{2} > h_m(w') > h_m(w) - \frac{\varepsilon}{2} \geq h(w) - \frac{\varepsilon}{2}$$

so that we conclude that $h(w') - h(w) > -\varepsilon$ or $h(w) - h(w') < \varepsilon$. Thus we have shown that for any point $w \in W$ and any positive number $\varepsilon > 0$ there is a neighborhood $V \subset \overline{V} \subset W$ of w such that $|h(w) - h(w')| < \varepsilon$ for any $w' \in V$, which shows that h is continuous on W.

Next we show the *superharmonicity* of h on W. Fix an arbitrary point $w \in W$ and a closed disc $\{w + re^{i\theta} \colon 0 \leq r \leq t,\ 0 \leq \theta \leq 2\pi\} \subset W$, where t is an arbitrary positive number less than the distance between w and the boundary ∂W of W. Thus

$$h_n(w) = \frac{1}{2\pi} \int_0^{2\pi} h_n(w + re^{i\theta}) d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} h(w + re^{i\theta}) d\theta$$

for every $n \in \mathbb{N}$ and every $0 < r < t$. Taking the infimum of $h_n(w)$ with respect to $n \in \mathbb{N}$ we deduce the super mean value property

$$h(w) \geq \frac{1}{2\pi} \int_0^{2\pi} h(w + re^{i\theta}) d\theta \quad (0 < r < t),$$

which assures the superharmonicity of h, and thus the above lemma has been completely proven.

As a direct consequence of the above lemma 5.1 we have the following

Claim 5.1 The function v defined by (5.2) is a continuous superharmonic function on $Y \setminus (E \cup \overline{B})$.

Choose an arbitrary point $c \in Y \setminus (E \cup \overline{B})$ and a distinguished closed disc $\overline{\Delta}(c, r)$ at c with $\overline{\Delta}(c, r) \subset Y \setminus (E \cup \overline{B})$ so that we have (4.1) and (4.2) with a replaced by c. Set $u_m := u|K_m$ and we view that $\mathcal{H} := \{u_n \colon n \in \mathbb{N}\}$ is a uniformly bounded (i.e. $0 < u_n < 1$) family of harmonic functions u_n on $\Delta(c, r)$ and

$$v(z) = \inf_{n \in \mathbb{N}} u_n(z) \quad (z \in \Delta(c, r)).$$

We see, by the above lemma, that v is continuous and superharmonic on $\Delta(c, r)$. The arbitrariness of $\Delta(c, r)$ assures the validity of the Claim 5.1.

We deform v on Y to a new function $\widehat{v}$ on Y as follows. Observe that

$Y \setminus \overline{B}$ is dense in Y since $\overline{B}$ is polar. Hence we can define

$$\widehat{v}(z) = \begin{cases} v(z) & (z \in Y \setminus \overline{B}), \\ \liminf\limits_{w \in Y \setminus \overline{B}, w \to z} v(w) & (z \in \overline{B}). \end{cases} \tag{5.6}$$

Claim 5.2 The function $\widehat{v}$ defined by (5.6) above is a superharmonic function on $Y \setminus E$ and $0 \leq \widehat{v} \leq v \leq 1$ on Y.

From $v \circ \pi \leq u$ on Y and the continuity of v on $Y \setminus (E \cup \overline{B})$ and that of u on $X \setminus \pi^{-1}(E)$ it follows that $\widehat{v} \circ \pi \leq u$ on X and a fortiori $\widehat{v} \circ \pi \leq v \circ \pi$ on X. Therefore $0 \leq \widehat{v} \leq v \leq 1$ on Y. In particular v is bounded from below on $Y \setminus (E \cup \overline{B})$ and $\overline{B}$ is polar. Hence the first part of the above claim is nothing but the standard superharmonic extention theorem (cf. [5]).

Let $\overline{\Delta}(a, r(a))$ be the distinguished closed disc in Assumption 5.1, K_n ($n \in \mathbb{N}$) be the closed disc in (4.1) and (4.2) with $\overline{\Delta}(a, r)$ replaced by $\overline{\Delta}(a, r(a))$ and $u_n := u|K_n$ ($n \in \mathbb{N}$) be viewed as functions on $\overline{\Delta}(a, r(a))$. Since $u_0 \leq u_n$ ($n \in \mathbb{N}$) on $\overline{\Delta}(a, r(a))$, we see that

$$u_0 \leq \widehat{v} = v \leq u_n \quad (n \in \mathbb{N}) \tag{5.7}$$

on $\overline{\Delta}(a, r(a))$. We denote by E_1 (E_0, resp.) the set of regular (irregular, resp.) points with respect to the Dirichlet problem for the region $\Delta(a, r(a)) \setminus E$ in $\partial(\Delta(a, r(a)) \setminus E) \setminus \partial\Delta(a, r(a))$. The nonpolarity of E assures that $E_1 \neq \emptyset$ and, needless to say, E_0 is a polar set. Let φ (φ_0, resp.) be the boundary function for $X \setminus \pi^{-1}(E)$ ($\Delta(a, r(a)) \setminus E$, resp.) such that $\varphi = 1$ ($\varphi_0 = 1$, resp.) on the boundary in $\pi^{-1}(E)$ (E, resp.) and $\varphi = 0$ ($\varphi_0 = 0$, resp.) at the ideal boundary of X ($\partial\Delta(a, r(a))$, resp.). Then $u_n = H_{\varphi}^{X \setminus \pi^{-1}(E)}|K_n$ ($n \in \mathbb{N}$) and $u_0 = H_{\varphi_0}^{\Delta(a, r(a)) \setminus E}$ and therefore

$$1 = \lim_{z \to w} u_0(z) \leq \liminf_{z \to w} \widehat{v}(z) \leq \limsup_{z \to w} \widehat{v}(z) \leq \lim_{z \to w} u_n(z) = 1,$$

where $z \in \Delta(a, r(a)) \setminus E$ and $w \in E_1$. Thus we see that

$$\lim_{z \in X \setminus E, z \to w} \widehat{v}(z) = 1 \quad (w \in E_1). \tag{5.8}$$

By this we can conclude that

$$0 < \widehat{v} \circ \pi(\zeta) \leq u(\zeta) < 1 \quad (\zeta \in X \setminus \pi^{-1}(E)) \tag{5.9}$$

so that we infer

$$0 < \widehat{v}(z) < 1 \quad (z \in Y \setminus E) \tag{5.10}$$

and a fortiori we have $0 < H_{\widehat{v}}^{\Delta(a,r(a))}$ on $\Delta(a, r(a))$. Since E_0 is polar, we can find a positive superharmonic function s on $\overline{\Delta}(a, r(a))$ such that $s|E_0 = +\infty$. For any positive number $\varepsilon > 0$ we see that

$$\liminf_{z \in \Delta(a,r(a)) \setminus E} \big(\widehat{v}(z) + \varepsilon s(z) - H_{\widehat{v}}^{\Delta(a,r(a))}(z)\big) \geq 0$$

for every $w \in \partial(\Delta(a, r(a)) \setminus E)$. By the comparison principle we deduce that $\widehat{v}(z) + \varepsilon s(z) - H_{\widehat{v}}^{\Delta(a,r(a))} \geq 0$ for every $z \in \Delta(a, r(a)) \setminus E$. On letting $\varepsilon \downarrow 0$ we conclude that

$$\widehat{v}(z) \geq H_{\widehat{v}}^{\Delta(a,r(a))}(z) \quad (z \in \Delta(a, r(a)) \setminus E). \tag{5.11}$$

Finally we set

$$p(z) = \begin{cases} \widehat{v}(z) & (z \in Y \setminus \Delta(a, r(a))), \\ H_{\widehat{v}}^{\Delta(a,r(a))}(z) & (z \in \Delta(a, r(a))). \end{cases} \tag{5.12}$$

Claim 5.3 The function p defined by (5.12) is a potential on Y.

Since $\widehat{v}$ is superharmonic on $Y \setminus E$ by Claim 5.2 and positive on $Y \setminus E$ by (5.10) and $H_{\widehat{v}}^{\Delta(a,r(a))}$ is positive and superharmonic (and actually harmonic) on $\Delta(a, r(a))$ with $\widehat{v} = H_{\widehat{v}}^{\Delta(a,r(a))}$ on $\partial\Delta(a, r(a))$, the inequality (5.11) assures that the new positive function p is superharmonic on Y by the pasting lemma (cf. e.g. [4]). On the other hand we see that

$$H_{\widehat{v}}^{\Delta(a,r(a))}(z) = \widehat{v}(z) = v(z) \leq u_n(z) \quad (n \in \mathbb{N})$$

for every $z \in \partial\Delta(a, r(a))$ and therefore $H_{\widehat{v}}^{\Delta(a,r(a)} \leq u_n$ on $\Delta(a, r(a))$ for every $n \in \mathbb{N}$ by the comparison principle. This shows that $H_{\widehat{v}}^{\Delta(a,r(a))} \circ \pi \leq u$ on $\pi^{-1}(\Delta(a, r(a)))$. We also have $\widehat{v} \circ \pi \leq u$ on $X \setminus \pi^{-1}(Y \setminus \Delta(a, r(a)))$ by (5.9). Hence

$$p \circ \pi(\zeta) \leq u(\zeta) \quad (\zeta \in X). \tag{5.13}$$

Choose an arbitrary nonnegative harmonic function q on Y with $0 \leq q \leq p$ on Y. Then $q \circ \pi$ is harmonic on X and $q \circ \pi \leq p \circ \pi$ on X. Thus (5.13) assures that $0 \leq q \circ \pi \leq u$ on X and, since u is a potential on X, we must conclude that $q \circ \pi = 0$ on X or $q = 0$ on Y. Thus p is a potential on Y.

Recall that Y is parabolic by the very assumption of Theorem 4.1. Since the parabolicity of Riemann surfaces is also characterized by the nonexistence of potentials on them, the existence of the potential p on Y is a contradiction and the proof of Theorem 4.1 is herewith complete. $\square$

6. Royden compactifications

Here we compile results relevant to the Royden compactification R^* of an open Riemann surface R which we use to complete the proof of our main theorem as necessary tools. The space R^* is characterized as the smallest compactification of R on which every bounded continuous Tonelli function f with finite Dirichlet integral $D(f, R) < +\infty$ are continuously extendable to R^*. It is convenient to call such an f as *Royden function* on R. A Royden function f on R is said to be a *Royden potential* on R if there is a sequence $(f_n)_{n\in\mathbb{N}}$ of Royden functions f_n with compact supports in R such that $\sup_{n\in\mathbb{N}}(\sup_R |f_n|) < \infty$ and the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f almost uniformly on R (i.e. uniformly on each compact subset of R) and at the same time $D(f-f_n, R) \to 0$ $(n \to \infty)$. The *Royden harmonic boundary* Δ of R is a part of the Royden boundary $R^* \setminus R$ consisting of points ζ such that $f(\zeta) = 0$ for every Royden potential f on R. The parabolicity of R, i.e. $R \in O_G$, is characterized by the fact that $\Delta = \emptyset$ (cf. e.g. [12, p. 158]). Given a continuum K in R such that $R \setminus K$ is connected and any Royden function f on R. Here the case $K = \emptyset$ is not excluded. Then we have the unique decomposition

$$f = h + g, \tag{6.1}$$

where h and g are Royden functions on R such that $h \in H(R \setminus K)$ and $g|\Delta \cup K = 0$ or equivalently $h = f$ on $\Delta \cup K$. The relation (6.1) is said to be a *orthogonal decomposition* or *Royden decomposition* of f (cf. [12, p. 162]). It may be impressive to call h the harmonic part of f (on $R \setminus K$) and g the potential part of f. We also need to recall the *minimum principle* of the following form (cf. [12, p. 168]). Let G be a subregion of R and s be a superharmonic function on G bounded from below. Here G may be identical with R. We denote by ∂G, as usual, the relative boundary of G with respect to R but we denote by $\overline{G}$ the closure of G taken in R^*. Then we see that $s \geq 0$ on G if (and only if)

$$\liminf_{z\in G, z\to\zeta} s(\zeta) \geq 0$$

for every $\zeta \in (\partial G) \cup (\overline{G} \cap \Delta)$. Actually we will mainly use this when $s \in H(G)$.

7. Existence of genuine potentials

We are now completing our proof of the main theorem on deriving the contradiction to the corollary to Theorem 4.1 by showing the possibility of finding a compact nonpolar subset E of S such that the balayage $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma}$ of 1 in S_Γ with respect to the closed set $\pi^{-1}(E)$ in S_Γ with $\pi = \pi_\Gamma$ is a potential on S_Γ.

Since (S_Γ, S, π) is sparsely branched, we have $\overline{S \setminus \overline{B}} = S$. Thus we can choose arbitrarily and then fix a point $a \in S \setminus \overline{B}$. Since (S_Γ, S, π) is complete, there is a distinguished closed disc $\overline{\Delta}(a, r)$ at a. As E we take the disc $\overline{\Delta}(a, r)$, i.e. $E := \overline{\Delta}(a, r)$, which is clearly a nonpolar compact subset of S. Let

$$\pi^{-1}(E) = \bigcup_{n \in \mathbb{N}} K_n$$

be the decomposition of $\pi^{-1}(E)$ into connected components $K_n \subset S_n$ ($n \in \mathbb{N}$). Then we may view that $K_n = E$ with the identification $S_n = S$. We consider functions u_n on S_Γ ($n \in \mathbb{N}$) defined as follows. Let $u_n \equiv 0$ on $S_\Gamma \setminus (S_n \setminus (\gamma_{n-1} \cup \gamma_n))$ and $u_n|(S_n \setminus (\gamma_{n-1} \cup \gamma_n))$ be given by

$$u_n \in C(S_n) \cap H(S_n \setminus (\gamma_{n-1} \cup \gamma_n \cup K_n))$$

with $u_n|K_n = 1$ and $u_n|\gamma_{n-1} \cup \gamma_n = 0$. Then clearly $u_n \in C(S_\Gamma)$. Here changing the meaning of γ_0 we mean only in this section that $\gamma_0 = \emptyset$ so that in case of u_1 it is given by

$$u_1 \in C(S_1) \cap H(S_1 \setminus (\gamma_1 \cup K_1))$$

with $u_1|K_1 = 1$ and $u_1|\gamma_1 = 0$. We simply write $D(\,\cdot\,, S_\Gamma) = D(\,\cdot\,)$. Observe that

$$\begin{aligned} D(u_n) &= D(u_n, S_n) = \operatorname{cap}(\gamma_{n-1} \cup \gamma_n, S_n \setminus K_n) \\ &= \operatorname{cap}(\gamma_{n-1} \cup \gamma_n, S \setminus E) \le \operatorname{cap}(\gamma_{n-1}, S \setminus E) + \operatorname{cap}(\gamma_n, S \setminus E), \end{aligned}$$

since $u_n|S_n$ may be identified with the capacity function for $\gamma_{n-1} \cup \gamma_n$ with respect to $S \setminus E$. Thus each u_n ($n \in \mathbb{N}$) is a Royden function with compact support in S_Γ. Using u_n $(n \in \mathbb{N})$ we set

$$v_m := \sum_{1 \le n \le m} u_n \quad (m \in \mathbb{N})$$

on S_Γ and, as the limit function of the above sequence, we set

$$v := \sum_{n \in \mathbb{N}} u_n = \lim_{m \to \infty} v_m \tag{7.1}$$

on S_Γ, where the convergences of the the above two limits on the right hand sides of the above identities are almost uniform on S_Γ. We see that

$$D(v_m) = \sum_{1 \le n \le m} D(u_n)$$
$$= \sum_{1 \le n \le m} D(u_n, S_n) \le 2 \sum_{1 \le n \le m} \mathrm{cap}(\gamma_n, S \setminus E)$$

and similarly

$$D(v) = \sum_{n \in \mathbb{N}} D(u_n) = \sum_{n \in \mathbb{N}} D(u_n, S_n) \le 2 \sum_{n \in \mathbb{N}} \mathrm{cap}(\gamma_n, S \setminus E) < +\infty.$$

Hence v_n $(n \in \mathbb{N})$ and v are Royden functions on S_Γ. Moreover $(v_m)_{m \in \mathbb{N}}$ is a uniformly bounded sequence of Royden functions with compact supports in S_Γ and converges to v almost uniformly on S_Γ. Therefore with

$$D(v - v_m) = \sum_{n \ge m} D(u_n) \le 2 \sum_{n \ge m} \mathrm{cap}(\gamma_n, S \setminus E) \to 0 \ (m \to \infty)$$

we can conclude the following.

Claim 7.1 The function v defined by (7.1) is a Royden potential on S_Γ; $v|\Delta = 0$ and $v|\pi^{-1}(E) = 1$.

We apply the Royden decomposition to v: $v = h + g$, where $h \in C((S_\Gamma)^*) \cap H(S_\Gamma \setminus \pi^{-1}(E))$ with $h|\Delta = 0$ and $h|\pi^{-1}(E) = 1$ as the result of $(v - h)|\Delta \cup \pi^{-1}(E) = g|\Delta \cup \pi^{-1}(E) = 0$ and the above Claim 7.1. Hence we can see that h is positive and superharmonic on S_Γ with $h \equiv 1$ on $\pi^{-1}(E)$. Hence it is clear that

$$\widehat{\mathbf{R}}_1^{\pi^{-1}(E), S_\Gamma} \le h \tag{7.2}$$

on S_Γ. On the other hand, since $\widehat{\mathbf{R}}_1^{\pi^{-1}(E), S_\Gamma}$ is a continuous function on S_Γ

with $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma} = 1$ on $\pi^{-1}(E)$ and harmonic on $S_\Gamma \setminus \pi^{-1}(E)$, we see that $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma} - h$ is continuous on S_Γ and a harmonic function bounded from below on $S_\Gamma \setminus \pi^{-1}(E)$ and

$$\liminf_{z\in S_\Gamma\setminus\pi^{-1}(E),z\to\zeta}\big(\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma}(z) - h(z)\big) \geq 0 \quad (\zeta \in \Delta \cup \pi^{-1}(E))$$

so that the minimum principle assures that

$$\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma} \geq h \tag{7.3}$$

on S_Γ. Putting (7.2) and (7.3) together we conclude that $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma} = h$ on S_Γ which can be rephrased as follows.

Claim 7.2 The balayaged function $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma}$ on S_Γ of the constant function 1 with respect to $\pi^{-1}(E)$ is a Royden potential on S_Γ so that $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma}$ is continuous on the Royden compactification $(S_\Gamma)^*$ of S_Γ and $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma}|\Delta = 0$.

Take an arbitrary harmonic function w on S_Γ such that $0 \leq w \leq \widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma}$ on S_Γ. Since $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma}|\Delta = 0$, we see that w is continuous on $S_\Gamma \cup \Delta = 0$ and $w|\Delta = 0$. In view of $0 \leq w \leq 1$ on S_Γ, appealing the minimum principle to $\pm w$ on S_Γ, we conclude that $\pm w \geq 0$ on S_Γ, which implies that $w \equiv 0$ on S_Γ. Therefore we have the following final result in our proof.

Claim 7.3 The balayage $\widehat{\mathbf{R}}_1^{\pi^{-1}(E),S_\Gamma}$ is a genuine potential.

The content of the above claim is nothing but the condition assuring that the projection $\pi = \pi_\Gamma\colon S_\Gamma \to S$ for the covering surface (S_Γ, S, π) is a Fatou mapping (cf. Fatou map criterion in §4). This contradicts the corollary to Theorem 4.1 in §4.

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