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Nonlinear scattering for a system of one dimensional nonlinear Klein-Gordon equations

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Abstract. We consider a system of nonlinear Klein-Gordon equations in one space dimension with quadratic nonlinearities

$$(\partial_t^2 - \partial_x^2 + m_j^2)u_j = \mathcal{N}_j(\partial u),$$

$j = 1, \ldots, l$. We show the existence of solutions in an analytic function space. When the nonlinearity satisfies a strong null condition introduced by Georgiev we prove the global existence and obtain the large time asymptotic behavior of small solutions.

Key words: systems of Klein Gordon equations, scattering problem, one dimension.

1. Introduction

We consider the Cauchy problem for the system of semi-linear Klein-Gordon equations

$$\begin{cases} (\partial_t^2 - \partial_x^2 + m_j^2)u_j = \mathcal{N}_j(\partial u), & (t, x) \in \mathbf{R} \times \mathbf{R}, \\ u_j(0, x) = \overset{\circ}{u}{}_j^{(1)}(x), \ \partial_t u_j(0, x) = \overset{\circ}{u}{}_j^{(2)}(x), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $j = 1, \ldots, l$, $m_j > 0$, the partial derivative $\partial = (\partial_t, \partial_x)$ and $u = (u_1, \ldots, u_l)$. We assume that $\mathcal{N}_j(\partial u)$ are quadratic nonlinearities. Our purpose is to prove global existence of small solutions and to consider a scattering problem for equation (1.1) under the strong null condition on the nonlinearities $\mathcal{N}_j$ introduced by [6] which is written as

$$\mathcal{N}_j(\partial u) = \sum_{p,q=1}^{l} A_{jpq}\big((\partial_t u_p)\partial_x u_q - (\partial_x u_p)\partial_t u_q\big), \tag{1.2}$$

where $A_{jpq} \in \mathbf{C}$. Condition (1.2) implies an additional time decay of order t^{-1} through the operator $\mathcal{Z} = x\partial_t + t\partial_x$, since the following identity is true

$$\big((\partial_t u_p)\partial_x u_q - (\partial_x u_p)\partial_t u_q\big) = \frac{1}{t}\big((\partial_t u_p)\mathcal{Z} u_q - (\mathcal{Z} u_p)\partial_t u_q\big).$$

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However we encounter a derivative loss difficulty with respect to the operator $\mathcal{Z}$. To overcome the derivative loss we use an analytic function space involving the operator $\mathcal{Z}$. The operator $\mathcal{Z}$ was used previously by Klainerman [12] to prove global existence theorem for the nonlinear Klein-Gordon equations with quadratic nonlinearities in three space dimensions (see also papers [1], [6], [7], [11], [15], [17]). Global existence of small solutions to cubic nonlinear Klein-Gordon equations in one space dimension was studied extensively. Non resonance cubic nonlinearities were studied in [11], [14] for a single equation and in [17] for a system of equations with different masses. In papers [4], [9], [18], the resonance cubic nonlinearities were also treated. For the case of quadratic nonlinearities there are few results. In papers [15], [3] an almost global existence of small solutions to a single semi-linear Klein-Gordon equation was studied. In [4], the global existence of small solutions was shown for some type of quadratic nonlinearities, by using the reduction of the original equation through the hyperbolic coordinates and the method of normal forms of [16]. By the method of normal forms the equation with a quadratic nonlinearity can be translated to a cubic one, for which to prove the global existence is more easy. However some suitable conditions on quadratic nonlinearities are required for applying this technique. Using hyperbolic coordinates implies the requirement of a compact support for the initial data. As far as we know there are no global results for systems of nonlinear Klein-Gordon equations with quadratic nonlinearities except of [5], where the authors generalized the method of paper [4] to a system of two Klein-Gordon equations. It was shown in [5] a global existence of small solutions under some mass conditions, a number of conditions for nonlinearities and the compactness for the initial data. Unfortunately, there were no any typical example of nonlinearity which satisfies an extensive list of conditions in Theorem 1.1 from paper [5]. However our condition (1.2) on the nonlinearities can not be compatible with that from [5] since our system includes the case which can not be translated to the cubic nonlinearities by the method of normal forms. On the other hand, we do not use here the hyperbolic coordinates and the method of normal forms by [16], therefore we do not need the compactness of the initial data and any mass condition.

In order to explain the analytic function space used in this paper we now state the notations. Let $\mathbf{L}^p$ be the usual Lebesgue space with the norm $\|\phi\|_{\mathbf{L}^p} = (\int_{\mathbf{R}} |\phi(x)|^p dx)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. Sobolev space is

$$\mathbf{H}_p^m = \Big\{\phi \in \mathbf{L}^p \colon \|\phi\|_{\mathbf{H}_p^m} \equiv \sum_{j=0}^{m} \|\partial_x^j \phi\|_{\mathbf{L}^p} < \infty\Big\},$$

where $m \in \mathbf{N}$, $1 \le p \le \infty$. We also write $\mathbf{H}^m = \mathbf{H}_2^m$ for simplicity. We let

$$\mathcal{Q} = (\partial_t, \partial_x, \mathcal{Z}),\ \mathcal{P} = (x, \partial_x, \partial_t, \mathcal{Y}, \mathcal{Z}),\ \mathcal{Y} = x\partial_x + t\partial_t,\ \mathcal{Z} = x\partial_t + t\partial_x$$

and

$$\mathbf{X}_n = \Big\{\phi \in \mathbf{L}^2 \colon \|\phi\|_{\mathbf{X}_n} = \sum_{|\alpha| \le n} \|\mathcal{Q}^\alpha \phi\|_{\mathbf{L}^2} < \infty\Big\},\ n \in \mathbf{N}.$$

We use the same notations for vector-functions, for example we write $\|f\|_{\mathbf{H}_p^m} = \sum_{j=1}^{l} \|f_j\|_{\mathbf{H}_p^m}$ for a vector $f = (f_1, \dots, f_l)$. Different positive constants we denote by the same letter C. We define an analytic function space as follows:

$$\mathbf{G}^{\mathbf{A}}(\mathcal{A}; \mathbf{X}) = \Big\{f \in \mathbf{X}; \|f\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{A};\mathbf{X})} = \sum_{\alpha \ge 0} \frac{A^\alpha}{\alpha!} \|\mathcal{A}^\alpha f\|_{\mathbf{X}} < \infty\Big\},$$

where $A = (A_1, \dots, A_N)$, $A_j > 0$, $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_N)$, $\alpha! = \prod_{j=1}^{N} \alpha_j!$, $|\alpha| = \sum_{j=1}^{N} \alpha_j$, $\alpha \ge 0$ means that $\alpha_j \ge 0$ for $1 \le j \le N$, and $\mathbf{X}$ is a Banach space. It is easy to see that

$$\mathbf{G}^{A_1 \cdots A_N}(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_N; \mathbf{X}) = \mathbf{G}^{A_2 \cdots A_N}(\mathcal{A}_2, \dots, \mathcal{A}_N; \mathbf{G}^{A_1}(\mathcal{A}_1; \mathbf{X})).$$

Our basic analytic function space is $\mathbf{G}^{\mathbf{a}}(\partial_t, \partial_x, \mathcal{Z}; \mathbf{L}^2)$, $\mathbf{a} = (a_1, a_2, a_3)$. To prove a-priori estimate of solutions in the neighborhood of $t = 0$ in the class $\mathbf{G}^{\mathbf{a}}(\partial_t, \partial_x, \mathcal{Z}; \mathbf{L}^2)$ we need to show for some small T

$$\sup_{t \in [0,T]} \|u(t)\|_{\mathbf{G}^{\mathbf{a}}(\partial_t, \partial_x, x\partial_t; \mathbf{L}^2)} < \infty.$$

Since ∂_t is equivalent to $\sqrt{m_j^2 - \partial_x^2}$ in the linear case, so this estimate is naturally related with a-priori estimate

$$\sup_{t \in [0,T]} \|u(t)\|_{\mathbf{G}^{\mathbf{a}}(x, \partial_x, x\partial_x; \mathbf{L}^2)} < \infty.$$

First we state the local existence result. Denote $\mathcal{B} = (x, \partial_x, \mathcal{Y})$.

Theorem 1.1 *Assume that for some constant vector* $\mathbf{A} = (A_1, A_2, A_3)$ *with* $A_1, A_2 > 0$, $0 < A_3 < 1$ *the norms*

$$\|\overset{\circ}{u}{}_j^{(1)}\|_{\mathbf{G^A}(x,\partial_x,x\partial_x;\mathbf{H}^2)} + \|\overset{\circ}{u}{}_j^{(2)}\|_{\mathbf{G^A}(x,\partial_x,x\partial_x;\mathbf{H}^1)} < \infty.$$

Then for some $T > 0$ *(which depends on the size of the initial data) there exists a unique solution of* (1.1) *which satisfies the estimates*

$$\sup_{0\le t\le T}\left(\|u(t)\|_{\mathbf{G^A}(\mathcal{B};\mathbf{H}^2)} + \|\partial_t u(t)\|_{\mathbf{G^A}(\mathcal{B};\mathbf{H}^1)}\right) < \infty.$$

Moreover for some constant vector $\mathbf{a}$ *the solution satisfies the estimate*

$$\sup_{0\le t\le T}\|u(t)\|_{\mathbf{G^a}(\mathcal{P};\mathbf{H}^2)} < \infty.$$

Remark 1.1 Typical example of the initial function is given by $\varepsilon\exp(-x^2)$ which decays exponentially at infinity and has an analytic continuation on the strip and on the sector. Therefore $\exp(-x^2) \in \mathbf{G^A}(x, \partial_x, x\partial_x; \mathbf{H}^2)$.

Remark 1.2 We do not need the condition $A_3 < 1$. However the result is changed as "there exists a unique solution of (1.1) and a vector $\mathbf{B}$ satisfy the estimates

$$\sup_{0\le t\le T}\left(\|u(t)\|_{\mathbf{G^B}(\mathcal{B};\mathbf{H}^2)} + \|\partial_t u(t)\|_{\mathbf{G^B}(\mathcal{B};\mathbf{H}^1)}\right) < \infty,$$

where $B_3 < 1$". This result follows from the fact that if $\sum_j(|\sin\theta|^j/j!) \times\|x^j\partial_x^j f\|_{\mathbf{L}^2} < \infty$, then f has an analytic continuation on the sector $\{z; z = x + iy, \tan\theta = y/x, x, y \in R\}$. If $A_3 \ge 1$, then we can take $\theta = \pi/2$ and $\sum_j(A_3^j/j!)\|x^j\partial_x^j f^2\|_{\mathbf{L}^2}$ can not be estimated by $\left(\sum_j(A_3^j/j!)\|x^j\partial_x^j f\|_{\mathbf{H}^1}\right)^2$ when $f \ne 0$.

Remark 1.3 The first estimate of Theorem 1.1 is also valid for the case of quasi-linear nonlinearities if nonlinear terms $\mathcal{N}_j(\partial u)$ satisfy the condition of hyperbolicity. However it seems that the second estimate of Theorem 1.1 is not valid for that case. The second estimate of Theorem 1.1 enables us to consider the problem in the time interval $[T, \infty)$ and in the class $\mathbf{G}^{\mathbf{b}(t)}(\mathcal{P}; \mathbf{H}^2)$, where $\mathbf{b}(t) = (b(t), b(t), b(t), b(t), b(t))$ and the function $b(t) = a(1+t^{-\delta})$ is a monotone decreasing function which compensates the derivative loss of the nonlinearities.

We now state a global existence and asymptotics of solutions.

Theorem 1.2 *Assume that for some constant vector* $\mathbf{A} = (A_1, A_2, A_3)$ *with* $A_1, A_2 > 0$, $0 < A_3 < 1$ *the norms*

$$\|\overset{\circ}{u}{}_j^{(1)}\|_{\mathbf{G}^{\mathbf{A}}(x,\partial_x,x\partial_x;\mathbf{H}^2)} + \|\overset{\circ}{u}{}_j^{(2)}\|_{\mathbf{G}^{\mathbf{A}}(x,\partial_x,x\partial_x;\mathbf{H}^1)} < \varepsilon$$

with some small $\varepsilon > 0$*. Furthermore suppose that the strong null condition* (1.2) *is fulfilled. Then the Cauchy problem* (1.1) *has a unique global solution* u *such that*

$$u_j \in \mathbf{C}\big([0,\infty);\mathbf{G}^{\mathbf{a}}(\mathcal{Q};\mathbf{X}_5)\big)$$

and

$$\|u(t)\|_{\mathbf{G}^a(\partial_x;\mathbf{L}^\infty)} \le C\langle t\rangle^{-1/2}$$

for all $t \ge 0$*, where* $\mathbf{a} = (a, a, a)$, $a > 0$ *is a small positive constant depending on* $\mathbf{A}$, ε*. Furthermore there exists a unique final state* $u_j^{+(1)}$, $u_j^{+(2)} \in \mathbf{G}^a(\partial_x;\mathbf{L}^2)$ *satisfying*

$$\left\| u_j(t) - \left(\cos\big(t\sqrt{m_j^2-\partial_x^2}\big)u_j^{+(1)} + \frac{\sin\big(t\sqrt{m_j^2-\partial_x^2}\big)}{\sqrt{m_j^2-\partial_x^2}}u_j^{+(2)}\right)\right\|_{\mathbf{G}^a(\partial_x;\mathbf{L}^2)} \le C\varepsilon^2\langle t\rangle^{-1/2}$$

for all $t \ge 0$, $1 \le j \le l$.

The rest of the paper is organized as follows. In Section 2 we give some preliminary estimates of the solutions. Section 3 is devoted to the proof of the local existence Theorem 1.1. We prove Theorem 1.2 in Section 4.

2. Lemmas

We denote $\alpha! = \prod_{j=1}^N \alpha_j!$ and

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\beta!}$$

for $0 \le \beta \le \alpha$. By Lemma 2.8 from paper [2] we have the estimate

$$\binom{l}{k}\binom{m}{j} \le \binom{l+m}{k+j} \tag{2.1}$$

for all $0 \leq k \leq l$ and $0 \leq j \leq m$. We next state the commutator relation between the operators ∂_t, ∂_x, $\mathcal{Y} = x\partial_x + t\partial_t$ and $\mathcal{Z} = x\partial_t + t\partial_x$. We denote $\partial_p = \partial_x$ when p is an even and $\partial_p = \partial_t$ when p is an odd.

Lemma 2.1 *The identities are true*

$$\partial_p \mathcal{Z}^l = \sum_{k=0}^{l} \binom{l}{k} \mathcal{Z}^k \partial_{p+l-k},\ \mathcal{Z}^l \partial_p = \sum_{k=0}^{l} \binom{l}{k} (-1)^k \partial_{p+l-k} \mathcal{Z}^k,$$
$$\partial_p \mathcal{Y}^l = \sum_{k=0}^{l} \binom{l}{k} \mathcal{Y}^k \partial_p,\ \mathcal{Y}^l \partial_p = \sum_{k=0}^{l} \binom{l}{k} (-1)^k \partial_p \mathcal{Y}^k,$$
$$[\partial_p^l,\, \mathcal{Z}] = l\partial_p^{l-1}\partial_{p+1},\ [\partial_p^l,\, \mathcal{Y}] = l\partial_p^l.$$

Proof. We prove the first identity by induction. When $l = 1$ we find $\partial_p \mathcal{Z} = \mathcal{Z}\partial_p + \partial_{p+1}$ and $\partial_p \mathcal{Y} = \mathcal{Y}\partial_p + \partial_p$ so the identities are valid. We suppose that the first identity is true for some $l \geq 1$, then we have

$$\begin{aligned}
&\partial_p \mathcal{Z}^{l+1} = (\partial_p \mathcal{Z})\mathcal{Z}^l = \mathcal{Z}\partial_p \mathcal{Z}^l + \partial_{p+1}\mathcal{Z}^l \\
&= \sum_{k=0}^{l} \binom{l}{k} \mathcal{Z}^{k+1} \partial_{p+l-k} + \sum_{k=0}^{l} \binom{l}{k} \mathcal{Z}^k \partial_{p+1+l-k} \\
&= \mathcal{Z}^{l+1}\partial_p + \partial_{p+1+l} + \sum_{k=1}^{l} \left(\binom{l}{k-1} + \binom{l}{k} \right) \mathcal{Z}^k \partial_{p+1+l-k} \\
&= \sum_{k=0}^{l+1} \binom{l+1}{k} \mathcal{Z}^k \partial_{p+l+1-k}.
\end{aligned}$$

Thus by induction the first identity is fulfilled for all $l \geq 1$. The other identities are considered in the same way. Lemma 2.1 is proved. □

In the following lemma we prove equivalence of the norms of the analytic functional spaces involving the operator $\mathcal{P} = (x,\, \partial_x,\, \partial_t,\, \mathcal{Y},\, \mathcal{Z})$.

Lemma 2.2 *The following inequalities are true*

$$\frac{1}{4e^{|\mathbf{A}|}} \|\mathcal{P}\phi\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X})} \leq \sum_{|\beta|=1} \sum_{\alpha \geq 0} \frac{\mathbf{A}^\alpha}{\alpha!} \|\mathcal{P}^{\beta+\alpha}\phi\|_{\mathbf{X}} \leq 4e^{|\mathbf{A}|} \|\mathcal{P}\phi\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X})}$$

and

$$\|\phi\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X}))} \leq \|\phi\|_{\mathbf{G}^{\mathbf{B}}(\mathcal{P};\mathbf{X})}$$

with $\mathbf{B} = (1 + 4e^{|\mathbf{A}|})\mathbf{A}$.

Proof. By Lemma 2.1 we have

$$\|\partial_p \phi\|_{\mathbf{G}^{A_2}(\mathcal{Z};\mathbf{Y})} = \sum_{l=0}^{\infty} \frac{A_2^l}{l!} \|\mathcal{Z}^l \partial_p \phi\|_{\mathbf{Y}}$$

$$\leq \sum_{l=0}^{\infty} \frac{A_2^l}{l!} \sum_{k=0}^{l} \binom{l}{k} \|\partial_{p+1-k} \mathcal{Z}^k \phi\|_{\mathbf{Y}}$$

$$= \sum_{k=0}^{\infty} \frac{A_2^k}{k!} \|\partial_{p+1-k} \mathcal{Z}^k \phi\|_{\mathbf{Y}} \sum_{l=k}^{\infty} \frac{A_2^{l-k}}{(l-k)!}$$

$$\leq e^{A_2} \sum_{|\beta|=1} \sum_{k=0}^{\infty} \frac{A_2^k}{k!} \|\partial^\beta \mathcal{Z}^k \phi\|_{\mathbf{Y}}. \tag{2.2}$$

We now take $\mathbf{Y} = \mathbf{G}^{A_1}(\mathcal{Y};\mathbf{X})$ and again apply Lemma 2.1 to get

$$\|\partial_p \mathcal{Z}^k \phi\|_{\mathbf{Y}} = \sum_{l=0}^{\infty} \frac{A_4^l}{l!} \|\mathcal{Y}^l \partial_p \mathcal{Z}^k \phi\|_{\mathbf{X}}$$

$$\leq \sum_{l=0}^{\infty} \frac{A_4^l}{l!} \sum_{j=0}^{l} \binom{l}{j} \|\partial_p \mathcal{Y}^j \mathcal{Z}^k \phi\|_{\mathbf{X}} \leq e^{A_4} \sum_{|\beta|=1} \sum_{j=0}^{\infty} \frac{A_4^j}{j!} \|\partial^\beta \mathcal{Y}^j \mathcal{Z}^k \phi\|_{\mathbf{X}}. \tag{2.3}$$

Substitution of (2.3) into (2.2) yields

$$\|\partial_p \phi\|_{\mathbf{G}^{A_4,A_5}(\mathcal{Y},\mathcal{Z};\mathbf{X})} = \|\partial_p \phi\|_{\mathbf{G}^{A_5}(\mathcal{Z};\mathbf{G}^{A_4}(\mathcal{Y};\mathbf{X}))}$$

$$\leq e^{A_5} \sum_{p=1,2} \sum_{k=0}^{\infty} \frac{A_5^k}{k!} \|\partial_p \mathcal{Z}^k \phi\|_{\mathbf{G}^{A_4}(\mathcal{Y};\mathbf{X})}$$

$$\leq e^{A_4+A_5} \sum_{|\beta|=1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{A_4^j A_5^k}{j!k!} \|\partial^\beta \mathcal{Y}^j \mathcal{Z}^k \phi\|_{\mathbf{X}}.$$

Since $[\mathcal{Z}, \mathcal{Y}] = 0$ and $[\partial_p, \partial] = 0$ we find

$$\|\partial_p \phi\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X})} = \sum_{\alpha \geq 0} \frac{\mathbf{A}^\alpha}{\alpha!} \|x^{\alpha_1} \partial_x^{\alpha_2} \partial_t^{\alpha_3} \mathcal{Y}^{\alpha_4} \mathcal{Z}^{\alpha_5} \partial_p \phi\|_{\mathbf{X}}$$

$$\leq e^{A_1+A_4+A_5} \sum_{|\beta|=1} \sum_{\alpha \geq 0}^{\infty} \frac{\mathbf{A}^\alpha}{\alpha!} \|\partial_p \mathcal{P}^\alpha \phi\|_{\mathbf{X}}.$$

In the same way as above via Lemma 2.1 we obtain

$$\begin{aligned}\|\mathcal{Z}\phi\|_{\mathbf{G}^{A_3}(\partial_x;\mathbf{Y})} &= \sum_{l=0}^{\infty}\frac{A_3^l}{l!}\|\partial_x^l\mathcal{Z}\phi\|_{\mathbf{Y}}\\ &\le \sum_{l=0}^{\infty}\frac{A_3^l}{l!}\|\mathcal{Z}\partial_x^l\phi\|_{\mathbf{Y}} + \sum_{l=1}^{\infty}\frac{A_3^l}{(l-1)!}\|\partial_t\partial_x^{l-1}\phi\|_{\mathbf{Y}}\\ &\le e^{A_3}\sum_{|\beta|=1}\sum_{l=0}^{\infty}\frac{A_3^l}{l!}\|(\mathcal{Z},\,\partial_t)^{\beta}\partial_x^l\phi\|_{\mathbf{Y}}.\end{aligned}$$

We now take $\mathbf{Y}=\mathbf{G}^{A_2}(\partial_t;\mathbf{X})$ and again use Lemma 2.1 to have

$$\begin{aligned}\|\mathcal{Z}\partial_x^l\phi\|_{\mathbf{Y}} &= \sum_{k=0}^{\infty}\frac{A_2^k}{k!}\|\partial_t^k\mathcal{Z}\partial_x^l\phi\|_{\mathbf{X}}\\ &\le \sum_{k=0}^{\infty}\frac{A_2^k}{k!}\|\mathcal{Z}\partial_t^k\partial_x^l\phi\|_{\mathbf{X}} + \sum_{k=1}^{\infty}\frac{A_2^k}{(k-1)!}\|\partial_x\partial_t^{k-1}\partial_x^l\phi\|_{\mathbf{X}}\\ &\le e^{A_2}\sum_{|\beta|=1}\sum_{k=0}^{\infty}\frac{A_2^k}{k!}\|(\mathcal{Z},\,\partial_x)^{\beta}\partial_t^k\partial_x^l\phi\|_{\mathbf{X}}.\end{aligned}$$

Since $[\mathcal{Z},\,\mathcal{Y}]=0$, we find

$$\begin{aligned}&\|\mathcal{Z}\phi\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X})} = \|\mathcal{Z}\phi\|_{\mathbf{G}^{A_2,A_3}(\partial_t,\partial_x;\mathbf{G}^{A_1,A_2}(\mathcal{Y},\mathcal{Z};\mathbf{X}))}\\ &\le e^{A_2}\sum_{l=0}^{\infty}\frac{A_3^l}{l!}\|(\partial_t,\mathcal{Z})\partial_x^l\phi\|_{\mathbf{G}^{A_3}(\partial_t;\mathbf{G}^{A_4,A_5}(\mathcal{Y},\mathcal{Z};\mathbf{X}))}\\ &\le e^{A_2+A_3}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{A_2^jA_3^k}{j!k!}\|(\partial_t,\,\partial_x,\,\mathcal{Z})\partial_t^k\partial_x^l\phi\|_{\mathbf{G}^{A_4,A_5}(\mathcal{Y},\,\mathcal{Z};\mathbf{X})}\\ &\le e^{|\mathbf{A}|}\sum_{|\beta|=1}\sum_{\alpha\ge 0}^{\infty}\frac{\mathbf{A}^{\alpha}}{\alpha!}\|\mathcal{P}^{\alpha+\beta}\phi\|_{\mathbf{X}}.\end{aligned}$$

In the same way we have

$$\|\mathcal{Y}\phi\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X})} \le e^{|\mathbf{A}|}\sum_{|\beta|=1}\sum_{\alpha\ge 0}^{\infty}\frac{\mathbf{A}^{\alpha}}{\alpha!}\|\mathcal{P}^{\alpha+\beta}\phi\|_{\mathbf{X}}.$$

Thus we get the first inequality of the lemma. The second inequality is considered in the same manner.

By the first estimate of the lemma we have

$$\sum_{|\beta|=1} \|\mathcal{P}^{\beta}\phi\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X})} \leq \sum_{|\beta|=1} (4e^{|\mathbf{A}|})^{|\beta|} \sum_{\alpha\geq 0} \frac{\mathbf{A}^{\alpha}}{\alpha!} \|\mathcal{P}^{\beta+\alpha}\phi\|_{\mathbf{X}}$$

hence

$$\begin{aligned}
&\|\phi\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X}))} = \sum_{\beta\geq 0}^{\infty} \frac{\mathbf{A}^{\beta}}{\beta!} \|\mathcal{P}^{\beta}\phi\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X})} \\
&\leq \sum_{\beta\geq 0}^{\infty} (4e^{|\mathbf{A}|})^{|\beta|} \sum_{\alpha\geq 0} \frac{\mathbf{A}^{\alpha+\beta}}{\alpha!\beta!} \|\mathcal{P}^{\alpha+\beta}\phi\|_{\mathbf{X}} \\
&= \sum_{\alpha\geq 0} \frac{\mathbf{A}^{\alpha}}{\alpha!} \|\mathcal{P}^{\alpha}\phi\|_{\mathbf{X}} \sum_{\beta\geq 0}^{\alpha} \binom{\alpha}{\beta} (4e^{|\mathbf{A}|})^{|\beta|} \\
&= \sum_{\alpha\geq 0} \frac{((1+4e^{|\mathbf{A}|})\mathbf{A})^{\alpha}}{\alpha!} \|\mathcal{P}^{\alpha}\phi\|_{\mathbf{X}} = \|\phi\|_{\mathbf{G}^{\mathbf{B}}(\mathcal{P};\mathbf{X})}.
\end{aligned}$$

Lemma 2.2 is proved. □

By Lemma 2.2 we can see that the ordering of the operators x, ∂_x, ∂_t, $\mathcal{Y}$, $\mathcal{Z}$ in the analytic spaces $\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X})$ is not so important, i.e. the analytic spaces $\mathbf{G}^{\mathbf{A}}(\mathcal{Y}, \mathcal{Z}, \partial_t, \partial_x, x;\mathbf{X})$ and $\mathbf{G}^{\mathbf{A}}(x, \partial_x, \partial_t, \mathcal{Y}, \mathcal{Z};\mathbf{X})$ are equivalent. Also by the definition of the analytic spaces we see that

$$\|\phi\|_{\mathbf{G}^{A}(\mathcal{A}+\mathcal{B};\mathbf{X})} \leq \|\phi\|_{\mathbf{G}^{A,A}(\mathcal{A},\mathcal{B};\mathbf{X})}$$

if $[\mathcal{A}, \mathcal{B}] = 0$.

By Lemma 2.2 of [8] we have the following result. Let

$$\mathbf{X}_n = \Big\{\phi \in \mathbf{L}^2 \colon \|\phi\|_{\mathbf{X}_n} = \sum_{|\alpha|\leq n} \|\mathcal{Q}^{\alpha}\phi\|_{\mathbf{L}^2} < \infty\Big\},\ n \in \mathbf{N}.$$

Lemma 2.3 *The estimate is true*

$$\|fg\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X}_n)} \leq C\|f\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X}_n)}\|g\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{P};\mathbf{X}_n)}$$

for $n \geq 2$.

We now state the time decay estimates of smooth and decaying functions through the operator $\mathcal{J}_m = \sqrt{m^2 - \partial_x^2}\mathcal{U}_m(t)x\mathcal{U}_m(-t)$, where

$$\mathcal{U}_m(t) = \begin{pmatrix} e^{-it\sqrt{m^2-\partial_x^2}} & 0 \\ 0 & e^{it\sqrt{m^2-\partial_x^2}} \end{pmatrix}.$$

Lemma 2.4 *Assume that $m > 0$. Then the estimate*

$$\|\phi\|_{\mathbf{L}^p} \le C\langle t\rangle^{-(1/2)(1-2/p)} \sum_{l=0}^{1} \|\mathcal{J}_m^l \phi\|_{\mathbf{H}^{\nu-1}}^{(1/2)(1-2/p)} \|\phi\|_{\mathbf{H}^{\nu}}^{1-(1/2)(1-2/p)}$$

is valid for all $t > 0$,for $2 \le p \le \infty$, where $\nu = (3/2)(1 - 2/p)$.

Proof. For the convenience of the reader we give the proof for $p = \infty$ according to Lemma 2.1 of [10]. For general p, see [10]. We have the $\mathbf{L}^\infty - \mathbf{L}^1$ time decay estimate for the free evolution group $\mathcal{U}_m(t)$ (see Lemma 1 in [13])

$$\|\phi\|_{\mathbf{L}^\infty} = \|\mathcal{U}_m(t)\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^\infty} \le Ct^{-1/2}\|\langle i\partial_x\rangle^{3/2}\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^1}$$

for all $t > 0$. Taking $\rho = \|x\phi\|_{\mathbf{L}^2}\|\phi\|_{\mathbf{L}^2}^{-1}$ we obtain by the Hölder inequality

$$\begin{aligned} &\|\phi\|_{\mathbf{L}^1} \le C\|(\rho + |x|)^{-1}\|_{\mathbf{L}^2}\|(\rho + |x|)\phi\|_{\mathbf{L}^2} \\ &\le C\rho^{-1/2}\||x|\phi\|_{\mathbf{L}^2} + C\rho^{1/2}\|\phi\|_{\mathbf{L}^2} \le C\||x|\phi\|_{\mathbf{L}^2}^{1/2}\|\phi\|_{\mathbf{L}^2}^{1/2} \end{aligned}$$

From these estimates

$$\begin{aligned} \|\phi\|_{\mathbf{L}^\infty} \le\, & Ct^{-1/2}\|\langle i\partial_x\rangle^{3/2}\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^2}^{1/2} \\ & \times\||x|\langle i\partial_x\rangle^{3/2}\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^2}^{1/2}. \end{aligned} \tag{2.4}$$

Since $x\langle i\partial_x\rangle^\alpha = \mathcal{F}^{-1}(i\partial_\xi\langle\xi\rangle^\alpha) = \langle i\partial_x\rangle^\alpha x + \alpha\langle i\partial_x\rangle^{\alpha-2}\partial_x$, we find by a direct computation $\mathcal{U}_m(t)x\langle i\partial_x\rangle^\alpha\mathcal{U}_m(-t)\phi = \alpha\langle i\partial_x\rangle^{\alpha-2}\partial_x\phi + \langle i\nabla\rangle^{\alpha-1}\mathcal{J}_m\phi$. Hence

$$\|x\langle i\partial_x\rangle^\alpha\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{H}^{\alpha-1}} + C\|\mathcal{J}_m\phi\|_{\mathbf{H}^{\alpha-1}}. \tag{2.5}$$

We apply (2.5) to (2.4) to obtain the result of the lemma. □

3. Proof of Theorem 1.1

Let us consider the linearized version of equation (1.1)

$$\begin{cases} (\Box + m_j^2)u_j = \mathcal{N}_j(\partial v), & (t,\, x) \in \mathbf{R}\times\mathbf{R}, \\ u_j(0,\, x) = \overset{\circ}{u}{}_j^{(1)}(x),\ \partial_t u_j(0,\, x) = \overset{\circ}{u}{}_j^{(2)}(x), & x \in \mathbf{R}, \end{cases} \tag{3.1}$$

for $0 \le j \le l$, where $v = (v_1, \ldots, v_l)$ is given, $\Box = \partial_t^2 - \partial_x^2$. Suppose that

$$\sup_{0\le t\le T}(\|\partial_t v(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)} + \|v(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}) \le \rho,$$

where $\mathcal{B} = (x, \partial_x, \mathcal{Y})$, $A = (A_1, A_2, A_3)$ with $A_1, A_2 > 0$, $0 < A_3 < 1$ and prove that

$$\sup_{0\le t\le T}(\|\partial_t u(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)} + \|u(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}) \le \rho$$

for small time $T > 0$.

We apply the operator $\mathcal{B} = (x, \partial_x, \mathcal{Y})$ to equation (3.1). By the commutator relations $[\Box, x^n] = (n(n-1)x^{n-2} - 2n\partial_x x^{n-1})$ and $[\Box, \mathcal{Y}] = 2\Box$ we get denoting $\beta = (0, 0, \beta_3)$, $\gamma = (1, 0, 0)$

$$\Box\mathcal{B}^\alpha = \sum_{\beta_3=0}^{\alpha_3}\binom{\alpha_3}{\beta_3}2^{\beta_3}\mathcal{B}^{\alpha-\beta}\Box + \alpha_1(\alpha_1-1)\mathcal{B}^{\alpha-2\gamma} - 2\alpha_1\partial_x\mathcal{B}^{\alpha-\gamma}.$$

Hence

$$\begin{aligned}(\Box + m_j^2)\mathcal{B}^\alpha u_j = &\sum_{\beta_3=0}^{\alpha_3}\binom{\alpha_3}{\beta_3}2^{\beta_3}\mathcal{B}^{\alpha-\beta}\mathcal{N}_j(\partial v)\\ &- m_j^2\sum_{\beta_3=0}^{\alpha_3-1}\binom{\alpha_3}{\beta_3}2^{\beta_3}\mathcal{B}^{\alpha-\beta}u_j\\ &+ \alpha_1(\alpha_1-1)\mathcal{B}^{\alpha-2\gamma}u_j - 2\alpha_1\partial_x\mathcal{B}^{\alpha-\gamma}u_j.\end{aligned}$$

Multiplying both sides of the above equation by $(1+\partial_x)^2\partial_t\mathcal{B}^\alpha u_j$, integrating the result with respect to space, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\partial_t\mathcal{B}^\alpha u_j(t)\|_{\mathbf{H}^1} + \|\partial_x\mathcal{B}^\alpha u_j(t)\|_{\mathbf{H}^1} + m_j\|\mathcal{B}^\alpha u_j(t)\|_{\mathbf{H}^1})\\ &\le 2\sum_{\beta_3=0}^{\alpha_3}\binom{\alpha_3}{\beta_3}2^{\beta_3}(\|\mathcal{B}^{\alpha-\beta}\mathcal{N}_j(\partial v)\|_{\mathbf{H}^1} + m_j\|\mathcal{B}^{\alpha-\beta}u_j\|_{\mathbf{H}^1})\\ &\quad + 2\alpha_1(\alpha_1-1)\|\mathcal{B}^{\alpha-2\gamma}u_j\|_{\mathbf{H}^1} + 4\alpha_1\|\partial_x\mathcal{B}^{\alpha-\gamma}u_j\|_{\mathbf{H}^1}.\end{aligned}$$

Multiplying this inequality by $\mathbf{A}^\alpha/\alpha!$ and taking a sum over $\alpha \ge 0$ we get

$$\frac{d}{dt}\sum_{\alpha\ge0}\frac{\mathbf{A}^\alpha}{\alpha!}(\|\partial_t\mathcal{B}^\alpha u_j(t)\|_{\mathbf{H}^1} + \|\partial_x\mathcal{B}^\alpha u_j(t)\|_{\mathbf{H}^1} + m_j\|\mathcal{B}^\alpha u_j(t)\|_{\mathbf{H}^1})$$

$$
\begin{aligned}
&\leq 2\sum_{\alpha\geq 0}\frac{\mathbf{A}^{\alpha-\beta}}{(\alpha-\beta)!}\sum_{\beta_3=0}^{\alpha_3}\frac{(2\mathbf{A})^{\beta}}{\beta!}(\|\mathcal{B}^{\alpha-\beta}\mathcal{N}_j(\partial v)\|_{\mathbf{H}^1}+m_j\|\mathcal{B}^{\alpha-\beta}u_j\|_{\mathbf{H}^1})\\
&\quad+2A_1^2\sum_{\alpha\geq 0}\frac{\mathbf{A}^{\alpha-2\gamma}}{(\alpha-2\gamma)!}\|\mathcal{B}^{\alpha-2\gamma}u_j\|_{\mathbf{H}^1}\\
&\quad+4A_1\sum_{\alpha\geq 0}\frac{\mathbf{A}^{\alpha-\gamma}}{(\alpha-\gamma)!}\|\partial_x\mathcal{B}^{\alpha-\gamma}u_j\|_{\mathbf{H}^1}.
\end{aligned}
$$

By Lemma 2.2 the norm

$$
\sum_{\alpha\geq 0}\frac{\mathbf{A}^{\alpha}}{\alpha!}(\|\partial_t\mathcal{B}^{\alpha}u_j(t)\|_{\mathbf{H}^1}+\|\partial_x\mathcal{B}^{\alpha}u_j(t)\|_{\mathbf{H}^1})
$$

is equivalent to the norm $\|\partial u(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)}$. Therefore we have

$$
\begin{aligned}
&\|\partial u_j(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)}+m_j\|u_j(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)}\\
&\leq C\|\overset{\circ}{u}{}_j^{(1)}\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}+C\|\overset{\circ}{u}{}_j^{(2)}\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)}\\
&\quad+C\int_0^t(\|\mathcal{N}_j(\partial v)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)}+\|u(\tau)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)}\\
&\qquad\qquad+\|\partial_x u(\tau)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)})d\tau
\end{aligned}
$$

We now use Lemma 2.3 to estimate the nonlinearity

$$
\begin{aligned}
&\int_0^t\|\mathcal{N}_j(\partial v)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)}d\tau\\
&\qquad\qquad\leq C\int_0^t(\|\partial_t v(\tau)\|^2_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)}+\|v(\tau)\|^2_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)})d\tau.
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
&\|\partial u_j(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)}+m_j\|u_j(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}\\
&\leq C\|\overset{\circ}{u}{}_j^{(1)}\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}+C\|\overset{\circ}{u}{}_j^{(2)}\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)}\\
&\quad+C\int_0^t\|u(\tau)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}d\tau+C\rho^2T\\
&\leq\frac{\rho}{4}+CT\sup_{t\in[0,T]}\|u(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}+C\rho^2T,
\end{aligned}
$$

provided that $C\|\overset{\circ}{u}_j^{(1)}\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)} + C\|\overset{\circ}{u}_j^{(2)}\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)} \le \rho/4$ and

$$\sup_{0\le t\le T}(\|\partial_t v(\tau)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)} + \|v(\tau)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}) \le \rho.$$

Therefore for some time T such that $\{1/(1-CT)\}(1/4+C\rho T) < 1$ we find

$$\sup_{0\le t\le T}(\|\partial_t u(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)} + \|u(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}) \le \rho. \tag{3.2}$$

Therefore the mapping $\mathcal{M}$ defined by $u = \mathcal{M}(v)$ transforms

$$\mathbf{X}_T = \big\{u \in C([0,\,T];\mathbf{L}^2);\\ \sup_{0\le t\le T}(\|\partial_t u(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)} + \|u(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}) < \infty\big\}$$

into itself if $T > 0$ is sufficiently small. In the same way we can prove that

$$\sup_{0\le t\le T}(\|\partial_t(\mathcal{M}(v_1)-\mathcal{M}(v_2))\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)} + \|\mathcal{M}(v_1)-\mathcal{M}(v_2)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)})\\ \le \frac{1}{2}\sup_{0\le t\le T}(\|\partial_t(v_1(t)-v_2(t))\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^1)} + \|v_1(t)-v_2(t)\|_{\mathbf{G}^{\mathbf{A}}(\mathcal{B};\mathbf{H}^2)}).$$

Therefore $\mathcal{M}$ is a contraction mapping and we have the first result of Theorem 1.1.

Let us prove the following estimate

$$\|\mathcal{B}^\alpha \partial_t^k u(t)\|_{\mathbf{H}^1} \le \rho A^{-k-|\alpha|}(|\alpha|+k)! \tag{3.3}$$

for all $\alpha \ge 0$ and $k \ge 0$. For $k = 0, 1$ estimate (3.3) follows from (3.2) and Lemma 2.2. By equation (3.1) we have

$$\partial_t^{k+1}u_j = \partial_t^{k-1}\partial_x^2 u_j + (\partial_t^{k-1}\mathcal{N}_j(\partial v) - m_j^2\partial_t^{k-1}u_j).$$

By induction we assume that (3.3) holds for some k, then

$$\|\mathcal{B}^\alpha\partial_t^{k+1}u(t)\|_{\mathbf{H}^1} \le \|\mathcal{B}^\alpha\partial_t^{k-1}\partial_x^2 u\|_{\mathbf{H}^1} + C\|\mathcal{B}^\alpha\partial_t^{k-1}u_j\|_{\mathbf{H}^1}\\ + \|\mathcal{B}^\alpha\partial_t^{k-1}\mathcal{N}_j(\partial v)\|_{\mathbf{H}^1}\\ \le C\rho A^{-1-k-|\alpha|}(|\alpha|+k+1)!.$$

Therefore (3.3) is true for all $k \ge 0$ and $\alpha \ge 0$. Then since $(|\alpha|+k)!/(\alpha!k!) \le 4^{|\alpha|+k}$

$$\|u\|_{\mathbf{G}^{\mathbf{B}}(\partial_t,\mathcal{B};\mathbf{H}^1)} = \sum_{k=0}^{\infty}\sum_{\alpha\ge 0}\frac{\mathbf{B}^\alpha}{\alpha!}\frac{B^k}{k!}\|\mathcal{B}^\alpha\partial_t^k u\|_{\mathbf{H}^1}$$

$$\le C\rho \sum_{k=0}^{\infty} \sum_{\alpha \ge 0} \frac{B^{|\alpha|+k}}{\alpha! k!} A^{-k-|\alpha|}(|\alpha|+k)!$$

$$\le C\rho \sum_{k=0}^{\infty} \sum_{\alpha \ge 0} \Big(\frac{4B}{A}\Big)^{|\alpha|+k} \le C\rho$$

if $B < A/4$.

In the same manner we prove the estimate

$$\|\mathcal{B}^{\alpha}\mathcal{Z}^{k}u(t)\|_{\mathbf{H}^1} \le \rho A^{-k-|\alpha|}(|\alpha|+k)!. \tag{3.4}$$

For $k = 0$, 1 estimate (3.4) follows from (3.2) and Lemma 2.2. Applying the identity $\mathcal{Z}^2 = \mathcal{Y}^2 + (x^2 - t^2)\Box$ we find

$$\mathcal{Z}^2 u_j = \mathcal{Y}^2 u_j + (x^2 - t^2)(\mathcal{N}_j(\partial v) - m_j^2 u_j).$$

Therefore we have

$$\mathcal{Z}^{k+1}u_j = \mathcal{Z}^{k-1}\mathcal{Y}^2 u_j - (t^2 - x^2)(\mathcal{Z}^{k-1}\mathcal{N}_j(\partial v) - m_j^2 \mathcal{Z}^{k-1}u_j)$$

for $k \ge 1$. Also we have $x\partial_t = \mathcal{Z} - t\partial_x$ and $x\partial_x = \mathcal{Y} - t\partial_t$

$$\begin{aligned} x^2\mathcal{N}_j(\partial u) = &\sum_{p,q=1}^{l} A_{jpq}((\mathcal{Z}u_p)\mathcal{Y}u_q - (\mathcal{Z}u_q)\mathcal{Y}u_p) - t^2\mathcal{N}_j(\partial u) \\ &- t\sum_{p,q=1}^{l} A_{jpq}((\mathcal{Z}u_p)\partial_t u_q - (\mathcal{Z}u_q)\partial_t u_p) \\ &- t\sum_{p,q=1}^{l} A_{jpq}((\partial_x u_p)\mathcal{Y}u_q - (\partial_x u_q)\mathcal{Y}u_p). \end{aligned}$$

By induction we assume that (3.4) holds for some k, then

$$\begin{aligned} \|\mathcal{B}^{\alpha}\mathcal{Z}^{k+1}u\|_{\mathbf{H}^1} &\le \|\mathcal{B}^{\alpha}\mathcal{Y}^2\mathcal{Z}^{k-1}u\|_{\mathbf{H}^1} + \|\mathcal{B}^{\alpha}(t^2 - x^2)\mathcal{Z}^{k-1}u\|_{\mathbf{H}^1} \\ &\quad + \|\mathcal{B}^{\alpha}(t^2 - x^2)\mathcal{Z}^{k-1}\mathcal{N}(\partial v)\|_{\mathbf{H}^1} \\ &\le C\rho A^{-1-k-|\alpha|}(|\alpha|+k+1)!. \end{aligned}$$

Therefore (3.4) is true for all $k \ge 0$ and $\alpha \ge 0$. As above since $(|\alpha| + k)!/\alpha! k! \le 4^{|\alpha|+k}$ we find

$$\|u\|_{\mathbf{G^B}(\mathcal{Z},\mathcal{B};\mathbf{H}^1)} = \sum_{k=0}^{\infty} \sum_{\alpha \ge 0} \frac{\mathbf{B}^{\alpha}}{\alpha!} \frac{B^k}{k!} \|\mathcal{B}^{\alpha}\mathcal{Z}^{k}u\|_{\mathbf{H}^1}$$

$$
\begin{aligned}
&\leq C\rho \sum_{k=0}^{\infty} \sum_{\alpha \geq 0} \frac{B^{|\alpha|+k}}{\alpha! k!} A^{-k-|\alpha|} (|\alpha|+k)! \\
&\leq C\rho \sum_{k=0}^{\infty} \sum_{\alpha \geq 0} \Big(\frac{4B}{A}\Big)^{|\alpha|+k} \leq C\rho
\end{aligned}
$$

if $B < A/4$. From which it follows that there exists a constant vector $\mathbf{a}$ such that

$$
\|u(t)\|_{\mathbf{G}^{\mathbf{a}}(x,\, \partial_x,\, \partial_t,\, \mathcal{Y},\, \mathcal{Z}; \mathbf{H}^2)} \leq C\rho.
$$

Theorem 1.1 is proved. $\square$

4. Proof of Theorem 1.2

As stated in the introduction we have

$$
\begin{aligned}
\mathcal{N}_j(\partial u) &= \sum_{p,q=1}^{l} A_{jpq}((\partial_t u_p)\partial_x u_q - (\partial_x u_p)\partial_t u_q) \\
&= \frac{1}{t} \sum_{p,q=1}^{l} A_{jpq}((\partial_t u_p)\mathcal{Z} u_q - (\mathcal{Z} u_p)\partial_t u_q).
\end{aligned}
$$

Denote $\langle i\partial_x \rangle_{m_j} = \sqrt{m_j^2 - \partial_x^2}$, $m_j > 0$. We translate the original equation to a system of evolution equations

$$
\begin{cases}
\mathcal{L}_{m_j} w_j = \langle i\partial_x \rangle_{m_j}^{-1} F_j(\partial w), & (t,\, x) \in [T,\, \infty) \times \mathbf{R}, \\
w_j(t,\, x) = \overset{\circ}{w}_j, & x \in \mathbf{R},
\end{cases}
\tag{4.1}
$$

where the vector-functions

$$
\begin{aligned}
w_j &= \begin{pmatrix} w_j^{(1)} \\ w_j^{(2)} \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} u_j + i\langle i\partial_x \rangle_{m_j}^{-1} \partial_t u_j \\ u_j - i\langle i\partial_x \rangle_{m_j}^{-1} \partial_t u_j \end{pmatrix}, \\
\overset{\circ}{w}_j &= \begin{pmatrix} \overset{\circ}{w}{}_j^{(1)} \\ \overset{\circ}{w}{}_j^{(2)} \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} \overset{\circ}{u}{}_j^{(1)} + i\langle i\partial_x \rangle_{m_j}^{-1} \overset{\circ}{u}{}_j^{(2)} \\ \overset{\circ}{u}{}_j^{(1)} - i\langle i\partial_x \rangle_{m_j}^{-1} \overset{\circ}{u}{}_j^{(2)} \end{pmatrix}, \\
F_j(\partial w) &= \begin{pmatrix} i\mathcal{N}_j(\partial(w^{(1)} + w^{(2)})) \\ -i\mathcal{N}_j(\partial(w^{(1)} + w^{(2)})) \end{pmatrix}
\end{aligned}
$$

and the linear operator

$$\mathcal{L}_m = \begin{pmatrix} \partial_t + i\langle i\partial_x\rangle_m & 0 \\ 0 & \partial_t - i\langle i\partial_x\rangle_m \end{pmatrix}.$$

We use the free Klein-Gordon evolution group

$$\mathcal{U}_m(t) = \begin{pmatrix} e^{-it\langle i\partial_x\rangle_m} & 0 \\ 0 & e^{it\langle i\partial_x\rangle_m} \end{pmatrix}$$

introduced in Lemma 2.4. Note that the operator

$$\begin{aligned} \mathcal{J}_m &= \langle i\partial_x\rangle_m \mathcal{U}_m(t) x \mathcal{U}_m(-t) \\ &= \langle i\partial_x\rangle_m \begin{pmatrix} x + t\langle i\partial_x\rangle_m^{-1}\partial_x & 0 \\ 0 & x - t\langle i\partial_x\rangle_m^{-1}\partial_x \end{pmatrix} \\ &= \begin{pmatrix} x\langle i\partial_x\rangle_m - i\langle i\partial_x\rangle_m^{-1}\partial_x + it\partial_x & 0 \\ 0 & x\langle i\partial_x\rangle_m - i\langle i\partial_x\rangle_m^{-1}\partial_x - it\partial_x \end{pmatrix} \end{aligned} \tag{4.2}$$

is useful for obtaining the time decay estimates of solutions as stated in Lemma 2.4, where we applied the commutator relations

$$[x,\, \langle i\partial_x\rangle_m^{\lambda}] = \mathcal{F}^{-1}[i\partial_\xi,\, \langle \xi\rangle_m^{\lambda}] = \lambda\langle i\partial_x\rangle_m^{\lambda-2}\partial_x. \tag{4.3}$$

By a direct calculation we see that $[\mathcal{L}_m,\, \mathcal{J}_m] = 0$. However the operator $\mathcal{J}_m$ does not act as the first order differential operator on the power nonlinearity. Therefore we use the first order differential operator

$$\mathcal{Z}E = \begin{pmatrix} \mathcal{Z} & 0 \\ 0 & \mathcal{Z} \end{pmatrix}$$

which is related to $\mathcal{J}_m$ by

$$\mathcal{Z}E = \mathcal{L}_m x - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathcal{J}_m \tag{4.4}$$

and it almost commutes with $\mathcal{L}_m$

$$\begin{aligned} [\mathcal{L}_m,\, \mathcal{Z}E] &= E\partial_x - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}[x,\, \langle i\partial_x\rangle_m]\partial_t \\ &= -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\langle i\partial_x\rangle_m^{-1}\partial_x\mathcal{L}_m. \end{aligned} \tag{4.5}$$

Now we construct the solution for all $t \geq T$. Denote $\mathcal{Q} = (\partial_t,\, \partial_x,\, \mathcal{Z})$. Since $[\partial_t^2 - \partial_x^2 + m_j^2,\, \mathcal{Q}] = 0$, applying the operator $\mathcal{Q}^\alpha$ to equation (1.1) we get

$$(\partial_t^2 - \partial_x^2 + m_j^2)\mathcal{Q}^\alpha u_j = \mathcal{Q}^\alpha \mathcal{N}_j(\partial u), \tag{4.6}$$

where

$$\begin{aligned}\mathcal{Q}^\alpha \mathcal{N}_j(\partial u) = \frac{1}{t} \sum_{p,q=1}^{l} A_{jpq} \sum_{0 \leq \beta \leq \alpha} \binom{\beta}{\alpha} \\ \times((\mathcal{Q}^{\alpha-\beta}\partial_t u_p)\mathcal{Q}^\beta \mathcal{Z} u_q - (\mathcal{Q}^{\alpha-\beta}\mathcal{Z} u_p)\mathcal{Q}^\beta \partial_t u_q).\end{aligned}$$

Therefore we have

$$\begin{cases} \begin{pmatrix} (\partial_t + i\langle i\partial_x\rangle_{m_j})\mathcal{Q}^\alpha w_j^{(1)} \\ (\partial_t - i\langle i\partial_x\rangle_{m_j})\mathcal{Q}^\alpha w_j^{(2)} \end{pmatrix} = \langle i\partial_x\rangle_{m_j}^{-1} \mathcal{Q}^\alpha F_j(\partial w), & \\ \qquad\qquad\qquad\qquad\qquad (t,\, x) \in [T,\, \infty) \times \mathbf{R}, & \\ \mathcal{Q}^\alpha w_j(t,\, x) = \mathcal{Q}^\alpha \overset{\circ}{w}_j(x), \qquad\qquad x \in \mathbf{R}. & \end{cases} \tag{4.7}$$

Note that by Theorem 1.1, we may assume that

$$\begin{aligned}&\sup_{t\in[0,\,T]} \|w(t)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)} \\ &\leq C(t)(\|\overset{\circ}{u}{}_j^{(1)}\|_{\mathbf{G}^{A_1A_2A_3}(x,\partial_x,x\partial_x;\mathbf{H}^2)} + \|\overset{\circ}{u}{}_j^{(2)}\|_{\mathbf{G}^{A_1A_2A_3}(x,\partial_x,x\partial_x;\mathbf{H}^1)}) \\ &\leq C\varepsilon,\end{aligned}$$

where $\mathbf{b}(t) = (b(t),\, b(t),\, b(t))$ and $b(t) = a(1 + t^{-\delta})$ for some small a. We consider now problem (4.7) with the nonlinearities replaced by $\langle i\partial_x\rangle_{m_j}^{-1}\mathcal{Q}^\alpha F_j(\partial v)$, where v is in the space

$$\begin{aligned}\mathbf{X} = \Big\{v;\, \sup_{t\in[T,\,\infty)} \|v(t)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)} \\ + a\delta \int_T^\infty t^{-1-\delta} \|\mathcal{Q}v(t)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)} dt \leq 2C\varepsilon\Big\}.\end{aligned}$$

We multiply both sides of equation (4.7) by the vector

$$b^{2|\alpha|}(t)(\alpha!)^{-2}\overline{\begin{pmatrix} \mathcal{Q}^\alpha w_j^{(1)} \\ \mathcal{Q}^\alpha w_j^{(2)} \end{pmatrix}}.$$

Then integrating in space and taking the real part of the result, we obtain for all $|\alpha| \geq 1$

$$\frac{d}{dt}\frac{1}{\alpha!}b^{|\alpha|}(t)\|\mathcal{Q}^{\alpha}w_j(t)\|_{\mathbf{L}^2} - \frac{|\alpha|}{\alpha!}b^{|\alpha|-1}(t)b'(t)\|\mathcal{Q}^{\alpha}w_j(t)\|_{\mathbf{L}^2}$$
$$= \frac{\alpha!}{2b^{|\alpha|}(t)\|\mathcal{Q}^{\alpha}w_j(t)\|_{\mathbf{L}^2}}$$
$$\times \operatorname{Re}\Big(\frac{1}{\alpha!}b^{|\alpha|}(t)\langle i\partial_x\rangle_{m_j}^{-1}\mathcal{Q}^{\alpha}F_j(\partial v),\ \frac{1}{\alpha!}b^{|\alpha|}(t)\mathcal{Q}^{\alpha}w_j\Big).$$

Hence using the fact that

$$\frac{d}{dt}\|w_j(t)\|_{\mathbf{L}^2} \leq \frac{1}{2}\|F_j(\partial v)\|_{\mathbf{L}^2}$$

and using the norm

$$\|w\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)} = \sum_{j=1}^{l}\sum_{\alpha\geq 0}\frac{1}{\alpha!}b^{|\alpha|}(t)\|\mathcal{Q}^{\alpha}w_j(t)\|_{\mathbf{X}_5},$$

we get

$$\frac{d}{dt}\|w\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)} + a\delta t^{-1-\delta}\sum_{|\beta|=1}\|\mathcal{Q}^{\beta}w\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}$$
$$\leq C\|\langle i\partial_x\rangle_{m_j}^{-1}F_j(\partial v)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}.$$

Applying Lemma 2.3, time decay estimate of Lemma 2.4 and (4.4) we get

$$\|\langle i\partial_x\rangle_{m_j}^{-1}F(\partial v)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}$$
$$\leq Ct^{-3/2}(\|\mathcal{Q}v\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)} + \|v\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)})$$
$$\times\Big(\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}b^{|\alpha|}(t)\sum_{|\beta|\leq 4}\|\mathcal{J}_{m_j}\mathcal{Q}^{\alpha+\beta}v\|_{\mathbf{L}^2} + \|v\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_4)}\Big)$$
$$\leq Ct^{-3/2}(\|\mathcal{Q}v\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)} + \|v\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)})$$
$$\times\Big(\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}b^{|\alpha|}(t)\sum_{|\beta|\leq 4}\|x\mathcal{L}_{m_j}\mathcal{Q}^{\alpha+\beta}v\|_{\mathbf{L}^2} + \|v\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}\Big).$$

Since by (4.7) and the strong null condition we get

$$\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}b^{|\alpha|}(t)\sum_{|\beta|\leq 4}\|x\mathcal{L}_{m_j}\mathcal{Q}^{\alpha+\beta}v\|_{\mathbf{L}^2} \leq C\|v\|_{\mathbf{G}^{b(t)}(\mathcal{Q};\mathbf{X}_5)}$$

from which it follows that

$$\begin{aligned}&\frac{d}{dt}\|w\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)} + a\delta(1+t)^{-1-\delta}\|\mathcal{Q}w\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}\\ &\le Ct^{-3/2}(\|\mathcal{Q}v\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)} + \|v\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)})\|v(t)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}.\end{aligned}$$

Thus we have the a-priori estimate

$$\begin{aligned}&\sup_{t\in[T,\infty)}\|w(t)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)} + a\delta\int_T^\infty t^{-1-\delta}\|\mathcal{Q}w(t)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}dt\\ &\le 2C\varepsilon.\end{aligned}$$

Therefore we find that the mapping $\mathcal{M}$ defined by $w = \mathcal{M}(v)$ transforms $\mathbf{X}$ into itself. In the same way we can prove

$$\begin{aligned}&\sup_{t\in[T,\infty)}\|\mathcal{M}(v_1) - \mathcal{M}(v_2)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}\\ &+ a\delta\int_T^\infty t^{-1-\delta}\|\mathcal{Q}(\mathcal{M}(v_1)(t) - \mathcal{M}(v_2)(t))\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}dt\\ &\le \frac{1}{2}\sup_{t\in[T,\infty)}\|v_1(t) - v_2(t)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}\\ &+ \frac{1}{2}a\delta\int_T^\infty t^{-1-\delta}\|\mathcal{Q}v_1(t) - \mathcal{Q}v_2(t)\|_{\mathbf{G}^{\mathbf{b}(t)}(\mathcal{Q};\mathbf{X}_5)}dt,\end{aligned}$$

which means that there exists a unique global solution

$$w_j \in \mathbf{C}([0,\,\infty);\mathbf{G}^{\mathbf{a}}(\mathcal{Q};\mathbf{X}_5))$$

to the Cauchy problem (4.7) satisfying the estimate

$$\|w_j(t)\|_{\mathbf{G}^a(\partial_x;\mathbf{L}^\infty)} \le C\langle t\rangle^{-1/2}$$

for all $t \ge 0$, $1 \le j \le l$.

We next consider the asymptotic behavior of solutions. By the integral equation we have

$$\begin{aligned}\|\mathcal{U}_{m_j}(-t)w_j(t) - \mathcal{U}_{m_j}(-s)w_j(s)\|_{\mathbf{G}^a(\partial_x;\mathbf{L}^2)} &\le C\varepsilon^2\int_s^t\langle\tau\rangle^{-1-1/2}d\tau\\ &\le C\varepsilon^2 s^{-1/2} \qquad (4.8)\end{aligned}$$

for all $t > s \ge 1$ with some $\delta > 0$. We let $t \to \infty$, then there exist unique final states $w_j^+ \in \mathbf{G}^a(\partial_x; \mathbf{L}^2)$ such that

$$\|w_j^+ - \mathcal{U}_{m_j}(-s) w_j(s)\|_{\mathbf{G}^a(\partial_x;\mathbf{L}^2)} \le C\rho^2 s^{-1/2}.$$

The asymptotic behavior stated in the theorem follows from the relations $u_j = w_j^{(1)} + w_j^{(2)}$, $\langle i\partial_x \rangle_{m_j}^{-1} \partial_t u_j = w_j^{(1)} - w_j^{(2)}$. Theorem 1.2 is proved. □

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