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Classification of marginally trapped surfaces of constant curvature in Lorentzian complex plane

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Abstract. A surface in the Lorentzian complex plane $\mathbf{C}_1^2$ is called *marginally trapped* if its mean curvature vector is light-like at each point on the surface. In this article, we classify marginally trapped surfaces of constant curvature in the Lorentzian complex plane $\mathbf{C}_1^2$. Our main results state that there exist twenty-one families of marginally trapped surfaces of constant curvature in $\mathbf{C}_1^2$. Conversely, up to rigid motions and dilations, marginally trapped surfaces of constant curvature in $\mathbf{C}_1^2$ are locally obtained from these twenty-one families.

Key words: Lorentz surfaces, marginally trapped surfaces, Lagrangian surfaces, Lorentzian complex plane, surface of constant curvature.

1. Introduction

Let $\mathbf{C}^n$ denote the complex number n-space with complex coordinates $z_1, \ldots, z_n$. The $\mathbf{C}^n$ endowed with $g_{i,n}$, i.e., the real part of the Hermitian form

$$b_{i,n}(z,w) = -\sum_{k=1}^{i} \bar{z}_k w_k + \sum_{j=i+1}^{n} \bar{z}_j w_j, \quad z, w \in \mathbf{C}^n,$$

defines a flat indefinite complex space form with complex index i. We simply denote the pair $(\mathbf{C}^n, g_{i,n})$ by $\mathbf{C}_i^n$. In particular, the flat indefinite complex n-space $\mathbf{C}_1^n$ with complex index $i = 1$ is called the Lorentzian complex n-space.

A vector v is called *space-like* (respectively, *time-like*) if $\langle v, v\rangle > 0$ (respectively, $\langle v, v\rangle < 0$). A vector v is called *light-like* if it is nonzero and it satisfies $\langle v, v\rangle = 0$.

The concept of *trapped surfaces* in $4D$ space-times, introduced by R. Penrose [15] plays a very important role in general relativity. In the theory of cosmic black holes, if there is a massive source inside the surface, then close enough to a massive enough source, the outgoing light rays be

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converging; a *trapped surface*. Everything inside is trapped within a shrinking area. Nothing can escape, not even light. In between, there will be a *marginally trapped surface* (up to the issue of differentiability) where the outgoing light rays are instantaneously parallel.

This is a *black hole*; its surface is located by the marginally surface, where outgoing light rays are instantaneously parallel, ingoing light rays are converging just inside, and outgoing light rays are diverging just outside.

In terms of mean curvature vector, a spatial surface is *future trapped* if its mean curvature vector is timelike and future-pointing at each point (similarly, for passed trapped); and *marginally trapped* if its mean curvature vector is light-like at each point on the surface (cf. for instance [5], [11], [12], [13]).

Every surface in the Lorentzian complex plane $\mathbf{C}^2_1$ *is automatically Lorentzian if its mean curvature vector is light-like* at each point. In this article, by a *marginally trapped surface in* $\mathbf{C}^2_1$ we mean a surface whose mean curvature vector is light-like at each point (see, for instance, [3], [5], [8]). Such surfaces are also known as quasi-minimal surfaces (cf. [9], [16], [17]).

In this article, we classify marginally trapped surfaces of constant curvature in the Lorentzian complex plane $\mathbf{C}^2_1$. Our main results state that there exist 21 families of marginally trapped surfaces of constant curvature in $\mathbf{C}^2_1$. Conversely, up to rigid motions and dilations, marginally trapped surfaces of constant curvature in $\mathbf{C}^2_1$ are locally obtained from these 21 families.

2. Preliminaries

2.1. Basic formulas, equation and definitions

Let M be a Lorentz surface of a Lorentzian Kähler surface $\tilde{M}^2_1$. Denote by $\tilde{g}$ the metric on $\tilde{M}^2_1$ and by $\langle\ ,\ \rangle$ the inner product associated with $\tilde{g}$. Let g be the the induced metric on M.

Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connection on M and $\tilde{M}^2_1$, respectively. Then the formulas of Gauss and Weingarten are given respectively by (cf. [2], [14])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to M and ξ normal to M, where h, A and D are the second fundamental form, the shape operator and the normal connection.

The shape operator and the second fundamental form are related by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle \tag{2.3}$$

for X, Y tangent to M and ξ normal to M.

For each normal vector ξ of M at $x \in M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_xM. The mean curvature vector is defined by

$$H = \frac{1}{2}\operatorname{trace} h. \tag{2.4}$$

For a Lorentz surface, the equations of Gauss, Codazzi and Ricci are given by

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = \langle \tilde{R}(X,Y)Z,W\rangle + \langle h(X,W),h(Y,Z)\rangle \\ - \langle h(X,Z),h(Y,W)\rangle,\end{aligned} \tag{2.5}$$

$$(\tilde{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{2.6}$$

$$\langle R^D(X,Y)\xi,\eta\rangle = \langle \tilde{R}(X,Y)\xi,\eta\rangle + \langle [A_\xi, A_\eta]X,Y\rangle, \tag{2.7}$$

where X, Y, Z, W are vector tangent to M, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.8}$$

For Lorentz surfaces in a Lorentzian Kähler surface $\tilde{M}^2_1$ we have the following general result from [7].

Theorem 2.1 *The equation of Ricci is a consequence of the equations of Gauss and Codazzi for any Lorentz surface in any Lorentzian Kaehler surface.*

3. Basics results for Lorentzian surfaces

Let M be a Lorentz surface in a Lorentzian Kähler surface $\tilde{M}^2_1$ with almost complex structure J. For each tangent vector X of M, we put

$$JX = PX + FX, \tag{3.1}$$

where PX and FX are the tangential and the normal components of JX. On M there exists a *pseudo-orthonormal* local frame $\{e_1, e_2\}$ such that

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0, \quad \langle e_1, e_2\rangle = -1. \tag{3.2}$$

For a pseudo-orthonormal frame $\{e_1, e_2\}$ satisfying (3.2), it follows from (3.1), (3.2), and $\langle JX, JY\rangle = \langle X, Y\rangle$ that

$$Pe_1 = \sinh\alpha e_1, \quad Pe_2 = -\sinh\alpha e_2 \tag{3.3}$$

for some function α. This function α is called the *Wirtinger angle* of M. When the Wirtinger angle α is constant, the Lorentz surface M is called a *slant surface* (cf. [9], [10]). In this case, α is called the *slant angle* and the surface is called α-slant. A α-slant surface is called *Lagrangian* if $\alpha = 0$. Slant surfaces in $\tilde{M}^2_1$ are Lorentzian.

If we put

$$e_3 = \operatorname{sech}\alpha F e_1, \quad e_4 = \operatorname{sech}\theta F e_2, \tag{3.4}$$

we find from (3.1)–(3.4) that

$$Je_1 = \sinh\alpha e_1 + \cosh\alpha e_3, \qquad Je_2 = -\sinh\alpha e_2 + \cosh\alpha e_4, \tag{3.5}$$

$$Je_3 = -\cosh\alpha e_1 - \sinh\alpha e_3, \quad Je_4 = -\cosh\alpha e_2 + \sinh\alpha e_4, \tag{3.6}$$

$$\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = 0, \quad \langle e_3, e_4\rangle = -1. \tag{3.7}$$

We call such a frame $\{e_1, e_2, e_3, e_4\}$ an *adapted pseudo-orthonormal frame.*

We recall the following lemmas from [4], [6].

Lemma 3.1 *If M is a Lorentz surface in a Lorentzian Kähler surface $\tilde{M}^2_1$, then with respect to an adapted pseudo-orthonormal frame we have*

$$\nabla_X e_1 = \omega(X)e_1, \quad \nabla_X e_2 = -\omega(X)e_2, \tag{3.8}$$

$$D_X e_3 = \Phi(X)e_3, \quad D_X e_4 = -\Phi(X)e_4 \tag{3.9}$$

for some 1-forms ω, Φ on M.

For a Lorentz surface M in $\tilde{M}^2_1$ with second fundamental form h, we put

$$h(e_i, e_j) = h^3_{ij} e_3 + h^4_{ij} e_4, \tag{3.10}$$

where e_1, e_2, e_3, e_4 is an adapted pseudo-orthonormal frame.

Lemma 3.2 *If M is a Lorentz surface in a Lorentzian Kähler surface $\tilde{M}^2_1$, then with respect to an adapted pseudo-orthonormal frame $\{e_1, e_2, e_3, e_4\}$ we have*

$$A_{e_3} e_j = h^4_{j2} e_1 + h^4_{1j} e_2, \quad A_{e_4} e_j = h^3_{j2} e_1 + h^3_{1j} e_2, \tag{3.11}$$

$$e_j \alpha = (\omega_j - \Phi_j) \coth \alpha - 2h^3_{1j}, \tag{3.12}$$

$$e_1 \alpha = h^4_{12} - h^3_{11}, \quad e_2 \alpha = h^4_{22} - h^3_{12}, \tag{3.13}$$

$$\omega_j - \Phi_j = (h^3_{1j} + h^4_{j2}) \tanh \alpha, \tag{3.14}$$

for $j = 1, 2$, where $\omega_j = \omega(e_j)$ and $\Phi_j = \Phi(e_j)$.

4. Marginally trapped flat surfaces in $\mathbf{C}^2_1$

The *light cone* $\mathcal{LC}$ in $\mathbf{C}^2_1$ is defined by $\mathcal{LC} = \{v \in \mathbf{C}^2_1 : \langle v, v \rangle = 0\}$.

Theorem 4.1 *There exist nine families of marginally trapped flat surfaces in the Lorentzian complex plane $\mathbf{C}^2_1$ given by the following*:

(1) *A Lagrangian surface defined by $L(x, y) = z(x)e^{iy}$, where $z(x)$ is a null curve lying in the light cone $\mathcal{LC}$ satisfying $\langle \mathrm{i}z', z \rangle = 1$.*

(2) *A slant surface with slant angle $\theta \neq 0$ given by $L(x, y) = z(x)y^{\frac{1}{2}(1 - \mathrm{i} \operatorname{csch} \theta)}$, where z is a null curve lying in the light cone $\mathcal{LC}$ satisfying $\langle \mathrm{i}z, z' \rangle = 2 \sinh \theta$.*

(3) *A surface given by*

$$L(x, y) = yz(x) + \int^x \frac{\mathrm{i} + w(x)}{q(x)} z'(x) dx,$$

where q, w are real-valued functions and z is a null curve lying in the light cone $\mathcal{LC}$ satisfying $\langle z', \mathrm{i}z \rangle = q$.

(4) *A surface given by*

$$L(x, y) = z(x)(1 - \mathrm{i} \sinh \alpha(y)) e^{\int^y (\mathrm{i} \operatorname{sech} \alpha(y) - \tanh \alpha(y)) \mu(y) dy},$$

where z is a null curve in $\mathcal{LC}$ satisfying $\langle z', \mathrm{i}z \rangle = b^{-1}$ for some real number $b \neq 0$, and α, μ are real-valued functions satisfying $\alpha' = \mu + be^{\int^y 2\mu \tanh \alpha dy} \operatorname{sech} \alpha \neq 0$.

(5) *A surface given by*

$$L(x,y)=\Big(x+\frac{y}{2}+\mathrm{i}p(y)-\mathrm{i}\int^x \sinh\alpha dx,\quad x-\frac{y}{2}+\mathrm{i}p(y)-\mathrm{i}\int^x \sinh\alpha dx\Big),$$

where α *and* p *are real-valued functions with* $\alpha_y \neq 0$.

(6) *A Lagrangian surface defined by*

$$L(x,y)=\Big(xe^{\mathrm{i}F(y)}+\int^y e^{\mathrm{i}F(y)}\Big(\mathrm{i}\psi(y)+\frac{1}{2}\Big)dy,\quad xe^{\mathrm{i}F(y)}+\int^y e^{\mathrm{i}F(y)}\Big(\mathrm{i}\psi(y)-\frac{1}{2}\Big)dy\Big),$$

where ψ *is a real-valued function and* F *is a non-constant real-valued function.*

(7) *A slant surface with slant angle* $\theta \neq 0$ *defined by*

$$\begin{aligned}L(x,y)=\;&\Big(xe^{(\mathrm{i}-\sinh\theta)F(y)}+\frac{\mathrm{i}}{2}\cosh\theta\coth\theta\int^y e^{(\mathrm{i}+\sinh\theta)F(y)}dy\\&+(\mathrm{i}\,\mathrm{sech}\,\theta-\tanh\theta)\int^y\Big(e^{(\mathrm{i}+\sinh\theta)F(y)}\int^y q(y)e^{-2F(y)\sinh\theta}dy\Big)dy,\\&xe^{(\mathrm{i}-\sinh\theta)F(y)}+\Big(\frac{\mathrm{i}}{2}\cosh\theta\coth\theta-\mathrm{i}\sinh\theta-1\Big)\int^y e^{(\mathrm{i}+\sinh\theta)F(y)}dy\\&+(\mathrm{i}\,\mathrm{sech}\,\theta-\tanh\theta)\int^y\Big(e^{(\mathrm{i}+\sinh\theta)F(y)}\int^y q(y)e^{-2F(y)\sinh\theta}dy\Big)dy\Big),\end{aligned}$$

where F *and* q *are real-valued functions with* F *being non-constant.*

(8) *A surface given by*

$$\begin{aligned}L(x,y)=\;&\Big(\int^y e^{\int^y(\mathrm{i}+\sinh\alpha(y))f(y)dy}\Big\{\frac{1}{2}+\mathrm{i}\int^y e^{-2\int^y f(y)\sinh\alpha(y)dy}k(y)dy\Big\}dy\\&+x(1-\mathrm{i}\sinh\alpha(y))e^{\int^y(\mathrm{i}-\sinh\alpha(y))f(y)dy},\end{aligned}$$

$$x(1-\mathrm{i}\sinh\alpha(y))e^{\int^y(\mathrm{i}-\sinh\alpha(y))f(y)dy}-\int^y e^{\int^y(\mathrm{i}+\sinh\alpha(y))f(y)dy}$$
$$\times\left\{\frac{1}{2}-\mathrm{i}\int^y e^{-2\int^y f(y)\sinh\alpha(y)dy}k(y)dy\right\}dy\bigg),$$

where α, f *and* k *are real-valued functions with* $\alpha'\neq 0$ *and* $f\neq 0$.

(9) *A surface given by*

$$\begin{aligned}L(x,y)=&\int^x(1-\mathrm{i}\sinh\alpha)z(x)e^{\int^y(\mathrm{i}-\sinh\alpha)f(y)dy}dx\\&+\mathrm{i}\int^y\left\{\int^x z(x)(\alpha_y\cosh\alpha-f(y)\cosh^2\alpha)e^{\int^y(\mathrm{i}-\sinh\alpha)f(y)dy}dx\right\}dy\\&+\frac{\mathrm{i}}{p(x)}\int^y e^{\int^y(\mathrm{i}+\sinh\alpha)fdy}\left\{z(x)\int^y f(y)\alpha_x\cosh\alpha dy-z'(x)\right\}dy,\end{aligned}$$

where α, f, p *are nonzero real-valued functions satisfying* $\alpha_x\alpha_y\neq 0$ *and* (4.72), *and* z *is a null curve lying in* $\mathcal{LC}$ *with* $\langle z, \mathrm{i}z'\rangle = p$.

Conversely, up to rigid motions and dilations, every marginally trapped flat surface in $\mathbf{C}_1^2$ *is locally an open portion of one of the surfaces given by the nine families.*

Proof. We show by examples that immersions given by cases (1)–(9) of this theorem define marginally trapped flat surfaces in $\mathbf{C}_1^2$.

For case (1) we have $L_x = z'(x)e^{iy}$, $L_y = \mathrm{i}z(x)e^{\mathrm{i}y}$, which imply that

$$\langle L_x, L_x\rangle = \langle z', z'\rangle,\quad \langle L_x, L_y\rangle = \langle z', \mathrm{i}z\rangle,\quad \langle L_y, L_y\rangle = \langle z, z\rangle. \tag{4.1}$$

Since $z(x)$ is a null curve lying in the light cone $\mathcal{LC}$ satisfying $\langle \mathrm{i}z', z\rangle = 1$, it follows from (4.1) that the metric tensor is given by

$$g = -(dx\otimes dy + dy\otimes dx). \tag{4.2}$$

Thus, the immersion L defines a flat Lorentz surface in $\mathbf{C}_1^2$. Moreover, since $L_{xy} = \mathrm{i}z'(x)e^{\mathrm{i}y} = \mathrm{i}L_x$, we also know from (2.4) and (4.2) that the mean curvature vector is given by $H = -L_{xy} = -\mathrm{i}L_x$, which is light-like. Therefore, $L(x,y) = z(x)e^{\mathrm{i}y}$ defines a marginally trapped flat surface in $\mathbf{C}_1^2$.

For case (3), we get

$$L_x = \left(y + \frac{\mathrm{i} + w(x)}{q(x)}\right) z'(x), \quad L_y = z(x), \tag{4.3}$$

which implies

$$\begin{aligned} \langle L_x, L_x \rangle &= \left(\left(y + \frac{w(x)}{q(x)}\right)^2 + \frac{1}{q^2(x)}\right)\langle z', z' \rangle, \\ \langle L_x, L_y \rangle &= \left(y + \frac{w(x)}{q(x)}\right)\langle z', z \rangle + \frac{1}{q(x)}\langle \mathrm{i}z', z \rangle, \\ \langle L_y, L_y \rangle &= \langle z, z \rangle. \end{aligned} \tag{4.4}$$

Because z is a null curve lying in the light cone $\mathcal{LC}$ satisfying $\langle z', \mathrm{i}z \rangle = q$, it follows from (4.4) that the induced metric is also given by (4.2). Moreover, it follows from (4.3) that

$$H = -L_{xy} = \frac{-q(x)}{\mathrm{i} + w(x) + q(x)y} L_x. \tag{4.5}$$

Thus, the mean curvature vector is light-like. Hence, this immersion also defines a marginally trapped flat surface.

For case (5), we know from direct computation that the induced metric tensor is given by (4.2). Moreover, by straight-forward computation, we have

$$L_{xy} = (\tanh \alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)\alpha_y L_x.$$

Since $H = -L_{xy}$ and α_y is nonzero, we know that the mean curvature vector is light-like. Hence, the immersion defines a marginally trapped flat surface in $\mathbf{C}^2_1$.

Similar computations show that the remaining cases give rise to marginally trapped flat surfaces in $\mathbf{C}^2_1$ as well.

Conversely, let $L : M \to \mathbf{C}^2_1$ be a marginally trapped immersion of a flat surface in $\mathbf{C}^2_1$. Then M is Lorentzian. So, we may assume that locally M is an open portion of the xy-plane equipped with the flat Lorentzian metric given by (4.2). Thus we have $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = 0$. If we put $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, then $\{e_1, e_2\}$ is a pseudo-orthonormal frame satisfying (3.2).

Let e_3, e_4 be the normal vector fields defined by (3.4). From (3.5) we find

$$e_3 = (\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)L_x, \quad e_4 = (\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh\alpha)L_y. \tag{4.6}$$

Since M is marginally trapped, in view of (2.4) and (3.2) we may assume that

$$h(e_1, e_1) = \beta e_3 + \gamma e_4, \quad h(e_1, e_2) = \delta e_3, \quad h(e_2, e_2) = \lambda e_3 + \mu e_4 \tag{4.7}$$

with $\delta \neq 0$ at each point.

In view of (3.5), (3.7), and (4.7), the equation of Gauss can be expressed as

$$\gamma\lambda + \beta\mu = 0. \tag{4.8}$$

Using Lemma 3.2 we find

$$\begin{aligned} D_{e_1} e_3 &= -\beta \tanh\alpha e_3, \quad & D_{e_2} e_3 &= -(\delta + \mu)\tanh\alpha e_3, \\ D_{e_1} e_4 &= \beta \tanh\alpha e_4, \quad & D_{e_2} e_3 &= (\delta + \mu)\tanh\alpha e_4. \end{aligned} \tag{4.9}$$

Hence, it follows from (4.7) and (4.9) that

$$\begin{aligned} (\bar\nabla_{e_1} h)(e_1, e_2) &= (\delta_x - \delta\beta\tanh\alpha)e_3, \\ (\bar\nabla_{e_2} h)(e_1, e_1) &= (\beta_y - \beta(\delta + \mu)\tanh\alpha)e_3 + (\gamma_y + \gamma(\delta + \mu)\tanh\alpha)e_4, \\ (\bar\nabla_{e_1} h)(e_2, e_2) &= (\lambda_x - \lambda\beta\tanh\alpha)e_3 + (\mu_x + \mu\beta\tanh\alpha)e_4, \\ (\bar\nabla_{e_2} h)(e_1, e_2) &= (\delta_y - \delta(\delta + \mu)\tanh\alpha)e_3. \end{aligned} \tag{4.10}$$

From (4.10) and the equation of Codazzi we obtain

$$\lambda_x - \delta_y = (\lambda\beta - \delta^2 - \delta\mu)\tanh\alpha, \tag{4.11}$$

$$\mu_x = -\beta\mu\tanh\alpha, \tag{4.12}$$

$$\beta_y - \delta_x = \beta\mu\tanh\alpha, \tag{4.13}$$

$$\gamma_y = -\gamma(\delta + \mu)\tanh\alpha. \tag{4.14}$$

Also, it follows from (4.7) and Lemma 3.2 that

$$\alpha_x = -\beta, \quad \alpha_y = \mu - \delta. \tag{4.15}$$

Case (a): $\mu = 0$. Equations (4.8) and (4.15) give

$$\gamma\lambda = 0, \quad \beta = -\alpha_x, \quad \delta = -\alpha_y. \tag{4.16}$$

Case (a.1): $\gamma = 0$. From (4.11) and $\mu = 0$ we get

$$\lambda_x - \delta_y = (\lambda\beta - \delta^2)\tanh\alpha. \tag{4.17}$$

We find from (4.16) and (4.17) that

$$\lambda_x + (\alpha_x \tanh\alpha)\lambda = -\alpha_{yy} - \alpha_y^2 \tanh\alpha. \tag{4.18}$$

Solving this equation, we give

$$\lambda = k(y)\operatorname{sech}\alpha - \operatorname{sech}\alpha \int^x \big(\alpha_{yy}\cosh\alpha + \alpha_y^2 \sinh\alpha\big)dx. \tag{4.19}$$

Therefore, we obtain from (4.2)–(4.7), (4.16), and formula (2.1) of Gauss that

$$\begin{aligned} L_{xx} &= \alpha_x(\tanh\alpha - \mathrm{i}\operatorname{sech}\alpha)L_x, \\ L_{xy} &= \alpha_y(\tanh\alpha - \mathrm{i}\operatorname{sech}\alpha)L_x, \\ L_{yy} &= \lambda(\mathrm{i}\operatorname{sech}\alpha - \tanh\alpha)L_x, \end{aligned} \tag{4.20}$$

where λ is given by (4.19). After solving the first equation in (4.20) we obtain

$$L(x,y) = A(y) + B(y)\bigg(x - \mathrm{i}\int^x \sinh\alpha dx\bigg) \tag{4.21}$$

for vector function A, B. Substituting this into the second equation in (4.20), we get $B' = 0$. So, $B = c_1$ for some vector $c_1 \in \mathbf{C}_1^2$. Substituting (4.21) with $B = c_1$ into the last equation into (4.20) we find $A'' = \mathrm{i}c_1 k$. Consequently, L is congruent to

$$L(x,y) = c_1 x + c_2 y + \mathrm{i}c_1 \int^y \bigg(\int^y k(y)dy\bigg)dy - \mathrm{i}c_1 \int^x \sinh\alpha dx$$

for some vectors c_1, c_2. Hence, after choosing suitable initial conditions, we obtain case (5) of the theorem with $p(y) = y + \int^y \big(\int^y k(y)dy\big)dy$.

Case (a.2): $\gamma \neq 0$ *and* $\lambda = 0$. In this case, (4.7), (4.11) and (4.14) reduce to

$$h(e_1, e_1) = \beta e_3 + \gamma e_4, \quad h(e_1, e_2) = \delta e_3, \quad h(e_2, e_2) = 0, \tag{4.22}$$

$$(\ln \delta)_y = -\alpha_y \tanh \alpha, \quad (\ln \gamma)_y = \alpha_y \tanh \alpha. \tag{4.23}$$

Solving the two equations in (4.23), we see that

$$\gamma = p(x) \cosh \alpha, \quad \delta = -q(x) \operatorname{sech} \alpha \tag{4.24}$$

for some real-valued functions p, q. After substituting this into the second equation in (4.15), we get $\alpha_y \cosh \alpha = q(x)$. Hence we have

$$\alpha = \sinh^{-1}(q(x)y + w(x)) \tag{4.25}$$

for some function w. Consequently, the immersion L satisfies

$$\begin{aligned} L_{xx} &= \frac{q'(x)y + w'(x)}{\mathrm{i} + q(x)y + w(x)} L_x + (\mathrm{i} + q(x)y + w(x))p(x)L_y, \\ L_{xy} &= \frac{q(x)}{\mathrm{i} + q(x)y + w(x)} L_x, \quad L_{yy} = 0. \end{aligned} \tag{4.26}$$

Solving the last two equations in (4.26) gives

$$L(x, y) = \int^x \frac{\mathrm{i} + w(x)}{q(x)} z'(x)dx + z(x)y \tag{4.27}$$

for a vector function z. Substituting this into the first equation in (4.26), we have

$$q(x)z''(x) - q'(x)z'(x) - p(x)q^2(x)z(x) = 0. \tag{4.28}$$

By applying (4.2) and (4.27), we find $\langle z, z\rangle = \langle z', z'\rangle = 0$ and $\langle z', \mathrm{i}z\rangle = q$. Therefore, we obtain case (3) of the theorem.

Case (b): $\mu \neq 0$ *and* $\lambda = 0$. Equation (4.8) gives $\beta = 0$. Thus, the first equation in (4.15) implies $\alpha = \alpha(y)$. So, we find from (4.7) and (4.11)–(4.14) that

$$h(e_1,e_1)=\gamma e_4,\quad h(e_1,e_2)=\delta e_3,\quad h(e_2,e_2)=\mu e_4,\quad \mu=\mu(y), \tag{4.29}$$

$$(\ln\delta)_y=(\delta+\mu)\tanh\alpha,\quad (\ln\gamma)_y=-(\delta+\mu)\tanh\alpha, \tag{4.30}$$

$$\delta=\mu(y)-\alpha'(y). \tag{4.31}$$

It follows from (4.30) and (4.31) that

$$\gamma=\frac{v(x)}{\mu(y)-\alpha'(y)} \tag{4.32}$$

for some real-valued function $v(x)\neq 0$. Therefore, the immersion L of M satisfies

$$\begin{aligned} L_{xx}&=\frac{v(x)(\mathrm{i}\,\mathrm{sech}\,\alpha-\tanh\alpha)}{\mu(y)-\alpha'(y)}L_y,\\ L_{xy}&=(\mu(y)-\alpha'(y))(\mathrm{i}\,\mathrm{sech}\,\alpha-\tanh\alpha)L_x,\\ L_{yy}&=\mu(y)(\mathrm{i}\,\mathrm{sech}\,\alpha+\tanh\alpha)L_y. \end{aligned} \tag{4.33}$$

By substituting (4.31) and (4.32) into the second equation in (4.30), we have

$$\mu'=(2\mu^2-3\alpha'\mu+\alpha'^2)\tanh\alpha+\alpha''. \tag{4.34}$$

Case (b.1): $\alpha=0$. In this case, M is Lagrangian. It follows from (4.34) that μ is a nonzero real number, say c. By applying a suitable dilation, we have $\mu=1$. Thus, (4.31) and (4.32) yields $\delta=1$ and $\gamma=v(x)$. Hence, (4.33) reduces to

$$L_{xx}=\mathrm{i}v(x)L_y,\quad L_{xy}=\mathrm{i}L_x,\quad L_{yy}=\mathrm{i}L_y. \tag{4.35}$$

Solving this system, we give

$$L(x,y)=z(x)e^{iy}. \tag{4.36}$$

It now follows from (4.2) and (4.36) that z is null curve lying in the light cone $\mathcal{LC}$ satisfying $\langle iz',z\rangle=1$. Consequently, we obtain case (1) of the theorem.

Case (b.2): *α is a nonzero real number.* In this case, M is α-slant. Thus, (4.34) reduces to $\mu' = 2\mu^2 \tanh\alpha$. After solving this equation we have

$$\mu(y) = \frac{1}{2b - 2y\tanh\alpha} \tag{4.37}$$

for some real number b. Thus, system (4.33) becomes

$$\begin{aligned} L_{xx} &= 2v(x)(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)(b - y\tanh\alpha)L_y, \\ L_{xy} &= \frac{\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha}{2b - 2y\tanh\alpha}L_x, \\ L_{yy} &= \frac{\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh\alpha}{2b - 2y\tanh\alpha}L_y. \end{aligned} \tag{4.38}$$

Solving the second equation in (4.38), we give

$$L(x,y) = A(y) + B(x)(y\tanh\alpha - b)^{\frac{1}{2}(1-\mathrm{i}\,\mathrm{csch}\,\alpha)} \tag{4.39}$$

for vector functions A, B. Substituting this into the last equation in (4.38), we find that

$$2(b - y\tanh\alpha)A''(y) = (\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh\alpha)A'(y).$$

After solving this equation we get

$$A(y) = c_1(y\tanh\alpha - b)^{\frac{1}{2}(1-\mathrm{i}\,\mathrm{csch}\,\alpha)} + c_2$$

with $c_1,\, c_2 \in \mathbf{C}^2_1$. Hence, it follows from (4.39) that L is congruent to

$$L(x,y) = z(x)(y - b\coth\alpha)^{\frac{1}{2}(1-\mathrm{i}\,\mathrm{csch}\,\alpha)}, \tag{4.40}$$

where $z(x) = (c_1 + B(x))(\tanh\alpha)^{\frac{1}{2}(1-\mathrm{i}\,\mathrm{csch}\,\alpha)}$. By substituting this into the first equation in (4.38), we find that

$$z''(x) = (\tanh\alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)^2 v(x)z(x). \tag{4.41}$$

After applying a suitable translation in y, we obtain from (4.40) that

$$L(x,y)=z(x)y^{\frac{1}{2}(1-\mathrm{i}\,\mathrm{csch}\,\alpha)}. \tag{4.42}$$

Now, by applying (4.2) and (4.42) we find $\langle z,z\rangle=\langle z',z'\rangle=0$, $\langle z',\mathrm{i}z\rangle=2\sinh\alpha$. Therefore, we obtain case (2) of the theorem.

Case (b.3): *α is non-constant.* Solving the second equation in (4.33), we give

$$L=A(y)+B(x)(1-\mathrm{i}\sinh\alpha)e^{\int^y \mu(y)(\mathrm{i}\,\mathrm{sech}\,\alpha-\tanh\alpha)dy} \tag{4.43}$$

for some vector functions A,B. So, after substituting (4.43) info the first equation in (4.33) we find

$$\frac{B''(x)}{v(x)}+B(x)=\frac{\mathrm{i}e^{\int^y \mu(y)(\tanh\alpha-\mathrm{i}\,\mathrm{sech}\,\alpha)dy}}{\mu(y)-\alpha'(y)}\,\mathrm{sech}\,\alpha(y)A'(y).$$

Hence, there exists a vector $c_1\in\mathbf{C}^2_1$ such that

$$B''(x)+v(x)(B(x)-c_1)=0, \tag{4.44}$$

$$A'(y)=\mathrm{i}c_1(\alpha'(y)-\mu(y))(\cosh\alpha)e^{\int^y(\mathrm{i}\,\mathrm{sech}\,\alpha-\tanh\alpha)\mu dy}. \tag{4.45}$$

By applying (4.43) and (4.45) we know that the immersion is congruent to

$$\begin{aligned}L(x,y)&=(z(x)+c_1)(1-\mathrm{i}\sinh\alpha)e^{\int^y(\mathrm{i}\,\mathrm{sech}\,\alpha-\tanh\alpha)\mu dy}\\&\quad+\mathrm{i}c_1\int^y(\alpha'(y)-\mu(y))(\cosh\alpha)e^{\int^y(\mathrm{i}\,\mathrm{sech}\,\alpha-\tanh\alpha)\mu dy}dy,\end{aligned} \tag{4.46}$$

where $z=B-c_1$ is a vector function satisfying $z''+vz=0$. From (4.46) we find

$$\begin{aligned}L_x&=(1-\mathrm{i}\sinh\alpha)e^{\int^y(\mathrm{i}\,\mathrm{sech}\,\alpha-\tanh\alpha)\mu dy}z'(x),\\L_y&=\mathrm{i}(\mu-\alpha')\cosh\alpha e^{\int^y(\mathrm{i}\,\mathrm{sech}\,\alpha-\tanh\alpha)\mu dy}z(x).\end{aligned} \tag{4.47}$$

It follows from (4.2) and (4.47) that $\langle z,z\rangle=\langle z',z'\rangle=0$ and

$$-1=\langle L_x,L_y\rangle=(\mu-\alpha')(\cosh\alpha)e^{-\int^y 2\mu\tanh\alpha dy}\langle \mathrm{i}z,z'\rangle. \tag{4.48}$$

On the other hand, it follows from (4.34) that

$$\frac{d}{dy}\big\{(\alpha'(y)-\mu(y))\cosh\alpha(y)e^{-2\int^y \mu(y)\tanh\alpha(y)dy}\big\}=0.$$

Hence, there exist a nonzero real number b such that

$$(\alpha'(y)-\mu(y))\cosh\alpha(y)=be^{2\int^y \mu(y)\tanh\alpha(y)dy}. \tag{4.49}$$

Substituting this into (4.46), we give

$$L(x,y)=\frac{(1-\mathrm{i}\sinh\alpha)(z(x)+c_1)}{e^{\int^y(\tanh\alpha-\mathrm{i}\,\mathrm{sech}\,\alpha)\mu dy}}+\mathrm{i}bc_1\int^y e^{\int^y(\mathrm{i}\,\mathrm{sech}\,\alpha+\tanh\alpha)\mu dy}dy \tag{4.50}$$

and $\langle z',\mathrm{i}z\rangle=b^{-1}$. Therefore, after replacing $z(x)+c_1$ simply by $z(x)$, we obtain

$$L(x,y)=z(x)(1-\mathrm{i}\sinh\alpha)e^{\int^y(\mathrm{i}\,\mathrm{sech}\,\alpha-\tanh\alpha)\mu dy}. \tag{4.51}$$

Consequently, we obtain case (4) of the theorem.

Case (c): $\lambda,\mu\neq 0$. In this case, we find from (4.12) and (4.15) that

$$(\ln\mu)_x=\alpha_x\tanh\alpha. \tag{4.52}$$

Thus, by using the second equation in (4.15) we have

$$\beta=-\alpha_x,\quad \mu=f(y)\cosh\alpha,\quad \delta=f(y)\cosh\alpha-\alpha_y \tag{4.53}$$

for some nonzero real-valued function $f(y)$.

Case (c.1): $\gamma=0$. From (4.8) we get $\beta=0$. Thus, $\alpha=\alpha(y)$ according to (4.15).

Case (c.1.i): $\alpha'(y)=0$. In this case, the surface is slant. From (4.53) we get

$$\mu=\delta=f(y)\cosh\alpha. \tag{4.54}$$

Hence, (4.11) becomes $\lambda_x=f'(y)\cosh\alpha-f^2(y)\sinh 2\alpha$, which implies that

$$\lambda=x(f'(y)\cosh\alpha-f^2(y)\sinh 2\alpha)+q(y) \tag{4.55}$$

for some real-valued function q. Thus, by (4.2)–(4.7), (4.54), and (4.55), we have

$$\begin{aligned} &L_{xx} = 0, \quad L_{xy} = f(y)(\mathrm{i} - \sinh\alpha)L_x, \\ &L_{yy} = \big\{xf'(y)\cosh\alpha - xf^2(y)\sinh 2\alpha + q(y)\big\}(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)L_x \\ &\qquad + f(y)(\mathrm{i} + \sinh\alpha)L_y. \end{aligned} \tag{4.56}$$

Solving the first two equations in (4.56), we give

$$L(x,y) = B(y) + c_1 x e^{(\mathrm{i}-\sinh\alpha)\int^y f dy} \tag{4.57}$$

for some vector c_1 and vector-valued function B. Substituting this into the last equation in (4.56), we know that

$$B'' - (\mathrm{i} + \sinh\alpha)f(y)B' = c_1(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)q(y)e^{(\mathrm{i}-\sinh\alpha)F} \tag{4.58}$$

with $F(y) = \int^y f(y)dy$. After solving this differential equation we have

$$\begin{aligned} B(y) = c_3 + c_2 \int^y e^{(\mathrm{i}+\sinh\alpha)F} dy \\ + c_1(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)\int^y \left(e^{(\mathrm{i}+\sinh\alpha)F}\int^y q e^{-2F\sinh\alpha} dy\right)dy \end{aligned} \tag{4.59}$$

for some vectors c_2, c_3. Hence, up to translations, the immersion is given by

$$\begin{aligned} L(x,y) = c_1 x e^{(\mathrm{i}-\sinh\alpha)F} + c_2 \int^y e^{(\mathrm{i}+\sinh\alpha)F} dy \\ + c_1(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)\int^y \left(e^{(\mathrm{i}+\sinh\alpha)F}\int^y q e^{-2F\sinh\alpha} dy\right)dy. \end{aligned} \tag{4.60}$$

From (4.60) we find

$$L_x = c_1 e^{(\mathrm{i}-\sinh\alpha)F},$$

$$L_y = e^{(\mathrm{i}+\sinh\alpha)F}\left\{c_2 + c_1(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)\int^y q e^{-2F\sinh\alpha}dy\right\} + c_1 x(\mathrm{i}-\sinh\alpha)f(y)e^{(\mathrm{i}-\sinh\alpha)F}. \tag{4.61}$$

If $\alpha = 0$, then (4.60) and (4.61) reduces to

$$L = c_1 x e^{\mathrm{i}F} + c_2\int^y e^{\mathrm{i}F}dy + \mathrm{i}c_1\int^y e^{\mathrm{i}F}\psi(y)dy, \tag{4.62}$$

$$L_x = c_1 e^{\mathrm{i}F}, \quad L_y = e^{\mathrm{i}F}(c_2 + \mathrm{i}c_1(xF' + \psi(y))), \quad \psi(y) = \int^y q(y)dy. \tag{4.63}$$

By applying (4.2) and (4.63), we obtain $\langle c_1, c_1\rangle = \langle c_2, c_2\rangle = \langle \mathrm{i}c_1, c_2\rangle = 0$ and $\langle c_1, c_2\rangle = -1$. After choosing suitable initial condition, we conclude that the surface is congruent to the one given by case (6) of the theorem.

If $\alpha \neq 0$, we find from (4.2) and (4.61) that $\mathrm{csch}\,\alpha\langle \mathrm{i}c_1, c_2\rangle = \langle c_1, c_2\rangle = -1$ and $\langle c_1, c_1\rangle = \langle c_2, c_2\rangle = 0$. Thus, we obtain case (7) of the theorem.

Case (c.1.ii): $\alpha'(y) \neq 0$. Substituting $\beta = \gamma = 0$ and (4.53) into (4.11), we give

$$\lambda_x = f'(y)\cosh\alpha - \alpha'' - f^2(y)\sinh(2\alpha) + 4f(y)\alpha'\sinh\alpha - \alpha'^2\tanh\alpha, \tag{4.64}$$

which implies

$$\lambda = x\big(f'\cosh\alpha - \alpha'' - f^2\sinh(2\alpha) + 4f\alpha'\sinh\alpha - \alpha'^2\tanh\alpha\big) + k(y)\,\mathrm{sech}\,\alpha \tag{4.65}$$

for some function $k(y)$. Hence, we obtain from (4.2)–(4.7), (4.53) and (4.65) that

$$\begin{aligned} &L_{xx} = 0, \quad L_{xy} = (f(y)\cosh\alpha - \alpha'(y))(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)L_x, \\ &L_{yy} = (\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)\big\{x(f'\cosh\alpha - \alpha'' - f^2\sinh(2\alpha) + 4f\alpha'\sinh\alpha \\ &\qquad - \alpha'^2\tanh\alpha) + k(y)\,\mathrm{sech}\,\alpha\big\}L_x + f(y)(\mathrm{i}+\sinh\alpha)L_y. \end{aligned} \tag{4.66}$$

Solving the first two equations in (4.66), we give

$$L(x,y) = B(y) + c_1 x(1 - \mathrm{i}\sinh\alpha)e^{\int^y(\mathrm{i}-\sinh\alpha)f(y)dy}. \tag{4.67}$$

Substituting this into the last equation in (4.66), we have

$$B''(y) - (\mathrm{i} + \sinh\alpha) f(y) B'(y) = \mathrm{i} c_1 k(y) e^{\int^y (\mathrm{i} - \sinh\alpha) f(y) dy}.$$

Solving this differential equation, we obtain

$$B(y) = c_3 + \int^y e^{\int^y (\mathrm{i}+\sinh\alpha) f dy} \left\{ c_2 + \mathrm{i} c_1 \int^y e^{-\int^y 2f \sinh\alpha dy} k(y) dy \right\} dy \tag{4.68}$$

for some vector c_3. Consequently, the immersion is congruent to

$$\begin{aligned} L(x,y) = {} & c_1 x (1 - \mathrm{i}\sinh\alpha) e^{\int^y (\mathrm{i}-\sinh\alpha) f(y) dy} + c_2 \int^y e^{\int^y (\mathrm{i}+\sinh\alpha) f dy} dy \\ & + \mathrm{i} c_1 \int^y e^{\int^y (\mathrm{i}+\sinh\alpha) f dy} \left\{ \int^y e^{-\int^y 2f \sinh\alpha dy} k(y) dy \right\} dy. \end{aligned} \tag{4.69}$$

It follows from (4.2) and (4.69) that $\langle c_1, c_1 \rangle = \langle c_2, c_2 \rangle = \langle c_1, \mathrm{i} c_2 \rangle = 0$, $\langle c_1, c_2 \rangle = -1$. Thus, after choosing suitable initial condition, we obtain case (8) of the theorem.

Case (c.2): $\gamma \neq 0$. We find from (4.8) that $\beta = -\alpha_x \neq 0$. Also, it follows from (4.14), (4.15) and (4.53) that $(\ln\gamma)_y = \alpha_y \tanh\alpha - 2f(y)\sinh\alpha$. So we have

$$\gamma = p(x)(\cosh\alpha) e^{-\int^y 2f(y)\sinh\alpha dy} \tag{4.70}$$

for some nonzero function $p(x)$. Substituting (4.53) and (4.70) into (4.8), we give

$$\lambda = \frac{f\alpha_x}{p(x)} e^{\int^y 2f(y)\sinh\alpha dy}. \tag{4.71}$$

From (4.11), (4.53), (4.70) and (4.71), we obtain

$$\begin{aligned} & f p e^{\int^y 2f \sinh\alpha dy} \big(\alpha_{xx} + \alpha_x^2 \tanh\alpha\big) + p^2 \big(\alpha_{yy} + \alpha_y^2 \tanh\alpha\big) \\ & - f e^{\int^y 2f \sinh\alpha dy} \left(p' - 2p \int^y f \alpha_x \cosh\alpha dy \right) \alpha_x - 4 f p^2 (\sinh\alpha) \alpha_y \\ & \quad = (f' - 2f^2 \sinh\alpha) p^2 \cosh\alpha, \end{aligned} \tag{4.72}$$

which implies that

$$G = k(x) \tag{4.73}$$

for some function $k(x)$, where G is a real-valued function defined by

$$G = \left(\int^y f\alpha_x \cosh\alpha dy\right)^2 - \frac{p'(x)}{p(x)}\int^y f\alpha_x \cosh\alpha dy + \frac{p(x)(\alpha_y \cosh\alpha - f\cosh^2\alpha)}{e^{\int^y 2f\sinh\alpha dy}} + \int^y f(\alpha_x^2 \sinh\alpha + \alpha_{xx}\cosh\alpha)dy. \tag{4.74}$$

On the other hand, from (4.2)–(4.7), (4.53), (4.70), and (4.71) we have

$$\begin{aligned} L_{xx} &= \alpha_x(\tanh\alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)L_x + (\mathrm{i} + \sinh\alpha)p(x)e^{-\int^y 2f(y)\sinh\alpha dy}L_y, \\ L_{xy} &= (f(y)\cosh\alpha - \alpha_y)(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)L_x, \\ L_{yy} &= \frac{\alpha_x f(y)}{p(x)}(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)e^{\int^y 2f(y)\sinh\alpha dy}L_x + f(y)(\mathrm{i} + \sinh\alpha)L_y. \end{aligned} \tag{4.75}$$

Solving the second equation in (4.75), we see that

$$L(x,y) = w(y) + \int^x (1 - \mathrm{i}\sinh\alpha)z(x)e^{\int^y(\mathrm{i}-\sinh\alpha)f(y)dy}dx \tag{4.76}$$

for some $\mathbf{C}_1^2$-valued functions z, w. From (4.76) we get

$$L_x = (1 - \mathrm{i}\sinh\alpha)z(x)e^{\int^y(\mathrm{i}-\sinh\alpha)fdy}. \tag{4.77}$$

Substituting (4.76) into the first equation in (4.75), we get

$$w'(y) = \mathrm{i}H, \tag{4.78}$$

where H is a $\mathbf{C}_1^2$-valued function defined by

$$H = \int^x z(x)(\alpha_y\cosh\alpha - f\cosh^2\alpha)e^{\int^y(\mathrm{i}-\sinh\alpha)f(y)dy}dx + \frac{e^{\int^y(\mathrm{i}+\sinh\alpha)fdy}}{p(x)}\left(z(x)\int^y f\alpha_x\cosh\alpha dy - z'(x)\right). \tag{4.79}$$

A direct computation shows that the function G in (4.74) and H are related by

$$p(x)\frac{\partial H}{\partial x} = e^{\int^y (\mathrm{i}+\sinh\alpha)f dy}\bigg(Gz(x) + \frac{p'(x)}{p(x)}z'(x) - z''(x)\bigg). \tag{4.80}$$

Thus, in view of (4.73) and (4.79), we obtain

$$p(x)z''(x) = p'(x)z'(x) + k(x)p(x)z(x). \tag{4.81}$$

Notice that, for a curve z satisfying (4.81) and with functions α, f, p satisfying (4.72), the vector function H in (4.79) is independent of x.

Combining (4.76), (4.78) and (4.79), we show that the immersion is congruent to

$$\begin{aligned} L(x,y) = & \int^x (1-\mathrm{i}\sinh\alpha)z(x)e^{\int^y (\mathrm{i}-\sinh\alpha)f(y)dy}dx \\ & + \mathrm{i}\int^y\bigg\{\int^x z(x)(\alpha_y\cosh\alpha - f\cosh^2\alpha)e^{\int^y(\mathrm{i}-\sinh\alpha)f(y)dy}dx\bigg\}dy \\ & + \frac{\mathrm{i}}{p(x)}\int^y e^{\int^y(\mathrm{i}+\sinh\alpha)fdy}\bigg(z(x)\int^y f\alpha_x\cosh\alpha dy - z'(x)\bigg)dy. \end{aligned} \tag{4.82}$$

It follows from (4.82) that

$$L_y = \frac{\mathrm{i}e^{\int^y(\mathrm{i}+\sinh\alpha)fdy}}{p(x)}\bigg(z(x)\int^y f\alpha_x\cosh\alpha dy - z'(x)\bigg). \tag{4.83}$$

We find from (4.2), (4.77) and (4.83) that $\langle z,z\rangle = \langle z',z'\rangle = 0$, $\langle z,\mathrm{i}z'\rangle = p$. Thus, z is a null curve lying in $\mathcal{LC}$ satisfying $\langle z,\mathrm{i}z'\rangle = p$. Notice that, for such a null curve, equation (4.81) holds automatically for some function $k(x)$. Consequently, we obtain case (9) of the theorem. $\square$

5. Marginally trapped surfaces of constant positive curvature

Theorem 5.1 *There exist six families of marginally trapped surfaces of constant curvature one in* $\mathbf{C}^2_1$ *given by the following*:

(1) *A surface defined by*

$$L(x,y)=\left(x+\frac{\mathrm{i}bx^2}{\sqrt{2}}+\frac{\sqrt{2}\mathrm{i}-2b(2x+y)}{2b(x+y)^2},\ x-\frac{\mathrm{i}bx^2}{\sqrt{2}}+\frac{\sqrt{2}\mathrm{i}+2b(2x+y)}{2b(x+y)^2}\right),$$

where b is a nonzero real number.

(2) *A surface defined by*

$$\begin{aligned}L=\Bigg(&\left(\frac{\mathrm{i}-\sqrt{2}b(x+y)}{\sqrt{2}b(x+y)^2}+\frac{\sqrt{2}\,\mathrm{i}b}{a^2}\right)\cosh(ax)\\&\qquad+\left(\frac{1}{a}-\frac{1}{a(x+y)^2}+\frac{\sqrt{2}\mathrm{i}a}{2b(x+y)}\right)\sinh(ax),\\&\left(\frac{\mathrm{i}+\sqrt{2}b(x+y)}{\sqrt{2}b(x+y)^2}-\frac{\sqrt{2}\,\mathrm{i}b}{a^2}\right)\cosh(ax)\\&\qquad+\left(\frac{1}{a}+\frac{1}{a(x+y)^2}+\frac{\sqrt{2}\mathrm{i}a}{2b(x+y)}\right)\sinh(ax)\Bigg),\end{aligned}$$

where a,b are nonzero real numbers.

(3) *A surface defined by*

$$\begin{aligned}L=\Bigg(&\left(\frac{\mathrm{i}-\sqrt{2}b(x+y)}{\sqrt{2}b(x+y)^2}-\frac{\sqrt{2}\,\mathrm{i}b}{a^2}\right)\cos(ax)\\&\qquad+\left(\frac{(x+y)^2-1}{a(x+y)^2}-\frac{\mathrm{i}a}{\sqrt{2}b(x+y)}\right)\sin(ax),\\&\left(\frac{\mathrm{i}+\sqrt{2}b(x+y)}{\sqrt{2}b(x+y)^2}+\frac{\sqrt{2}\,\mathrm{i}b}{a^2}\right)\cos(ax)\\&\qquad+\left(\frac{(x+y)^2+1}{a(x+y)^2}-\frac{\mathrm{i}a}{\sqrt{2}b(x+y)}\right)\sin(ax)\Bigg),\end{aligned}$$

where a,b are nonzero real numbers.

(4) *A surface defined by*

$$L(x,y)=\frac{\mathrm{i}z(x)+\mathrm{i}(x+y)z'(x)}{\sqrt{2}p(x)(x+y)^2}-\int^x(1+\mathrm{i}k(x))z(x)dx$$
$$+\frac{\mathrm{i}}{\sqrt{2}}\int^x\left(\frac{f(x)}{x+y}-\frac{2p(x)+(x+y)p'(x)}{(x+y)^3p^2(x)}\right)z(x)dx$$
$$+\frac{\mathrm{i}}{\sqrt{2}}\int^y\left(\int^x\left\{\frac{f(x)}{(x+y)^2}-\frac{6p(x)+2(x+y)p'(x)}{(x+y)^4p^2(x)}\right\}z(x)dx\right)dy,$$

where z is a null curve lying in $\mathcal{LC}$ satisfying $z''-(\ln p)'z'=fz$, and p,f,k are real valued functions with $p\neq 0$.

(5) *A surface defined by*

$$L(x,y)=z(y)+\frac{\mathrm{i}c_1(1+(x+y)(\mathrm{i}+p(y))q(y))}{(x+y)^2q(y)}e^{\int^y(\mathrm{i}-p(y))q(y)dy}$$

for some real-valued functions $q\neq 0$ and p, where c_1 is a null vector, z is a null curve satisfying $\langle c_1,z'\rangle=-2e^{\int^y pqdy}\cos(\int^y qdy)$ and $\langle c_1,\mathrm{i}z'\rangle=2e^{\int^y pqdy}\sin(\int^y qdy)$.

(6) *A surface defined by*

$$L(x,y)=\int^x\frac{1-\mathrm{i}\sinh\alpha}{e^{\int^y q(y)(\sinh\alpha-\mathrm{i})dy}}z(x)dx-\frac{\mathrm{i}z'(x)}{\sqrt{2}p(x)}\int^y\frac{e^{\int^y q(y)(\mathrm{i}+\sinh\alpha)dy}}{(x+y)^2}dy$$
$$+\frac{z(x)}{\sqrt{2}\,\mathrm{i}p(x)}\int^y\frac{e^{\int^y q(y)(\mathrm{i}+\sinh\alpha)dy}}{(x+y)^2}\left(\frac{2}{x+y}-\int^y q(y)(\sinh\alpha)_x dy\right)dy$$
$$-\mathrm{i}\int^y\int^x\frac{(q(y)\cosh\alpha-\alpha_y)z(x)}{e^{\int^y q(y)(\sinh\alpha-\mathrm{i})dy}\operatorname{sech}\alpha}dxdy,$$

where α,p,q are real-valued functions with $q\neq 0$ and $\alpha_y\neq q\cosh\alpha$, and z is a null curve lying in the light cone $\mathcal{LC}$ satisfying $\langle z,\mathrm{i}z'\rangle=2\sqrt{2}p$.

Conversely, up to rigid motions and dilations, every marginally trapped surface of constant positive curvature in $\mathbf{C}^2_1$ is locally an open portion of one of the surfaces given by the six families.

Proof. As for Theorem 4.1, we show by examples that immersions (1)–(6) give marginally trapped surfaces of constant curvature one in $\mathbf{C}^2_1$.

For case (1) we find from direct computation that the induced metric tensor of the immersion is given by

$$g = \frac{-2}{(x+y)^2}(dx \otimes dy + dy \otimes dx). \tag{5.1}$$

Thus, the immersion L defines a Lorentz surface of constant curvature one in $\mathbf{C}^2_1$. Moreover, by straight-forward computation, we have

$$L_{xy} = (\tanh\alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)\alpha_y L_x, \quad \alpha = \sinh^{-1}\left(\frac{2}{b(x+y)^3}\right)$$

which is light-like. Therefore, the immersion defines a marginally trapped surface of constant curvature one in $\mathbf{C}^2_1$.

Cases (2)–(6) can be proved in similar way.

Conversely, if $L : M \to \mathbf{C}^2_1$ is a marginally trapped immersion of a surface of constant positive curvature K in $\mathbf{C}^2_1$, then M is Lorentzian. Moreover, after applying a suitable dilation, we have $K = 1$. Thus, we may assume that locally M is an open portion of the xy-plane equipped with the Lorentzian metric (5.1). Hence, we have

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -\frac{2}{x+y}\frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0, \quad \nabla_{\frac{\partial}{\partial yx}}\frac{\partial}{\partial y} = -\frac{2}{x+y}\frac{\partial}{\partial y}. \tag{5.2}$$

If we put

$$e_1 = \frac{x+y}{\sqrt{2}}\frac{\partial}{\partial x}, \quad e_2 = \frac{x+y}{\sqrt{2}}\frac{\partial}{\partial y}, \tag{5.3}$$

then $\{e_1, e_2\}$ is a pseudo-orthonormal frame in M satisfying (3.2) and

$$\nabla_{e_1}e_1 = -\frac{e_1}{\sqrt{2}}, \quad \nabla_{e_2}e_1 = \frac{e_1}{\sqrt{2}}, \quad \nabla_{e_1}e_2 = \frac{e_2}{\sqrt{2}}, \quad \nabla_{e_2}e_2 = -\frac{e_2}{\sqrt{2}}. \tag{5.4}$$

Let e_3, e_4 be the normal vector fields defined by (3.4). From (3.5) we find

$$\begin{aligned} e_3 &= \frac{x+y}{\sqrt{2}}(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)L_x, \\ e_4 &= \frac{x+y}{\sqrt{2}}(\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh\alpha)L_y. \end{aligned} \tag{5.5}$$

Since M is marginally trapped, we may assume as in section 4 that

$$h(e_1,e_1)=\beta e_3+\gamma e_4,\quad h(e_1,e_2)=\delta e_3,\quad h(e_2,e_2)=\lambda e_3+\mu e_4 \tag{5.6}$$

with $\delta\neq 0$ on M. From (5.3), (5.6) and Lemma 3.2 we obtain

$$\alpha_x=-\frac{\sqrt{2}\beta}{x+y},\quad \alpha_y=\frac{\sqrt{2}(\mu-\delta)}{x+y}. \tag{5.7}$$

It follows from (5.3), (5.4), (5.6), Lemma 3.2, and the equation of Codazzi that

$$\lambda_x-\delta_y=\frac{3\lambda+\delta}{x+y}+\frac{\sqrt{2}(\beta\lambda-\delta^2-\delta\mu)}{x+y}\tanh\alpha, \tag{5.8}$$

$$\mu_x=\frac{\mu}{x+y}+\mu\alpha_x\tanh\alpha, \tag{5.9}$$

$$\gamma_y=\frac{3\gamma}{x+y}-\frac{\sqrt{2}\gamma(\delta+\mu)}{x+y}\tanh\alpha. \tag{5.10}$$

In view of (3.5), (3.7), and (5.5), equation (2.5) of Gauss can be expressed as

$$\gamma\lambda+\beta\mu=1. \tag{5.11}$$

Case (a): $\mu=0$. In this case, (5.7), (5.8), (5.10) and (5.11) reduce to

$$\alpha_x=-\frac{\sqrt{2}\beta}{x+y},\quad \alpha_y=-\frac{\sqrt{2}\delta}{x+y}, \tag{5.12}$$

$$\lambda_x-\delta_y=\frac{3\lambda+\delta}{x+y}+\frac{\sqrt{2}(\beta\lambda-\delta^2)}{x+y}\tanh\alpha, \tag{5.13}$$

$$\gamma_y=\frac{3\gamma}{x+y}+\gamma\alpha_y\tanh\alpha \tag{5.14}$$

with $\lambda=\gamma^{-1}\neq 0$. Since $\delta\neq 0$, we get $\alpha_y\neq 0$. It follows from (5.14) that

$$\gamma=p(x)(x+y)^3\cosh\alpha,\quad \lambda=\frac{\operatorname{sech}\alpha}{p(x)(x+y)^3} \tag{5.15}$$

for some nonzero real-valued function p.

Case (a.1): $\delta = \beta$. From (5.12) we get $\alpha_x = \alpha_y$. Thus, we have

$$\beta = \delta = \frac{x+y}{\sqrt{2}}\alpha_x = -\frac{x+y}{\sqrt{2}}\alpha_y. \tag{5.16}$$

Substituting (5.15) and (5.16) into (5.13) yields

$$\alpha'' + \alpha'^2 \tanh\alpha + \frac{2\alpha'}{u} = \frac{\sqrt{2}(6p(x) + up'(x))\,\mathrm{sech}\,\alpha}{u^5 p^2(x)} \tag{5.17}$$

with $\alpha = \alpha(u)$, $u = x + y$. By differentiating (5.17) with respect to x we find

$$p''(x) = 2\left(\frac{3}{u} + \frac{p'(x)}{p(x)}\right)p'(x),$$

which is impossible unless $p' = 0$. Thus, we get $p = b$ for some nonzero real number b. Hence, (5.17) reduces to

$$\alpha'' + \alpha'^2 \tanh\alpha + \frac{2\alpha'}{u} = \frac{6\sqrt{2}\,\mathrm{sech}\,\alpha}{bu^5}. \tag{5.18}$$

By combining $p = b$ with (5.15) and (5.16) we have

$$\beta = \delta = -\frac{u}{\sqrt{2}}\alpha', \quad \gamma = bu^3\cosh\alpha, \quad \lambda = \frac{\mathrm{sech}\,\alpha}{bu^3}. \tag{5.19}$$

Therefore, the immersion satisfies

$$\begin{aligned} L_{xx} &= (\tanh\alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)\alpha_x L_x - \frac{2L_x}{x+y} + \sqrt{2}b(x+y)^2(\mathrm{i} + \sinh\alpha)L_y, \\ L_{xy} &= (\tanh\alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)\alpha_y L_x, \\ L_{yy} &= \frac{\sqrt{2}\,\mathrm{sech}\,\alpha(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)}{b(x+y)^4}L_x - \frac{2}{x+y}L_y. \end{aligned} \tag{5.20}$$

Solving the second equation in (5.20) gives

$$L(x,y) = A(y) + \int^x z(x)(1 - \mathrm{i}\sinh\alpha)dx \tag{5.21}$$

for vector functions A, z. Substituting this into the first equation in (5.20), we get

$$A'(y) = \mathrm{i}\bigg\{ \int^x z(x)\alpha_x \cosh \alpha dx - \frac{2z(x) + (x+y)z'(x)}{\sqrt{2}b(x+y)^3} \bigg\}. \tag{5.22}$$

By differentiating (5.22) with respect to x we find

$$z''(x) = \left(\frac{6}{u^2} + \sqrt{2}bu^2\alpha'(u)\cosh\alpha(u) \right) z(x), \quad u = x+y,$$

which is impossible unless we have

$$z''(x) = cz(x), \tag{5.23}$$

$$\sqrt{2}bu^6\alpha'(u)\cosh\alpha(u) = cu^2 - 6 \tag{5.24}$$

for a real number c. Solving (5.24) shows that up to suitable translation, we have

$$\alpha = \sinh^{-1}\left(\frac{2 - cu^2}{\sqrt{2}bu^3} \right). \tag{5.25}$$

Combining this with (5.21) yields

$$L(x,y) = A(y) + \int^x \left(1 - \frac{\mathrm{i}(2 - c(x+y)^2)}{\sqrt{2}b(x+y)^3} \right) z(x) dx. \tag{5.26}$$

Case (a.1.i): $c = 0$. From (5.23) we have $z(x) = c_1 + 2c_2x$ for some vectors c_1, c_2. Thus, (5.26) becomes

$$L(x,y) = A(y) + c_1\left(x + \frac{\mathrm{i}}{\sqrt{2}b(x+y)^2} \right) + c_2\left(x^2 + \frac{\mathrm{i}\sqrt{2}(2x+y)}{b(x+y)^2} \right). \tag{5.27}$$

substituting this into the first equation in (5.21) gives $A' = 0$. Thus, the immersion of the surface is congruent to

$$L(x,y) = c_1\left(x + \frac{\mathrm{i}}{\sqrt{2}b(x+y)^2} \right) + c_2\left(x^2 + \frac{\mathrm{i}\sqrt{2}(2x+y)}{b(x+y)^2} \right). \tag{5.28}$$

From (5.1) and (5.28) we have $\langle c_1, c_1 \rangle = \langle c_1, c_2 \rangle = \langle c_2, c_2 \rangle = 0$, $\langle c_1, \mathrm{i}c_2 \rangle =$

$\sqrt{2}b$. Thus, after choosing suitable initial conditions, we obtain case (1) of the theorem.

Case (a.1.ii): $c = a^2 > 0$. From (5.23) we get $z(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$ for some vectors c_1, c_2. Thus, (5.26) becomes

$$L(x,y) = A(y) + \left(\frac{\mathrm{i}c_1}{\sqrt{2}b(x+y)^2} + \frac{c_2}{a} + \frac{\mathrm{i}ac_2}{\sqrt{2}b(x+y)} \right) \cosh(ax) + \left(\frac{c_1}{a} + \frac{\mathrm{i}ac_1}{\sqrt{2}b(x+y)} + \frac{\mathrm{i}c_2}{\sqrt{2}b(x+y)^2} \right) \sinh(ax).$$

Substituting this into the first equation in (5.20), we get $A' = 0$. Therefore, the immersion is congruent to

$$L(x,y) = \left(\frac{\mathrm{i}c_1}{\sqrt{2}b(x+y)^2} + \frac{c_2}{a} + \frac{\mathrm{i}ac_2}{\sqrt{2}b(x+y)} \right) \cosh(ax) + \left(\frac{c_1}{a} + \frac{\mathrm{i}ac_1}{\sqrt{2}b(x+y)} + \frac{\mathrm{i}c_2}{\sqrt{2}b(x+y)^2} \right) \sinh(ax).$$

Thus, by using (5.1) we find $\langle c_1, c_1 \rangle = \langle c_1, c_2 \rangle = \langle c_2, c_2 \rangle = 0$, $\langle c_1, \mathrm{i}c_2 \rangle = 2\sqrt{2}b/a$. Consequently, after choosing suitable initial conditions, we obtain case (2).

Case (a.1.iii): $c = -a^2 < 0$. From (5.23) we have $z(x) = c_1 \cos(ax) + c_2 \sin(ax)$. Thus, (5.26) becomes

$$L(x,y) = A(y) + \left(\frac{\mathrm{i}c_1}{\sqrt{2}b(x+y)^2} - \frac{c_2}{a} + \frac{\mathrm{i}ac_2}{\sqrt{2}b(x+y)} \right) \cos(ax) + \left(\frac{c_1}{a} - \frac{\mathrm{i}ac_1}{\sqrt{2}b(x+y)} + \frac{\mathrm{i}c_2}{\sqrt{2}b(x+y)^2} \right) \sin(ax).$$

Substituting this into the first equation in (5.20), we find $A' = 0$. Therefore, the immersion is congruent to

$$L(x,y) = \left(\frac{\mathrm{i}c_1}{\sqrt{2}b(x+y)^2} - \frac{c_2}{a} + \frac{\mathrm{i}ac_2}{\sqrt{2}b(x+y)} \right) \cos(ax) + \left(\frac{c_1}{a} - \frac{\mathrm{i}ac_1}{\sqrt{2}b(x+y)} + \frac{\mathrm{i}c_2}{\sqrt{2}b(x+y)^2} \right) \sin(ax).$$

After choosing suitable initial conditions, we obtain case (3) of the theorem.

Case (a.2): $\delta \neq \beta$. From (5.12) and (5.15) we have

$$\beta = -\frac{x+y}{\sqrt{2}}\alpha_x, \quad \delta = -\frac{x+y}{\sqrt{2}}\alpha_y, \quad \alpha_x \neq \alpha_y, \tag{5.29}$$

$$\gamma = p(x)(x+y)^3 \cosh\alpha, \quad \lambda = \frac{\operatorname{sech}\alpha}{p(x)(x+y)^3}. \tag{5.30}$$

Substituting (5.29) and (5.30) into (5.13), we get

$$p' + \frac{6p}{x+y} - \frac{(x+y)^3}{\sqrt{2}}\big\{(x+y)\alpha_y^2 \sinh\alpha + ((x+y)\alpha_{yy} + 2\alpha_y)\cosh\alpha\big\}p^2 = 0,$$

which implies that

$$\frac{\partial}{\partial y}\left(\frac{6p+2(x+y)p'}{(x+y)^2} + \sqrt{2}p^2(x+y)^2\alpha_y\cosh\alpha\right) = 0.$$

Hence, there exists a real-valued function f such that

$$6p + 2(x+y)p' + \sqrt{2}p^2(x+y)^4(\sinh\alpha)_y = f(x)p^2(x)(x+y)^2. \tag{5.31}$$

Solving this equation for $(\sinh\alpha)_y$, we have

$$(\sinh\alpha)_y = \frac{(x+y)^2 f(x)p^2(x) - 2(x+y)p'(x) - 6p(x)}{\sqrt{2}(x+y)^4p^2(x)}.$$

So, we have

$$\alpha = \sinh^{-1}\left(k(x) - \frac{f(x)}{\sqrt{2}(x+y)} + \frac{2p(x)+(x+y)p'(x)}{\sqrt{2}(x+y)^3p^2(x)}\right) \tag{5.32}$$

for some function k. In views of (5.1)–(5.6), (5.29) and (5.30) we obtain

$$\begin{aligned} L_{xx} &= (\tanh\alpha - \mathrm{i}\operatorname{sech}\alpha)\alpha_x L_x - \frac{2L_x}{x+y} + \sqrt{2}p(x)(x+y)^2(\mathrm{i}+\sinh\alpha)L_y, \\ L_{xy} &= (\tanh\alpha - \mathrm{i}\operatorname{sech}\alpha)\alpha_y L_x, \end{aligned} \tag{5.33}$$

$$L_{yy} = \frac{\sqrt{2}\,\mathrm{sech}\,\alpha(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)}{p(x)(x+y)^4} L_x - \frac{2L_y}{x+y}.$$

Solving the second equation in (5.33), we obtain

$$L(x,y) = A(y) + \int^x z(x)(1 - \mathrm{i}\sinh\alpha)dx \tag{5.34}$$

for some vector functions A, z. From (5.34) we have

$$L_x = z(x)(1 - \mathrm{i}\sinh\alpha). \tag{5.35}$$

It follows from (5.1) and (5.35) that $\langle z, z\rangle = 0$. So, z is a curve lying in the light cone $\mathcal{LC} \subset \mathbf{C}^2_1$. By substituting (5.34) into the first equation in (5.33) we find

$$A'(y) = \mathrm{i}\int^x z(x)(\sinh\alpha)_y dx - \mathrm{i}\frac{2z(x) + (x+y)z'(x)}{\sqrt{2}p(x)(x+y)^3}. \tag{5.36}$$

Differentiating (5.36) with respect to x, we have

$$pz'' - p'z' = z(x)\left(\frac{6p + 2(x+y)p'}{(x+y)^2} + \sqrt{2}p^2(x+y)^2(\sinh\alpha)_y\right). \tag{5.37}$$

Combining this with (5.31), we get

$$p(x)z''(x) - p'(x)z(x) = f(x)p(x)z(x). \tag{5.38}$$

Now, by applying $\langle z, z\rangle = \langle z', z\rangle = 0$, we derive from (5.38) that $\langle z'', z\rangle = 0$.

On the other hand, it follows from $\langle z, z'\rangle = 0$ that $\langle z', z'\rangle = -\langle z'', z\rangle$. Hence, we get $\langle z', z'\rangle = 0$. Therefore, z is a null curve lying in $\mathcal{LC}$.

From (5.34) and (5.36) we conclude that the immersion is congruent to

$$\begin{aligned} L(x,y) = {} & \frac{\mathrm{i}(z(x) + (x+y)z'(x))}{\sqrt{2}p(x)(x+y)^2} + \int^x z(x)(1 - \mathrm{i}\sinh\alpha)dx \\ & + \mathrm{i}\int^y\int^x (\sinh\alpha)_y z(x)dxdy. \end{aligned}$$

Consequently, by applying (5.32) we obtain case (4) of the theorem.

Case (b): $\mu \neq 0$. It follows from (5.9) that

$$\mu = \frac{u}{\sqrt{2}} q(y) \cosh \alpha, \quad u = x + y \tag{5.39}$$

for a real-valued nonzero function q. Hence, we find from (5.7) that

$$\beta = -\frac{u}{\sqrt{2}} \alpha_x, \quad \delta = \frac{u}{\sqrt{2}} (q(y) \cosh \alpha - \alpha_y) \neq 0. \tag{5.40}$$

Case (b.1): $\gamma = 0$. It follows from (5.11), (5.39) and (5.40) that

$$\alpha_x \cosh \alpha = -\frac{2}{u^2 q(y)},$$

which implies

$$\alpha = \sinh^{-1} \left(p(y) + \frac{2}{uq(y)} \right), \quad u = x + y, \tag{5.41}$$

for some function p. Substituting this into (5.39) and (5.40), we get

$$\begin{aligned} \mu &= \frac{\sqrt{u^2 q^2 + (2 + upq)^2}}{\sqrt{2}} \quad \beta = \frac{\sqrt{2}}{\sqrt{u^2 q^2 + (2 + upq)^2}}, \\ \delta &= \frac{6q + u^2 q^2 ((1 + p^2) q - p') + 2u(2pq^2 + q')}{\sqrt{2} q \sqrt{u^2 q^2 + (2 + upq)^2}}. \end{aligned} \tag{5.42}$$

It follows from (5.1)–(5.3), (5.5), (5.6), (5.41) and (5.42) that

$$\begin{aligned} L_{xx} &= -\frac{2}{u} L_x - \frac{4 + 2u(p - \mathrm{i})q}{u(u^2 q^2 + (2 + upq)^2)} L_x, \\ L_{xy} &= -\frac{6q + u^2 q^2 ((1 + p^2) q - p') + 2u(2pq^2 + q')}{uq(2 + u(\mathrm{i} + p)q)} L_x, \\ L_{yy} &= \frac{\sqrt{2}\{(\mathrm{i} - p)qu - 2\}\lambda}{u\sqrt{u^2 q^2 + (2 + upq)^2}} L_x + (\mathrm{i} + p) q L_y. \end{aligned} \tag{5.43}$$

Solving the second equation in (5.43), we obtain

$$L(x,y)=z(y)-\frac{2\mathrm{i}e^{\int^y(\mathrm{i}-p)qdy}}{q(y)}\int^x\frac{w(x)dx}{(x+y)^3}+\frac{(1-\mathrm{i}p(y))}{e^{\int^y(p-\mathrm{i})qdy}}\int^x\frac{w(x)dx}{(x+y)^2} \tag{5.44}$$

for vector functions z, w. By substituting this into the first equation in (5.43), we get $w'(x)=0$. Thus, $w=c_1$ for a constant vector c_1. Hence, (5.44) reduces to

$$L(x,y)=z(y)+\frac{\mathrm{i}c_1(1+(x+y)(\mathrm{i}+p(y))q(y))}{(x+y)^2q(y)}e^{\int^y(\mathrm{i}-p(y))q(y)dy}. \tag{5.45}$$

From (5.45), we get

$$\begin{aligned} L_x&=\frac{c_1e^{\mathrm{i}\int^y qdy}((x+y)q(y)-\mathrm{i}((x+y)p(y)q(y)+2)}{(x+y)^3q(y)e^{\int^y pqdy}},\\ L_y&=z'(y)-\frac{\mathrm{i}c_1e^{\mathrm{i}\int^y q(y)dy}}{(x+y)^3q^2(y)e^{\int^y pqdy}}\big\{(x+y)^2(1+p^2)q^3\\ &\quad+(x+y)(2p-(x+y)p')q^2+2q+(x+y)q'\big\}. \end{aligned} \tag{5.46}$$

Now, it follows from (5.1) and (5.46) that

$$\begin{aligned} &\langle c_1,c_1\rangle=\langle z',z'\rangle=0,\quad (\sin F)\langle c_1,z'\rangle=(\cos F)\langle \mathrm{i}c_1,z'\rangle,\\ &\big\{(x+y)q\cos F+((x+y)pq+2)\sin F)\big\}\langle c_1,z'\rangle\\ &+\big\{(x+y)q\sin F-((x+y)pq+2)\cos F)\big\}\langle \mathrm{i}c_1,z'\rangle\\ &\quad=-2(x+y)qe^{\int^y pqdy} \end{aligned} \tag{5.47}$$

with $F=\int^y q(y)dy$. Solving the last two equations in (5.47), we find

$$\langle c_1,z'\rangle=-2e^{\int^y pqdy}\cos\Big(\int^y qdy\Big),\quad \langle \mathrm{i}c_1,z'\rangle=-2e^{\int^y pqdy}\sin\Big(\int^y qdy\Big).$$

Consequently, we obtain case (5) of the theorem.

Case (b.2): $\gamma\neq 0$. It follows from (5.10), (5.39) and (5.40) that

$$(\ln\gamma)_y=\frac{3}{x+y}+\alpha_y\tanh\alpha-2q(y)\sinh\alpha. \tag{5.48}$$

From (5.11), (5.40) and (5.48) we have

$$\gamma=\frac{p(x)(x+y)^3\cosh\alpha}{e^{2\int^y q(y)\sinh\alpha dy}},\quad \lambda=\frac{2\operatorname{sech}\alpha+(x+y)^2q(y)\alpha_x}{2p(x)(x+y)^3e^{-2\int^y q(y)\sinh\alpha dy}} \tag{5.49}$$

for some real-valued nonzero function $p(x)$.

In views of (5.1)–(5.6), (5.39), (5.40) and (5.49) we obtain

$$\begin{aligned} L_{xx}&=(\tanh\alpha-\mathrm{i}\operatorname{sech}\alpha)\alpha_xL_x-\frac{2L_x}{x+y}+\frac{\sqrt{2}p(x)(x+y)^2(\mathrm{i}+\sinh\alpha)}{e^{2\int^y q(y)\sinh\alpha dy}}L_y,\\ L_{xy}&=(q(y)\cosh\alpha-\alpha_y)(\mathrm{i}\operatorname{sech}\alpha-\tanh\alpha)L_x, \qquad (5.50)\\ L_{yy}&=\frac{(\mathrm{i}\operatorname{sech}\alpha-\tanh\alpha)(2\operatorname{sech}\alpha+(x+y)^2q(y)\alpha_x)}{\sqrt{2}p(x)(x+y)^4e^{-2\int^y q(y)\sinh\alpha dy}}L_x-\frac{2L_y}{x+y}\\ &\quad+(\mathrm{i}+\sinh\alpha)q(y)L_y. \end{aligned}$$

Solving the second equation in (5.50), we get

$$L(x,y)=A(y)+\int^x\frac{1-\mathrm{i}\sinh\alpha}{e^{\int^y q(\sinh\alpha-\mathrm{i})dy}}z(x)dx \tag{5.51}$$

for some vector functions $A(y), z(x)$. Thus, we have

$$\begin{aligned} L_x&=(1-\mathrm{i}\sinh\alpha)e^{\int^y q(y)(\mathrm{i}-\sinh\alpha)dy}z(x),\\ L_y&=A'(y)+\mathrm{i}\int^x\frac{(q(y)\cosh\alpha-\alpha_y)\cosh\alpha}{e^{\int^y q(\sinh\alpha-\mathrm{i})dy}}z(x)dx. \end{aligned} \tag{5.52}$$

By substituting (5.51) into the first equation in (5.50), we have

$$\begin{aligned} A'(y)&=\frac{e^{\int^y q(\mathrm{i}+\sinh\alpha)dy}}{\sqrt{2}\,\mathrm{i}(x+y)^2p(x)}\left(z'(x)+\frac{2z(x)}{x+y}-z(x)\int^y q(y)(\sinh\alpha)_xdy\right)\\ &\quad-\mathrm{i}\int^x\frac{(q(y)\cosh\alpha-\alpha_y)z(x)}{e^{\int^y q(\sinh\alpha-\mathrm{i})dy}\operatorname{sech}\alpha}dx. \end{aligned} \tag{5.53}$$

By combining (5.51) and (5.53) we know that the immersion is congruent to

$$\begin{aligned} L(x,y) = & \int^x \frac{1-\mathrm{i}\sinh\alpha}{e^{\int^y q(\sinh\alpha-\mathrm{i})dy}} z(x)dx \\ & + \frac{z(x)}{\sqrt{2}\,\mathrm{i}p(x)} \int^y \frac{e^{\int^y q(\mathrm{i}+\sinh\alpha)dy}}{(x+y)^2} \left(\frac{2}{x+y} - \int^y q(y)(\sinh\alpha)_x dy \right) dy \\ & - \mathrm{i} \int^y \int^x \frac{(q(y)\cosh\alpha - \alpha_y) z(x)}{e^{\int^y q(\sinh\alpha-\mathrm{i})dy} \operatorname{sech}\alpha} dx dy \\ & - \frac{\mathrm{i}z'(x)}{\sqrt{2}p(x)} \int^y \frac{e^{\int^y q(\mathrm{i}+\sinh\alpha)dy}}{(x+y)^2} dy. \end{aligned} \tag{5.54}$$

Also, substituting (5.53) into (5.52), we find

$$L_y = \frac{\mathrm{i}e^{\int^y q(\mathrm{i}+\sinh\alpha)dy}}{\sqrt{2}(x+y)^2 p(x)} \left\{ z(x) \int^y q(y)(\sinh\alpha)_x dy - z'(x) - \frac{2z(x)}{x+y} \right\}. \tag{5.55}$$

It follows from (5.39), (5.52) and (5.55) that $\langle z, z\rangle = \langle z', z'\rangle = 0$, $\langle z, \mathrm{i}z'\rangle = 2\sqrt{2}p$. Consequently, we obtain case (6) of the theorem. □

6. Marginally trapped surfaces of constant negative curvature

Theorem 6.1 *There exist six families of marginally trapped surfaces of constant curvature* -1 *in* $\mathbf{C}^2_1$ *given by the following*:

(1) *A surface defined by*

$$\begin{aligned} L(x,y) = & \\ & \left(\frac{(\mathrm{i}+k)(bx^2-4\mathrm{i}x)}{4} + \frac{4\mathrm{i} - 2bx + \sqrt{2}b\sinh\left(\sqrt{2}(x+y)\right)}{4b} \operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right), \right. \\ & \left. - \frac{(\mathrm{i}+k)(bx^2+4\mathrm{i}x)}{4} + \frac{4\mathrm{i} + 2bx - \sqrt{2}b\sinh\left(\sqrt{2}(x+y)\right)}{4b} \operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) \right), \end{aligned}$$

where b *is a nonzero real number and* k *is an arbitrary real number.*

(2) *A surface defined by*

$$L(x,y) =$$

$$\frac{1}{2a^2b}\Bigg(a\bigg(2b(1-\mathrm{i}k) - b\,\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) - 2\sqrt{2}\mathrm{i}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\bigg)\sinh(ax)$$

$$+\left(b^2(\mathrm{i}+k) + 2\mathrm{i}a^2\,\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) + \sqrt{2}a^2b\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right)\cosh(ax),$$

$$a\left(b\,\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) + 2b(1-\mathrm{i}k) - 2\sqrt{2}\mathrm{i}a^2\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right)\sinh(ax)$$

$$-\left(b^2(\mathrm{i}+k) - 2\mathrm{i}a^2\,\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) + \sqrt{2}a^2b\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right)\cosh(ax)\Bigg),$$

where a, b are nonzero real numbers and k is an arbitrary real number.

(3) *A surface defined by*

$$L(x,y) =$$

$$\frac{1}{2a^2b}\Bigg((b\cos(ax) + 2\mathrm{i}a\sin(ax))\left(\sqrt{2}a^2\tanh\left(\frac{x+y}{\sqrt{2}}\right) - b(\mathrm{i}+k)\right)$$

$$+\frac{2\mathrm{i}a^2\cos(ax) - ab\sin(ax)}{\cosh^2((x+y)/\sqrt{2})},$$

$$\frac{2\mathrm{i}a^2\cos(ax) + ab\sin(ax)}{\cosh^2((x+y)/\sqrt{2})} + (b\cos(ax) - 2\mathrm{i}a\sin(ax))$$

$$\times\left(b(\mathrm{i}+k) - \sqrt{2}a^2\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right)\Bigg),$$

where a, b are nonzero real numbers and k is an arbitrary real number.

(4) *A surface defined by*

$$L(x,y) = \int^x (1-\mathrm{i}\sinh\alpha)z(x)dx + \mathrm{i}\int^y\int^x (\sinh\alpha)_y\, z(x)dxdy$$

$$-\frac{\mathrm{i}}{p(x)}\left(\sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)z'(x) - \mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)z(x)\right),$$

where z is a null curve in $\mathcal{LC}$ satisfying $z''(x) - (\ln p)'(x)z'(x) = (2 + f(x))z(x)$,

$$\alpha = \sinh^{-1}\Bigg(k(x) + \frac{\sqrt{2}(2+f(x))}{p(x)\coth((x+y)/\sqrt{2})} + \frac{p'(x)+\sqrt{2}p(x)\tanh((x+y)/\sqrt{2})}{p^2(x)\cosh^2((x+y)/\sqrt{2})}\Bigg),$$

and f, p, k are real-valued functions with $p \neq 0$.

(5) *A surface defined by*

$$L(x,y) = z(y) + \frac{\sqrt{2}c_1(1-\mathrm{i}p(y))}{e^{\int^y (p(y)-\mathrm{i})q(y)dy}}\tanh\left(\frac{x+y}{\sqrt{2}}\right) + \frac{\mathrm{i}c_1 \operatorname{sech}^2\left((x+y)/\sqrt{2}\right)}{q(y)e^{\int^y (p(y)-\mathrm{i})q(y)dy}},$$

where p, q are real-valued functions with $q \neq 0$, c_1 is a null vector, and z is a null curve satisfying $\langle c_1, z'\rangle = -e^{\int^y pqdy}\cos\left(\int^y qdy\right)$ and $\langle c_1, \mathrm{i}z'\rangle = e^{\int^y pqdy}\sin\left(\int^y qdy\right)$.

(6) *A surface defined by*

$$\begin{aligned} L(x,y) = &\int^x \frac{1-\mathrm{i}\sinh\alpha}{e^{\int^y q(y)(\sinh\alpha-\mathrm{i})dy}} z(x)dx - \frac{\mathrm{i}z'(x)}{p(x)}\int^y \frac{e^{\int^y q(y)(\mathrm{i}+\sinh\alpha)dy}}{\cosh^2\left((x+y)/\sqrt{2}\right)}dy \\ &+ \frac{\mathrm{i}z(x)}{p(x)}\int^y \frac{e^{\int^y q(y)(\mathrm{i}+\sinh\alpha)dy}}{\cosh^2\left((x+y)/\sqrt{2}\right)} \\ &\times \left(\int^y q(y)(\sinh\alpha)_x dy - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right)dy \\ &- \mathrm{i}\int^y\int^x \frac{(q(y)\cosh\alpha - \alpha_y)z(x)}{e^{\int^y q(y)(\sinh\alpha-\mathrm{i})dy}\operatorname{sech}\alpha}dxdy, \end{aligned}$$

where α, p, q are real-valued functions with $q \neq 0$ and $\alpha_y \neq q\cosh\alpha$, and z is a null curve lying in the light cone $\mathcal{LC}$ satisfying $\langle z, \mathrm{i}z'\rangle = p$.

Conversely, up to rigid motions and dilations, every marginally trapped surface of constant negative curvature in $\mathbf{C}^2_1$ is locally an open portion of one of the surfaces given by the six families.

Proof. For case (1), we know from direct computation that the induced metric tensor is given by

$$g = -\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)(dx\otimes dy + dy\otimes dx). \tag{6.1}$$

Thus, the immersion defines a Lorentz surface of constant curvature -1 in $\mathbf{C}^2_1$. Moreover, by straight-forward computation, we have

$$L_{xy} = (\tanh\alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)\alpha_y L_x$$

with

$$\alpha = \sinh^{-1}\left(k + \frac{\sqrt{2}}{b}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)\right).$$

This implies that the mean curvature vector of the immersion is light-like. Thus, the immersion defines a marginally trapped surface of constant curvature -1.

Similar computations show that the remaining cases give rise to marginally trapped surfaces of constant curvature -1 in $\mathbf{C}^2_1$ as well.

Conversely, if $L : M \to \mathbf{C}^2_1$ is a marginally trapped immersion of a surface of constant negative curvature, then M is Lorentzian. Moreover, after applying a suitable dilation, we have $K = -1$. Thus, we may assume that locally M is an open portion of the xy-plane with the Lorentzian metric given by (6.1). Hence, we have

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -\sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0, \\ &\nabla_{\frac{\partial}{\partial yx}}\frac{\partial}{\partial y} = -\sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\frac{\partial}{\partial y}. \end{aligned} \tag{6.2}$$

If we put

$$e_1 = \cosh\left(\frac{x+y}{\sqrt{2}}\right)\frac{\partial}{\partial x}, \quad e_2 = \cosh\left(\frac{x+y}{\sqrt{2}}\right)\frac{\partial}{\partial y}, \tag{6.3}$$

then $\{e_1, e_2\}$ is a pseudo-orthonormal frame in M satisfying (3.2) and

$$\begin{aligned} &\nabla_{e_1}e_1 = -\frac{\sinh\left(\sqrt{2}(x+y)\right)}{2\sqrt{2}}e_1, \quad \nabla_{e_2}e_1 = \frac{\sinh\left(\sqrt{2}(x+y)\right)}{2\sqrt{2}}e_1, \\ &\nabla_{e_1}e_2 = \frac{\sinh\left(\sqrt{2}(x+y)\right)}{2\sqrt{2}}e_2, \qquad \nabla_{e_2}e_2 = -\frac{\sinh\left(\sqrt{2}(x+y)\right)}{2\sqrt{2}}e_2. \end{aligned} \tag{6.4}$$

Let e_3, e_4 be the normal vector fields defined by (3.4). From (3.5) we find

$$\begin{aligned} e_3 &= \cosh\left(\frac{x+y}{\sqrt{2}}\right)(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)L_x, \\ e_4 &= \cosh\left(\frac{x+y}{\sqrt{2}}\right)(\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh\alpha)L_y. \end{aligned} \tag{6.5}$$

Since M is marginally trapped, we may assume as in section 4 that

$$h(e_1, e_1) = \beta e_3 + \gamma e_4, \quad h(e_1, e_2) = \delta e_3, \quad h(e_2, e_2) = \lambda e_3 + \mu e_4 \tag{6.6}$$

with $\delta \neq 0$ on M. From (6.4), (6.6) and Lemma 3.2 we obtain

$$\alpha_x = -\beta\,\mathrm{sech}\left(\frac{x+y}{\sqrt{2}}\right), \quad \alpha_y = (\mu - \delta)\,\mathrm{sech}\left(\frac{x+y}{\sqrt{2}}\right). \tag{6.7}$$

By using (6.4), (6.6), (6.7), Lemma 3.2 and the equation of Codazzi, we find

$$\lambda_x - \delta_y = \frac{3\lambda + \delta}{\sqrt{2}}\tanh\left(\frac{x+y}{\sqrt{2}}\right) + (\beta\lambda - \delta^2 - \delta\mu)\,\mathrm{sech}\left(\frac{x+y}{\sqrt{2}}\right)\tanh\alpha, \tag{6.8}$$

$$\mu_x = \frac{\mu}{\sqrt{2}}\tanh\left(\frac{x+y}{\sqrt{2}}\right) + \mu\alpha_x\tanh\alpha, \tag{6.9}$$

$$\gamma_y = \frac{3\gamma}{\sqrt{2}}\tanh\left(\frac{x+y}{\sqrt{2}}\right) - \gamma(\delta + \mu)\,\mathrm{sech}\left(\frac{x+y}{\sqrt{2}}\right)\tanh\alpha. \tag{6.10}$$

In view of (3.7) and (6.6), equation (2.5) of Gauss can be expressed as

$$\gamma\lambda + \beta\mu = -1. \tag{6.11}$$

Case (a): $\mu = 0$. In this case, (6.7), (6.8), (6.10), and (6.11) reduce to

$$\alpha_x = -\beta\,\mathrm{sech}\left(\frac{x+y}{\sqrt{2}}\right), \quad \alpha_y = -\delta\,\mathrm{sech}\left(\frac{x+y}{\sqrt{2}}\right) \neq 0, \tag{6.12}$$

$$\lambda_x - \delta_y = \frac{3\lambda + \delta}{\sqrt{2}}\tanh\left(\frac{x+y}{\sqrt{2}}\right) + (\beta\lambda - \delta^2)\,\mathrm{sech}\left(\frac{x+y}{\sqrt{2}}\right)\tanh\alpha, \tag{6.13}$$

$$\gamma_y = \frac{3\gamma}{\sqrt{2}} \tanh\left(\frac{x+y}{\sqrt{2}}\right) + \gamma\alpha_y \tanh\alpha, \quad \gamma = -\frac{1}{\lambda} \neq 0. \tag{6.14}$$

It follows from (6.14) that

$$\gamma = p(x)\cosh^3\left(\frac{x+y}{\sqrt{2}}\right)\cosh\alpha, \quad \lambda = -\frac{\operatorname{sech}\alpha}{p(x)}\operatorname{sech}^3\left(\frac{x+y}{\sqrt{2}}\right) \tag{6.15}$$

for some nonzero real-valued function p.

Case (a.1): $\delta = \beta$. From (6.12) we get $\alpha_x = \alpha_y$. Thus, we have

$$\beta = \delta = -\alpha'(u)\cosh\left(\frac{u}{\sqrt{2}}\right), \quad u = x+y. \tag{6.16}$$

Substituting (6.15) and (6.16) into (6.13), we get

$$\begin{aligned} 0 &= \alpha''(u) + \alpha'^2(u)\tanh\alpha(u) + \sqrt{2}\tanh\left(\frac{u}{\sqrt{2}}\right)\alpha'(u) \\ &\quad + \frac{\operatorname{sech}\alpha}{p^2(x)}\operatorname{sech}^4\left(\frac{u}{\sqrt{2}}\right)\left(p'(x) + 3\sqrt{2}p\tanh\left(\frac{u}{\sqrt{2}}\right)\right). \end{aligned} \tag{6.17}$$

By regarding x and u as independent variables and by taking partial differentiation of (6.17) with respect to x, we obtain

$$p(x)p''(x) = 2p'^2(x) + 3\sqrt{2}\tanh\left(\frac{u}{\sqrt{2}}\right)p(x)p'(x),$$

which is impossible unless $p'(x) = 0$. Thus, (6.15) reduces to

$$\gamma = b\cosh^3\left(\frac{x+y}{\sqrt{2}}\right)\cosh\alpha, \quad \lambda = -\frac{\operatorname{sech}\alpha}{b}\operatorname{sech}^3\left(\frac{x+y}{\sqrt{2}}\right)$$

for some nonzero real number b. Hence, the immersion satisfies

$$\begin{aligned} L_{xx} &= (\tanh\alpha - \mathrm{i}\operatorname{sech}\alpha)\alpha_x L_x - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_x \\ &\quad + b(\mathrm{i} + \sinh\alpha)\cosh^2\left(\frac{x+y}{\sqrt{2}}\right)L_y, \end{aligned}$$

$$L_{xy} = (\tanh\alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)\alpha_y L_x, \tag{6.18}$$
$$L_{yy} = \frac{\mathrm{sech}^4\left((x+y)/\sqrt{2}\right)}{b(\mathrm{i}+\sinh\alpha)} L_x - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right) L_y.$$

Solving the second equation in (6.18), we get

$$L(x,y) = A(y) + \int^x z(x)(1 - \mathrm{i}\sinh\alpha)dx. \tag{6.19}$$

Substituting this into the first equation in (6.18), we find

$$A'(y) = \mathrm{i}\int^x z(x)\alpha_x \cosh\alpha dx - \frac{2\mathrm{i}(z'(x) + \sqrt{2}\tanh((x+y)/\sqrt{2})z(x)}{b(1+\cosh(\sqrt{2}(x+y))}. \tag{6.20}$$

By differentiating (6.20) with respect to x, we obtain

$$z''(x) = \left(2 - 3\,\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) + b\cosh^2\left(\frac{x+y}{\sqrt{2}}\right)\alpha'\cosh\alpha\right) z(x),$$

which is impossible unless we have

$$z''(x) = cz(x), \tag{6.21}$$
$$2 - 3\,\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) + b\cosh^2\left(\frac{x+y}{\sqrt{2}}\right)\alpha'\cosh\alpha = c \tag{6.22}$$

for a real number c. Solving (6.22) shows that up to suitable translation, we have

$$\alpha = \sinh^{-1}\left(k + \frac{\sqrt{2}}{b}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\left(c + \mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)\right)\right) \tag{6.23}$$

for a real number k. Combining this with (6.19), we obtain

$$L = A(y) + \int^x \left\{1 - \mathrm{i} - \frac{\sqrt{2}\mathrm{i}}{b}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\left(c + \mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)\right)\right\} z(x)dx. \tag{6.24}$$

Case (a.1.i): $c = 0$. From (6.21) and (6.23) we obtain $z(x) = c_1 + 2c_2 x$

and

$$\alpha = \sinh^{-1}\left(k + \frac{\sqrt{2}}{b}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)\right) \tag{6.25}$$

for some vectors c_1, c_2. Hence, we find from (6.24) that

$$\begin{aligned} L(x,y) &= A(y) + (1-\mathrm{i}k)(x+y)(c_1 + c_2(x-y)) \\ &\quad + \frac{\mathrm{i}}{b}\left((c_1 + 2c_2 x)\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) - 2\sqrt{2}c_2\tanh\left(\sqrt{2}(x+y)\right)\right). \end{aligned}$$

Substituting this into the first equation in (6.18) and by using (6.25), we find $A'(y) = (1-\mathrm{i}k)(2c_2 y - c_1)$. Therefore, the immersion is congruent to

$$\begin{aligned} L(x,y) &= (1-\mathrm{i}k)(c_2 y^2 - c_1 y) + (1-\mathrm{i}k)(x+y)(c_1 + c_2(x-y)) \\ &\quad + \frac{\mathrm{i}}{b}\left((c_1 + 2c_2 x)\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) - 2\sqrt{2}c_2\tanh\left(\sqrt{2}(x+y)\right)\right). \end{aligned} \tag{6.26}$$

From (6.1) and (6.26) we have $\langle c_1, c_1\rangle = \langle c_1, c_2\rangle = \langle c_2, c_2\rangle = 0$, $\langle \mathrm{i}c_1, c_2\rangle = -b/2$. Thus, after choosing suitable initial conditions, we obtain case (1) of the theorem.

Case (a.1.ii): $c = a^2 > 0$. From (6.21) we get $z(x) = c_1\cosh(ax) + c_2\sinh(ax)$ for some vectors c_1, c_2. Thus, (6.24) becomes

$$\begin{aligned} L(x,y) &= \left\{\frac{1-\mathrm{i}k}{a}c_2 + \frac{2c_1 - \sqrt{2}c_2\sinh\left(\sqrt{2}(x+y)\right)}{2b\cosh^2\left((x+y)/\sqrt{2}\right)}\right\}\cosh(ax) \\ &\quad + \left\{\frac{1-\mathrm{i}k}{a}c_1 + \frac{2c_2 - \sqrt{2}c_1\sinh\left(\sqrt{2}(x+y)\right)}{2b\cosh^2\left((x+y)/\sqrt{2}\right)}\right\}\sinh(ax) + A(y). \end{aligned}$$

Substituting this into the first equation in (6.18), we get $A' = 0$. Therefore, after choosing suitable initial conditions, we obtain case (2) of the theorem.

Case (a.1.iii): $c = -a^2 < 0$. From (6.22) we get $z(x) = c_1\cos(ax) + c_2\sin(ax)$ for some vectors c_1, c_2. Thus, (6.24) becomes

$$L(x,y) = A(y) + \frac{\mathrm{i}}{b}(c_1\cos(ax) + c_2\sin(ax))\,\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)$$
$$+ \frac{\mathrm{i}(c_2\cos(ax) - c_1\sin(ax))}{ab}\left(b(\mathrm{i}+k) - \sqrt{2}a^2\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right).$$

Substituting this into the first equation in (6.18), we find $A' = 0$. Hence, after choosing suitable initial conditions, we obtain case (3) of the theorem.

Case (a.2): $\delta \neq \beta$. From (6.12) and (6.15) we have

$$\begin{aligned}
\beta &= -\alpha_x \cosh\left(\frac{x+y}{\sqrt{2}}\right),\\
\delta &= -\alpha_y \cosh\left(\frac{x+y}{\sqrt{2}}\right), \quad \alpha_x \neq \alpha_y,\\
\gamma &= p(x)\cosh^3\left(\frac{x+y}{\sqrt{2}}\right)\cosh\alpha,\\
\lambda &= -\frac{\mathrm{sech}\,\alpha}{p(x)}\,\mathrm{sech}^3\left(\frac{x+y}{\sqrt{2}}\right).
\end{aligned} \tag{6.27}$$

In views of (6.1)–(6.3), (6.5), (6.6), and (6.27), we obtain

$$\begin{aligned}
L_{xx} &= (\tanh\alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)\alpha_x L_x - \sqrt{2}p(x)\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_x\\
&\quad + p(x)\cosh^2\left(\frac{x+y}{\sqrt{2}}\right)(\mathrm{i}+\sinh\alpha)L_y,\\
L_{xy} &= (\tanh\alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)\alpha_y L_x,\\
L_{yy} &= \frac{\mathrm{sech}\,\alpha(\tanh\alpha - \mathrm{i}\,\mathrm{sech}\,\alpha)}{p(x)\cosh^4\left((x+y)/\sqrt{2}\right)}L_x - \sqrt{2}p(x)\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_y.
\end{aligned} \tag{6.28}$$

Solving the second equation in (6.28), we get

$$L(x,y) = A(y) + \int^x z(x)(1 - \mathrm{i}\sinh\alpha)dx \tag{6.29}$$

for some vector functions A, z. From (6.29) we have

$$L_x = z(x)(1 - \mathrm{i} \sinh \alpha). \tag{6.30}$$

Thus, it follows from (6.1) and (6.30) that z is a curve lying in $\mathcal{LC} \subset \mathbf{C}^2_1$.

Next, by substituting (6.29) into the first equation in (6.28), we find

$$A'(y) = \mathrm{i} \int^x z(x)(\sinh \alpha)_y dx - \mathrm{i} \frac{z'(x) + \sqrt{2} z(x) \tanh((x+y)/\sqrt{2})}{p(x) \cosh^2 \left((x+y)/\sqrt{2}\right)}. \tag{6.31}$$

By differentiating (6.31) with respect to x, we obtain the following

$$\begin{aligned} z'' - \frac{p'}{p} z' - 2z = \Bigg\{ & \sqrt{2} \frac{p'(x)}{p(x)} \tanh\left(\frac{x+y}{\sqrt{2}}\right) - 3 \operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) \\ & + p(x) \cosh^2\left(\frac{x+y}{\sqrt{2}}\right) \alpha_y \cosh \alpha \Bigg\} z(x). \end{aligned} \tag{6.32}$$

On the other hand, by substituting (6.27) into (6.13), we find

$$\begin{aligned} & p' + 3\sqrt{2} \tanh\left(\frac{x+y}{\sqrt{2}}\right) p(x) + p^2(x) \cosh^3\left(\frac{x+y}{\sqrt{2}}\right) \\ & \times \Bigg\{ \sqrt{2} \alpha_y \cosh \alpha \sinh\left(\frac{x+y}{2\sqrt{2}}\right) + \alpha_y^2 \cosh\left(\frac{x+y}{\sqrt{2}}\right) \sinh \alpha \\ & \quad + \alpha_{yy} \cosh\left(\frac{x+y}{\sqrt{2}}\right) \cosh \alpha \Bigg\} = 0, \end{aligned}$$

which implies that

$$\frac{\partial}{\partial y} \Bigg\{ \sqrt{2} \frac{p'}{p} \tanh\left(\frac{x+y}{\sqrt{2}}\right) - 3 \operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) + p \cosh^2\left(\frac{x+y}{\sqrt{2}}\right) (\sinh \alpha)_y \Bigg\} = 0.$$

Hence, there exists a real-valued function $f(x)$ such that

$$f(x) = \sqrt{2} \frac{p'}{p} \tanh\left(\frac{x+y}{\sqrt{2}}\right) - 3 \operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) + p \cosh^2\left(\frac{x+y}{\sqrt{2}}\right) (\sinh \alpha)_y. \tag{6.33}$$

Combining this with (6.32), we get

$$p z'' - p' z' = (2 + f(x)) p(x) z(x). \tag{6.34}$$

Also, by solving (6.33) for $(\sinh\alpha)_y$, we find

$$(\sinh\alpha)_y = \frac{p(x)f(x) + 3p(x)\,\text{sech}^2\big((x+y)/\sqrt{2}\big) - \sqrt{2}p'(x)\tanh\big((x+y)/\sqrt{2}\big)}{p^2(x)\cosh^2\big((x+y)/\sqrt{2}\big)}.$$

Thus, we obtain

$$\alpha = \sinh^{-1}\left(k(x) + \frac{\sqrt{2}(2+f(x))}{p(x)\coth\big((x+y)/\sqrt{2}\big)} + \frac{p'(x) + \sqrt{2}p(x)\tanh((x+y)/\sqrt{2})}{p^2(x)\cosh^2\big((x+y)/\sqrt{2}\big)}\right)$$

for some function k. By applying $\langle z,z\rangle = \langle z',z\rangle = 0$, we get $\langle z'',z\rangle = 0$ from (6.34). On the other hand, since $\langle z,z'\rangle = 0$, we have $\langle z',z'\rangle = -\langle z'',z\rangle$. So, we also have $\langle z',z'\rangle = 0$. Thus, z is a null curve lying in $\mathcal{LC}$.

It follows from (6.29) and (6.31) that the immersion is congruent to

$$L(x,y) = \int^x (1 - \mathrm{i}\sinh\alpha) z(x)dx + \mathrm{i}\int^y\int^x (\sinh\alpha)_y z(x)dxdy - \frac{\mathrm{i}}{p(x)}\left(\sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)z'(x) - \text{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)z(x)\right). \quad (6.35)$$

Consequently, we obtain case (4) of the theorem.

Case (b): $\mu \neq 0$. It follows from (6.9) that

$$\mu = q(y)\cosh\alpha\cosh\left(\frac{x+y}{\sqrt{2}}\right) \quad (6.36)$$

for a real-valued nonzero function q. Hence, (6.7) yields

$$\beta = -\alpha_x\cosh\left(\frac{x+y}{\sqrt{2}}\right), \qquad \delta = \cosh\left(\frac{x+y}{\sqrt{2}}\right)(q(y)\cosh\alpha - \alpha_y) \neq 0. \quad (6.37)$$

Case (b.1): $\gamma = 0$. From (6.11), (6.36) and (6.37) we find

$$(\sinh\alpha)_x = \frac{1}{q(y)}\,\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right),$$

which implies that

$$\alpha = \sinh^{-1}\left(p(y) + \frac{\sqrt{2}}{q(y)}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right) \tag{6.38}$$

for some function p. Substituting (6.38) into (6.36) and (6.37), we get

$$\mu = \sqrt{q^2 + \big(pq + \sqrt{2}\tanh((x+y)/\sqrt{2})\big)^2}\,\cosh\left(\frac{x+y}{\sqrt{2}}\right),$$

$$\beta = -\frac{\mathrm{sech}\,\big((x+y)/\sqrt{2}\big)}{\sqrt{q^2 + \big(pq + \sqrt{2}\tanh\big((x+y)/\sqrt{2}\big)\big)^2}},$$

$$\begin{aligned}\delta &= \sqrt{q^2 + \big(pq + \sqrt{2}\tanh\big((x+y)/\sqrt{2}\big)\big)^2}\,\cosh\left(\frac{x+y}{\sqrt{2}}\right)\\ &\quad - \frac{p'q + \mathrm{sech}^2\big((x+y)/\sqrt{2}\big) - \sqrt{2}(\ln q)'\tanh\big((x+y)/\sqrt{2}\big)}{\sqrt{q^2 + \big(pq + \sqrt{2}\tanh\big((x+y/\sqrt{2}\big)\big)^2}}\cosh\left(\frac{x+y}{\sqrt{2}}\right).\end{aligned}$$

Hence, the immersion satisfies

$$\begin{aligned}L_{xx} &= \frac{\Big((p-\mathrm{i})q + \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\Big)\,\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)}{q^2 + \big(pq + \sqrt{2}\tanh\big((x+y)/\sqrt{2}\big)\big)^2}L_x - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_x,\\ L_{xy} &= \frac{-\sqrt{2}\Big((p-\mathrm{i})q + \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\Big)}{q^2 + \big(pq + \sqrt{2}\tanh\big((x+y)/\sqrt{2}\big)\big)^2}\\ &\quad\times\left\{\sqrt{2}(2pq + (\ln q)')\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right.\\ &\quad\left. + 2 + (1+p^2)q^2 - p'q - 3\,\mathrm{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)\right\}L_x,\end{aligned} \tag{6.39}$$

$$L_{yy}=\frac{\lambda\{(p-\mathrm{i})q+\sqrt{2}\tanh\big((x+y)/\sqrt{2}\big)\}}{\sqrt{q^2+\big(pq+\sqrt{2}\tanh\big((x+y)/\sqrt{2}\big)^2}}\operatorname{sech}\left(\frac{x+y}{\sqrt{2}}\right)L_x+(\mathrm{i}+p)qL_y.$$

Solving the second equation in (6.39), we have

$$L=z(y)+e^{\int^y(\mathrm{i}-p)qdy}\int^x\left(1-\mathrm{i}p-\frac{\mathrm{i}\sqrt{2}}{q}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right)\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)B(x)dx \tag{6.40}$$

for some vector functions z and B. By substituting this into the first equation in (6.39), we get $B'=0$. Thus, $B=c_1$ for some $c_1\in\mathbf{C}^2_1$. Hence, (6.40) reduces to

$$L(x,y)=z(y)+\frac{\sqrt{2}c_1(1-\mathrm{i}p(y))}{e^{\int^y(p-\mathrm{i})qdy}}\tanh\left(\frac{x+y}{\sqrt{2}}\right)+\frac{\mathrm{i}c_1\operatorname{sech}^2\big(((x+y)/\sqrt{2}\big)}{q(y)e^{\int^y(p-\mathrm{i})qdy}}. \tag{6.41}$$

From (6.41), we get

$$\begin{aligned}L_x&=c_1\left(1-\mathrm{i}p(y)-\frac{\mathrm{i}\sqrt{2}}{q(y)}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right)e^{\int^y(\mathrm{i}-p)qdy}\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right),\\ L_y&=\frac{\mathrm{i}c_1e^{\int^y(\mathrm{i}-p)qdy}}{q^2(y)}\Bigg\{\sqrt{2}q^2(y)((1+p^2(y))q(y)-p'(y))\tanh\left(\frac{x+y}{\sqrt{2}}\right)\\ &\quad-\left(2p(y)q^2(y)+q'(y)+\sqrt{2}q(y)\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right)\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)\Bigg\}\\ &\quad+z'(y).\end{aligned} \tag{6.42}$$

It follows from (6.1) and (6.42) that $\langle c_1,c_1\rangle=\langle z',z'\rangle=0$ and

$$\langle c_1,z'\rangle=-e^{\int^y pqdy}\cos\left(\int^y qdy\right),\quad \langle \mathrm{i}c_1,z'\rangle=-e^{\int^y pqdy}\sin\left(\int^y qdy\right).$$

Consequently, we obtain case (5) of the theorem.

Case (b.2): $\gamma\neq 0$. It follows from (6.10), (6.36) and (6.37) that

$$(\ln\gamma)_y=\frac{3}{\sqrt{2}}\tanh\left(\frac{x+y}{\sqrt{2}}\right)+\alpha_y\tanh\alpha-2q(y)\sinh\alpha.$$

After solving this equation, we get

$$\begin{aligned}\gamma &= \frac{p(x)\cosh\alpha}{e^{2\int^y q(y)\sinh\alpha dy}}\cosh^3\left(\frac{x+y}{\sqrt{2}}\right),\\ \lambda &= \frac{e^{2\int^y q(y)\sinh\alpha dy}}{p(x)\cosh\big((x+y)/\sqrt{2}\big)}\left(\alpha_x q(y) - \operatorname{sech}\alpha\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)\right)\end{aligned} \tag{6.43}$$

for some real-valued nonzero function p. Hence, it follows from (6.1), (6.2), (6.3), (6.5) (6.36), (6.37) and (6.43) that

$$\begin{aligned} L_{xx} &= (\tanh\alpha - \mathrm{i}\operatorname{sech}\alpha)\alpha_x L_x - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_x \\ &\quad + \frac{p(x)(\mathrm{i}+\sinh\alpha)}{e^{2\int^y q(y)\sinh\alpha dy}}\cosh^2\left(\frac{x+y}{\sqrt{2}}\right)L_y,\\ L_{xy} &= (q(y)\cosh\alpha - \alpha_y)(\mathrm{i}\operatorname{sech}\alpha - \tanh\alpha)L_x, \\ L_{yy} &= (\mathrm{i}+\sinh\alpha)q(y)L_y - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_y \\ &\quad + \frac{(\tanh\alpha - \mathrm{i}\operatorname{sech}\alpha)\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)}{p(x)e^{-2\int^y q(y)\sinh\alpha dy}}\left(\operatorname{sech}\alpha\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) - q(y)\alpha_x\right)L_x.\end{aligned} \tag{6.44}$$

After solving the second equation in (6.44), we find

$$L(x,y) = A(y) + e^{\mathrm{i}\int^y q(y)dy}\int^x \frac{1-\mathrm{i}\sinh\alpha}{e^{\int^y q\sinh\alpha dy}}z(x)dx \tag{6.45}$$

for some vector functions A and z. Thus, we have

$$\begin{aligned} L_x &= (1-\mathrm{i}\sinh\alpha)e^{\int^y q(y)(\mathrm{i}-\sinh\alpha)dy}z(x),\\ L_y &= A'(y) + \mathrm{i}e^{\mathrm{i}\int^y q(y)dy}\int^x \frac{(q(y)\cosh\alpha - \alpha_y)\cosh\alpha}{e^{\int^y q\sinh\alpha dy}}z(x)dx.\end{aligned} \tag{6.46}$$

By substituting (6.45) into the first equation in (6.44), we obtain

$$A' = -\mathrm{i}\int^x \frac{q(y)\cosh\alpha - \alpha_y}{e^{\int^y q(\sinh\alpha - \mathrm{i})dy}\operatorname{sech}\alpha} z(x)dx + \frac{\mathrm{i}e^{\int^y q(\mathrm{i}+\sinh\alpha)dy}}{p(x)\cosh^2\left((x+y)/\sqrt{2}\right)}$$
$$\times\left\{z(x)\int^y q(\sinh\alpha)_x dy - \sqrt{2}z(x)\tanh\left(\frac{x+y}{\sqrt{2}}\right) - z'(x)\right\}. \quad (6.47)$$

By combining this with (6.45), we show that the immersion is congruent to

$$L(x,y) =$$
$$e^{\mathrm{i}\int^y q(y)dy}\int^x \frac{1-\mathrm{i}\sinh\alpha}{e^{\int^y q(y)\sinh\alpha dy}} z(x)dx + \frac{\mathrm{i}z(x)}{p(x)}\int^y \frac{e^{\int^y q(y)(\mathrm{i}+\sinh\alpha)dy}}{\cosh^2\left((x+y)/\sqrt{2}\right)}$$
$$\times\left(\int^y q(y)(\sinh\alpha)_x dy - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right)dy$$
$$-\mathrm{i}\int^y\!\!\int^x \frac{(q(y)\cosh\alpha - \alpha_y)z(x)}{e^{\int^y q(y)(\sinh\alpha - \mathrm{i})dy}\operatorname{sech}\alpha}dxdy - \frac{\mathrm{i}z'(x)}{p(x)}\int^y \frac{e^{\int^y q(y)(\mathrm{i}+\sinh\alpha)dy}}{\cosh^2\left((x+y)/\sqrt{2}\right)}dy.$$
$$(6.48)$$

Also, by substituting (6.47) into (6.46), we find

$$L_y = \frac{\mathrm{i}e^{\int^y q(\mathrm{i}+\sinh\alpha)dy}}{p(x)\cosh^2\left((x+y)/\sqrt{2}\right)}$$
$$\times\left\{z(x)\left(\int^y q(\sinh\alpha)_x dy - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\right) - z'(x)\right\}.$$

This, together with (6.1) and (6.46), implies that $\langle z,z\rangle = \langle z',z'\rangle = 0$, $\langle z,\mathrm{i}z'\rangle = p$. Consequently, we obtain case (6) of the theorem. □

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