Existence and decay of solutions
to a semilinear Schrödinger equation with magnetic field

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Abstract. In this paper we study the decay properties of solutions to a semilinear Schrödinger equation, $-(\nabla - iA)^2 u + (V - E)u = Q|u|^{p-2}u$, on $\mathbb{R}^n$, where $n \geq 2$ and $2 < p < 2^*$. We give a lower bound estimate of nontrivial solutions at infinity. In two-dimensional case, we give super-exponential decay estimates of solutions at infinity. Moreover, we show the existence of a nontrivial solution under additional assumptions on potentials.

Key words: Gaussian decay of stationary solutions, nonlinear Schrödinger equation, magnetic field.

1. Introduction and results

We study the decay properties and the existence of solutions to the semilinear Schrödinger equation

$$-(\nabla - iA)^2 u + (V - E)u = Q|u|^{p-2}u \tag{1.1}$$

on $\mathbb{R}^n$, where $p > 2$ and $n \geq 2$. Here, A is a magnetic vector potential, V is an electric scalar potential, E is a real constant, and Q is a real-valued function.

We fix some notation. We denote the standard inner product and norm on $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively. We denote by $\mathbb{N}$ the set of *non-negative* integers. By $A + B =: C + D$, we mean that C and D are defined by A and B, respectively. We denote by $L^2(N, M)$ the space of all M-valued L^2-functions on N, and denote $L^2(\mathbb{R}^n, \mathbb{C})$ by $L^2(\mathbb{R}^n)$, etc. We denote by $C_0^\infty(\mathbb{R}^n)$ the space of all (complex-valued) smooth functions on $\mathbb{R}^n$ with compact support. We denote by $\rho(T)$ and $\mathrm{Spec}_{(\mathrm{ess,disc})}(T)$ the resolvent set and the (essential, discrete) spectrum of any operator T, respectively. The symbol 2^* stands for $2n/(n-2)$ if $n \geq 3$ and for ∞ if $n = 1, 2$.

We define the magnetic field $B(x) = dA(x)$ by the $n \times n$ matrix $(\partial_j A_k(x) - \partial_k A_j(x))_{j,k=1}^n$ for any magnetic vector potential $A = (A_1, \ldots, A_n)$.

Throughout this paper, we use the notations $\nabla_A = \nabla - iA$ and $\Delta_A = (\nabla_A)^2$, where ∇ is the standard gradient. We introduce the function space

$$H^1_{A,V}(\mathbb{R}^n) = \{u \mid u \in L^2(\mathbb{R}^n),\ \nabla_A u \in L^2(\mathbb{R}^n),\ V|u|^2 \in L^1(\mathbb{R}^n)\}$$

equipped with inner product

$$(u,\, v)_{H^1_{A,V}} = (\nabla_A u,\, \nabla_A v)_{L^2} + (u,\, Vv)_{L^2} + (u,\, v)_{L^2}.$$

The equation (1.1) (possibly with more general nonlinear terms or with semi-classical parameters) has been studied extensively by many authors (see, e.g., Esteban and Lions [Es-Li], Arioli and Szulkin [Ar-Sz], Chabrowski and Szulkin [Ch-Sz], Cingolani [Cin], Cingolani and Secchi [Ci-Se1], [Ci-Se2], Schindler and Tintarev [Sc-Ti], Pankov [Pan], Kurata [Kur], Bartsch, Dancer and Norman [B-D-N], and references therein).

As well as the existence of nontrivial solutions, the exponential decay property of the solutions is an interesting problem in the theory of nonlinear Schrödinger equations (see, e.g., Pankov [Pan2], [Pan3], Fukuizumi and Ozawa [Fu-Oz], Rabier and Stuart [Ra-St], Section 8 in Cazenave [Caz], and references therein). To the author's knowledge, the resulting weight functions ρ in the decay estimate $|u(x)| \le Ce^{-\rho(x)}$ are given essentially by the so-called Agmon metric (see Agmon [Agm]), which reflects no magnetic effect.

The main purpose of the paper is to show that the magnetic field in fact affects the decay properties at infinity of solutions to (1.1); we obtain an $L^2(\mathbb{R}^n)$-averaged lower bound estimate (Theorem 1.1 below) and super-exponential decay estimates in two-dimensional case (Theorems 1.3 and 1.4 below). Moreover, in Theorem 1.6 below, we show the existence of a nontrivial solution for a special class of potentials.

To formulate the result concerning the lower bound estimate, we make the following conditions on A and V:

(A.1) The magnetic vector potential A belongs to $C^1(\mathbb{R}^n, \mathbb{R}^n)$. Moreover, the asymptotic strength of the magnetic field

$$\|B\|_\infty = \limsup_{|y| \to \infty}\ \sup_{\hat{x} \in S^{n-1}} |B(y)\hat{x}|$$

is finite.

(A.2) The scalar potential V is a measurable function on $\mathbb{R}^n$. Moreover, V is bounded outside a compact set in $\mathbb{R}^n$.

We have the following $L^2(\mathbb{R}^n)$-averaged lower bound for the solutions:

Theorem 1.1 *Let $n \geq 2$ and $p > 2$. Assume (A.1) and (A.2). Assume that the function Q is measurable on $\mathbb{R}^n$. Let u be a solution to (1.1) in $H^2_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying the condition:*
(N.1) *The function $Q|u|^{p-2}$ is bounded outside a compact set in $\mathbb{R}^n$.*
Assume further that the support of u is non-compact. Then the function $\exp(\kappa|x|^2)u(x)$ does not belong to $L^2(\mathbb{R}^n)$ if $\kappa > \|B\|_\infty/4$.

Remark 1.2 The lower bound given in Theorem 1.1 is optimal in the following sense. On the one hand, under the assumption as in the theorem, no nontrivial solution satisfies the pointwise Gaussian estimate $|u(x)| \leq C\exp(-\kappa|x|^2)$ if $\kappa > \|B\|_\infty/4$. On the other hand, we can find a nontrivial solution to (1.1) which satisfies the pointwise Gaussian estimate with $\kappa = \|B\|_\infty/4$; in fact, such a solution is given by a ground state of the two-dimensional Schrödinger operator with constant magnetic field in the case of $Q = V = 0$.

The next two results concern the upper bound estimate of solutions. We restrict ourselves to the two-dimensional case. The magnetic field $B = (B_{jk})_{j,k=1,2}$ is identified with the function B_{12} because B is anti-symmetric. In what follows we shall adopt this identification.

We say that a (vector-valued) function f on $\mathbb{R}^n$ *decays at infinity* if for any $\varepsilon > 0$ there exists a compact set K of $\mathbb{R}^n$ such that $\|f\|_{L^\infty(\mathbb{R}^n \setminus K)} \leq \varepsilon$ holds.

To formulate the results we make the following conditions on A, V, and Q:

(A.3) The magnetic vector potential A belongs to $C^1(\mathbb{R}^2, \mathbb{R}^2)$. Moreover, the magnetic field has a decomposition $B = B_0 + B_1$, where B_0 is a non-zero constant, $B_1 \in C(\mathbb{R}^2, \mathbb{R})$, and B_1 decays at infinity.

(A.4) The scalar potential has a decomposition $V = V_1 + V_2$, where $V_1 \in L^2(\mathbb{R}^2, \mathbb{R})$, $V_2 \in L^\infty(\mathbb{R}^2, \mathbb{R})$, and V_2 decays at infinity.

(A.5) The scalar potential V is bounded from below.

(A.6) There exist positive constants δ, β, and C such that $0 < \beta < 2$ and

$$|V(x)| \leq C\exp(-\delta|x|^\beta)$$

holds outside a compact set.

(N.2) The function Q belongs to $L^\infty(\mathbb{R}^n, \mathbb{R})$.

The following theorems both show that asymptotically non-zero con-

stant magnetic fields can create super-exponentially decaying solutions.

Theorem 1.3   *Let $p > 2$. Assume (A.3)–(A.5) and (N.2). Let u be a solution to (1.1) in $H^1_{A,V}(\mathbb{R}^2)$. Assume that the real number E does not belong to the set $\{(2k+1)|B_0| \mid k \in \mathbb{N}\}$. Then for any $\alpha > 0$ there exists a positive constant C_α such that $|u(x)| \le C_\alpha e^{-\alpha|x|}$ holds for all $x \in \mathbb{R}^2$.*

Theorem 1.4   *Let $p > 2$. Assume (A.3)–(A.6) and (N.2), and assume that the function B_1 in (A.3) is compactly supported. Let u be a solution to (1.1) in $H^1_{A,V}(\mathbb{R}^2)$. Assume that the real number E does not belong to the set $\{(2k+1)|B_0| \mid k \in \mathbb{N}\}$. Then there exist positive constants μ and C such that $|u(x)| \le C \exp\left(-\mu|x|^{1+\beta/2}\right)$ holds for all $x \in \mathbb{R}^2$.*

   If, in addition, both V and Q have the pointwise Gaussian decay, then u has the same property.

Remark 1.5

1. The conditions (A.3) and (A.4) ensure the essential self-adjointness of $-\Delta_A + V$ on $C^\infty_0(\mathbb{R}^2)$ and the relative compactness of the perturbation V with respect to $-\Delta_A$. The set $\{(2k+1)|B_0| \mid k \in \mathbb{N}\}$ is often called the Landau levels, which is the essential spectrum of (the self-adjoint extension of) $-\Delta_A + V$.

2. The question whether or not the solutions have the Gaussian decay property has a subtle nature even when $Q = 0$. Erdös [Erd] gives an example of an eigenfunction which decays strictly slower than a Gaussian at infinity under some mild assumptions. For the Gaussian decay of eigenfunctions, we refer to Erdös [Erd], Nakamura [Nak], Sordoni [Sor], and Cornean and Nenciu [Co-Ne].

3. As is well known, the additional growth condition on the electromagnetic fields improves the decay rate of the solutions. In Theorems 1.3 and 1.4, however, we are interested in the phenomena that *bounded* electro-magnetic fields can create super-exponential decaying solutions.

The existence of a nontrivial solution is assumed in the theorems above. We now discuss the existence of a nontrivial solution to (1.1). To formulate the result we make the following conditions on A, V, and Q:

(E.1)   The vector potential A is expressed as $A = A_0 + A_1$ for some smooth vector potentials A_0 and A_1. Moreover, dA_0 is $\mathbb{Z}^n$-periodic and A_1 decays at infinity.

(E.2) The scalar potential V is expressed as $V = V_0 + V_1$ for some smooth scalar potentials V_0 and V_1. Moreover, V_0 is $\mathbb{Z}^n$-periodic and V_1 decays at infinity.

(E.3) The function Q is positive, bounded, and measurable. Moreover, Q decays at infinity.

Here, we say that a (vector-valued) function f on $\mathbb{R}^n$ is $\mathbb{Z}^n$-*periodic* if $f(x + \gamma) = f(x)$ holds for any $x \in \mathbb{R}^n$ and any $\gamma \in \mathbb{Z}^n$.

The conditions (E.1) and (E.2) ensure the essential self-adjointness of $H_0 = -\Delta_{A_0} + V_0$ on $C_0^\infty(\mathbb{R}^n)$ (see Leinfelder and Simader [Le-Si]). In the sequel, we shall identify any closable operator with its operator closure.

Our existence result is the following theorem.

Theorem 1.6 *Let $n \geq 2$ and $2 < p < 2^*$. Assume (E.1)–(E.3). Assume that the real number E belongs to the resolvent set of H_0. Then the equation (1.1) has a solution in $H_{A,V}^1(\mathbb{R}^n) \setminus \{0\}$. Moreover, the solution is bounded and decays at infinity.*

Remark 1.7 All the conditions (A.1)–(A.6), (N.1), (N.2), and (E.1)–(E.3) are satisfied if, e.g., $n = 2$, the magnetic field B is non-zero constant, the scalar potential V is smooth and is compactly supported, the function Q is positive and has the Gaussian decay, and $E \notin \mathrm{Spec}_{\mathrm{ess}}(-\Delta_A + V)$.

The proof of Theorem 1.6 is more or less standard; the method is based on a linking theorem and a concentration-compactness type argument. We give, however, a proof for the sake of completeness. A similar argument can be found, e.g., in Bartsch and Ding [Ba-Di], [Ba-Di3], Chabrowski and Szulkin [Ch-Sz], Pankov [Pan2], and Willem and Zou [Wi-Zo].

The organization of this paper is as follows. We give a proof of Theorem 1.1 in Section 2 and give proofs of Theorems 1.3 and 1.4 in Section 3. In Sections 4–7 we devote ourselves to proving Theorem 1.6. In Section 4 we recall an abstract linking theorem due to Bartsch and Ding [Ba-Di]. In Sections 5 and 6 we formulate a variational setting associated with the equation (1.1) and apply the linking theorem. In Section 7 we give a proof of Theorem 1.6.

2. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. The argument is similar to that used in the proof of Theorem 1.2 in Uchiyama [Uch].

Throughout this paper we denote by "C" (possibly with some super- or subscripts) various constants in estimates, which may vary from line to line. In this section, for simplicity, we write $\| \cdot \|$ and $(\cdot, \cdot)$ for $\| \cdot \|_{L^2(\mathbb{R}^n)}$ and $(\cdot, \cdot)_{L^2(\mathbb{R}^n)}$, respectively.

For any s, $t > 0$, we set $S(t) = \{x \in \mathbb{R}^n \mid |x| = t\}$ and $B(s, t) = \{x \in \mathbb{R}^n \mid s < |x| < t\}$. We denote by dS the standard Haar measure on $S(t)$ (normalized by $dx = dt dS$). For $x \in \mathbb{R}^n$, we set $r = |x|$ and $\hat{x} = x/r$.

Let $\kappa > 0$ and $m \geq 1$. Let u be a solution satisfying the assumption of Theorem 1.1. We set $\rho(r) = \rho(r; \kappa, m) = \kappa r^2 + mr$, $w = e^\rho u$, and $W = V - E - Q|u|^{p-2}$. Following Uchiyama [Uch], we introduce the quantities

$$k_1(r) = -(\rho'(r))^2 = -(4\kappa^2 r^2 + 4\kappa mr + m^2),$$
$$k_2(r) = \rho''(r) + (n-1)r^{-1}\rho'(r) = 2n\kappa + m(n-1)r^{-1},$$
$$g(r) = (n-1)r^{-1} = \operatorname{div}(\hat{x}),$$
$$F(t) = \int_{S(t)} \big(2|\langle \hat{x}, \nabla_A w\rangle|^2 - (|\nabla_A w|^2 + k_1|w|^2)$$
$$+ g\operatorname{Re}[\langle \hat{x}, \overline{\nabla_A w}\rangle w]\big) dS,$$
$$G(t; \kappa, m) = \int_{S(t)} \big(|\nabla_A u|^2 + ((\rho'(t))^2 + 1)|u|^2\big) dS,$$

where $\operatorname{Re}[\,\cdot\,]$ stands for the real part, and then $\rho'(r) = 2\kappa r + m$, $\rho''(r) = 2\kappa$,

$$\nabla k_1(r) = -r^{-1}(8\kappa^2 r^2 + 4\kappa mr)\hat{x},$$
$$\nabla g(r) = g'(r)\hat{x} = -(n-1)r^{-2}\hat{x}.$$

Lemma 2.1   *There exists $R > 0$ such that if $t > s > R$ then we have*

$$F(t) - F(s)$$
$$= \int_{B(s,t)} \Big(2(2\rho' - r^{-1})|\langle \hat{x}, \nabla_A w\rangle|^2 + 2r^{-1}|\nabla_A w|^2$$
$$+ 2\operatorname{Re}\Big[\Big(W + k_2 + g\rho' + \frac{1}{2}g'\Big)\langle \hat{x}, \overline{\nabla_A w}\rangle w\Big]$$
$$- 2\operatorname{Im}[\langle B\hat{x}, \overline{\nabla_A w}\rangle w]$$
$$+ \{-\langle \hat{x}, \nabla k_1\rangle + g(\operatorname{Re}[W] + k_2)\}|w|^2\Big) dx, \qquad (2.1)$$

where $\operatorname{Im}[\,\cdot\,]$ stands for the imaginary part.

Proof.   This is a special case of Lemma 2.1 of [Uch] with $f = 1$, $A = \operatorname{Id}$, $q_1 = 0$, $q_2 = W = V - E - Q|u|^{p-2}$ and $\lambda = 0$ in his notation. Note that the assumption $u \in H^2_{\text{loc}}$ is used in the proof in [Uch]. $\qquad\qquad \square$

Lemma 2.2 *For any $\kappa > \|B\|_\infty/4$ there exists $R_1 = R_1(\kappa, Q, V, u) > 0$ such that $F(t) \geq F(s)$ holds for any $t > s > R_1$ and any $m \geq 1$.*

Proof. Let W be as above. Put $T = \sup_{|x| \geq s} |W(x)|$, which is uniformly bounded with respect to (large) s by (A.2) and (N.1). We divide and estimate the integrand on the right-hand side of (2.1) as follows:

$$\mathrm{I} = 2(2\rho' - r^{-1})|\langle \hat{x}, \nabla_A w \rangle|^2 = r^{-1}(8\kappa r^2 + 4mr - 2)|\langle \hat{x}, \nabla_A w \rangle|^2,$$

$$\mathrm{II} = 2r^{-1}|\nabla_A w|^2,$$

$$\mathrm{III} = 2\,\mathrm{Re}\Big[\Big(k_2 + W + g\rho' + \frac{1}{2}g'\Big)\langle \hat{x}, \overline{\nabla_A w}\rangle w\Big]$$

$$= 2r^{-1}\,\mathrm{Re}\Big[\Big(((4n-2)\kappa + W)r + 2m(n-1) - \frac{n-1}{2r}\Big)\langle \hat{x}, \overline{\nabla_A w}\rangle w\Big]$$

$$\geq -2r^{-1}\Big(((4n-2)\kappa + T)r + 2m(n-1) + \frac{n-1}{2r}\Big)|\langle \hat{x}, \overline{\nabla_A w}\rangle||w|$$

$$\geq -r^{-1}\big((4n-2)\kappa + T\big)\big(\varepsilon_1 r^2|\langle \hat{x}, \nabla_A w \rangle|^2 + \varepsilon_1^{-1}|w|^2\big)$$

$$\quad - 2r^{-1}m(n-1)\big(\varepsilon_2|\langle \hat{x}, \nabla_A w \rangle|^2 + \varepsilon_2^{-1}|w|^2\big)$$

$$\quad - r^{-1}\frac{n-1}{2r}\big(|\langle \hat{x}, \nabla_A w \rangle|^2 + |w|^2\big)$$

$$\geq -r^{-1}\Big(8\kappa r^2|\langle \hat{x}, \nabla_A w \rangle|^2 + \frac{((4n-2)\kappa + T)^2}{8\kappa}|w|^2\Big)$$

$$\quad - r^{-1}\Big(3mr|\langle \hat{x}, \nabla_A w \rangle|^2 + \frac{4m(n-1)^2}{3r}|w|^2\Big)$$

$$\quad - r^{-1}\frac{n-1}{2r}\big(|\langle \hat{x}, \nabla_A w \rangle|^2 + |w|^2\big)$$

$$= -r^{-1}\Big(8\kappa r^2 + 3mr + \frac{n-1}{2r}\Big)|\langle \hat{x}, \nabla_A w \rangle|^2$$

$$\quad - r^{-1}\Big(\frac{((4n-2)\kappa + T)^2}{8\kappa} + \frac{4(n-1)^2 m}{3r} + \frac{n-1}{2r}\Big)|w|^2,$$

where we set $((4n-2)\kappa + T)\varepsilon_1 = 8\kappa$, $2(n-1)\varepsilon_2 = 3r$, and

$$\mathrm{IV} = -2\,\mathrm{Im}[\langle B\hat{x}, \overline{\nabla_A w}\rangle w] \geq -r^{-1}\Big(2|\nabla_A w|^2 + \frac{1}{2}|B\hat{x}|^2 r^2|w|^2\Big),$$

$$\mathrm{V} = \big(-\langle \hat{x}, \nabla k_1 \rangle + g(\mathrm{Re}[W] + k_2)\big)|w|^2$$

$$\geq r^{-1}\Big(8\kappa^2 r^2 + 4\kappa mr + 2n(n-1)\kappa + \frac{m(n-1)^2}{r} - (n-1)T\Big)|w|^2,$$

Then there exists $R_1 = R_1(n, \kappa, T) > 0$ such that

$$r(\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V})$$

$$\geq \frac{1}{2}(2mr - 4 - (n-1)r^{-1})|\langle \hat{x},\, \nabla_A w \rangle|^2$$

$$+ \left(8\left(\kappa^2 - \left| \frac{B\hat{x}}{4} \right|^2 \right) r^2 + 4\kappa \left(r - \frac{(n-1)^2}{12\kappa r} \right) m \right.$$

$$\left. + 2n(n-1)\kappa - \frac{((4n-2)\kappa + T)^2}{8\kappa} - \frac{n-1}{2r} - (n-1)T \right) |w|^2$$

$$\geq \frac{1}{2}\left(2r - 4 - (n-1)r^{-1} \right) |\langle \hat{x},\, \nabla_A w \rangle|^2$$

$$+ \left(4\kappa \left(r - \frac{(n-1)^2}{12\kappa r} \right) - \frac{((4n-2)\kappa + T)^2}{8\kappa} - 1 - (n-1)T \right) |w|^2$$

$$\geq \left(4\kappa \left(r - \frac{(n-1)^2}{12\kappa r} \right) - \frac{((4n-2)\kappa + T)^2}{8\kappa} - 1 - (n-1)T \right) |w|^2 \ (2.2)$$

holds for any $m \geq 1$ if $r > R_1$, where we used the condition $\kappa > \|B\|_\infty/4$. This shows the lemma. $\qquad\Box$

Lemma 2.3   *Let $\rho(r) = \rho(r; \kappa,\, m)$ and $G(t; \kappa,\, m)$ be as before. For any $\kappa > \|B\|_\infty/4$ there exists $m \geq 1$ such that*

$$\liminf_{t\to\infty} e^{2\rho(t;\kappa,\, m)} G(t; \kappa,\, m) > 0.$$

Proof.   We show this by contradiction.   Assume that there exists $\kappa > \|B\|_\infty/4$ such that

$$\liminf_{t\to\infty} e^{2\rho(t)} G(t; \kappa,\, m) = 0$$

holds for any $m \geq 1$.

Using the definition of $F(t)$ and the relation $w = e^\rho u$, we have

$$F(t) = e^{2\rho(t)} \int_{S(t)} \left(2|\langle \hat{x},\, \nabla_A u \rangle|^2 - |\nabla_A u|^2 + \frac{n-1}{r} \rho'|u|^2 \right.$$

$$\left. + 2(\rho')^2 |u|^2 + \left(2\rho' + \frac{n-1}{r} \right) \operatorname{Re}[\langle \hat{x},\, \overline{\nabla_A u} \rangle u] \right) dS. \ (2.3)$$

Then it follows that

$$F(t) \leq C e^{2\rho(t)} \int_{S(t)} \left(|\nabla_A u|^2 + ((\rho'(t))^2 + 1)|u|^2 \right) dS$$

$$= C e^{2\rho(t)} G(t; \kappa,\, m)$$

holds for some $C > 0$, independent of t and m. By the assumption, we have

$\liminf_{t\to\infty} F(t) \le 0$ for each $m \ge 1$. Thus it follows that $F(t) \le 0$ holds for any $t > R_1$ and any $m \ge 1$ because $F(t)$ is monotone increasing with respect to $t \ge R_1$ by Lemma 2.2.

On the other hand, it follows from (2.3) and the definition of ρ that $e^{-2\rho(t)}F(t)$ is a quadratic in m and the coefficient of m^2 is

$$2\int_{S(t)} |u|^2 dS. \tag{2.4}$$

By the non-compactness of the support of u, there exists $R \, (> R_1)$ such that the coefficient (2.4) is positive at $t = R$. Note that R is independent of m. Hence $F(R) > 0$ holds for some $m \ge 1$. This contradicts the non-positivity of F. □

In what follows we fix the constant $m = m(\kappa)$ found in the preceding lemma and we denote $G(t; \kappa, m(\kappa))$ simply by $G(t; \kappa)$.

Lemma 2.4 *Assume that $\kappa_0 > \kappa_1 > \|B\|_\infty/4$. Then we have*

$$\lim_{t\to\infty} e^{2\kappa_0 t^2} G(t; \kappa_1) = \infty.$$

Proof. Let $\kappa_0 > \kappa_1 > \|B\|_\infty/4$. Setting $\varepsilon = (\kappa_0 - \kappa_1)/2$, we have

$$\rho(t; \kappa_1) = \kappa_1 t^2 + m(\kappa_1)t \le (\kappa_1 + \varepsilon)t^2 + C_\varepsilon$$
$$= \kappa_0 t^2 + C_\varepsilon - \frac{(\kappa_0 - \kappa_1)t^2}{2}$$

for some constant C_ε. Then it follows from Lemma 2.3 that

$$\lim_{t\to\infty} e^{2\kappa_0 t^2} G(t, \kappa_1) \ge e^{-2C_\varepsilon} \liminf_{t\to\infty} e^{(\kappa_0 - \kappa_1)t^2} e^{2\rho(t;\kappa_1)} G(t, \kappa_1) = \infty.$$

This shows the lemma. □

Let ζ be a smooth function on $\mathbb{R}$ satisfying the conditions: $0 \le \zeta \le 1$ on $\mathbb{R}$, $\zeta(t) = 1$ if $1/3 \le t \le 2/3$, and $\mathrm{supp}(\zeta) \subset (0, 1)$. For any $R > 0$, we set $\zeta_R(r) = \zeta(r - R)$ with $r = |x|$.

Lemma 2.5 *Let u be the solution. There exist positive constants R_2 and C such that*

$$\|\zeta_R \nabla_A u\|^2 \le C \int_{B(R,R+1)} |u(x)|^2 dx$$

holds for any $R > R_2$.

Proof. Let $W = V - E - Q|u|^{p-2}$. By (A.2) and (N.1), there exist $R_2 > 0$ and $T > 0$ such that $|((\zeta_R)^2 Wu, u)| \leq T\|\zeta_R u\|^2$ holds for any $R > R_2$. Then, using the equation (1.1), we have

$$
\begin{aligned}
0 &= \left((\zeta_R)^2(-\Delta_A + W)u, \, u\right) \\
&= \|\zeta_R \nabla_A u\|^2 + \left([(\zeta_R)^2, \nabla_A]\nabla_A u, \, u\right) + \left((\zeta_R)^2 Wu, \, u\right) \\
&\geq \frac{1}{2}\|\zeta_R \nabla_A u\|^2 - (C + T)\int_{B(R,R+1)} |u(x)|^2 dx,
\end{aligned}
$$

where we used the identity $[(\zeta_R)^2, \nabla_A] = -2\hat{x}(\zeta_R)'\zeta_R$. Here, the constant C is independent of R. This proves the lemma. □

Proof of Theorem 1.1. Fix $\kappa > \|B\|_\infty/4$. Let $\kappa > \kappa_0 > \kappa_1 > \|B\|_\infty/4$. By Lemma 2.4, for any $M > 0$ there exists $R_3 > 0$ such that

$$
G(t; \kappa_1) \geq Me^{-2\kappa_0 t^2} \tag{2.5}
$$

holds for any $t > R_3$. Then for $R > R_3$ we have

$$
\begin{aligned}
&M\left(\int_0^1 \zeta(t)^2 dt\right)e^{-2\kappa_0(R+1)^2} \\
&\leq M\int_R^{R+1} \zeta_R(t)^2 e^{-2\kappa_0 t^2} dt \leq \int_R^{R+1} \zeta_R(t)^2 G(t; \kappa_1) dt \\
&\leq \int_{B(R,R+1)} |\zeta_R(|x|)(\nabla_A u)(x)|^2 dx + \int_{B(R,R+1)} (1 + (\rho'(r; \kappa_1))^2)|u(x)|^2 dx \\
&\leq C\int_{B(R,R+1)} (1 + |x|^2)|u(x)|^2 dx \tag{2.6}
\end{aligned}
$$

for some $C > 0$, independent of R, where we used ((2.5)) in the second inequality and used Lemma 2.5 in the last inequality. Then it follows from ((2.6)) that there exists $C > 0$ such that

$$
CM \leq e^{2\kappa_0(R+1)^2}\int_{B(R,R+1)} (1 + |x|^2)|u(x)|^2 dx. \tag{2.7}
$$

Hence, for any $\varepsilon > 0$ we have

$$
\begin{aligned}
\|e^{\kappa|x|^2}u\|^2 &\geq \int_{B(R,R+1)} e^{2\kappa|x|^2}|u(x)|^2 dx \\
&\geq C_\varepsilon \int_{B(R,R+1)} e^{2(\kappa-\varepsilon)|x|^2}(1 + |x|^2)|u(x)|^2 dx
\end{aligned}
$$

$$\geq C_\varepsilon e^{2(\kappa-\varepsilon)R^2} e^{-2\kappa_0(R+1)^2} e^{2\kappa_0(R+1)^2} \int_{B(R,R+1)} (1+|x|)^2 |u(x)|^2 dx$$

$$\geq C_\varepsilon M \exp\left(2(\kappa-\varepsilon)R^2 - 2\kappa_0(R+1)^2\right), \qquad (2.8)$$

where we used (2.7) in the last inequality. If we set $\varepsilon = (\kappa - \kappa_0)/2 \ (> 0)$, the right-hand side of (2.8) is bounded from below by $C_\varepsilon M$ for any large R. This completes the proof of Theorem 1.1 because $M > 0$ is arbitrary. $\qquad \square$

3. Proof of Theorem 1.3 and Theorem 1.4

In this section we show Theorem 1.3 and Theorem 1.4. Our proof is based on the simple (and obvious) fact that if u is a solution to (1.1) then u solves also the linear equation (3.9) below. Therefore we can reduce the problem to the decay estimate of solutions to a linear elliptic equations. The same argument can be found in Chabrowski and Szulkin [Ch-Sz], Pankov [Pan3], etc.

The crucial step is to establish a priori L^∞-estimate for the solutions (Corollary 3.4 below). This is done by a bootstrap argument. Although this kind of argument is standard in the theory of elliptic equation, we give proofs for the sake of completeness.

Lemma 3.1 *Let $n \geq 2$. Assume that the vector potential A belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$. Then we have the following assertions*:
(i) *The map $H^1_{A,0}(\mathbb{R}^n) \ni u \mapsto |u| \in H^1(\mathbb{R}^n)$ is continuous.*
(ii) *The embedding $H^1_{A,0}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ is continuous if $2 \leq q < 2^*$.*
(iii) *The restriction $H^1_{A,0}(\mathbb{R}^n)$ to $L^q(\Omega)$ is compact if $2 \leq q < 2^*$ and Ω is a bounded open set.*

Proof. The dia-magnetic inequality $\big|\nabla|v|\big| \leq |\nabla_A v|$ holds if $v \in L^2(\mathbb{R}^n)$, $\nabla_A v \in L^2(\mathbb{R}^n, \mathbb{R}^n)$, and $A \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ (see, e.g., Lieb and Loss [Li-Lo], Theorem 7.21). This implies the assertion (i). The assertions (ii) and (iii) follow from the Sobolev inequality and the Rellich-Kondrashov theorem.
$\qquad \square$

Before proceeding to the proof of Theorems 1.3 and 1.4, we consider the linear Schrödinger equation

$$-\Delta_A \psi + V_+ \psi = g\psi \qquad (3.1)$$

in the sense of distribution on $\mathbb{R}^n$. We make the following conditions on V_+ and g:

(C.1) The function V_+ is non-negative and belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

(C.2) The function g belongs to $L^\infty(\mathbb{R}^n)$.

(C.3) There exists $\gamma > n/2$ such that g belongs to $L^\infty(\mathbb{R}^n) + L^\gamma(\mathbb{R}^n)$.

Lemma 3.2 ([Ch-Sz]) *Let $n \geq 2$. Assume that A belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$. Assume (C.1) and (C.2). Assume that ψ is a solution to (3.1) in $H^1_{A,V_+}(\mathbb{R}^n)$. Then ψ is bounded and decays at infinity.*

Proof. This result can be found in Chabrowski and Szulkin [Ch-Sz], Proposition 2.2 and Remark 2.4. A similar argument can be found in the proof of Theorem 5.1 in Agmon [Agm]. However, we give a proof for the sake of completeness.

Let ψ be as above and let $\beta > 1$ and $L > 0$. Let $\eta \in C^1(\mathbb{R}^n, \mathbb{R}) \cap L^\infty(\mathbb{R}^n)$ and $\nabla\eta \in L^\infty(\mathbb{R}^n)$. Set $\phi(x) = \eta(x)^2\psi(x)\min\{|\psi(x)|^{\beta-1}, L\}$. We note that $\psi \in L^q(\mathbb{R}^n)$ for some $q > 2$ by Lemma 3.1.

As in the same way to deduce the formula (2.2) in [Ch-Sz] (or by a direct calculation using $\mathrm{Re}(\bar{\psi}\nabla_A\psi) = |\psi||\nabla|\psi||$), we find that

$$
\begin{aligned}
&\mathrm{Re}(\nabla_A\psi\overline{\nabla_A\phi}) \\
&= |\nabla_A\psi|^2\eta^2\min\{|\psi|^{\beta-1}, L\} + 2\eta\nabla\eta(\nabla|\psi|)|\psi|\min\{|\psi|^{\beta-1}, L\} \\
&\quad + (\beta-1)\eta^2|\psi|^{\beta-1}\big|\nabla|\psi|\big|^2\chi_{\{|\psi|^{\beta-1}<L\}} \\
&\geq |\nabla_A\psi|^2\eta^2\min\{|\psi|^{\beta-1}, L\} + 2\eta\nabla\eta(\nabla|\psi|)|\psi|\min\{|\psi|^{\beta-1}, L\}, \quad (3.2)
\end{aligned}
$$

where χ_Ω is the characteristic function on Ω.

By (C.2), there exists $a > 0$ such that $|g(x)| \leq a$ for all $x \in \mathbb{R}^n$. Testing the equation (3.1) with ϕ and using (3.2), we obtain the estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla_A\psi|^2\eta^2\min\{|\psi|^{\beta-1}, L\}dx \\
&\qquad + 2\int_{\mathbb{R}^n} \eta\nabla\eta(\nabla|\psi|)|\psi|\min\{|\psi|^{\beta-1}, L\}dx \\
&\leq \int_{\mathbb{R}^n} a|\psi|^2\eta^2\min\{|\psi|^{\beta-1}, L\}dx. \qquad\qquad (3.3)
\end{aligned}
$$

By the dia-magnetic inequality we have

$$
\begin{aligned}
\frac{1}{2}\eta^2\big|\nabla|\psi|\big|^2 - 2|\psi|^2|\nabla\eta|^2 &\leq \eta^2\big|\nabla|\psi|\big|^2 + (2|\psi||\nabla\eta|)(\eta\nabla|\psi|) \\
&\leq \eta^2|\nabla_A\psi|^2 + 2\eta\nabla\eta(\nabla|\psi|)|\psi|.
\end{aligned}
$$

Then by (3.3) we have

$$\frac{1}{2}\int_{\mathbb{R}^n}\left|\nabla|\psi|\right|^2\eta^2\min\{|\psi|^{\beta-1},\,L\}dx$$

$$\leq 2\int_{\mathbb{R}^n}|\nabla\eta|^2|\psi|^2\min\{|\psi|^{\beta-1},\,L\}dx+\int_{\mathbb{R}^n}a|\psi|^2\eta^2\min\{|\psi|^{\beta-1},\,L\}dx.$$

Letting $L\to\infty$ we obtain

$$\frac{1}{2}\int_{\mathbb{R}^n}\left|\nabla|\psi|\right|^2\eta^2|\psi|^{\beta-1}dx\leq 2\int_{\mathbb{R}^n}|\nabla\eta|^2|\psi|^{\beta+1}dx+\int_{\mathbb{R}^n}a\eta^2|\psi|^{\beta+1}dx.$$

Substituting $w=|\psi|^{(\beta+1)/2}$ in this inequality, we obtain

$$\frac{2}{(\beta+1)^2}\int_{\mathbb{R}^n}|\nabla w|^2\eta^2 dx\leq 2\int_{\mathbb{R}^n}|\nabla\eta|^2w^2 dx+\int_{\mathbb{R}^n}aw^2\eta^2 dx.$$

Then, since

$$\int_{\mathbb{R}^n}|\nabla(w\eta)|^2 dx\leq 2\int_{\mathbb{R}^n}|\nabla w|^2\eta^2 dx+2\int_{\mathbb{R}^n}|\nabla\eta|^2w^2 dx,$$

we obtain

$$\int_{\mathbb{R}^n}|\nabla(w\eta)|^2 dx+\int_{\mathbb{R}^n}w^2\eta^2 dx$$

$$\leq 2\big((\beta+1)^2+1\big)\int_{\mathbb{R}^n}|\nabla\eta|^2w^2 dx+\int_{\mathbb{R}^n}\big(a(\beta+1)^2+1\big)w^2\eta^2 dx. \quad (3.4)$$

The Sobolev embedding theorem yields

$$\|w\eta\|_{H^1(\mathbb{R}^n)}^2\geq S_r^2\|w\eta\|_{L^r(\mathbb{R}^n)}^2, \tag{3.5}$$

where $S_r^2=\inf\{\|\psi\|_{H^1(\mathbb{R}^n)}^2\mid\|\psi\|_{L^r(\mathbb{R}^n)}=1\}$ for any r with $2\leq r<2^*$. From (3.4) and (3.5), we obtain

$$S_r^2\|w\eta\|_{L^r(\mathbb{R}^n)}^2\leq 2\big((\beta+1)^2+1\big)\int_{\mathbb{R}^n}|\nabla\eta|^2w^2 dx$$

$$+\int_{\mathbb{R}^n}\big(a(\beta+1)^2+1\big)w^2\eta^2 dx. \tag{3.6}$$

Let $x_0\in\mathbb{R}^n$. We now make the following additional assumption on η: $0\leq\eta\leq 1$ on $\mathbb{R}^n$, $\eta(x)=1$ in $B(x_0,\rho_1)$, $\eta(x)=0$ outside $B(x_0,\rho_2)$, $|\nabla\eta(x)|\leq 2/(\rho_2-\rho_1)$ on $\mathbb{R}^n$ and $1\leq\rho_1<\rho_2\leq 2$. Fix r with $2<r<2^*$

and set $t = r/2$. Then it follows from (3.6) with $w = |\psi|^{(\beta+1)/2}$, $q = \beta + 1$ and $r = 2t$ that

$$\left(\int_{B(x_0,\rho_1)} |\psi|^{qt} dx\right)^{1/(qt)} \leq \left[\frac{Aq}{(\rho_2 - \rho_1)}\right]^{2/q} \left(\int_{B(x_0,\rho_2)} |\psi|^q dx\right)^{1/q}$$

for some constant A, independent of q, ρ_1, ρ_2. Iterating this inequality with $s_m = 1 + 2^{-m}$, ρ_1 replaced by s_m, ρ_2 replaced by s_{m-1} for $m \geq 1$ and q replaced by qt^{m-1} (> 2), we obtain

$$\left(\int_{B(x_0,s_m)} |\psi|^{qt^m} dx\right)^{1/(qt^m)}$$

$$\leq \left[\frac{Aqt^{m-1}}{(\rho_2 - \rho_1)}\right]^{1/(qt^{m-1})} \left(\int_{B(x_0,s_{m-1})} |\psi|^{qt^{m-1}} dx\right)^{1/(qt^{m-1})}$$

$$= (Aq)^{1/qt^{m-1}} 2^{m/qt^{m-1}} t^{(m-1)/qt^{m-1}} \left(\int_{B(x_0,s_{m-1})} |\psi|^{qt^{m-1}} dx\right)^{1/(qt^{m-1})}$$

$$\leq \dots$$

$$\leq (Aq)^{\sum_{j=0}^{m-1} 1/qt^j} 2^{\sum_{j=0}^{m-1}(j+1)/qt^j} t^{\sum_{j=0}^{m-1} j/qt^j} \left(\int_{B(x_0,s_0)} |\psi|^q dx\right)^{1/q}.$$

Since all the sums above converge, we have

$$\|\psi\|_{L^{qt^m}(B(x_0,1))} \leq C\|\psi\|_{L^q(B(x_0,2))}$$

for some $C = C_{t,q} > 0$, independent of x_0. By letting $m \to \infty$, we conclude that

$$\sup_{B(x_0,1)} |\psi(x)| \leq C\|\psi\|_{L^q(B(x_0,2))}.$$

This implies the boundedness and the decay of ψ at infinity because $\psi \in L^q(\mathbb{R}^n)$ as we mentioned at the beginning of this proof. $\qquad\square$

Lemma 3.3   *Let $n \geq 3$. Assume that A belongs to $L_{\text{loc}}^2(\mathbb{R}^n, \mathbb{R}^n)$. Assume (C.1) and (C.3). Assume that ψ is a solution to (3.1) in $H_{A,V_+}^1(\mathbb{R}^n)$. Then ψ belongs to $L^q(\mathbb{R}^n)$ for any $q \geq 2$.*

Proof. We mimic the proof of Lemma 2.1 in Chabrowski and Szulkin [Ch-Sz]. Let $\beta > 1$ and let γ be as in (C.3). Set $w = |\psi|^{(\beta+1)/2}$. We may assume that $|g(x)| \leq a + b(x)$ for some $a \in \mathbb{R}$ and $b \in L^\gamma$. Repeating the argument used to derive (3.6) in the proof of Lemma 3.2 replaced a by

$a + b$, η by 1, we have

$$S_r^2 \|w\|_{L^r}^2 \leq (\beta+1)^2 \int bw^2 dx + \left(a(\beta+1)^2 + 1\right) \int w^2 dx \qquad (3.7)$$

for any r with $2 \leq r < 2^*$. Here, the constant S_r^2 is as before. The Hölder inequality yields

$$\int bw^2 dx \leq N \int w^2 dx + \left(\int_{b>N} b^\gamma dx\right)^{1/\gamma}\left(\int w^{2\gamma/(\gamma-1)} dx\right)^{(\gamma-1)/\gamma} (3.8)$$

for any $N > 0$.

Setting $r = 2\gamma/(\gamma-1)$ we deduce from (3.7) and (3.8) that

$$\left(S_r^2 - (\beta+1)^2\left(\int_{b>N} b^\gamma dx\right)^{1/\gamma}\right)\|\psi\|_{L^{\gamma(\beta+1)/(\gamma-1)}}^{\beta+1}$$
$$\leq \left((a+N)(\beta+1)^2 + 1\right)\|\psi\|_{L^{\beta+1}}^{\beta+1}$$

for any $N > 0$. Note that the condition $2 \leq r < 2^*$ is equivalent to $\gamma > n/2$.

This shows that if $\psi \in L^q(\mathbb{R}^n)$ for some $q\,(= \beta + 1) > 2$ then $\psi \in L^{q\gamma/(\gamma-1)}(\mathbb{R}^n)$. Iterating this procedure we deduce that $\psi \in L^q(\mathbb{R}^n)$ for all $q > 2$ by interpolation. This completes the proof because $\psi \in H^1_{A,V_+}$ is assumed. $\qquad\square$

Corollary 3.4 *Let $n \geq 2$. Let $p > 2$ if $n = 2$, and $2 < p < 2^*$ if $n \geq 3$. Assume that A belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$. Assume (A.4), (A.5) and (N.2). Then every solution to (1.1) in $H^1_{A,V}(\mathbb{R}^n)$ is bounded and decays at infinity.*

Proof. Let u be such a solution to (1.1) and let p be as above. First, we show the boundedness of u. Obviously $\psi = u$ solves (3.1) with $V_+ = V_+$ and $g = (E + V_-) + Q|u|^{p-2}$. The condition (A.4) implies that the non-negative part V_+ of V belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ because $V_+ \leq |V| \leq |V_1| + |V_2|$ holds. Thus the condition (C.1) holds.

We claim that $|u| + |u|^{p-1} \in L^q(\mathbb{R}^n)$ for all $q \geq 2$. If $n = 2$, this claim follows from Lemma 3.1 because $\psi = u \in H^1_{A,V}(\mathbb{R}^n)$ and $p > 2$. If $n \geq 3$, the condition (C.3) holds because $g = (E + V_-) + Q|u|^{p-2} \in L^\infty(\mathbb{R}^n) + L^{p/(p-2)}(\mathbb{R}^n)$ and $p/(p-2) > n/2$, which is equivalent to $2 < p < 2^*$. Thus it follows from Lemma 3.3 that $\psi = u \in L^q(\mathbb{R}^n)$ for all $q \geq 2$ and therefore $|u| + |u|^{p-1} \in L^q(\mathbb{R}^n)$ for all $q \geq 2$ because $p > 2$. This shows the claim.

Under the assumptions on A and V above, the space $C_0^\infty(\mathbb{R}^n)$ is a form core for the self-adjoint operator $-\Delta_A + V_+ + 1$ with form domain $H^1_{A,V}(\mathbb{R}^n)$,

and moreover the following version of the dia-magnetic inequality

$$|(-\Delta_A + V_+ + 1)^{-1}f| \leq (-\Delta + 1)^{-1}|f|$$

holds for all $f \in C_0^\infty(\mathbb{R}^n)$ (see Lemmata 1 and 6, Theorem 1 in Leinfelder and Simader [Le-Si]). This inequality is still valid for any $f \in L^q(\mathbb{R}^n)$ for any $q > 1$ because $C_0^\infty(\mathbb{R}^n)$ is dense in $L^q(\mathbb{R}^n)$ and the operator $(-\Delta+1)^{-1}$ is bounded on $L^q(\mathbb{R}^n)$.

By the equation (1.1), we have

$$(-\Delta_A + V_+ + 1)u = (E + 1 + V_-)u + Q|u|^{p-2}u$$

and then, by the dia-magnetic inequality above,

$$|u| \leq (-\Delta + 1)^{-1}(|E + 1 + V_-||u| + \|Q\|_{L^\infty}|u|^{p-1}).$$

By the claim above, the right-hand side belongs to $(-\Delta + 1)^{-1}L^q(\mathbb{R}^n) = W^{2,q}(\mathbb{R}^n)$ for any $q \geq 2$. Then the boundedness of u follows from the Sobolev embedding theorem.

Finally, we show the decay of u. The function $\psi = u$ solves the linear equation (3.1) with $V_+ = V_+$ and $g = E + V_- - Q|u|^{p-2}$. The condition (C.1) is satisfied as we mentioned above. The condition (C.2) is satisfied by (N.2), (A.5) and the boundedness of u proven above. Then the assertion follows from Lemma 3.2.   □

Proof of Theorem 1.3. Let u be a solution to (1.1) in $H_{A,V}^1(\mathbb{R}^2)$ and let $V = V_1 + V_2$ be the decomposition as in (A.4). The function $\psi = u$ solves the linear equation

$$-\Delta_A\psi + (W - E)\psi = 0, \tag{3.9}$$

where $W = V - Q|u|^{p-2} = V_1 + (V_2 - Q|u|^{p-2})$. By Corollary 3.4, the L^∞-part $V_2 - Q|u|^{p-2}$ of the potential W decays at infinity, and therefore the theorem follows from Theorem 4.1 in Cornean and Nenciu [Co-Ne].   □

Proof of Theorem 1.4. Let u be the solution as in Theorem 1.4 and let $W = V - Q|u|^{p-2}$. By (A.6) and Theorem 1.3, there exists $\delta > 0$ and $C > 0$ such that $|W(x)| \leq C(e^{-\delta|x|^\beta} + e^{-\alpha|x|}) \leq C\exp(-\delta|x|^{\beta \wedge 1})$ holds outside a compact set. Here, we denote by $a \wedge b$ the minimum of a and b.

We now apply Theorem 4.2 in Cornean and Nenciu [Co-Ne] to the equation (3.9). Note that u is bounded by Corollary 3.4. Then there exist $\mu_1 > 0$ and $C_1 > 0$ such that $|u(x)| \leq C_1\exp(-\mu_1|x|^{\beta_1})$ holds with $\beta_1 =$

$1 + (\beta \wedge 1)/2$. This estimate of u yields the estimate

$$
\begin{aligned}
|W(x)| &\leq C(e^{-\delta|x|^\beta} + e^{-(p-2)\mu_1|x|^{\beta_1}}) \\
&\leq C \exp\left(-(\delta \wedge (p-2)\mu_1)|x|^{\beta \wedge \beta_1}\right),
\end{aligned}
$$

and again by Theorem 4.2 in [Co-Ne] there exist $\mu_2 > 0$ and $C_2 > 0$ such that $|u(x)| \leq C_2 \exp\left(-\mu_2|x|^{\beta_2}\right)$ holds with $\beta_2 = 1 + (\beta \wedge \beta_1)/2$. Repeating this procedure we can deduce that there exist $\mu_j > 0$ and $C_j > 0$ such that $|u(x)| \leq C_j \exp\left(-\mu_j|x|^{\beta_j}\right)$ holds with $\beta_{j+1} = 1 + (\beta \wedge \beta_j)/2$ for any $j \in \mathbb{N}$, where we set $\beta_0 = 1$.

We claim that there exists j such that $\beta_j > \beta$. Otherwise, we have $\beta \geq \beta_j$ for all j, and hence $\beta_{j+1} = 1 + (\beta \wedge \beta_j)/2 = 1 + \beta_j/2$. Then we have $\beta_j = 2 - 2^{-j}$, and therefore $\beta_j > \beta$ for large j since $0 < \beta < 2$. This is a contradiction and we have the claim. Therefore the first assertion in Theorem 1.4 obeys.

When both V and Q have the Gaussian decay, we find that $|W(x)| \leq C \exp\left(-\delta|x|^2\right)$ and then the second assertion in the theorem follows from Theorem 4.3 in [Co-Ne], which is valid also for $\beta = 2$. $\qquad\square$

4. A linking theorem

In this section we recall a linking theorem due to Bartsch and Ding [Ba-Di] (see also [Ba-Di2], [Kr-Sz]). This result is needed in Section 6 below.

For any (real) Banach space X, we denote by w and w^* the weak topology on X and the weak-$*$ topology on the dual space X', respectively. For any $\rho \geq 0$, we write $B_\rho(X) = \{u \in X \,|\, \|u\|_X \leq \rho\}$ and $S_\rho(X) = \{u \in X \,|\, \|u\|_X = \rho\}$. We write $u_k \rightharpoonup u$ for the weak convergence of a sequence $\{u_k\}$ to u.

Let X be a Banach space with direct sum decomposition $X = X_1 \oplus X_2$ and P_{X_j} the corresponding projection onto X_j for $j = 1, 2$. Assume that X_1 is separable and reflexible. For a functional Φ, we write $\Phi_a = \{u \in X \,|\, \Phi(u) \geq a\}$. Recall that a sequence $\{u_k\}$ in X is said to be a $(C)_c$-*sequence for* Φ if $\Phi(u_k) \to c$ and $(1 + \|u_k\|_X)\|\Phi'(u_k)\|_{X'} \to 0$ as $k \to \infty$.

Let $\mathcal{S}$ be a dense subset of X_1'. For each $s \in \mathcal{S}$ we define a seminorm on X by $p_s(u) = |s(u_1)| + \|u_2\|_X$ for any $u = u_1 + u_2 \in X$. We denote by $\mathcal{T}_\mathcal{S}$ the induced topology by the family $\{p_s\}_{s \in \mathcal{S}}$.

To formulate the result we make the following conditions.

(Φ_0) For any $c \in \mathbb{R}$, the set Φ_c is $\mathcal{T}_\mathcal{S}$-closed and the map $\Phi' \colon (\Phi_c, \mathcal{T}_\mathcal{S}) \to$

(X^*, w^*) is continuous.

(Φ_1) For any $c > 0$, there exists $\zeta > 0$ such that $\|u\|_X < \zeta\|P_{X_2}u\|_X$ for any $u \in \Phi_c$.

(Φ_2) There exists $\rho > 0$ such that $\kappa = \inf \Phi(S_\rho(X_2)) > 0$ holds.

Theorem 4.1 ([Ba-Di], Theorem 5.1) *Assume* (Φ_0)–(Φ_2). *Assume further that there exist a positive number R and a unit vector $e \in X_2$ such that $0 < \rho < R$ and $\sup \Phi(\partial U) \leq \kappa$, where $U = \{u = x + te \mid x \in X_1, t \geq 0, \|u\|_X < R\}$. Then Φ has a $(C)_c$-sequence with $\kappa \leq c \leq \sup \Phi(U)$.*

A sufficient condition to (Φ_0) is given by the following lemma.

Lemma 4.2 ([Ba-Di], Proposition 5.4) *Let $X = X_1 \oplus X_2$ be as above. Assume that $\Phi \in C^1(X, \mathbb{R})$ and Φ is of the form $\Phi(u) = (1/2)(\|u_2\|_X^2 - \|u_1\|_X^2) - \Psi(u)$ for any $u = u_1 + u_2 \in X = X_1 \oplus X_2$. Assume*
(i) *The functional Ψ is bounded from below and $\Psi \in C^1(X, \mathbb{R})$.*
(ii) *The map $\Psi \colon (X, w) \to \mathbb{R}$ is sequentially lower semi-continuous, i.e.,*
    $\Psi(u) \leq \liminf \Psi(u_k)$ holds whenever $u_k \rightharpoonup u$ in X.
(iii) *The map $\Psi' \colon (X, w) \to (X', w^*)$ is sequentially continuous.*
(iv) *The map $\nu \colon X \ni u \mapsto \|u\|_X^2 \in \mathbb{R}$ is C^1 and $\nu' \colon (X, w) \to (X', w^*)$ is sequentially continuous.*
Then Φ satisfies the condition (Φ_0).

5.   Spectral property of the linear part

In this section we recall some spectral property of the linear part of the equation (1.1). For the theory of the magnetic Schrödinger operators, we refer to Mohamed and Raikov [Mo-Ra].

In the following we always assume (E.1)–(E.3) and that that $n \geq 2$ and $2 < p < 2^*$. For simplicity, we write H_A^1 for the space $H_{A,0}^1(\mathbb{R}^n)$. Without loss of generality, we may assume that any inner-product on a complex Hilbert space is linear with respect to the first component, i.e., $(\alpha u, \beta v) = \alpha\bar{\beta}(u, v)$ holds for any $\alpha, \beta \in \mathbb{C}$.

Corresponding to the decompositions of A and V as in (E.1) and (E.2), we set

$$H = -(\nabla - iA)^2 + V, \qquad H_0 = -(\nabla - iA_0)^2 + V_0.$$

Both of the operators are essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ under (E.1) and (E.2), and the operator $H - H_0$ is relatively compact with respect to

H_0 because of the decay of A_1 and V_1 (see, e.g., Hempel [Hem], [Mo-Ra]). In particular, the essential spectrum of the operator H coincides with that of H_0. In other words, the operator H may have discrete spectra in the spectral gaps of H_0.

The spectral theory of the magnetic Schrödinger operator H_0 has a rich structure. The Bloch-Floquet analysis tells us that the spectrum of H_0 is the locally finite union of closed intervals if the magnetic flux of $B_0 = dA_0$ over a unit cell $[0,\,1] \times [0,\,1]$ is integer.

Lemma 5.1 *Let $E \in \mathbb{R}$. Assume* (E.1) *and* (E.2). *Then the space H_A^1 coincides with the domain $D(|H-E|^{1/2})$ equipped with the graph norm $|||u|||_E = (|||H-E|^{1/2}u\|_{L^2}^2 + \|u\|_{L^2}^2)^{1/2}$. Moreover, we have*

$$C(E,\,V)^{-1}\|u\|_{H_A^1}^2 \le |||u|||_E^2 \le C(E,\,V)\|u\|_{H_A^1}^2$$

for any $u \in H_A^1$, where $C(E,\,V) = \|V\|_{L^\infty} + 2\max\{E - \inf \mathrm{Spec}(H),\,0\}$.

Proof. Let $P_H(I)$ be the spectral projection of H on I. The self-adjoint operator $|H-E|^{1/2}$ is defined by the spectral representation $\int |\lambda - E|^{1/2} dP_H(\lambda)$. Note that $C_0^\infty(\mathbb{R}^n)$ is a core for both H and $|H-E|^{1/2}$ under (E.1) and (E.2). We denote $\inf \mathrm{Spec}(H)$ by E_0 for simplicity. For any $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$
\begin{aligned}
&|||u|||_E^2 \\
&= \int_{\mathbb{R}} |\lambda - E| d(P_H(\lambda)u,\,u)_{L^2} + \|u\|_{L^2}^2 \\
&= \int_{\mathbb{R}} (\lambda - E) d(P_H(\lambda)u,\,u)_{L^2} - 2\int_{-\infty}^{E} (\lambda - E) d(P_H(\lambda)u,\,u)_{L^2} + \|u\|_{L^2}^2 \\
&= (u,\,(H-E)u)_{L^2} - 2(u,\,(H-E)P_H((-\infty,\,E])u)_{L^2} + \|u\|_{L^2}^2 \\
&= \|u\|_{H_A^1}^2 + (u,\,(V-E)u)_{L^2} - 2(u,\,(H-E)P_H((-\infty,\,E])u)_{L^2},
\end{aligned}
$$

and

$$
\begin{aligned}
&|(u,\,(V-E)u)_{L^2} - 2(u,\,(H-E)P_H((-\infty,\,E])u)_{L^2}| \\
&\le (\|V\|_{L^\infty} + |E| + 2(E - E_0) \vee 0)\|u\|_{L^2}^2.
\end{aligned}
$$

Here, we used the fact that support of P_H is contained in $[E_0,\,\infty)$. This completes the proof. $\qquad\square$

We introduce the magnetic translation with respect to A_0. For each

$\gamma \in \mathbb{Z}^n$ there exists a real-valued smooth function φ_γ on $\mathbb{R}^n$ such that

$$A_0(x + \gamma) - A_0(x) = d\varphi_\gamma(x) \tag{5.1}$$

holds for any $x \in \mathbb{R}^n$ because $d(A_0(x+\gamma) - A_0(x)) = dA_0(x+\gamma) - dA_0(x) = 0$ by (E.1) and $\mathbb{R}^n$ is simply connected. We find that for each $\gamma \in \mathbb{Z}^n$ there exists a real constant $C(\gamma)$ such that the cocycle condition $\varphi_{-\gamma}(x) = -\varphi_\gamma(x - \gamma) + C(\gamma)$ holds for all $x \in \mathbb{R}^n$. It is not hard to verify that $C(\gamma) = C(-\gamma)$ and $2\varphi_0(x) = C(0)$. We may assume that $\varphi_0 = 0$ and $C(0) = 0$ without loss of generality.

For any $\gamma \in \mathbb{Z}^n$ we define the magnetic translation S_γ by

$$(S_\gamma u)(x) = e^{-i\varphi_\gamma(x)} u(x + \gamma) \tag{5.2}$$

for any $u \in H_A^1$. We find that $S_\gamma^{-1} = e^{iC(\gamma)} S_{-\gamma}$ and therefore S_γ is a unitary operator on $L^2(\mathbb{R}^n)$.

Lemma 5.2 *Let S_γ be as above. For any $\gamma \in \mathbb{Z}^n$ we have the following assertions*:
(i) $\nabla_{A_0} S_\gamma = S_\gamma \nabla_{A_0}$ *and* $H_0 S_\gamma = S_\gamma H_0$ *hold on* $C_0^\infty(\mathbb{R}^n)$.
(ii) $H S_\gamma = S_\gamma H_0 + S_\gamma R_1(\gamma)$ *holds on* $C_0^\infty(\mathbb{R}^n)$, *where we set*

$$\begin{aligned}
R_1(\gamma) &= i\nabla_{A_0} \circ A_1(\cdot - \gamma) + iA_1(\cdot - \gamma) \circ \nabla_{A_0} + (A_1^2 + V_1)(\cdot - \gamma) \\
&= i\nabla_A \circ A_1(\cdot - \gamma) + iA_1(\cdot - \gamma) \circ \nabla_A + (A_1^2 + V_1)(\cdot - \gamma) \\
&\quad - 2A_1 \circ (A_1(\cdot - \gamma)),
\end{aligned}$$

where the notation "$\circ$" stands for the composition of operators. (For example, $(A_1 \circ (A_1(\cdot - \gamma))u)(x) = A_1(x)A_1(x - \gamma)u(x)$.)
(iii) *The operator S_γ defines an isometric isomorphism on $L^q(\mathbb{R}^n)$ for any q.*

Proof.   It is easy to see that the operator identity $S_\gamma^{-1} F S_\gamma = F(\cdot - \gamma)$ holds for any multiplication operator F and any $\gamma \in \mathbb{Z}^n$. In particular, every S_γ commutes with any multiplication operators by $\mathbb{Z}^n$-periodic functions. Then the assertion (i) follows from (5.1) and this fact. The assertion (ii) follows from a direct computation and the identity $H - H_0 = i\nabla_{A_0} \circ A_1 + iA_1 \nabla_{A_0} + A_1^2 + V_1$. The assertion (iii) follows from the identity $|S_\gamma u(x)| = |u(x + \gamma)|$.
$\square$

Lemma 5.3 *Every S_γ defines a homeomorphism on H_A^1. Moreover, the operator S_γ is uniformly bounded on H_A^1 with respect to γ.*

Proof. By Lemma 5.2 (i) we have

$$\nabla_A S_\gamma = (\nabla_{A_0} - iA_1)S_\gamma = S_\gamma\big(\nabla_A + i(A_1 - A_1(\,\cdot\, - \gamma))\big).$$

Then for any $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}
\|S_\gamma u\|_{H_A^1}^2 &= \|\nabla_A S_\gamma u\|_{L^2}^2 + \|S_\gamma u\|_{L^2}^2 \\
&\leq (\|\nabla_A u\|_{L^2} + 2\|A_1\|_{L^\infty}\|u\|_{L^2})^2 + \|u\|_{L^2}^2 \\
&\leq (8\|A_1\|_{L^\infty}^2 + 2)\|u\|_{H_A^1}^2,
\end{aligned}$$

where we used the unitarity of S_γ on L^2. This shows that S_γ maps H_A^1 to itself and $\sup_\gamma \|S_\gamma\| \leq 8\|A_1\|_{L^\infty}^2 + 2$. The rest of the assertion follows from $S_\gamma^{-1} = e^{iC(\gamma)}S_{-\gamma}$. $\square$

6. The energy functional and its properties

In this section we define an energy functional associated with the equation (1.1) and show that the functional possesses the linking geometry as in Section 4. The functional has consequently a $(C)_c$-sequence (Proposition 6.7 below).

We first recall that any complex Hilbert space $\mathcal{H}$ with inner product $(\,\cdot\,,\,\cdot\,)_\mathcal{H}$ has a natural real Hilbert space structure, i.e., $\mathcal{H}_r$ ($= \mathcal{H}$ as a set) with real inner product $\text{Re}(\,\cdot\,,\,\cdot\,)_\mathcal{H}$. In the sequel we often write $\mathcal{H}$ also for $\mathcal{H}_r$ if there is no fear of confusion.

In the rest of this paper we assume that the real constant E does not belong to the essential spectrum of H. We introduce the new norm $(\||H - E|^{1/2}u\|_{L^2}^2 + \|P_H(\{E\})u\|_{L^2})^{1/2}$ on $D(|H - E|^{1/2})$, which is equivalent to the graph norm $\||\,\cdot\,\||$ as in Lemma 5.1. In what follows we adopt this new norm on $D(|H - E|^{1/2})$.

Let X be the real Hilbert space $\mathcal{H}_r$ for $\mathcal{H}_c = D(|H - E|^{1/2})$ with the norm (and the induced inner product) as above. More precisely, X coincides with $H_A^1 = D(|H - E|^{1/2})$ as a set and the inner product $(\,\cdot\,,\,\cdot\,)_X$ is given by

$$(u, v)_X = \text{Re}(|H - E|^{1/2}u, |H - E|^{1/2}v)_{L^2} + \text{Re}(P_H(\{E\})u, v)_{L^2}$$

for any $u, v \in X$.

Corresponding to the spectral decomposition

$$H_A^1 = P_H\big((-\infty, E)\big)H_A^1 \oplus P_H(\{E\})H_A^1 \oplus P_H\big((E, \infty)\big)H_A^1,$$

we have the orthogonal decomposition $X = X^- \oplus X^0 \oplus X^+$. The spectral theorem yields $\|u\|_X^2 = \|u^-\|_X^2 + \|u^+\|_X^2 + \|u^0\|_{L^2}^2$ for any $u = u^- + u^0 + u^+ \in X$.

Let $2 < p < 2^*$. We define two functionals Φ and Ψ on X by

$$\Phi(u) = \frac{1}{2}\big(\|u^+\|_X^2 - \|u^-\|_X^2\big) - \frac{1}{p}\int_{\mathbb{R}^n} Q|u|^p dx, \tag{6.1}$$

$$\Psi(u) = \frac{1}{p}\int_{\mathbb{R}^n} Q|u|^p dx,$$

where $u = u^- + u^0 + u^+ \in X = X^- \oplus X^0 \oplus X^+$. The functionals are well-defined on X by Lemmata 5.1 and 3.1 because Q is bounded.

Lemma 6.1 *We have the following assertions*:
(i) *The functional Φ belongs to $C^1(X, \mathbb{R})$ and has the Fréchet derivative Φ' given by*

$$_{X'}\langle\Phi'(u),\, h\rangle_X = \mathrm{Re}(u^+,\, h^+)_X$$
$$- \mathrm{Re}(u^-,\, h^-)_X - \mathrm{Re}\int Q|u|^{p-2}u\bar{h}dx, \tag{6.2}$$

   where $h = h^- + h^0 + h^+ \in X$. Here, the notation $_{X'}\langle\,\cdot\,,\,\cdot\,\rangle_X$ stands for the pairing of X' and X.
(ii) *The functional Ψ belongs to $C^1(X, \mathbb{R})$ and satisfies*

$$\|\Psi'(u)\|_{X'} \le C\|Q\|_{L^\infty}^{1/p}\Big(\int Q|u|^p dx\Big)^{(p-1)/p} \tag{6.3}$$

   for any $u \in X$ and some $C > 0$.

Proof.   The proof of the assertion (i) and the first part of the assertion (ii) is standard; in fact, one can show the continuity of the Gâteaux derivative of Φ. We omit the detail and refer to Willem [Wil].

The inequality (6.3) follows from the estimate

$$|_{X'}\langle\Psi'(u),\, h\rangle_X| \le \int |Q^{1/p}u|^{p-1}|Q^{1/p}h|dx$$
$$\le \|Q\|_{L^\infty}^{1/p}\Big(\int (|Q^{1/p}u|^{p-1})^{p/(p-1)}dx\Big)^{(p-1)/p}\|h\|_{L^p}$$
$$\le C_p\|Q\|_{L^\infty}^{1/p}\Big(\int Q|u|^p dx\Big)^{(p-1)/p}\|h\|_{H_A^1},$$

where we used the Hölder inequality in the second inequality and Lemma 3.1

in the last inequality. □

Lemma 6.2 *For any $p \geq 2$, there exists a positive constant C_p such that*

$$\big||u + v|^{p-2}(u + v) - |u|^{p-2}u\big| \leq C_p|v|(|u|^{p-2} + |v|^{p-2})$$

holds for any $u, v \in \mathbb{C}$.

Proof. The assertion for $p = 2$ is obvious. Let $p > 2$ and set $f(x, y) = (x^2 + y^2)^{(p-2)/2}x$ for $(x, y) \in \mathbb{R}^2$. We have $f_x(x, y) = (p-2)(x^2+y^2)^{(p-2)/2-1}x^2 + (x^2 + y^2)^{(p-2)/2}$ and $f_y(x, y) = (p-2)(x^2 + y^2)^{(p-2)/2-1}xy$. By the Taylor expansion

$$\begin{aligned} f(x + h, y + k) &- f(x, y) \\ &= hf_x(x + \theta h, y + \theta k) + kf_y(x + \theta h, y + \theta k), \end{aligned}$$

where $\theta \in [0, 1]$, there exists $C_p > 0$ such that

$$\begin{aligned} |f(x + h, y + k) &- f(x, y)| \\ &\leq C_p|h + ik|(|x + iy|^{p-2} + |h + ik|^{p-2}) \quad (6.4) \end{aligned}$$

holds for all $x, y, h, k \in \mathbb{R}$.

The lemma follows from (6.4) and (6.5) because

$$\begin{aligned} |u + v|^{p-2}&(u + v) - |u|^{p-2}u \\ &= (f(x + h, y + k) - f(x, y)) + i(f(y + k, x + h) - f(y, x)) \quad (6.5) \end{aligned}$$

for any $u = x + iy$, $v = h + ik$. □

Lemma 6.3 *The functional Φ satisfies the condition (Φ_0) in Section 4.*

Proof. We verify (i)–(iv) in Lemma 4.2 with $X = X$, $X_1 = X^- \oplus X^0$, $X_2 = X^+$, $\Phi = \Phi$, and $\Psi = \Psi$. The property (i) follows from the positivity of Q and the proof of Lemma 6.1. Let $u_k \rightharpoonup u$ in X. Lemma 3.1 (iii) implies that $u_k(x) \to u(x)$ a.e. Then the property (ii) follows from Fatou's lemma.

We show the property (iii) in Lemma 4.2. Let $u_k \rightharpoonup u$ in X and let $r > 0$. For any $h \in C_0^\infty(\{x \in \mathbb{R}^n \mid |x| < r\})$, we have

$$\begin{aligned} |_{X'}\langle\Psi'&(u_k), h\rangle_X - _{X'}\langle\Psi'(u), h\rangle_X| \\ &\leq \|Q\|_{L^\infty}\int_{|x|\leq r}\big||u_k|^{p-2}u_k - |u|^{p-2}u\big||h|dx \end{aligned}$$

$$\leq C_p \|Q\|_{L^\infty} \int_{|x|\leq r} |u_k - u|(|u|^{p-2} + |u_k - u|^{p-2})|h|dx$$

$$\leq C\|u_k - u\|_{L^p(|x|\leq r)}(\|u_k - u\|_{L^p(|x|\leq r)}^{p-2} + \|u\|_{L^p(|x|\leq r)}^{p-2})\|h\|_{H_A^1} \quad (6.6)$$

for some $C > 0$, where we used Lemma 6.2 in the second inequality, the Hölder inequality, and Lemma 3.1 in the third inequality. The rightmost of (6.6) tends to zero as $k \to \infty$ because $u_k \to u$ in $L^p(|x| \leq r)$ by Lemma 3.1.

For any $h \in X$ there exists a sequence $\{h_j\}$ in $C_0^\infty(\mathbb{R}^n)$ such that $h_j \to h$ in X. Then we have

$$|_{X'}\langle \Psi'(u_k), h\rangle_X - {}_{X'}\langle \Psi'(u), h\rangle_X|$$

$$\leq |_{X'}\langle \Psi'(u_k), h - h_j\rangle_X| + |_{X'}\langle \Psi'(u_k) - \Psi'(u), h_j\rangle_X|$$

$$+ |_{X'}\langle \Psi'(u), h - h_j\rangle_X|$$

$$\leq \sup_k \|\Psi'(u_k)\|_{X'}\|h - h_j\|_X + |_{X'}\langle \Psi'(u_k) - \Psi'(u), h_j\rangle_X|$$

$$+ \|\Psi'(u)\|_{X'}\|h - h_j\|_X. \quad (6.7)$$

The first and third terms in the rightmost of (6.7) tend to zero as $j \to \infty$ because $\sup_k \|\Psi'(u_k)\|_{X'}$ is finite by (6.3) and the boundedness of $\{u_k\}$ in X. The second term in the rightmost of (6.7) tends to zero as $k \to \infty$ for each $h_j \in C_0^\infty$ by (6.6). This implies (iii).

We show the property (iv). Let $\nu(u) = \|u\|_X^2$. We have $_{X'}\langle \nu'(u), h\rangle_X = 2\,\mathrm{Re}(u, h)_X$ for any $u, h \in X$, and therefore

$$|_{X'}\langle \nu'(u + v), h\rangle_X - {}_{X'}\langle \nu'(u), h\rangle_X| \leq 2|(v, h)_X| \leq 2\|v\|_X\|h\|_X.$$

These two inequalities show the continuity of ν' and the continuity of the map $\nu' \colon (X, w) \to (X', w^*)$. This completes the proof. $\qquad \square$

Lemma 6.4 *Let F be a finite-dimensional subspace of X and let P_F be the projection from X onto F. There exists a positive constant C_F such that*

$$C_F^{-1}\|P_F u\|_X^p \leq \int Q|P_F u|^p dx \leq C_F \int Q|u|^p dx$$

holds for any $u \in X$.

Proof. Note that any finite-dimensional subspace of a Banach space is topologically complemented. We introduce a Banach space $X_Q = \{u \mid \|u\|_{X_Q} < \infty\}$ with norm $\|u\|_{X_Q} = (\int Q|u|^p dx)^{1/p}$. By Lemma 3.1 and

the boundedness of Q, the inclusion $X \subset X_Q$ is continuous and then F is also regarded as a finite-dimensional subspace of the Banach space $\overline{X}^{X_Q}$, the closure of X in X_Q. It is not hard to see that P_F coincides with the projection $\tilde{P}_F$ from $\overline{X}^{X_Q}$ onto F restricted to X. Then the second inequality in the lemma follows from the continuity of $\tilde{P}_F$. The first inequality follows because any norms are equivalent on a finite-dimensional subspace. $\qquad\square$

Lemma 6.5   *The functional Φ satisfies the condition (Φ_1) in Section 4.*

Proof.   The condition $0 < c \le \Phi(u)$ implies that

$$2c + \|u^-\|_X^2 + \frac{2}{p}\int Q|u|^p dx \le \|u^+\|_X^2. \tag{6.8}$$

By Lemma 6.4 with $F = X^0$ there exists $C > 0$ such that

$$C^{-1}\|u^0\|_X^p \le \int Q|u^0|^p dx \le C\int Q|u|^p dx. \tag{6.9}$$

Thus it follows from (6.8) and (6.9) that

$$\begin{aligned}\|u\|_X^2 &= \|u^-\|_X^2 + \|u^0\|_X^2 + \|u^+\|_X^2 \le 2\|u^+\|_X^2 + C\|u^+\|_X^{4/p}\\ &\le C_c\|u^+\|_X^2\end{aligned} \tag{6.10}$$

because $4/p < 2$ and $c > 0$. This completes the proof. $\qquad\square$

Lemma 6.6   *The functional Φ satisfies the condition (Φ_2) in Section 4.*

Proof.   For any $u^+ \in X^+$ with $\|u^+\|_X = \rho$, we have

$$\Phi(u^+) \ge \frac{1}{2}\rho^2 - \frac{1}{p}C_p\|Q\|_{L^\infty}\rho^p, \tag{6.11}$$

where we used $\int Q|u|^p dx \le \|Q\|_{L^\infty}\|u\|_{L^p}^p \le C_p\|Q\|_{L^\infty}\|u\|_X^p$. An elementary calculation shows that the right-hand side of (6.11) takes the maximum $\kappa = (1/2 - 1/p)(C_p\|Q\|_{L^\infty})^{-2/(p-2)}$ at $\rho = (C_p\|Q\|_{L^\infty})^{-1/(p-2)}$ (with common C_p). This completes the proof. $\qquad\square$

Proposition 6.7   *There exist positive constants C_p and M such that Φ has a $(\mathrm{C})_c$-sequence with*

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)(C_p\|Q\|_{L^\infty})^{-2/(p-2)} \le c \le M.$$

Proof. We now apply Theorem 4.1. Take and fix $e \in X^+$ with $\|e\|_X = 1$. Let $R > 0$ and $U_R = \{u = te + z \mid z = u^- + u^0 \in X^- \oplus X^0,\, t \geq 0,\, \|u\|_X < R\}$.

By Lemma 6.4 with $F =$ "the one-dimensional subspace spanned by e", there exists $C > 0$ such that

$$\int Q|te|^p dx \leq C \int Q|u|^p dx \tag{6.12}$$

holds for any $u = te + u^- + u^0 \in U_R$. Similarly, by Lemma 6.4 with $F = X^0$, there exists $C > 0$ such that

$$\|u^0\|_X^p \leq C \int Q|u|^p dx \tag{6.13}$$

holds for any $u = te + u^- + u^0 \in U_R$.

We show that $\Phi(\partial U_R) \leq 0$ for some $R\ (> \rho > 0)$, where ρ is as in the proof of Lemma 6.6. Note that $\partial U_R = \{u = te + z \mid t > 0,\, \|u\| = R\} \cup \{u \in X^- \oplus X^0 \mid \|u\| \leq R\}$ and obviously Φ is non-positive on the second component. On the first component of ∂U_R above, by (6.13), we have

$$\Phi(u) \leq \frac{1}{2}(t^2 - \|u^-\|_X^2) - C\|u^0\|_X^p. \tag{6.14}$$

If $\|u^0\|_X \geq 1$, then $\|u^0\|_X^2 \leq \|u^0\|_X^p$ holds and the right-hand side of (6.14) is less than or equal to

$$\frac{1}{2}(t^2 - \|u^-\|_X^2) - C\|u^0\|_X^2 \leq \frac{t^2}{2} - \min\left\{\frac{1}{2}, C\right\}(R^2 - t^2)$$
$$\leq C(t^2 - C'R^2),$$

where we used the relation $t^2 + \|u^-\|_X^2 + \|u^0\|_X^2 = R^2$ in the first inequality. If $\|u^0\|_X \leq 1$, then $t^2 + \|u^-\|_X^2 + 1 \geq R^2$ holds and the right-hand side of (6.14) is less than or equal to

$$\frac{1}{2}(t^2 - \|u^-\|_X^2) \leq \frac{t^2}{2} - \frac{R^2 - t^2 - 1}{2} \leq C(t^2 - C'R^2).$$

Hence, if $0 \leq t^2 \leq C'R^2$, we have $\Phi(u) \leq 0$ for any $u = te + z \in \partial U_R$.

By (6.12), we have

$$\Phi(u) \leq \frac{1}{2}(t^2 - \|u^-\|_X^2) - \frac{C''}{p}t^p \int Q|e|^p dx \tag{6.15}$$

for any $u \in U_R$. Let $t = t_0 > 0$ be the larger zero of $t^2/2 - C''t^p \int Q|e|^p dx/p$.

We take and fix R so that $C'R^2 \geq t_0^2$ for the constant C' above and $R > \rho$. Hence, if $t^2 \geq C'R^2$, we have $\Phi(u) \leq 0$ for any $u = te + z \in U_R$. Thus it follows that $\Phi(u) \leq 0$ holds for any $u \in U_R$.

Finally, the bound $M = \sup \Phi(U_R)$ is given by the estimate

$$\Phi(u) \leq \frac{1}{2}R^2 + \frac{1}{p}C_p\|Q\|_{L^\infty}R^p$$

for any $u \in U_R$. We have now verified all the assumptions as in Theorem 4.1 and then have the conclusion. $\qquad\square$

In the concentration-compactness argument, the invariance of the functional under a certain group action plays an important role. Unfortunately, the functional Φ is not invariant and therefore the derivative Φ' is not equivariant under the magnetic translations. This is caused by the fact that the magnetic translation S_γ (for A_0) does not commute with H. The invariance and the equivariance, however, remain alive with small perturbation because $H - H_0$ is relatively compact with respect to H_0.

Lemma 6.8 *For any u, $h \in X$ and any $\gamma \in \mathbb{Z}^n$, we have*

$$_{X'}\langle \Phi'(S_\gamma u), h \rangle_X = {}_{X'}\langle \Phi'(u), S_\gamma^{-1}h \rangle_X + \mathrm{Re}(u, S_\gamma^{-1}R_2(\gamma)h)_{L^2}$$
$$+ \mathrm{Re}\int_{\mathbb{R}^n} (Q(x+\gamma) - Q(x))|S_\gamma u(x)|^{p-2}S_\gamma u(x)\overline{h(x)}dx,$$

where we set

$$R_2(\gamma) = i(A_1 - A_1(\cdot + \gamma)) \circ \nabla_{A_0} + i\nabla_{A_0} \circ (A_1 - A_1(\cdot + \gamma))$$
$$+ (A_1^2 + V_1) - (A_1^2 + V_1)(\cdot + \gamma).$$

Here, the notation "$\circ$" stands for the composition of operators.

Proof. By Lemma 5.2 (ii) we have

$$\Phi(S_\gamma u) = \Phi(u) + \frac{1}{2}(u, (R_1(\gamma) + H_0 - H)u)_{L^2}$$
$$+ \frac{1}{p}\int (Q(x) - Q(x-\gamma))|u(x)|^p dx \qquad (6.16)$$

for any $u \in C_0^\infty$. A simple calculation shows that

$$(R_1(\gamma) + H_0 - H)S_\gamma^{-1} = S_\gamma^{-1}R_2(\gamma). \qquad (6.17)$$

By differentiating (6.16) at u in the direction $S_\gamma^{-1}h$, we have

$$
\begin{aligned}
{}_{X'}\langle \Phi'(S_\gamma u), h\rangle_X \\
= {}_{X'}\langle \Phi'(u), S_\gamma^{-1}h\rangle_X + \mathrm{Re}(u, S_\gamma^{-1}R_2(\gamma)h)_{L^2} \\
+ \mathrm{Re}\int (Q(x+\gamma) - Q(x))|S_\gamma u(x)|^{p-2}S_\gamma u(x)\overline{h(x)}dx,
\end{aligned}
$$

where we used (6.17) and the unitarity of S_γ on $L^2(\mathbb{R}^n)$.   □

7.   Proof of Theorem 1.6

In this section we give a proof of Theorem 1.6. To the purpose, we show the boundedness of the $(\mathrm{C})_c$-sequence obtained in Proposition 6.7 and then use a concentration-compactness type argument.

Lemma 7.1   *Let $c > 0$. Any $(\mathrm{C})_c$-sequence for Φ is bounded in X.*

Proof.   Let $\{u_k\}$ be a $(\mathrm{C})_c$-sequence for Φ for $c > 0$. By Lemma 6.5, it is enough to show the boundedness of $\{u_k^+\}$. From

$$
{}_{X'}\langle \Phi'(u_k), u_k^+\rangle_X = \|u_k^+\|_X^2 - \mathrm{Re}\int Q|u_k|^{p-2}u_k\overline{u_k^+}dx,
$$

it follows that

$$
\begin{aligned}
\|u_k^+\|_X^2 &\le \|\Phi'(u_k)\|_{X'}\|u_k^+\|_X \\
&\quad + C\|Q\|_{L^\infty}^{1/p}\left(\int Q|u_k|^p dx\right)^{(p-1)/p}\|u_k^+\|_X,
\end{aligned}
\tag{7.1}
$$

where we used the Hölder inequality and Lemma 3.1. From

$$
\Phi(u_k) - \frac{1}{2}{}_{X'}\langle \Phi'(u_k), u_k\rangle_X = \left(\frac{1}{2} - \frac{1}{p}\right)\int Q|u_k|^p dx,
\tag{7.2}
$$

it follows that there exists $C > 0$ such that

$$
\int Q|u_k|^p dx \le C(1 + \|\Phi'(u_k)\|_{X'}\|u_k\|_X)
\tag{7.3}
$$

holds for large k.

By (7.1) and (7.3), we have

$$
\begin{aligned}
\|u_k^+\|_X &\le \|\Phi'(u_k)\|_{X'} + C\|Q\|_{L^\infty}^{1/p}(1 + \|\Phi'(u_k)\|_{X'}\|u_k\|_X)^{(p-1)/p} \\
&\le \|\Phi'(u_k)\|_{X'} + C'\|Q\|_{L^\infty}^{1/p}(1 + \|\Phi'(u_k)\|_{X'}\|u_k\|_X) \\
&\le \|\Phi'(u_k)\|_{X'} + C_c''\|Q\|_{L^\infty}^{1/p}(1 + \|\Phi'(u_k)\|_{X'}\|u_k^+\|_X),
\end{aligned}
$$

where we used the fact that $(p-1)/p < 1$ in the second inequality, and used (6.10) in the last inequality. This shows the boundedness of $\{u_k^+\}$ because $\|\Phi'(u_k)\|_{X'}$ tends to zero as $k \to \infty$. $\qquad\square$

The following magnetic version of the Lions lemma is needed in the concentration-compactness type argument below.

Lemma 7.2 *Fix $r > 0$. Let $2 \le q < 2^*$. Assume that $\{u_k\}$ is bounded in H_A^1 and satisfies $\lim_{k\to\infty} \sup_{y\in\mathbb{R}^n} \int_{|x-y|\le r} |u_k(x)|^q dx = 0$. Then $\{u_k\}$ converges to 0 in $L^t(\mathbb{R}^n)$ if $2 < t < 2^*$.*

Proof. If $\{u_k\}$ is bounded in H_A^1 then $\{|u_k|\}$ is bounded in $H^1(\mathbb{R}^n)$ by Lemma 3.1. Then the result follows from the standard Lions lemma (see, e.g., Willem [Wil]). $\qquad\square$

Lemma 7.3 *Let $\{u_k\}$ be a $(\mathrm{C})_c$-sequence for Φ with $c > 0$. For each $r > 0$ there exist a positive number η and a sequence $\{y_k\}$ in $\mathbb{Z}^n$ such that*

$$\liminf_{k\to\infty} \int_{|x-y_k|\le r+\sqrt{n}} |u_k(x)|^2 dx \ge \eta.$$

Proof. By Lemma 7.1, the sequence $\{u_k\}$ is bounded in X. By (7.2) we have

$$\Phi(u_k) - \frac{1}{2}{}_{X'}\langle\Phi'(u_k), u_k\rangle_X = \left(\frac{1}{2} - \frac{1}{p}\right)\int Q|u_k|^p dx \le \|Q\|_{L^\infty}\|u_k\|_{L^p}^p.$$

By taking a limit $k \to \infty$ we have

$$\liminf_{k\to\infty} \|u_k\|_{L^p} \ge C > 0 \tag{7.4}$$

because $\Phi(u_k) \to c \ (> 0)$ and $\Phi'(u_k) \to 0$ as $k \to \infty$. In particular, the sequence $\{u_k\}$ cannot converge to 0 in $L^p(\mathbb{R}^n)$. Then it follows from Lemma 7.2 that, for any $r > 0$ and q with $2 \le q < 2^*$, there exists $\eta > 0$ such that $\lim_{k\to\infty} \sup_z \int_{|x-z|<r} |u_k(x)|^q dx \ge 2\eta$ holds. This implies that for any $r > 0$ and q with $2 \le q < 2^*$ there exists a sequence $\{z_k\}$ in $\mathbb{R}^n$ such that $\int_{|x-z_k|<r} |u_k(x)|^q dx \ge \eta$ for large k.

For any $z_k \in \mathbb{R}^n$, there exists $y_k \in \mathbb{Z}^n$ such that $|y_k - z_k| < \sqrt{n}$. Since

$$\int_{|x-y_k|\le r+\sqrt{n}} |u_k|^q dx \ge \int_{|x-z_k|\le r} |u_k|^q dx,$$

putting $q = 2$, we have the lemma. $\qquad\square$

Let $\{u_k\}$ be the $(\mathrm{C})_c$-sequence as in Proposition 6.7 and let $\{y_k\}$ be the sequence as in Lemma 7.3 for this $(\mathrm{C})_c$-sequence.

First, we consider the case where $\{y_k\}$ is bounded in $\mathbb{R}^n$. Because $\{u_k\}$ is bounded in X by Lemma 7.1 and X is reflexible, there exists a subsequence, which is denoted also by $\{u_k\}$, such that $u_k \rightharpoonup u$ in X, and $u_k \to u$ in L_{loc}^t if $2 \leq t < 2^*$, and $u_k(x) \to u(x)$ a.e. for some $u \in X$.

We may assume that $\{y_k\}$ converge to a point in $\mathbb{R}^n$ because of the boundedness of $\{y_k\}$. It follows from Lemma 7.3 that the limit u of $\{u_k\}$ does not vanish near the point. Thus $u \neq 0$ in X because $\|u\|_{L^2} \leq C\|u\|_X$.

For any $h \in C_0^\infty(\mathbb{R}^n)$, we have

$$_{X'}\langle \Phi'(u_k), h\rangle_X = \mathrm{Re}(u_k, (H-E)h)_{L^2} - \mathrm{Re}\int Q|u_k|^{p-2}u_k\overline{h}dx. \quad (7.5)$$

Letting $k \to \infty$ on the both sides, we obtain

$$0 = \mathrm{Re}(u, (H-E)h)_{L^2} - \mathrm{Re}\int Q|u|^{p-2}u\overline{h}dx \quad (7.6)$$

because the derivatives Φ' and Ψ' are both weakly sequentially continuous as is shown in the proof of Lemma 6.3. (See (6.6).) We can eliminate "Re" by considering ih instead of h. Therefore u is a nontrivial solution to the equation (1.1). (In fact, u belongs to $C^\infty(\mathbb{R}^n)$ by elliptic regularity.)

Second, we consider the case where $\{y_k\}$ is unbounded (and we show that it is impossible). By passing to a subsequence, we may assume that $\lim_{k\to\infty}|y_k| = \infty$. Let S_γ be the magnetic translation as in (5.2) and set $v_k = S_{y_k}u_k$. The sequence $\{v_k\}$ is also bounded in X by Lemmata 5.1 and 5.3. By the same reason as in the first case, we may assume that $v_k \rightharpoonup v$ in X, $v_k \to v$ in L_{loc}^p, and $v_k(x) \to v(x)$ a.e. for some $v \in X$, and we conclude that $v \neq 0$ in X because $\|v_k\|_{L^2} = \|u_k\|_{L^2} \geq \eta/2$ holds for large k by Lemma 7.3.

By the same argument used to derive (7.6) from (7.5) (replaced u_k by v_k), we have

$$\lim_{k\to\infty} {}_{X'}\langle \Phi'(v_k), h\rangle_X = \mathrm{Re}(v, (H-E)h)_{L^2} - \mathrm{Re}\int Q|v|^{p-2}v\overline{h}dx \quad (7.7)$$

for any $h \in C_0^\infty(\mathbb{R}^n)$.

On the other hand, by Lemma 6.8 with $\gamma = y_k$ and $u = u_k$, we have

$$_{X'}\langle \Phi'(v_k),\, h\rangle_X$$
$$= {}_{X'}\langle \Phi'(u_k),\, S_{y_k}^{-1}h\rangle_X + \mathrm{Re}(v_k,\, R_2(y_k)h)_{L^2}$$
$$+ \mathrm{Re}\int (Q(x+y_k) - Q(x))|v_k(x)|^{p-2}v_k(x)\overline{h(x)}dx \qquad (7.8)$$

for any $h \in C_0^\infty(\mathbb{R}^n)$. The first term on the right-hand side of (7.8) tends to zero as $k \to \infty$ because $\{u_k\}$ is a $(C)_c$-sequence and $\sup_\gamma \|S_\gamma\|$ is finite by Lemma 5.3. For the second term, using estimates similar to (6.6) in the proof of Lemma 6.3, we have

$$\lim_{k\to\infty}(v_k,\, R_2(y_k)h)_{L^2}$$
$$= \lim_{k\to\infty}(v_k,\, i(A_1 - A_1(\,\cdot\, + y_k))\circ \nabla_{A_0}h)_{L^2}$$
$$+ \lim_{k\to\infty}(\nabla_{A_0}v_k,\, i(A_1 - A_1(\,\cdot\, + y_k))h)_{L^2}$$
$$+ \lim_{k\to\infty}(v_k,\, [(A_1^2 + V_1) - (A_1^2 + V_1)(\,\cdot\, + y_k)]h)_{L^2}$$
$$= (v,\, iA_1\nabla_{A_0}h)_{L^2} + (\nabla_{A_0}v,\, iA_1h)_{L^2} + (v,\, (A_1^2 + V_1)h)_{L^2}$$
$$= (v,\, (H - H_0)h)_{L^2}$$

for any $h \in C_0^\infty(\mathbb{R}^n)$, because both V_1 and A_1 are bounded and decay at infinity. Similarly, the third term on the right-hand side of (7.8) tends to $-\mathrm{Re}\int Q|v|^{p-2}v\overline{h}dx$ as $k \to \infty$. Hence, by taking a limit on both sides of (7.8), we have

$$\lim_{k\to\infty}{}_{X'}\langle \Phi'(v_k),\, h\rangle_X = \mathrm{Re}(v,\, (H - H_0)h)_{L^2} - \mathrm{Re}\int Q|v|^{p-2}v\overline{h}dx. \quad (7.9)$$

Then it follows from (7.7) and (7.9) that $(v,\, (H_0 - E)h)_{L^2} = 0$ for any $h \in C_0^\infty(\mathbb{R}^n)$. This implies that E is an eigenvalue of H_0, which is impossible. Therefore, we have shown the existence of a nontrivial solution.

Finally, we have the boundedness and the decay of the solution by Corollary 3.4. This completes the proof of Theorem 1.6.

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