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A note on hemivariational inequalities

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Abstract. The purpose of this note is to establish some uniqueness results for hemivariational inequality.

Key words: hemivariational inequality, nonlinear operators, strongly monotone, Lipschitz.

1. Introduction

Verma [2] considered the following problem:
Let C be a closed subset of a reflexive Banach space X which is star-shaped with respect to a ball $B(u_0\ \rho)$. Let $g \in X^*$, $A, B\colon X \to X^*$ be nonlinear operators. Then the problem is to find $u \in C$ such that

CNHVI: $[(A - B)u - g, \vartheta] \geq 0 \quad \forall \vartheta \in T_c(u)$, where $T_c(u)$ is the Clarke's tangent cone at u of C.

$$T_c(x) = \{\vartheta \in X\colon \lfloor \vartheta, z^* \rfloor \leq 0 \quad \forall z^* \in N_c(x)\}$$
$$N_c(x) = \{x^* \in X^*\colon \lfloor x, x^* \rfloor \leq 0\}.$$

Verma proved (see Verma [2]: Theorem 4.1) that CNHVI has at least one solution under certain conditions which include the operator A strongly monotone and B strongly Lipschitz.

2. Results

We observe

Theorem 1 CNHVI *has at most one solution if A is strongly monotone with constant $a > 0$ and B strongly Lipschitz with constant $c \geq 0$.*

In other words under the conditions of Theorem 4.1 of Verma [2], CNHVI has unique solution.

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Proof. Assume to the contrary that u_1 and u_2 are two solutions. Then we have

$$[(A-B)u_1 - g,\, \vartheta] \geq 0 \quad \forall \vartheta \in T_c(u_1) \tag{1}$$
$$[(A-B)u_2 - g,\, \vartheta] \geq 0 \quad \forall \vartheta \in T_c(u_2). \tag{2}$$

Taking $\vartheta = u_1$ in (2) and $\vartheta = u_2$ in (1)

$$[(A-B)u_1 - g,\, u_2] \geq 0$$
$$[(A-B)u_2 - g,\, u_1] \geq 0$$

Thus we have

$$[(A-B)u_1 - (A-B)u_2,\, u_1 - u_2] \leq 0. \tag{3}$$

since A is strongly monotone and B is strongly Lipschitz, we have $A-B$ is strongly monotone and

$$[(A-B)u_1 - (A-B)u_2,\, u_1 - u_2] \geq (a+c)\|u_1 - u_2\|^2 \tag{4}$$

Now (3) cannot happen because of (4) and hence the result follows. □

We now state another inequality as follows:

$$[(A-B)u - g,\, u] \geq (a+c)\|u - \vartheta\|^2 \quad \forall \vartheta \in T_c(u). \tag{5}$$

We have

Theorem 2 *If A is strongly monotone and B is strongly Lipschitz, then u satisfies* CNHVI *iff u satisfies* (5).

Proof. Given that A is strongly monotone with constant $a > 0$ and B is strongly Lipschitz with constant $c \geq 0$. Therefore

$$[(A-B)u - (A-B)\vartheta,\, u - \vartheta] \geq (a+c)\|u - \vartheta\|^2.$$

Therfore

$$\begin{aligned}[(A-B)\vartheta,\, u-\vartheta] + (a+c)\|u-\vartheta\|^2 &\leq [(A-B)u,\, u-\vartheta] \\ &\leq [g,\, u-\vartheta] \quad \text{by CHNVI.}\end{aligned}$$

Now changing the role of u and ϑ

$$\begin{aligned}&[(A-B)u,\, \vartheta-u] + (a+c)\|u-\vartheta\|^2 \leq [g,\, \vartheta-u] \\ \Rightarrow\ &[(A-B)u - g,\, \vartheta-u] + (a+c)\|u-\vartheta\|^2 \leq 0\end{aligned}$$

$$\Rightarrow [(A-B)u-g,\, u] \geq [(A-B)u-g,\, \vartheta] + (a+c)\|u-\vartheta\|^2$$
$$\geq (a+c)\|u-\vartheta\|^2$$

Hence (5) holds. □

We observe that

Theorem 3 *The problem in* Lemma 7.11 *in Naniewiez and Panagistopoulos* [1] *has unique solution. The formal statement is as follows*:
Let A be strongly monotone with constant $m > 0$, $f\colon V = \mathrm{dom}A \to R$ a locally Lipschitz function which satisfies relaxed monotonicity condition

$$(u^* - \vartheta^*,\, v-\vartheta) \geq -a\|u-\vartheta\|^2,\ \ \forall u,\, \vartheta \in V$$

for $u^* \epsilon f(u)$, $\vartheta \epsilon f(\vartheta)$ with $a > 0$, $a < m$. Then the problem: *find $\vartheta \epsilon V$ st*

$$(Au-g,\, \vartheta-u) + f^0(u,\, \vartheta-u) \geq 0 \quad \forall \vartheta \epsilon V$$

has a unique solution.

Proof. Existence was proved by Naniewicz and Panagietopoulos [1] in Lemma 7.11. We only prove the uniqueness.
Suppose u_1 and u_2 are two solutions. then

$$(Au_1-g,\, \vartheta-u_1) + f^0(u_1,\, \vartheta-u_1) \geq 0$$
$$(Au_2-g,\, \vartheta-u_2) + f^0(u_2,\, \vartheta-u_2) \geq 0$$

Putting $\vartheta = u_2$ and $\vartheta = u_1$ successively, we get

$$(Au_1-g,\, u_2-u_1) + f^0(u_1,\, u_2-u_1) \geq 0 \tag{6}$$
$$(Au_2-g,\, u_1-u_2) + f^0(u_2,\, u_1-u_2) \geq 0$$

i.e.

$$(-Au_2+g,\, u_2-u_1) + f^0(-u_2,\, u_2-u_1) \geq 0 \tag{7}$$

Adding (6) and (7) we get

$$(Au_1-Au_2,\, u_2-u_1) + f^0(u_1,\, u_2,\, u_2-u_1) \geq 0$$

$\Rightarrow$

$$(Au_2-Au_1,\, u_2-u_1) + f^0(u_2,\, u_1,\, u_2-u_1) \leq 0.$$

This constant m, f is relaxed monotone with constant a and $a < m$. This completes the proof. $\square$

Theorem 4 *Under the conditions of* Theorem 3*, the* HVI

$$(Au - g, \vartheta - u) \geq (m - a)\|u - \vartheta\|^2.$$

Proof. Suppose

$$(Au - g, \vartheta - u) + f^0(u, \vartheta - u) \geq 0$$

By the hypothesis,

$$(Au - A\vartheta, u - \vartheta) \geq m\|u - \vartheta\|^2$$

and

$$f^0(u - \vartheta, u - \vartheta) \geq -a\|u - \vartheta\|^2.$$

Therefore,

$$\begin{aligned}
&(m - a)\|u - \vartheta\|^2 \\
&\leq (Au - A\vartheta, u - \vartheta) + f^0(u - \vartheta, u - \vartheta) \\
&= (Au - g, u - \vartheta) + f^0(u, u - \vartheta) - (A\vartheta - g, u - \vartheta) - f^0(\vartheta, u - \vartheta) \\
&= -(Au - g, \vartheta - u) + f^0(\vartheta, \vartheta - u) \\
&\quad + (A\vartheta - g, \vartheta - u) - f^0(u, \vartheta - u) \\
&\Rightarrow (A\vartheta - g, \vartheta - u) + f^0(\vartheta, \vartheta - u) \\
&\geq (Au - g, \vartheta - u) + f^0(u, \vartheta - u) + (m - a)\|u - \vartheta\|^2 \\
&\geq (m - a)\|u - \vartheta\|^2
\end{aligned}$$

$\square$

References

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