*Hokkaido Mathematical Journal Vol. 37 (2008) p. 669–688*

# The existence of the global solutions to semilinear wave equations with a class of cubic nonlinearities in 2-dimensional space

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(Received October 29, 2007; Revised April 14, 2008)

**Abstract.** This paper deals with the Cauchy problem of the semilinear wave equation with a small initial data in 2-dimensional space. When the nonlinearity is cubic, we can not expect the global existence of smooth solutions, in general. However, Godin [1] showed that if the nonlinearity has the null-form, the solution exists globally. In this paper, we will show the global solvability for the other type of nonlinearities which does not have null-form.

*Key words*: semilinear wave equation, null-condition, global solvability.

## 1. Introduction

Let us consider the following Cauchy problem;

$$\Box u = \partial_t^2 u - \triangle u = F(\partial u) \quad (x,\, t) \in \mathbf{R}^2 \times (0,\, \infty), \tag{1.1}$$

$$u(x,\, 0) = \varepsilon f(x),\ \partial_t u(x,\, 0) = \varepsilon g(x) \quad x \in \mathbf{R}^2. \tag{1.2}$$

Here, $\partial = (\partial_0,\, \partial_1,\, \partial_2)$, $\partial_0 = \partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1,\, 2)$ and $\varepsilon$ is a positive small parameter. We assume $f,\, g \in C_0^\infty(\mathbf{R}^2;\, \mathbf{R})$, $|f| + |g| \not\equiv 0$ and $\mathrm{supp}\{f,\, g\} \subset \{x \in \mathbf{R}^2; |x| \le M\}$. We also assume that

$$F \in C^\infty(\mathbf{R}^3;\, \mathbf{R}),$$
$$F(\partial u) = O(|\partial u|^3) \quad \text{near} \quad \partial u = 0.$$

More presicely, we assume

$$F(\partial u) = \sum_{\alpha,\beta,\gamma=0}^{2} A^{\alpha\beta\gamma} \partial_\alpha u \partial_\beta u \partial_\gamma u + O(|\partial u|^4) \quad \text{near} \quad \partial u = 0, \tag{1.3}$$

where $A^{\alpha\beta\gamma}$ are real constants.

The aim of this paper is to estimate the lifespan $T_\varepsilon$ of the smooth solution to the Cauchy problem (1.1) and (1.2), which is defined for each

*2000 Mathematics Subject Classification* : 35L05, 35L15, 35L70.

$\varepsilon > 0$ as follows;

$$T_\varepsilon = \sup\{T > 0; \text{there exists a smooth solution to (1.1) and (1.2) in } C^\infty(\mathbf{R}^2 \times [0, T); \mathbf{R})\}.$$

In order to state the results which we have already known about the lifespan, we introduce some notations.

For vectors $X = (X_0, X_1, X_2) \in \mathbf{R}^3$, we define

$$C(X) = \sum_{\alpha,\beta,\gamma=0}^{2} A^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma.$$

This function charaterizes the essential cubic terms of $F(\partial u)$ along the light cone.

On the other hand, let $u^0 = u^0(x, t)$ be the solution to the Cauchy problem;

$$\Box u^0 = 0 \quad (x, t) \in \mathbf{R}^2 \times (0, \infty),$$
$$u^0(x, 0) = f(x), \ \partial_t u^0(x, 0) = g(x)$$

and set $r = |x| \geq 0$, $\omega = x/r \in S^1$ and $\rho = r - t \in \mathbf{R}$. Then we define

$$\mathcal{F}(\omega, \rho) = \lim_{r\to\infty} r^{1/2} u^0(r\omega, r - \rho),$$

which is called the Friedlander radiation field. Hörmander showed in [2] the following properties of $\mathcal{F}$.

$$|\partial_\rho^k \mathcal{F}(\omega, \rho)| \leq C(1 + |\rho|)^{-1/2-k} \quad \rho \in \mathbf{R}, \tag{1.4}$$
$$\mathcal{F}(\omega, \rho) = 0 \quad \text{for} \quad \rho \geq M. \tag{1.5}$$

By (1.4) and (1.5), we find that the constant

$$H = \max_{\rho\in\mathbf{R},\ \omega\in S^1} \left\{-C(-1, \omega)(\partial_\rho \mathcal{F}(\omega, \rho))^2\right\} \tag{1.6}$$

is well-defined and nonnegative.

Then Godin proved the following **(a)** and **(b)** in [1].

**(a)** *If $H > 0$, then*

$$\liminf_{\varepsilon\to+0} \varepsilon^2 \log T_\varepsilon \geq \frac{1}{H}$$

*holds.*

**(b)** *If $C(-1,\omega)\equiv 0$ holds for $\omega\in S^1$, then $T_\varepsilon=\infty$ holds for sufficiently small $\varepsilon>0$.*

The condition $C(-1,\omega)\equiv 0$ is called the *null-condition*. If the null-condition is satisfied, then we can write the cubic part of the nonlinear term $F(\partial u)$ as a linear combination of terms $\partial_\alpha u((\partial_0 u)^2-|\nabla u|^2)$, i.e.,

$$F(\partial u)=\sum_{\alpha=0}^{2}C_\alpha\partial_\alpha u((\partial_0 u)^2-|\nabla u|^2)+O(|\partial u|^4),$$

where $C_\alpha$ are real constants. It follows from (1.6) that the null-condition is a sufficient condition of $H=0$ and not a necessary condition. In the case where $|f|+|g|\not\equiv 0$, we find that $H=0$ is equivalent to the condition

$$C(-1,\omega)\geq 0 \qquad \text{for any} \quad \omega\in S^1. \tag{1.7}$$

This means that there is a nonlinearity $F(\partial u)$ which does not satisfy the assumptions of both **(a)** and **(b)**. For example,

$$F(\partial u)=-(\partial_0 u)^3,\quad F(\partial u)=-\partial_0 u|\nabla u|^2,$$
$$F(\partial u)=-(\partial_0 u)^3+\partial_0 u\partial_1 u\partial_2 u$$

and so on. For the above $F(\partial u)$, we can easily show that

$$C(X)=-X_0^3,\quad C(X)=-X_0(X_1^2+X_2^2),$$
$$C(X)=-X_0^3-X_0X_1X_2,$$

respectively. Hence we find that these $F(\partial u)$ satisfy (1.7). From the results **(a)** and **(b)**, we can expect the solution exists longer than the time $\exp(C/\varepsilon^2)$ for any constant $C>0$ in such cases. Our purpose of this paper is to show the global existence of the smooth solution to (1.1) and (1.2), when the condition (1.7) holds.

## 2. Statement of the main theorem

We introduce generalized differential operators;

$$\Omega=x_1\partial_2-x_2\partial_1,\quad L_i=t\partial_i+x_i\partial_0,\quad S=t\partial_0+x_1\partial_1+x_2\partial_2$$

and denote

$$\Gamma_0=S,\ \Gamma_1=\Omega,\ \Gamma_2=L_1,\ \Gamma_3=L_2,\ \Gamma_4=\partial_0,\ \Gamma_5=\partial_1,\ \Gamma_6=\partial_2.$$

We can verify the following commutator relations;

$$
\begin{aligned}
&[\partial_\alpha,\, \partial_\beta] = 0,\ [S,\, \partial_\alpha] = -\partial_\alpha,\ [\Omega,\, \partial_1] = -\partial_2,\ [\Omega,\, \partial_2] = \partial_1,\\
&[\Omega,\, \partial_0] = 0,\ [L_i,\, \partial_j] = -\delta_{ij}\partial_0,\ [L_i,\, \partial_0] = -\partial_i,\ [S,\, \Omega] = 0,\\
&[\Omega,\, L_1] = -L_2,\ [\Omega,\, L_2] = L_1,\ [S,\, L_i] = 0, \quad [L_1,\, L_2] = \Omega,\\
&[\Gamma_\sigma,\, \Box] = -2\delta_{0\sigma}\Box
\end{aligned}
\tag{2.1}
$$

for $\alpha,\, \beta = 0,\, 1,\, 2,\ i,\, j = 1,\, 2$ and $\sigma = 0,\, 1,\, \dots,\, 6$. Here $[\ ,\ ]$ denotes the usual commutator of linear operators and $\delta_{\alpha\beta}$ is the Kronecker delta. We also write $\Gamma^a u = \Gamma_0^{a_0}\Gamma_1^{a_1}\cdots\Gamma_6^{a_6}u$ for a multi-index $a = (a_0,\, a_1,\, \dots,\, a_6)$.

Next we define some generalized Sobolev norms as

$$
\begin{aligned}
|v(x,\, t)|_k &= \sum_{|a|\le k} |\Gamma^a v(x,\, t)|\\
|v(t)|_k &= \sum_{|a|\le k} \|\Gamma^a v(\,\cdot\,,\, t)\|_{L^\infty_x(\mathbf{R}^2;\mathbf{R})}\\
[v(t)]_k &= \sum_{|a|\le k} \|(1+|\,\cdot\,|+t)^{1/2}(1+||\,\cdot\,|-t|)^{1/2}\Gamma^a v(\,\cdot\,,\, t)\|_{L^\infty_x(\mathbf{R}^2;\mathbf{R})}\\
\|v(t)\|_k &= \sum_{|a|\le k} \|\Gamma^a v(\,\cdot\,,\, t)\|_{L^2_x(\mathbf{R}^2;\mathbf{R})}.
\end{aligned}
\tag{2.2}
$$

Note that by (2.1) and the definition of norms, we have

$$
|v(x,\, t)| \le \frac{[v(t)]_0}{(1+t)^{1/2}} \tag{2.3}
$$

and

$$
\begin{aligned}
\alpha|\partial v(t)|_k &\le \sum_{|a|\le k} |\partial\Gamma^a v(t)|_0 \le \beta|\partial v(t)|_k\\
\alpha[\partial v(t)]_k &\le \sum_{|a|\le k} [\partial\Gamma^a v(t)]_0 \le \beta[\partial v(t)]_k\\
\alpha\|\partial v(t)\|_k &\le \sum_{|a|\le k} \|\partial\Gamma^a v(t)\|_0 \le \beta\|\partial v(t)\|_k
\end{aligned}
\tag{2.4}
$$

for some positive constants $\alpha$ and $\beta$. Now we state the main theorem.

**Theorem 2.1** *Assume that* (1.7) *holds. Then there exists a constant* $\varepsilon_* > 0$ *such that* $T_\varepsilon = \infty$ *holds for* $\varepsilon \in (0,\, \varepsilon_*)$. *Moreover, for any integer* $k \ge 5$, *there exists constants* $c_l > 0$ *and* $m_l \ge 0$ $(l = 0,\, 1,\, \dots,\, k)$ *such that the* *solution satisfies*

$$\|\partial u(t)\|_l \le c_l\varepsilon(1+\varepsilon^2\log(1+t))^{m_l} \quad 0\le t<\infty,\ l=0,1,\ldots,k \tag{2.5}$$

*for* $\varepsilon \in (0, \varepsilon_*)$. *Here* $c_l > c_{l-1}$ *and* $c_0 = c_0(f, g)$, $m_l > m_{l-1}$ *and* $m_0 = 0$.

**Remark** Kubo proved the same result in [8], in which he also showed an asymptotic behavior of solutions.

## 3. Proof of the main theorem

First of all, the following local existence result has been shown in Theorem 6.5.3 in [3].

**Proposition 3.1** *There exist constants* $D > 0$ *and* $\varepsilon' > 0$ *such that for* $0 < \varepsilon < \varepsilon'$ *the Cauchy problem* (1.1) *and* (1.2) *has a local solution* $u \in C^\infty(\mathbf{R}^2 \times [0, \varepsilon^{-1}]; \mathbf{R})$ *which satisfies*

$$\|\partial u(t)\|_k \le D\varepsilon \quad 0 \le t \le \frac{1}{\varepsilon} \tag{3.1}$$

*for* $k = 0, 1, 2, \ldots$.

Combining Proposition 3.1 with the following lemma, we can show the Theorem 2.1.

**Lemma 3.1** *Assume that* (1.7) *holds, choose an integer* $k \ge 5$. *Let* $u \in C^\infty(\mathbf{R}^2 \times [0, T); \mathbf{R})$ *be a solution to* (1.1) *and* (1.2). *Then, there exist constants* $\varepsilon_0 > 0$, $c_l > 0$ *and* $m_l \ge 0$ ($c_l > c_{l-1}$, $c_0 > D$, $m_l > m_{l-1}$, $m_0 = 0$, $l = 0, 1, \ldots, k$) *independent of* $T$ *such that if*

$$\|\partial u(t)\|_l \le 2c_l\varepsilon(1+\varepsilon^2\log(1+t))^{m_l} \qquad 0 \le t < T,\ l = 0, 1, \ldots, k \tag{3.2}$$

*holds for an* $\varepsilon \in (0, \varepsilon_0)$, *then*

$$\|\partial u(t)\|_l \le c_l\varepsilon(1+\varepsilon^2\log(1+t))^{m_l} \qquad 0 \le t < T,\ l = 0, 1, \ldots, k \tag{3.3}$$

*holds for the same* $\varepsilon$.

*Proof of* Theorem 2.1. Choose an integer $k \ge 5$ and define a set

$$U_\varepsilon = \{t; t \le T_\varepsilon,\ \|\partial u(t)\|_l \le 2c_l\varepsilon(1+\varepsilon^2\log(1+t))^{m_l},$$

$$l = 0,\, 1,\, \dots,\, k\} \quad (\subset [0,\, \infty))$$

for each $\varepsilon \in (0,\, \varepsilon_*)$, where $\varepsilon_* = \min\{\varepsilon_0,\, \varepsilon'\}$. Proposition 3.1 implies that $U_\varepsilon$ is not empty. Furthermore, by Lemma 3.1, we can easily show that $U_\varepsilon$ is open and closed in $[0,\, \infty)$ from the usual argument. Namely we have $U_\varepsilon = [0,\, \infty)$ and therefore we find that $T_\varepsilon = \infty$ and (2.5) hold for any $\varepsilon \in (0,\, \varepsilon_*)$. This completes the proof of Theorem 2.1. □

In what follows we concentrate on showing (3.3) under the assumption (3.2). For this purpose, we will use the following propositions.

**Proposition 3.2** *Let $v \in C^2(\mathbf{R}^2 \times [0,\, T); \mathbf{R})$ be a function satisfying $\sup_{0 \le t < T} \|v(t)\|_2 < \infty$. Then there exists a constant $K > 0$ such that*

$$[\partial v(t)]_0 \le K \|\partial v(t)\|_2 \quad 0 \le t < T \tag{3.4}$$

*holds.*

**Proposition 3.3** *Let $v \in C^1(\mathbf{R}^2 \times [0,\, T); \mathbf{R})$ be a function satisfying $v(x,\, t) = 0$ when $|x| \ge t + R$ for some constant $R > 0$. Then there exists a constant $L = L(R) > 0$ such that*

$$\left\| \frac{v(t)}{1 + |t - |x||} \right\|_0 \le L \|\partial v(t)\|_0 \quad 0 \le t < T \tag{3.5}$$

*holds.*

**Proposition 3.4** *Let $u \in C^2(\mathbf{R}^2 \times [0,\, T); \mathbf{R})$ be a solution to* (1.1) *and* (1.2). *Then*

$$u(x,\, t) = 0 \quad \text{for} \quad |x| \ge t + M \tag{3.6}$$

*holds.*

See Corollary 1 in [7], Lemma 3.2 in [6] and Theorem 4a in [5] for the proof of Propositions 3.2, 3.3 and 3.4, respectively.

*Proof of* Lemma 3.1. Now we show Lemma 3.1 by 3 steps.

*Step* 1. *There exist constants $C' > 0$ and $\varepsilon_1 > 0$ such that*

$$|\partial u(x,\, t)|_j \le \frac{C' \varepsilon (1 + \varepsilon^2 \log(1 + t))^j}{(1 + |x| + t)^{1/2}} \quad (x,\, t) \in \mathbf{R}^2 \times [0,\, T) \tag{3.7}$$

*holds for* $\varepsilon \in (0, \varepsilon_1)$ *and* $j = 0, 1, 2$. *Here* $C'$ *is independent of the constants* $c_l$ *and* $m_l$ $(l = 0, 1, 2, \ldots, k)$.

Firstly, by (3.1), we know that

$$|\partial u(x, t)|_2 \leq \frac{KD\varepsilon}{(1+t)^{1/2}} \quad 0 \leq t \leq \frac{1}{\varepsilon} \tag{3.8}$$

holds for $\varepsilon \in (0, \varepsilon')$. This implies that (3.7) is true when $0 \leq t \leq 1/\varepsilon$. Secondly, by (3.2) and Proposition 3.2, we find that

$$\begin{aligned}(1+|x|+t)^{1/2}(1+||x|-t|)^{1/2}|\partial u(t, x)|_{l-2} &\leq [\partial u(t)]_{l-2} \\ &\leq K\|\partial u(t)\|_l \\ &\leq 2Kc_l\varepsilon(1+\varepsilon^2\log(1+t))^{m_l}\end{aligned} \tag{3.9}$$

holds for $(x, t) \in \mathbf{R}^2 \times [0, T)$ and $l = 2, 3, \ldots, k$. Then, setting

$$\Lambda_0 = \Big\{(y, s) \;\Big|\; y \in \mathbf{R}^2,\ \frac{1}{\varepsilon} \leq s < T,\ ||y| - s| \geq \frac{s}{2}\Big\},$$
$$\Lambda_1 = \Big\{(y, s) \;\Big|\; y \in \mathbf{R}^2,\ \frac{1}{\varepsilon} \leq s < T,\ ||y| - s| \leq \frac{s}{2}\Big\},$$

we find that (3.7) is true for $(x, t) \in \Lambda_0$. In fact, by (3.9) and the fact that

$$1 + ||x| - t| \geq 1 + \frac{t}{2} \geq \frac{1}{2\varepsilon} \quad (x, t) \in \Lambda_0,$$

we have

$$\begin{aligned}(1+|x|+t)^{1/2}|\partial u(t, x)|_2 &\leq \frac{2Kc_4\varepsilon(1+\varepsilon^2\log(1+t))^{m_4}}{(1+t/2)^{1/2}} \\ &\leq \frac{2^{5/4}Kc_4\varepsilon^{5/4}(1+\log(1+t))^{m_4}}{(1+t/2)^{1/4}} \\ &\leq \varepsilon\end{aligned} \tag{3.10}$$

for $\varepsilon \in (0, \varepsilon'')$, if we take $\varepsilon''$ as

$$\varepsilon'' < \min\Big\{1,\ \Big(2^{5/4}Kc_4 \sup_{0\leq s} \frac{(1+\log(1+s))^{m_4}}{(1+s/2)^{1/4}}\Big)^{-4}\Big\}.$$

In order to show (3.7) for $(x, t) \in \Lambda_1$, we prepare an estimate of $u$. It follows from (3.9) and Proposition 3.4 that

$$\begin{aligned}
|\Gamma^a u(x,\, t)| &= \Big| -\int_{|x|}^{t+M} \partial_r \Gamma^a u(\lambda\omega,\, t)\, d\lambda \Big| \\
&\le [\partial u(t)]_4 \int_{|x|}^{t+M} \frac{1}{(1+t+\lambda)^{1/2}(1+|\lambda - t|)^{1/2}} d\lambda \\
&\le 2Kc_4\varepsilon(1+\varepsilon^2\log(1+t))^{m_6} \\
&\quad \times \int_{|x|-t}^{M} \frac{1}{(1+2t+\mu)^{1/2}(1+|\mu|)^{1/2}} d\mu \\
&\le 2Kc_4\varepsilon(1+\varepsilon^2\log(1+t))^{m_6} \Big( \int_0^M d\mu + B\Big(\frac{1}{2},\, \frac{1}{2}\Big)\Big) \\
&\le 2Kc_4(M+\pi)\varepsilon(1+\varepsilon^2\log(1+t))^{m_6}
\end{aligned}$$

for $|a| \le 4$. Here, $B(p,\, q)$ stands for the beta function. Hence we have

$$|u(x,\, t)|_4 \le \tilde{C}_* \varepsilon (1+\log(1+t))^{m_6} \quad (x,\, t) \in \Lambda_1, \tag{3.11}$$

if $\varepsilon \in (0,\, 1)$. Here $\tilde{C}_*$ is a constant depending on $c_4$.

Since the operator $\partial_j$ $(j = 1,\, 2)$ can be written as

$$\partial_j = -\omega_j \partial_0 + \frac{1}{t} L_j + \frac{\omega_j}{t+r} S - \sum_{i=1}^{2} \frac{r\omega_i\omega_j}{t(t+r)} L_i,$$

we find

$$\partial_\alpha v = -\omega_\alpha \partial_0 v + O\Big(\sum_{|a|=1} \frac{|\Gamma^a v|_0}{t}\Big) \quad \text{with} \quad \omega_0 = -1, \tag{3.12}$$

$$(\partial_0 + \partial_r) v = O\Big(\sum_{|a|=1} \frac{|\Gamma^a v|_0}{t}\Big), \tag{3.13}$$

$$(\partial_0 + \partial_r)^2 v = O\Big(\sum_{|a|=2} \frac{|\Gamma^a v|_0}{t^2}\Big) \tag{3.14}$$

and therefore we obtain

$$\begin{aligned}
\Box u &= -C(-1,\, \omega)(\partial_0 u)^3 \\
&\quad + O\Big(\sum_{|b|=1} \Big\{ \frac{|\partial u|_0^2 |\Gamma^b u|_0}{t} + \frac{|\partial u|_0 |\Gamma^b u|_0^2}{t^2} + \frac{|\Gamma^b u|_0^3}{t^3} \Big\} + |\partial u|_0^4 \Big).
\end{aligned} \tag{3.15}$$

We also introduce characteristic lines of (1.1). By (1.1), we can show that

$$r^{-1/2}(\partial_0+\partial_r)(r^{1/2}\partial_0 v) = \frac{1}{2}\Box v + \frac{1}{2}(\partial_0+\partial_r)^2 v + \frac{1}{2r}(\partial_0+\partial_r)v + \frac{1}{2r^2}\Omega^2 v \quad (3.16)$$

holds for a function $v(x, t)$. Then, setting $r_\lambda(s) = s+\lambda$ for each $\lambda \in \mathbf{R}$ and denoting $v(s) = v(r_\lambda(s)\omega, s)$, we obtain

$$2\frac{d}{ds}(r_\lambda(s)^{1/2}\partial_0 v(s)) = r_\lambda(s)^{1/2}\Box v + r_\lambda(s)^{1/2}(\partial_0+\partial_r)^2 v + \frac{1}{r_\lambda(s)^{1/2}}(\partial_0+\partial_r)v + \frac{1}{r_\lambda(s)^{3/2}}\Omega^2 v \quad (3.17)$$

for each $\omega \in S^1$ and $\lambda \in \mathbf{R}$. For each $(x, t) \in \Lambda_1$, settting $x = r\omega$ and $\lambda = r-t$, we find that $r_\lambda(t) = r$. Then we call the line $(r_\lambda(s)\omega, s)$ $(0 \le s < T)$ the characteristic line of (1.1) passing through the point $(x, t)$. Moreover, we denote

$$t_\lambda = \inf\{s \mid (r_\lambda(s)\omega, s) \in \Lambda_1\}$$

which is the time when $(r_\lambda(s), s) \in \partial\Lambda_1$. Note that $t_\lambda \ge 1/\varepsilon$ and $(r_\lambda(s)\omega, s) \in \Lambda_1$ holds for $t_\lambda \le s < T$.

Hence, for any $(x, t) \in \Lambda_1$, by multiplying

$$P(s) = \exp\Big(\frac{1}{2}\int_{t_\lambda}^{s} C(-1, \omega)(\partial_0 u(\tau))^2 d\tau\Big)$$

to the both sides of (3.17) with $v = u$ and integrating it from $t_\lambda$ to $t$, we have

$$\begin{aligned} &P(t)|x|^{1/2}|\partial_0 u(x, t)| \\ &\le r_\lambda(t_\lambda)^{1/2}|\partial_0 u(r_\lambda(t_\lambda)\omega, t_\lambda)| \\ &\quad + C_1\int_{t_\lambda}^{t} P(s)\Big(\frac{|\partial u|_0^2|u|_1}{(1+s)^{1/2}} + r_\lambda(s)^{1/2}|\partial u|_0^4 + \frac{|u|_2}{(1+s)^{3/2}}\Big)ds, \end{aligned}$$

where we have used (3.13), (3.14), (3.15) and the fact

$$\frac{1}{As} \le \frac{1}{r_\lambda(s)} \le \frac{A}{1+s} \qquad (r_\lambda(s)\omega, s) \in \Lambda_1 \quad (3.18)$$

for some constnat $A > 0$. Since $P(t)$ is monotonously increasing and $P(t) \ge$

1, we have by (3.8), (3.9), (3.10) and (3.11),

$$
\begin{aligned}
&|x|^{1/2}|\partial_0 u(x,\,t)| \\
&\le \frac{1}{P(t)}\Big\{r_\lambda(t_\lambda)^{1/2}|\partial_0 u(r_\lambda(t_\lambda)\omega,\,t_\lambda)| \\
&\qquad\qquad + C_1\int_{t_\lambda}^{t} P(s)\Big(\frac{|\partial u|_0^2|u|_1}{r_\lambda(s)^{1/2}} + r_\lambda(s)^{1/2}|\partial u|_0^4 + \frac{|u|_2}{r_\lambda(s)^{3/2}}\Big)ds\Big\} \\
&\le (1+KD)\varepsilon + \tilde{C}_1\int_{t_\lambda}^{t}\frac{\varepsilon(1+\log(1+s))^{4m_6}}{(1+s)^{3/2}}ds \\
&\le (1+KD)\varepsilon + \frac{\tilde{C}_2\varepsilon}{(1+t_\lambda)^{1/4}} \\
&\le (1+KD)\varepsilon + \tilde{C}_2\varepsilon^{1+1/4} \quad (x,\,t)\in\Lambda_1.
\end{aligned}
$$

Hereafter, $C_j$ stands for constnats independent of $c_l$ and $m_l$, while $\tilde{C}_j$ stands for constants depending on $c_l$ or $m_l$ $(l = 0,\,1,\,2,\,\ldots,\,k)$. Therefore, taking

$$
\varepsilon_0''' \le \min\Big\{\varepsilon',\,\varepsilon'',\,\frac{1}{\tilde{C}_2^4}\Big\},
$$

we have

$$
|x|^{1/2}|\partial_0 u(x,\,t)| \le (2+KD)\varepsilon \quad (x,\,t)\in\Lambda_1 \tag{3.19}
$$

for $\varepsilon\in(0,\,\varepsilon_0''')$. Moreover, by (3.11), (3.12) and (3.19), we have

$$
\begin{aligned}
|x|^{1/2}|\partial_j u(x,\,t)| &= |x|^{1/2}|\partial_0 u(x,\,t)| + O\Big(\frac{|u|_1}{t}\Big) \\
&\le (3+KD)\varepsilon \quad (x,\,t)\in\Lambda_1
\end{aligned} \tag{3.20}
$$

for $\varepsilon\in(0,\,\varepsilon_0''')$, taking $\varepsilon_0'''$ smaller if necessary. Therefore, by (3.18), (3.19) and (3.20), we find that

$$
(1+t)^{1/2}|\partial u(x,\,t)|_0 \le C_0'\varepsilon \quad (x,\,t)\in\Lambda_1 \tag{3.21}
$$

holds for $\varepsilon\in(0,\,\varepsilon_0''')$, if we take $C_0' > 3(3+KD)A^{1/2}$.

Next we take $v=\Gamma u$ in (3.16). Here $\Gamma$ stands for any one of $\Gamma_\alpha$ $(\alpha = 0,\,1,\,\ldots,\,6)$. By (1.3), (2.1) and (3.15), we have

$$
\begin{aligned}
\Box\Gamma u &= \Gamma\Box u + C\Box u \\
&= -3C(-1,\omega)(\partial_0 u)^2(\partial_0\Gamma u) +
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
&+O\Bigg(|\partial u|_0^3 + |\partial u|_0^3|\partial u|_1 \\
&\qquad + \sum_{\substack{|b|=1\\|c|=1}}\bigg\{\frac{|\partial u|_0^2|\Gamma^b u|_1 + |\partial u|_0|\Gamma^b u|_0|\partial u|_1}{t} + \\
&\qquad + \frac{|\partial u|_1|\Gamma^b u|_0^2 + |\partial u|_0|\Gamma^b u|_0|\Gamma^c u|_1}{t^2} + \frac{|\Gamma^b u|_0^2|\Gamma^c u|_1}{t^3}\bigg\}\Bigg).
\end{aligned}
$$

Hence, for any $(x,\, t) \in \Lambda_1$, by multiplying

$$
P(s) = \exp\Big(\frac{3}{2}\int_{t_\lambda}^{s} C(-1,\, \omega)(\partial_0 u(\tau))^2 d\tau\Big)
$$

to the both sides of (3.17)) with $v = \Gamma u$ and integrating it from $t_\lambda$ to $t$, we have

$$
\begin{aligned}
&P(t)|x|^{1/2}|\partial_0\Gamma u(x,\, t)| \\
&\le r_\lambda(t_\lambda)^{1/2}|\partial_0\Gamma u(r_\lambda(t_\lambda)\omega,\, t_\lambda)| \\
&\quad + C_2\int_{t_\lambda}^{t} P(s)\Big(r_\lambda(s)^{1/2}(|\partial u|_0^3 + |\partial u|_0^3|\partial u|_1) \\
&\qquad\qquad + \frac{|\partial u|_0^2|u|_2 + |\partial u|_0|u|_1|\partial u|_1}{(1+s)^{1/2}} + \frac{|u|_3}{(1+s)^{3/2}}\Big)ds,
\end{aligned}
$$

where we have used (3.13), (3.14), (3.15) and (3.18). Since $P(t)$ is monotonously increasing and $P(t) \ge 1$, we have by (3.8), (3.9), (3.10), (3.11), (3.21) and (3.22),

$$
\begin{aligned}
&|x|^{1/2}|\partial_0\Gamma u(x,\, t)| \\
&\le \frac{1}{P(t)}\Big\{r_\lambda(t_\lambda)^{1/2}|\partial_0\Gamma u(r_\lambda(t_\lambda)\omega,\, t_\lambda)| \\
&\qquad + C_2\int_{t_\lambda}^{t} P(s)\Big(r_\lambda(s)^{1/2}(|\partial u|_0^3 + |\partial u|_0^3|\partial u|_1) \\
&\qquad\qquad + \frac{|\partial u|_0^2|u|_2 + |\partial u|_0|u|_1|\partial u|_1}{(1+s)^{1/2}} + \frac{|u|_3}{(1+s)^{3/2}}\Big)ds\Big\} \\
&\le (1+KD)\varepsilon + \int_{t_\lambda}^{t}\Big(\frac{C_3\varepsilon^3}{1+s} + \frac{\tilde{C}_3\varepsilon(1+\log(1+s))^{4m_6}}{(1+s)^{3/2}}\Big)ds \\
&\le (1+KD)\varepsilon + C_3\varepsilon^3\log(1+t) + \frac{\tilde{C}_4\varepsilon}{(1+t_\lambda)^{1/4}}
\end{aligned}
$$

$$\leq (1 + KD + C_3)\varepsilon(1 + \varepsilon^2 \log(1+t)) + \tilde{C}_4 \varepsilon^{1+1/4}$$
$$\leq (2 + KD + C_3)\varepsilon(1 + \varepsilon^2 \log(1+t)) \quad (x,\, t) \in \Lambda_1$$

for $\varepsilon \in (0,\, \varepsilon_1''')$, if we take

$$\varepsilon_1''' < \min\Big\{\varepsilon_0''',\, \frac{1}{\tilde{C}_4^4}\Big\}.$$

Namely we have

$$|x|^{1/2}|\partial_0 u(x,\, t)|_1 \leq 7(2 + KD + C_3)\varepsilon(1 + \varepsilon^2 \log(1+t)) \quad (x,\, t) \in \Lambda_1 \tag{3.23}$$

and therefore by using (3.11), (3.12) and (3.23), we have

$$|x|^{1/2}|\partial_j u(x,\, t)|_1 \leq (7(2 + KD + C_3) + 1) \times \varepsilon(1 + \varepsilon^2 \log(1+t)) \quad (x,\, t) \in \Lambda_1 \tag{3.24}$$

for $\varepsilon \in (0,\, \varepsilon_1''')$, taking $\varepsilon_1'''$ smaller if necessary. Hence, by (3.18) and (3.24), we have

$$(1+t)^{1/2}|\partial u(x,\, t)|_1 \leq C_1' \varepsilon(1 + \varepsilon^2 \log(1+t)) \quad (x,\, t) \in \Lambda_1 \tag{3.25}$$

for $\varepsilon \in (0,\, \varepsilon_1''')$, if we take $C_1' > (21(2 + KD + C_3) + 2)A^{1/2}$.

Repeating the same argument, we find that there exist positive constants $C_2'$ and $\varepsilon_2'''$ $(< \varepsilon_1''')$ such that

$$(1+t)^{1/2}|\partial u(x,\, t)|_2 \leq C_2' \varepsilon(1 + \varepsilon^2 \log(1+t))^2 \quad (x,\, t) \in \Lambda_1$$

for $\varepsilon \in (0,\, \varepsilon_2''')$. Therefore, taking

$$C' = \max\{KD,\, 1,\, C_0',\, C_1',\, C_2'\} \quad \text{and} \quad \varepsilon_1 = \varepsilon_2''',$$

we obtain (3.7).

Next we show the following

*Step 2. Let $\nu$ be a small positive number. Then, there exist constants $\tilde{C}'' > 0$ and $\varepsilon_2 > 0$ such that*

$$\|\partial u(t)\|_{k+1} \leq \tilde{C}'' \varepsilon(1+t)^{\nu} \quad 0 \leq t < T \tag{3.26}$$

*holds for $\varepsilon \in (0,\, \varepsilon_2)$. Here $\tilde{C}''$ depends on the constants $m_l$ and $c_l$ $(l = 0,\, 1,\, 2,\, \ldots,\, k)$.*

By (1.1) and (2.1), we have

$$\Box\Gamma^a u = \Gamma^a F(\partial u) + \sum_{|b|<|a|}{}' \Gamma^b F(\partial u). \tag{3.27}$$

Here $\sum'_{|b|<|a|} A_b = \sum_{b<|a|} \gamma_b A_b$ with certain constants $\gamma_b$. Furthermore, by (1.3), we have

$$\begin{aligned}
&\Gamma^a F(\partial u) + \sum_{|b|<|a|}{}' \Gamma^b F(\partial u) \\
&= \sum_{\alpha,\beta,\gamma=0,1,2}{}' \partial_\alpha u \partial_\beta u \Gamma^a \partial_\gamma u + \sum_{\substack{|c|+|d|+|e|\le|a| \\ |c|,|d|,|e|\le|a|-1}}{}' \Gamma^c \partial u \Gamma^d \partial u \Gamma^e \partial u \\
&\quad + O\Big( \sum_{|c|+|d|+|e|+|f|\le|a|}{}' \Gamma^c \partial u \Gamma^d \partial u \Gamma^e \partial u \Gamma^f \partial u \Big) \qquad (3.28) \\
&= O\Big( \big( |\partial u(t)|_0^2 + |\partial u(t)|^3_{[|a|/2]} \big) |\partial u(x,t)|_{|a|} + |\partial u(t)|^2_{[|a|/2]} |\partial u(x,t)|_{|a|-1} \Big).
\end{aligned}$$

Thus, multiplying $\partial_0 \Gamma^a u$ to both sides of (3.27), integrating it over $\mathbf{R}^2$ and summing up with respect to $a$ over $|a| \le k+1$, we have

$$\begin{aligned}
\frac{d}{dt}\|\partial u(t)\|_{k+1} \le C_4 \Big( &\big( |\partial u(t)|_0^2 + |\partial u(t)|^3_{[(k+1)/2]} \big) \|\partial u(t)\|_{k+1} \\
&+ |\partial u(t)|^2_{[(k+1)/2]} \|\partial u(t)\|_k \Big). \qquad (3.29)
\end{aligned}$$

Therefore, by (2.3), (2.4), (3.2), (3.7), (3.9), (3.29), the Gronwall inequality and the fact that $[(k+1)/2] \le k-2$ holds for $k \ge 4$, we have

$$\begin{aligned}
\|\partial u(t)\|_{k+1} &\le C_5 \Big( \|\partial u(0)\|_{k+1} + \int_0^t \frac{[\partial u(s)]^2_{[(k+1)/2]}}{1+s} \|\partial u(s)\|_k ds \Big) \\
&\qquad \times \exp\Big( C_4 \int_0^t \Big( |\partial u(s)|_0^2 + \frac{[\partial u(s)]^3_{[(k+1)/2]}}{(1+s)^{3/2}} \Big) ds \Big) \\
&\le \Big( C_6 \varepsilon + \tilde{C}_5 \varepsilon^3 \int_0^t \frac{(1+\varepsilon^2 \log(1+s))^{3m_k}}{1+s} ds \Big) \\
&\qquad \times \exp\Big( \int_0^t \Big( \frac{C_7 \varepsilon^2}{1+s} + \frac{\tilde{C}_6 \varepsilon^3 (1+\varepsilon^2 \log(1+s))^{3m_k}}{(1+s)^{3/2}} \Big) ds \Big) \\
&\le (C_6 \varepsilon + \tilde{C}_7 \varepsilon (1+\varepsilon^2 \log(1+t))^{3m_k+1})
\end{aligned}$$

$$\begin{aligned}&\times \exp(C_7\varepsilon^2\log(1+t)+\tilde{C}_8\varepsilon^3)\\ &\leq \tilde{C}_9\varepsilon(1+\varepsilon^2\log(1+t))^{3m_k+1}\times(1+t)^{C_7\varepsilon^2}\end{aligned}$$

for $\varepsilon \in (0,\, \varepsilon_1)$. Hence, setting

$$\begin{aligned}&\tilde{C}_{10}=\sup_{0\leq t}\Big(\frac{(1+\log(1+t))^{3m_k+1}}{(1+t)^{\nu/2}}\Big), \quad \tilde{C}''=\tilde{C}_9\tilde{C}_{10},\\ &\varepsilon_2=\min\Big\{\varepsilon_1,\, \frac{\sqrt{\nu}}{\sqrt{2C_7}}\Big\},\end{aligned}$$

we have

$$\begin{aligned}\|\partial u(t)\|_{k+1}&\leq \tilde{C}_9\tilde{C}_{10}\varepsilon(1+t)^{\nu/2+C_7\varepsilon^2}\\ &\leq \tilde{C}''\varepsilon(1+t)^{\nu}\end{aligned}$$

for $\varepsilon \in (0,\, \varepsilon_2)$. This implies (3.26).

Finally, we show (3.3).

*Step* 3. *We can determine constants $c_l$, $m_l$ $(l=0,\,1,\,2,\,\ldots,\,k)$ and $\varepsilon_0$ so that* (3.3) *holds under the assumption* (3.2).

By (3.1), we know that

$$\|\partial u(t)\|_k \leq D\varepsilon \quad 0\leq t<\frac{1}{\varepsilon} \tag{3.30}$$

for $\varepsilon \in (0,\, \varepsilon')$. Hence we have only to consider the case $1/\varepsilon \leq t < T$.

In order to estimate $\|\partial u(t)\|_l$, we make use of (3.27) again, but estimate the right hand side more precisely this time.

When $(x,\, t)\in \Lambda_0$, we find that $P(1+||x|-t|)\geq(1+|x|+t)$ holds for a certain constant $P>0$. Hence we have

$$|\partial u(x,\, t)|_l \leq \frac{P}{1+|x|+t}[\partial u(t)]_l \quad (x,\, t)\in\Lambda_0. \tag{3.31}$$

On the other hand, when $(x,\, t)\in\Lambda_1$, we find that

$$\frac{1}{Q}(1+|x|+t)\leq r\leq Qt \quad (x,\, t)\in\Lambda_1 \tag{3.32}$$

for a certain constant $Q>0$. Thus we have by (3.12) and (3.32),

$$\sum_{\alpha,\beta,\gamma=0}^{2} A^{\alpha\beta\gamma}(\partial_\alpha\Gamma^a u\partial_\beta u\partial_\gamma u+\partial_\alpha u\partial_\beta\Gamma^a u\partial_\gamma u+\partial_\alpha u\partial_\beta u\partial_\gamma\Gamma^a u)$$

$$
\begin{aligned}
&= -3C(-1,\,\omega)(\partial_0 u)^2\partial_0\Gamma^a u \\
&\quad + O\Big(\sum_{\substack{|b|=1\\|c|=1}}\Big\{\frac{|\partial u|_0^2|\Gamma^b u|_{|a|} + |\partial u|_0|\Gamma^b u|_0|\partial u|_{|a|}}{t} \\
&\quad + \frac{|\partial u|_0|\Gamma^b u|_0|\Gamma^c u|_{|a|} + |\Gamma^b u|_0^2|\partial u|_{|a|}}{t^2} + \frac{|\Gamma^b u|_0^2|\Gamma^c u|_{|a|}}{t^3}\Big\}\Big) \\
&= -3C(-1,\,\omega)(\partial_0 u)^2\partial_0\Gamma^a u \\
&\quad + O\Big(\frac{|\partial u|_0^2|u|_{|a|+1} + |\partial u|_0|u|_1|\partial u|_{|a|}}{1+|x|+t}\Big) \quad (x,\,t)\in\Lambda_1.
\end{aligned} \tag{3.33}
$$

Hence, combining (3.28) and (3.33), we obtain

$$
\begin{aligned}
&\Gamma^a F(\partial u) + \sum_{|b|<|a|}{}' \Gamma^b F(\partial u) \\
&= -3C(-1,\,\omega)(\partial_0 u)^2\partial_0\Gamma^a u \\
&\quad + O\Big(\frac{|\partial u|_0^2|u|_{|a|+1} + |\partial u|_0|u|_1|\partial u|_{|a|}}{1+|x|+t} \\
&\quad + |\partial u|_{[|a|/2]}^3|\partial u|_{|a|} + (1-\delta_{0|a|})|\partial u|_{[|a|/2]}^2|\partial u|_{|a|-1}\Big) \quad (x,\,t)\in\Lambda_1.
\end{aligned} \tag{3.34}
$$

Firstly, we consider the case $l = 0$. By (1.7), (3.27), (3.30), (3.31), (3.34) with $a = 0$ and Propositions 3.3 and 3.4, we have

$$
\begin{aligned}
&\frac{d}{dt}\|\partial u(t)\|_0^2 \\
&\le \int_{\mathbf{R}^2} F(\partial u)\partial_0 u dx \\
&\le C_8\int_{||x|-t|\ge t/2}|\partial u(x,\,t)|_0^4 dx - 3\int_{||x|-t|\le t/2} C(-1,\,\omega)|\partial u(x,\,t)|_0^4 dx \\
&\quad + C_8\int_{||x|-t|\le t/2}\frac{|\partial u(x,\,t)|_0^3|u(x,\,t)|_1}{1+|x|+t}dx + C_8\int_{\mathbf{R}^2}|\partial u(x,\,t)|_0^5 dx \\
&\le C_9\int_{||x|-t|\ge t/2}\frac{[\partial u(t)]_0^2|\partial u(x,\,t)|_0^2}{(1+t)^2}dx \\
&\quad + C_8\int_{||x|-t|\le t/2}\frac{[\partial u(t)]_0^2|u(x,\,t)|_1|\partial u(x,\,t)|_0}{(1+t)^2(1+||x|-t|)}dx \\
&\quad + C_{10}\int_{\mathbf{R}^2}\frac{(C')^3\varepsilon^3|\partial u(x,\,t)|_0^2}{(1+t)^{3/2}}dx
\end{aligned}
$$

$$\leq C_{11}\Big(\frac{\varepsilon^2}{(1+t)^{3/2}}\|\partial u(t)\|_0^2+\sum_{|c|=1}\frac{[\partial u(t)]_0^2}{(1+t)^2}\Big\|\frac{\Gamma^c u(t)}{1+||x|-t|}\Big\|_0\|\partial u(t)\|_0\Big)$$
$$\leq C_{12}\Big(\frac{\varepsilon^2}{(1+t)^{3/2}}\|\partial u(t)\|_0^2+\frac{[\partial u(t)]_0^2}{(1+t)^2}\|\partial u(t)\|_1\|\partial u(t)\|_0\Big),$$

which implis

$$\frac{d}{dt}\|\partial u(t)\|_0\leq C_{13}\Big(\frac{\varepsilon^2}{(1+t)^{3/2}}\|\partial u(t)\|_0+\frac{[\partial u(t)]_0^2}{(1+t)^2}\|\partial u(t)\|_1\Big). \quad (3.35)$$

Therefore, it follows from (3.2), (3.9), (3.30), (3.35) and the Gronwall inequality that

$$\begin{aligned}\|\partial u(t)\|_0 &\leq \Big(\|\partial u(\varepsilon^{-1})\|_0+C_{13}\int_{1/\varepsilon}^t\frac{[\partial u(s)]_0^2}{(1+s)^2}\|\partial u(s)\|_1 ds\Big)\\
&\quad\times\exp\Big(\int_{1/\varepsilon}^t\frac{C_{13}\varepsilon^2}{(1+s)^{3/2}}ds\Big)\\
&\leq\Big(C_{14}\varepsilon+\tilde{C_{11}}\int_{1/\varepsilon}^t\frac{\varepsilon^3(1+\varepsilon^2\log(1+s))^{3m_2}}{(1+s)^2}ds\Big)\\
&\quad\times\exp(C_{15}\varepsilon^{5/2})\\
&\leq\Big(C_{14}\varepsilon+\int_{1/\varepsilon}^t\frac{\tilde{C}_{12}\varepsilon^3}{(1+s)^{3/2}}ds\Big)\exp(C_{15}\varepsilon^{5/2})\\
&\leq(C_{14}\varepsilon+\tilde{C}_{13}\varepsilon^{7/2})\exp(C_{15}\varepsilon^{5/2})\\
&\leq(C_{14}+1)e\varepsilon\end{aligned}$$

for $\varepsilon\in(0,\,\varepsilon_{00})$, if we take $\varepsilon_{00}=\min\{\varepsilon_2,\,1/\tilde{C}_{13}^{2/5},\,1/C_{15}^{2/5}\}$. Hence we find that (3.3) holds for $l=0$, if we take $c_0=(C_{14}+1)e$ and $m_0=0$.

Next, we estimate $\|\partial u(t)\|_l$ assuming $c_i$ and $m_i$ $(i=0,\,1,\,2,\,\ldots,\,l-1)$ are determined. By (1.7), (2.4), (3.27), (3.31), (3.34) and Proposition 3.3, we have

$$\begin{aligned}&\frac{d}{dt}\|\partial u(t)\|_l^2\\
&\leq\sum_{|a|\leq l}{}'\int_{\mathbf{R}^2}(\Gamma^a F(\partial u)+\sum_{|b|<|a|}{}'\Gamma^b F(\partial u))\partial_0\Gamma^a u dx\\
&\leq\sum_{|a|\leq l}{}'\Big(\int_{||x|-t|\geq t/2}|\partial u(x,\,t)|_{[|a|/2]}^2|\partial u(x,\,t)|_{|a|}^2 dx\end{aligned}$$

$$
\begin{aligned}
&- 3\int_{||x|-t|\le t/2} C(-1,\,\omega)|\partial u(x,\,t)|_0^2|\partial u(x,\,t)|_{|a|}^2 dx\\
&+ \int_{||x|-t|\le t/2} \frac{1}{1+|x|+t}(|\partial u(x,\,t)|_0^2|u(x,\,t)|_{|a|+1}\\
&\qquad\qquad + |\partial u(x,\,t)|_0|u(x,\,t)|_1|\partial u(x,\,t)|_{|a|})|\partial u(x,\,t)|_{|a|}dx\\
&+ \int_{||x|-t|\le t/2} |\partial u(x,\,t)|_{[|a|/2]}^2|\partial u(x,\,t)|_{|a|-1}|\partial u(x,\,t)|_{|a|}dx\\
&+ \int_{\mathbf{R}^2} |\partial u(x,\,t)|_{[|a|/2]}^3|\partial u(x,\,t)|_{|a|}^2 dx\Big)\\
\le{}& \sum_{|a|\le l}{}' \Big(\int_{||x|-t|\ge t/2} \frac{[\partial u(t)]_{[|a|/2]}^2|\partial u(x,\,t)|_{|a|}^2}{(1+t)^2}dx\\
&+ \int_{||x|-t|\le t/2} \frac{[\partial u(t)]_0^2|u(x,\,t)|_{|a|+1}|\partial u(x,\,t)|_{|a|}}{(1+||x|-t|)(1+t)^2}dx\\
&+ \int_{||x|-t|\le t/2} \frac{[\partial u(t)]_0|u(x,\,t)|_1|\partial u(x,\,t)|_{|a|}^2}{(1+t)^{3/2}}dx\\
&+ \int_{||x|-t|\le t/2} |\partial u(t)|_{[|a|/2]}^2|\partial u(x,\,t)|_{|a|-1}|\partial u(x,\,t)|_{|a|}dx\\
&+ \int_{\mathbf{R}^2} \frac{[\partial u(t)]_{[|a|/2]}^3|\partial u(x,\,t)|_{|a|}^2}{(1+t)^{3/2}}dx\Big)\\
\le{}& C_{16}\Big(\frac{[\partial u(t)]_{[l/2]}^2}{(1+t)^2}\|\partial u(t)\|_l^2 + \sum_{|c|\le l+1}\frac{[\partial u(t)]_0^2}{(1+t)^2}\Big\|\frac{\Gamma^c u(x,\,t)}{1+||x|-t|}\Big\|_0\|\partial u(t)\|_l\\
&+ \frac{[\partial u(t)]_0|u(t)|_1}{(1+t)^{3/2}}\|\partial u(t)\|_l^2 + |\partial u(t)|_{[l/2]}^2\|\partial u(t)\|_{l-1}\|\partial u(t)\|_l\\
&+ \frac{[\partial u(t)]_{[l/2]}^3}{(1+t)^{3/2}}\|\partial u(t)\|_l^2\Big)\\
\le{}& C_{17}\Big(\frac{[\partial u(t)]_{[l/2]}^2 + [\partial u(t)]_{[l/2]}^3 + [\partial u(t)]_0|u(t)|_1}{(1+t)^{3/2}}\|\partial u(t)\|_l^2\\
&+ \frac{[\partial u(t)]_0^2}{(1+t)^2}\|\partial u(t)\|_{l+1}\|\partial u(t)\|_l\\
&+ |\partial u(t)|_{[l/2]}^2\|\partial u(t)\|_{l-1}\|\partial u(t)\|_l\Big),
\end{aligned}
$$

which implies

$$\frac{d}{dt}\|\partial u(t)\|_l \le C_{18}\Big(\frac{[\partial u(t)]_{[l/2]}^2 + [\partial u(t)]_{[l/2]}^3 + [\partial u(t)]_0 |u(t)|_1}{(1+t)^{3/2}}\|\partial u(t)\|_l + \frac{[\partial u(t)]_0^2}{(1+t)^2}\|\partial u(t)\|_{l+1} + |\partial u(t)|_{[l/2]}^2 \|\partial u(t)\|_{l-1}\Big). \tag{3.36}$$

Therefore, when $l = 1, 2, 3, 4$, it follows from (3.2), (3.7), (3.11), (3.26) with $\nu < 1/4$, (3.30), (3.36), Proposition 3.4 and the Gronwall inequality that

$$\begin{aligned}
\|\partial u(t)\|_l &\le \Big\{\|\partial u(\varepsilon^{-1})\|_l + \int_{1/\varepsilon}^t \Big(\frac{C_{18}[\partial u(s)]_0^2}{(1+s)^2}\|\partial u(s)\|_{l+1} \\
&\qquad + \frac{C_{19}[\partial u(s)]_{[l/2]}^2}{1+s}\|\partial u(s)\|_{l-1}\Big)ds\Big\} \\
&\qquad \times \exp\Big(C_{18}\int_{1/\varepsilon}^t \frac{[\partial u(s)]_2^2 + [\partial u(s)]_2^3 + [\partial u(s)]_0|u(s)|_1}{(1+s)^{3/2}}ds\Big) \\
&\le \Big\{C_{20}\varepsilon + \int_{1/\varepsilon}^t \Big(\frac{\tilde{C}_{14}\varepsilon^3(1+\varepsilon^2\log(1+s))^{2m_2}}{(1+s)^{2-\nu}} \\
&\qquad + \frac{C_{21}c_{l-1}\varepsilon^3(1+\varepsilon^2\log(1+s))^{m_{l-1}}}{1+s}\Big)ds\Big\} \\
&\qquad \times \exp\Big(\int_{1/\varepsilon}^t \frac{\tilde{C}_{15}(1+\log(1+s))^{3m_6}\varepsilon^2}{(1+s)^{3/2}}ds\Big) \\
&\le \Big\{C_{20}\varepsilon + \tilde{C}_{16}\varepsilon^3 + \frac{C_{21}c_{l-1}\varepsilon}{m_{l-1}+1}(1+\varepsilon^2\log(1+t))^{m_{l-1}+1}\Big\} \\
&\qquad \times \exp(\tilde{C}_{17}\varepsilon^2) \\
&\le c_l\varepsilon(1+\varepsilon^2\log(1+t))^{m_l}
\end{aligned}$$

for $\varepsilon \in (0, \varepsilon_{0l})$, if we take

$$\varepsilon_{0l} = \min\Big\{\varepsilon_2, \frac{1}{\tilde{C}_{16}^{1/2}}, \frac{1}{\tilde{C}_{17}^{1/2}}\Big\}, \quad c_l = \Big(C_{20} + 1 + \frac{C_{21}c_{l-1}}{m_{l-1}+1}\Big)e,$$
$$m_l = m_{l-1} + 1.$$

On the other hand, when $l \ge 5$, it follows from (3.2), (3.9), (3.26) with $\nu < 1/4$, (3.30), (3.36), Proposition 3.4, the Gronwall inequality and the fact $[l/2] + 2 \le l - 1$ that

$$\|\partial u(t)\|_l \le \Big\{\|\partial u(\varepsilon^{-1})\|_l + \int_{1/\varepsilon}^t C_{22}\Big(\frac{[\partial u(s)]_0^2}{(1+s)^2}\|\partial u(s)\|_{l+1}$$

$$
\begin{aligned}
&\qquad\qquad + \frac{[\partial u(s)]_{[l/2]}^2}{1+s}\|\partial u(s)\|_{l-1}\Big) ds\Big\} \\
&\times \exp\Big(C_{23}\int_{1/\varepsilon}^t \frac{[\partial u(s)]_{[l/2]}^2 + [\partial u(s)]_{[l/2]}^3 + [\partial u(s)]_0 |u(s)|_1}{(1+s)^{3/2}} ds\Big) \\
&\le \Big\{C_{24}\varepsilon + \int_{1/\varepsilon}^t \Big(\frac{\tilde{C}_{18}\varepsilon^3(1+\varepsilon^2\log(1+s))^{2m_2}}{(1+s)^{2-\nu}} \\
&\quad + \frac{C_{25}c_{l-1}^3\varepsilon^3(1+\varepsilon^2\log(1+s))^{3m_{l-1}}}{1+s}\Big) ds\Big\} \\
&\quad \times \exp\Big(\int_{1/\varepsilon}^t \frac{\tilde{C}_{19}(1+\log(1+s))^{3m_{l-1}+2m_6}\varepsilon^2}{(1+s)^{3/2}} ds\Big) \\
&\le \Big\{C_{24}\varepsilon + \tilde{C}_{20}\varepsilon^3 + \frac{C_{25}c_{l-1}^3\varepsilon}{3m_{l-1}+1}(1+\varepsilon^2\log(1+t))^{3m_{l-1}+1}\Big\} \\
&\quad \times \exp(\tilde{C}_{21}\varepsilon^2) \\
&\le c_l\varepsilon(1+\varepsilon^2\log(1+t))^{m_l}
\end{aligned}
$$

holds for $\varepsilon \in (0, \varepsilon_{0l})$, if we take

$$
\begin{aligned}
&\varepsilon_{0l} = \min\Big\{\varepsilon_2, \frac{1}{\tilde{C}_{20}^{1/2}}, \frac{1}{\tilde{C}_{21}^{1/2}}\Big\}, \quad c_l = \Big(C_{24} + 1 + \frac{C_{25}c_{l-1}^3}{3m_{l-1}+1}\Big)e, \\
&m_l = 3m_{l-1} + 1.
\end{aligned}
$$

Hence, we obtain (3.3) if we take $\varepsilon_0 = \min\{\varepsilon_{00}, \varepsilon_{01}, \dots, \varepsilon_{0k}\}$. This completes the proof of Lemma 3.1.

□

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